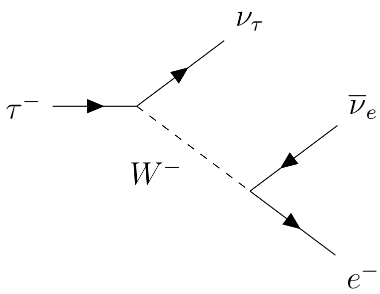
Problem 1

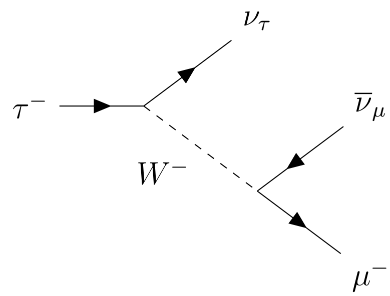
(a) The two largest branching ratio leptonic decays are:

$\tau^- \to \nu_\tau + e^- + \overline{\nu}_e$



$\Gamma_{\tau\to e} = (17.82 \pm 4.00)\,\%$

$\tau^- \to \nu_\tau + \mu^- + \overline{\nu}_\mu$



$\Gamma_{\tau\to \mu} = (17.39 \pm 4.00)\,\%$

(b) We are given the rate for muon decays:

$$\Gamma_{\mu\to e} = \left(\frac{m_\mu g_w}{M_W}\right)^4 \frac{m_\mu c^2}{12\hbar(8\pi)^3}$$

In calculating this rate, we assumed that $m_\mu \gg m_e$. Thus, for the $\tau \to \mu$ decay, we can use this same equation, as $m_\tau \gg m_\mu$:

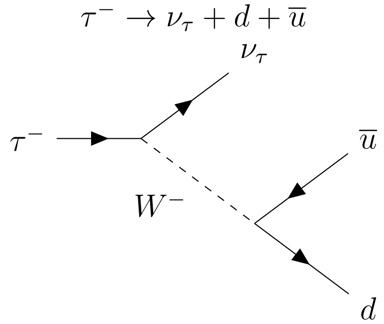
$$\Gamma_{\tau\to \mu} = \left(\frac{m_\tau g_w}{M_W}\right)^4 \frac{m_\tau c^2}{12\hbar(8\pi)^3}$$

The same goes for the $\tau \to e$ decay, since $m_\tau \gg m_e$. Summing them gives the total rate of leptonic tau decays:

$$\boxed{\Gamma_{\tau\to l} = \left(\frac{m_\tau g_w}{M_W}\right)^4 \frac{m_\tau c^2}{6\hbar(8\pi)^3} \sim 2\left(\frac{m_\tau}{m_\mu}\right)^5 (\Gamma_{\mu\to e})} \tag{1b}$$

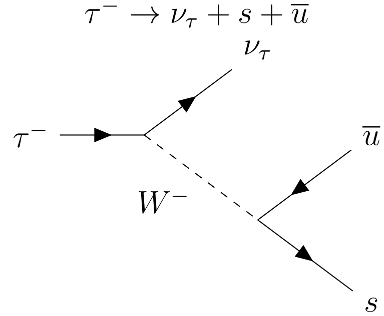
(c) The two diagrams are essentially the same, except with quarks instead of the final leptons:

$\tau^- \to \nu_\tau + d + \overline{u}$



(1)

$\tau^- \to \nu_\tau + s + \overline{u}$



(2)

(d) The $\overline{u}d$ vertex in (1) adds a factor of $\cos\theta_C$ to $\mathcal{M}$, and the $\overline{u}s$ vertex in (2) adds a factor of $\sin\theta_C$. Since $\mathcal{M}$ gets squared, the rates become (without accounting for color):

$$\Gamma_1 = \cos^2\theta_c \Gamma_{\tau\to\mu} \qquad \Gamma_2 = \sin^2\theta_c \Gamma_{\tau\to\mu}$$

However, there are 3 colors, so we have to multiply the rates by 3. Thus:

$$\boxed{\frac{\Gamma_1}{\Gamma_{\tau\to\mu}} = 3\cos^2\theta_C \qquad\qquad \frac{\Gamma_2}{\Gamma_{\tau\to\mu}} = 3\sin^2\theta_C} \tag{1d}$$

(e) The total rate is just the sum of each individual rate:

$$\begin{aligned}\Gamma_{\text{tot}} &= \Gamma_{\tau\to e} + \Gamma_{\tau\to\mu} + \Gamma_1 + \Gamma_2 \\ &= \left[1 + 1 + 3\cos^2\theta_C + 3\sin^2\theta_C\right]\Gamma_{\tau\to\mu} \\ \Gamma_{\text{tot}} &= \boxed{5\Gamma_{\tau\to\mu}}\end{aligned}$$

The expected branching ratios are:

$$\boxed{\frac{\Gamma_{\text{leptonic}}}{\Gamma_{\text{tot}}} = \frac{2}{5} = 40\,\% \qquad\qquad \frac{\Gamma_{\text{hadronic}}}{\Gamma_{\text{tot}}} = \frac{3}{5} = 60\,\%} \tag{1e}$$

These compare fairly well to the experimental values. The leptonic rate is $17.4 + 17.8 = 35.2\,\%$ and the hadronic rate is $100 - 35.2 \sim 65\,\%$, so it seems we are off by about $5\,\%$ on each decay mode.

(f) We know that $\Gamma_{\tau\to\mu} = (m_\tau/m_\mu)^5(\Gamma_{\mu\to e})$ (from Problem 1b), so:

$$\begin{aligned}\Gamma_{\text{tot}} &= 5\Gamma_{\tau\to\mu} \\ &= 5\left(\frac{m_\tau}{m_\mu}\right)^5(\Gamma_{\mu\to e}) \\ &= 5(16.8)^5\left(\frac{1}{2.2\times10^{-6}\,\text{s}}\right) \\ &= 3.06\times10^{12}\,\text{s}^{-1}\end{aligned}$$

The lifetime is:

$$\boxed{\tau = \frac{1}{\Gamma} = 327\times10^{-15}\,\text{s}} \tag{1f}$$

This is reasonably close to the observed lifetime of $(290.3 \pm 0.5)\times10^{-15}\,\text{s}$.

Problem 2

The W must carry off the extra momentum between the p and n, as $\mathbf{p}_p = \mathbf{p}_n + \mathbf{p}_W$. Assuming the neutron is at rest, the maximum W momentum occurs when $\mathbf{p}_p$ is at a maximum. This occurs when the proton and electron are back-to-back, so the neutrino has no energy. We can treat this then as the two-body decay $n \to e^- + p$ (since the neutrino is essentially massless), for which we have already found the momentum of the decay products as a practice problem in Week 3.

The proton momentum is:

$$
\begin{aligned}
|\mathbf{p}_p| &= \frac{\sqrt{\lambda(m_n^2, m_e^2, m_p^2)}c}{2m_n} \\
&= \frac{\sqrt{(939.6)^4 + (0.511)^4 + (938.3)^4 - 2[(939.6)^2(0.511)^2 + (939.6)^2(938.3)^2 + (938.3)^2(0.511)^2]}}{2(939.6)} \\
&= \frac{2244.761\,\mathrm{MeV}^2\,\mathrm{c}^{-1}}{2(939.6\,\mathrm{MeV\,c}^{-2})} \\
|\mathbf{p}_p| &= 1.194\,\mathrm{MeV\,c}^{-1} = |\mathbf{p}_W|
\end{aligned}
$$

The de Broglie wavelength is:

$$
\begin{aligned}
\lambda_W &= \frac{h}{|\mathbf{p}_W|} \\
&= \frac{2\pi\hbar c}{|\mathbf{p}_W|c} \\
&= \frac{2\pi(197.326\,\mathrm{MeV\,fm})}{(1.194\,\mathrm{MeV})} \\
\lambda_W &= \boxed{1038\,\mathrm{fm}}
\end{aligned} \tag{2}
$$

This is about 1000 times the diameter of the neutron, so the W really can't see inside neutrons or protons – allowing us to drop the form factor.

Problem 3

The rate for $K^- \to l^- + \overline{\nu}_l$ was given in class:

$$
\Gamma(K^- \to l^- + \overline{\nu}_l) = \frac{f_K^2 \sin^2\theta_C}{\pi\hbar m_K^2}\left(\frac{g_W}{4M_W}\right)^4 m_l^2\left(m_K^2 - m_l^2\right)^2
$$

Calculating the ratio, we note that the constants in front vanish, so:

$$
\begin{aligned}
\frac{\Gamma(K^- \to e^- + \overline{\nu}_e)}{\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)} &= \frac{m_e^2(m_K^2 - m_e^2)^2}{m_\mu^2\left(m_K^2 - m_\mu^2\right)^2} \\
&= \frac{(0.511\,\mathrm{MeV\,c}^{-2})^2\left((493.7\,\mathrm{MeV\,c}^{-2})^2 - (0.511\,\mathrm{MeV\,c}^{-2})^2\right)^2}{(105.7\,\mathrm{MeV\,c}^{-2})^2\left((493.7\,\mathrm{MeV\,c}^{-2})^2 - (105.7\,\mathrm{MeV\,c}^{-2})^2\right)^2} \\
\frac{\Gamma(K^- \to e^- + \overline{\nu}_e)}{\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)} &= \boxed{2.567 \times 10^{-5}}
\end{aligned} \tag{3}
$$

The PDG booklet gives:

$$
\frac{\Gamma(K^- \to e^- + \overline{\nu}_e)}{\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)} = \frac{1.582 \times 10^{-5}}{0.6356} = 2.48 \times 10^{-5}
$$

so our estimate agrees with experiment very well.

Problem 4

(b) For the top quark to be a possible decay product, its mass has to be less than half of the Z's mass (since it would be $Z_0 \to t + \bar{t}$). The t mass is $\sim 170\,\mathrm{GeV\,c^{-2}}$, which is clearly more than half of the Z_0 mass of $\sim 90\,\mathrm{GeV}$. Thus, $\boxed{\text{the } t \text{ decay is not possible.}}$ I'll first write out the values of c_V and c_A:

f	c_V	c_A	$(\lvert c_V\rvert^2 + \lvert c_A\rvert^2)_f$
ν_e, ν_μ, ν_τ	0.5	0.5	0.5
e^-, μ^-, τ^-	-0.0372	-0.5	0.2514
u, c	0.1915	0.5	0.2867
d, s, b	-0.3457	-0.5	0.3695

Accounting for color, the total decay rate goes as:

$$\Gamma_{\text{tot}} \propto 3(0.5) + 3(0.2514) + 3 \times 2(0.2867) + 3 \times 3(0.3695) \approx 7.3$$

The branching ratios are:

f	$\Gamma_f/\Gamma_{\text{tot}}$
ν_e, ν_μ, ν_τ	3.4 %
e^-, μ^-, τ^-	6.9 %
u, c	11.8 %
d, s, b	15.2 %

(4b)

(c) The total decay rate is:

$$\Gamma_{\text{tot}} = \frac{g_z^2 M_Z c^2}{48\pi\hbar}(7.3)$$

The coupling factor is given by Equation 9.91:

$$g_z = \frac{g_e}{\sin\theta_w \cos\theta_w} = 0.718$$

The total decay rate is then:

$$\begin{aligned}\Gamma_{\text{tot}} &= \frac{g_z^2 M_Z c^2}{48\pi\hbar}(7.3) \\ &= \frac{(0.718)^2(91\,188\,\mathrm{MeV})}{48\pi(6.58 \times 10^{-22}\,\mathrm{MeV\,s})}(7.3) \\ &= 3.458 \times 10^{24}\,\mathrm{s^{-1}}\end{aligned}$$

Inverting this gives the lifetime:

$$\boxed{\tau = 2.891 \times 10^{-25}\,\mathrm{s}} \tag{4c}$$

Adding another generation of quarks/leptons would just add another factor of the $(\lvert c_V\rvert^2 + \lvert c_A\rvert^2)$ for that particular type of particle to the decay rate. This would make the lifetime decrease.