UGP-Time Series

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1 Introduction

Definition 1.1. *A time series* $\{X_t\}$ *is an ordered collection (indexed by time) of random variables.*

Definition 1.2. *A time series is called* ***stationary*** *if the joint distribution of any finite collection of data points is time-independent. Formally, for all positive integers* $h, n_1, n_2, \ldots, n_k$*,*

$$X_{n_1}, X_{n_2}, \ldots, X_{n_k} \stackrel{d}{=} X_{n_1+h}, X_{n_2+h}, \ldots, X_{n_k+h}.$$

Definition 1.3. *A time series is called* ***covariance-stationary*** *if the following conditions are satisfied for all positive integers* $h, n, n_1,$ *and* n_2*:*

$$\mathbb{E}[X_n] = \mu < \infty$$

$$Cov(X_{n_1}, X_{n_2}) = Cov(X_{n_1+h}, X_{n_2+h}) < \infty$$

All stationary time series are covariance-stationary, but the converse is not generally true.

Definition 1.4. *A time series is called a* ***Gaussian process*** *if the joint distribution of any finite collection of data points follows a multivariate normal distribution.*

Theorem 1.1. *A covariance-stationary Gaussian process is stationary.*

Definition 1.5. *For a covariance-stationary time series, the* ***autocovariance function*** *is defined as:*

$$\rho(h) = Cov(X_n, X_{n+h}),$$

where h *is called the* ***lag****.*

Definition 1.6. *White noise* ε_t *is a sequence of uncorrelated random variables with zero mean and constant variance* σ^2*.*

1.1 Time series Decomposition

Usually, a time series is broken down into 3 components.

1. **Trend Component** (m_t) It is the long-term tendency of a time series to show average growth or decline as a function of time.

2. **Seasonal Component** (s_t) It is the distinguishable periodic(most commonly annual) variation in a time series.

3. **Random Component** (e_t) It is the stochastic component of the time series

A general approach in time series analysis is to decompose the time series into individual components and then fit a suitable model on the random component.

Decomposition can be done in an additive, multiplicative, or hybrid manner.

Additive Decomposition $X_t = m_t + s_t + e_t$

Multiplicative Decomposition $X_t = m_t * s_t * e_t$

1.2 ARIMA Model Explaination

1.2.1 AR Process

Definition 1.7. *A time series $\{X_t\}$ is said to follow an **AR(p)** process if it satisfies the equation:*

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \varepsilon_t$$

*where $\phi_1, \ldots, \phi_p$ are the **parameters** of the model, and ε_t is **white noise**.*

Here p is called the order of the AR process.

The $AR(p)$ process can be equivalently written using the backshift operator B as:

$$X_t = \sum_{i=1}^{p} \phi_i B^i X_t + \varepsilon_t$$

where the backshift operator B is defined by:

$$B^i X_t = X_{t-i}$$

1.2.2 MA Process

Definition 1.8. *A time series $\{X_t\}$ is said to follow an **MA(q)** process if it satisfies the equation:*

$$X_t = \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

where $\theta_1, \ldots, \theta_q$ are the coefficients of the model, and ε_t's are white noise error terms.

Here q is called the order of the MA process. The MA(q) process can be equivalently written in terms of the backshift operator B as:

$$X_t = (1 + \sum_{i=1}^{q} \theta_i B^i)\varepsilon_t.$$

1.2.3 ARMA Process

Definition 1.9. *A time series $\{X_t\}$ is said to follow an **ARMA(p,q)** process if it satisfies the equation:*

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

where $\phi_1, \ldots, \phi_p$, $\theta_1, \ldots, \theta_q$ are the coefficients of the model, and ε_t are white noise the error terms.

Here p, q are called the order of the ARMA process. The ARMA(p,q) process can be equivalently written in terms of the backshift operator B as:

$$(1 - \sum_{i=1}^{p} \phi_i B^i)X_t = (1 + \sum_{i=1}^{q} \theta_i B^i)\varepsilon_t$$

or

$$\Phi(B)X_t = \Theta(B)\varepsilon_t$$

Theorem 1.2. *All MA(q) processes of finite order q are covariance stationary and $\rho(h) = 0$ for $|h| > q$*

Theorem 1.3. *An AR(p) process $\{X_t\}$ defined by $\Phi(B)X_t = \varepsilon_t$ is covariance stationary if the roots of the polynomial $\Phi(z) = 0$ lie outside the unit circle.*

Theorem 1.4. *An ARMA(p, q) process $\{X_t\}$ defined by $\Phi(B)X_t = \Theta(B)\varepsilon_t$ is covariance stationary if the roots of the polynomial $\Phi(z) = 0$ lie outside the unit circle.*

1.2.4 ARIMA Process

We define the differencing operator ∇ as $\nabla X_t = X_t - X_{t-1}$ or $\nabla = 1 - B$.

Definition 1.10. *A time series X_t is said to be an ARIMA(p, d, q) process if $Y_t = \nabla^d X_t$ is a stationary ARMA(p, q) process.*

1.3 Seasonal Adjustment

Seasonal adjustment or deseasonalization is a statistical method for removing the seasonal component of a time series. It is usually done when wanting to analyse the trend, and cyclical deviations from trend, of a time series independently of the seasonal components. Many economic phenomena have seasonal cycles, such as agricultural production, (crop yields fluctuate with the seasons) and consumer consumption (increased personal spending leading up to Christmas). It is necessary to adjust for this component in order to understand underlying trends in the economy, so official statistics are often adjusted to remove seasonal components.

1.3.1 Filter based methods

[2] This method applies a set of fixed filters (moving averages) to decompose the time series into a trend, seasonal and irregular (stochastic) component.

Filter-based methods are often known as X11 style methods. These include X11 (developed by U.S Census Bureau), X11ARIMA (developed by Statistics Canada), X12ARIMA (developed by U.S Census Bureau), STL, SABL, and SEASABS (the package used by the ABS)..

Computational differences between various methods in X11 family are chiefly the result of different techniques used at the ends of the time series. For example, some methods use asymmetric filters at the ends, while other methods extrapolate the time series and apply symmetric filters to the extended series.

1.3.2 Model based methods

[2] This approach requires the time series's trend, seasonal and irregular components to be modelled separately. It assumes the irregular component is "white noise" - that is all cycle lengths are equally represented. The irregulars have zero mean and a constant variance. The seasonal component has its own noise element.

STAMP and SEATS/TRAMO (developed by the Bank of Spain) are two widely used software packages that apply model-based methods.

2 X13-ARIMA-SEATS

[1]X-13ARIMA-SEATS is a software for performing seasonal adjustments to time series data developed and implemented by the US Census Bureau. It is based on the U.S. Census Bureau's earlier X-11 program, the X-11-ARIMA program developed at Statistics Canada, the X-12-ARIMA program developed by the U.S. Census Bureau, and the SEATS program developed at the Bank of Spain.

X-13ARIMA-SEATS offers two algorithms to perform seasonal adjustment

1. Model-based: Signal Extraction in ARIMA Time Series (SEATS)

2. Filter based: Enhanced version of the X-11 method.

2.1 TRAMO (Time Series Regression with ARIMA Noise, Missing Observations, and Outliers)

TRAMO is responsible for pre-processing the time series by identifying and correcting outliers, modeling calendar effects (e.g., trading day effects), and handling missing observations. It then fits an ARIMA model to the cleaned data.
Given a time series $\{y_t\}$ with $t = 1, 2, \ldots, n$, TRAMO models it as:

$$y_t = x_t'\beta + \epsilon_t$$

where:

- x_t is a vector of regressors (e.g., calendar effects, outliers).
- β is the vector of coefficients.
- ϵ_t is the residual component modeled as an ARIMA process.

The residuals ϵ_t follow an ARIMA model:

$$\Phi(B)\epsilon_t = \Theta(B)a_t$$

where:

- $\Phi(B)$ and $\Theta(B)$ are polynomials in the backshift operator B representing autoregressive (AR) and moving average (MA) components, respectively.
- a_t is white noise with mean zero and variance σ^2.

2.1.1 Outlier Detection

TRAMO detects three main types of outliers:

1. **Additive Outlier (AO)**: A sudden, short-term shock affecting only one or a few observations.
2. **Level Shift (LS)**: A sudden, permanent shift in the level of the time series.
3. **Transitory Change (TC)**: A shock that gradually diminishes over time.

TRAMO identifies outliers by modeling the time series as:

$$y_t = x_t'\beta + \sum_j \omega_j \cdot I_j(t) + \epsilon_t$$

where:

- y_t is the observed time series at time t.
- $x_t'\beta$ represents the deterministic part of the model (including regressors like calendar effects).
- $I_j(t)$ is an indicator variable that equals 1 if an outlier occurs at time t, and 0 otherwise.
- ω_j represents the magnitude of the outlier's impact.
- ϵ_t is the ARIMA noise component.

For each type of outlier, $I_j(t)$ is defined as:

1. **Additive Outlier (AO)**

$$I_j(t) = \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

   The outlier affects the time series only at t_0, causing a temporary deviation.

2. **Level Shift (LS)**

$$I_j(t) = \begin{cases} 1 & \text{if } t \geq t_0 \\ 0 & \text{otherwise} \end{cases}$$

   The outlier causes a permanent shift in the level of the series from t_0 onward.

3. **Transitory Change (TC)**

$$I_j(t) = \begin{cases} \delta^{(t-t_0)} & \text{if } t \geq t_0 \\ 0 & \text{otherwise} \end{cases}$$

   The impact of the outlier diminishes over time, with δ typically between 0 and 1.

Detection Algorithm:

1. **Initial Model Estimation**: TRAMO first fits a preliminary ARIMA model to the time series, assuming no outliers.
2. **Outlier Detection**:
   - The residuals $\hat{\epsilon}_t$ from the fitted model are examined.
   - For each observation, TRAMO computes a test statistic that measures the likelihood of an outlier at that point.
   - The test statistic for an AO, for example, is typically the residual $\hat{\epsilon}_t$ scaled by its standard error.

- If the test statistic exceeds a critical threshold, an outlier is flagged at that time point.

3. **Iterative Refinement**:

   - Once an outlier is detected, it is included as an additional regressor in the model.
   - The ARIMA model is re-estimated with the outlier-adjusted series.
   - The process iterates, detecting and incorporating additional outliers until no significant outliers remain.

4. **Final Model**:

   - The final ARIMA model includes terms for all detected outliers, which are now accounted for in the model.

For an Additive Outlier (AO) detected at time t_0:

$$y_t = x_t'\beta + \omega \cdot I_{AO}(t) + \epsilon_t$$

where:

$$I_{AO}(t) = \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

After detecting an AO at t_0, the model is re-estimated to include this AO, adjusting the coefficients β and ω accordingly.

2.1.2 Estimation of Missing Values

The ARIMA model can be represented in a state-space form, which is essential for applying the Kalman filter. The state-space representation consists of two equations:

State Equation

$$\mathbf{x}_{t+1} = \mathbf{F}_t\mathbf{x}_t + \mathbf{G}_t\mathbf{w}_t$$

where:

- $\mathbf{x}_t$ is the state vector at time t, which includes both observed and missing values.
- $\mathbf{F}_t$ is the state transition matrix that defines how the state evolves over time.
- $\mathbf{G}_t$ is the control matrix that affects the state evolution.
- $\mathbf{w}_t$ is the process noise, assumed to be normally distributed with mean zero and covariance $\mathbf{Q}_t$.

Observation Equation

$$y_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{v}_t$$

where:

- y_t is the observed value at time t (or missing data if y_t is not observed).
- $\mathbf{H}_t$ is the observation matrix that links the state vector to the observed data.
- $\mathbf{v}_t$ is the observation noise, assumed to be normally distributed with mean zero and covariance $\mathbf{R}_t$.

Kalman Filter Application
The Kalman filter is applied in a two-step recursive process: prediction and update.

1. **Prediction Step** The filter predicts the state at time $t+1$ given the state at time t:
$$\hat{\mathbf{x}}_{t+1|t} = \mathbf{F}_t \hat{\mathbf{x}}_t$$
The error covariance matrix is also predicted:
$$\mathbf{P}_{t+1|t} = \mathbf{F}_t \mathbf{P}_t \mathbf{F}_t' + \mathbf{G}_t \mathbf{Q}_t \mathbf{G}_t'$$
2. **Update Step** When a new observation y_{t+1} becomes available (or remains missing), the filter updates the state estimate:
$$\hat{\mathbf{x}}_{t+1} = \hat{\mathbf{x}}_{t+1|t} + \mathbf{K}_{t+1} \left(y_{t+1} - \mathbf{H}_{t+1} \hat{\mathbf{x}}_{t+1|t} \right)$$
Here, $\mathbf{K}_{t+1}$ is the Kalman gain matrix:
$$\mathbf{K}_{t+1} = \mathbf{P}_{t+1|t} \mathbf{H}_{t+1}' \left(\mathbf{H}_{t+1} \mathbf{P}_{t+1|t} \mathbf{H}_{t+1}' + \mathbf{R}_{t+1} \right)^{-1}$$
The error covariance matrix is updated as:
$$\mathbf{P}_{t+1} = \left(\mathbf{I} - \mathbf{K}_{t+1} \mathbf{H}_{t+1} \right) \mathbf{P}_{t+1|t}$$

Once the entire time series has been processed through the Kalman filter, a smoothing algorithm can be applied to refine the estimates of the missing values. The Kalman smoother works backward through the time series to adjust the state estimates.After the initial estimation and smoothing, the ARIMA model is re-estimated using the completed time series. This iterative process continues until the estimates converge.

2.2 SEATS Algorithm and Wiener-Kolmogorov Filtering

The SEATS (Signal Extraction in ARIMA Time Series) algorithm is designed to decompose a time series y_t into its unobserved components: trend-cycle (C_t), seasonal (S_t), and irregular (I_t) components. The process involves fitting an ARIMA model to the time series and using this model to extract the components.

1. Decomposition

The time series y_t is decomposed as:

$$y_t = C_t + S_t + I_t$$

where:

- C_t is the **trend-cycle component** capturing long-term movements and cycles.
- S_t is the **seasonal component** reflecting regular, periodic fluctuations.
- I_t is the **irregular component** representing noise or residuals.

Each component is modeled using ARIMA processes derived from the original time series model.

2. Model Specification

The ARIMA model for the time series y_t is generally given by:

$$\Phi(B)y_t = \Theta(B)a_t$$

where:

- $\Phi(B)$ is the autoregressive polynomial.
- $\Theta(B)$ is the moving average polynomial.
- a_t is white noise with mean zero and variance σ^2.

For each component, SEATS specifies corresponding ARIMA models:

- **Trend-Cycle Component**:

$$\Phi_C(B)C_t = \Theta_C(B)a_{C,t}$$

- **Seasonal Component**:

$$\Phi_S(B)S_t = \Theta_S(B)a_{S,t}$$

- **Irregular Component**:

$$\Phi_I(B)I_t = \Theta_I(B)a_{I,t}$$

Wiener-Kolmogorov Filters

Wiener-Kolmogorov filters are used to extract the components by maximizing the signal-to-noise ratio in the frequency domain.

Spectral Density

The spectral density $F(\omega)$ of a time series y_t is defined as:

$$F(\omega) = \frac{1}{2\pi} \left| \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\omega k} \right|$$

where:

- $\gamma(k)$ is the autocovariance function of y_t at lag k.
- ω is the frequency.

Filter for Trend-Cycle Component $W_C(\omega)$

$$W_C(\omega) = \frac{|F_C(\omega)|^2}{|F_C(\omega)|^2 + |F_S(\omega)|^2 + |F_I(\omega)|^2}$$

where:

- $F_C(\omega)$, $F_S(\omega)$, and $F_I(\omega)$ are the spectral densities of the trend-cycle, seasonal, and irregular components at frequency ω.

Filter for Seasonal Component $W_S(\omega)$

$$W_S(\omega) = \frac{|F_S(\omega)|^2}{|F_C(\omega)|^2 + |F_S(\omega)|^2 + |F_I(\omega)|^2}$$

Filter for Irregular Component $W_I(\omega)$

$$W_I(\omega) = \frac{|F_I(\omega)|^2}{|F_C(\omega)|^2 + |F_S(\omega)|^2 + |F_I(\omega)|^2}$$

Summary

The SEATS algorithm decomposes a time series by:

- Fitting an ARIMA model to the series.
- Deriving separate ARIMA models for trend-cycle, seasonal, and irregular components.
- Applying Wiener-Kolmogorov filters to extract these components, with each filter designed to maximize the extraction accuracy by balancing the spectral densities of the components.

References

[1] JDemetra. https://jdemetradocumentation.github.io/JDemetra-documentation/pages/theory/index.html.

[2] Australian Bureau of Statistics. https://www.abs.gov.au/websitedbs/d3310114.nsf/4a256353001af3ed4b2562bb00121564/5fc845406def2c3dca256ce100188f8e.