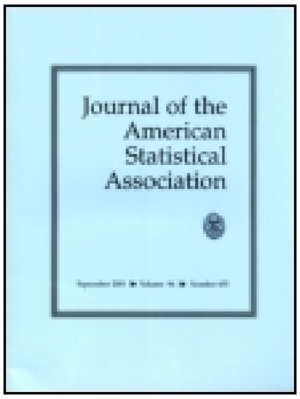


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# Modeling Time Series with Calendar Variation

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# Modeling Time Series With Calendar Variation

W. R. BELL and S. C. HILLMER*

The modeling of time series data that include calendar variation is considered. Autocorrelation, trends, and seasonality are modeled by ARIMA models. Trading day variation and Easter holiday variation are modeled by regression-type models. The overall model is a sum of ARIMA and regression models. Methods of identification, estimation, inference, and diagnostic checking are discussed. The ideas are illustrated through actual examples.

KEY WORDS: Calendar variation; Trading day variation; Easter holiday variation; ARIMA models; Monthly time series.

## 1. INTRODUCTION

Suppose we observe a time series $Z_t$ that follows the model (perhaps after transformation)

$$Z_t = f(\mathbf{X}_t;\boldsymbol{\xi}) + N_t. \tag{1.1}$$

Here $f$ is a function of $\boldsymbol{\xi}$, a vector of parameters, and of $\mathbf{X}_t$, a vector of fixed independent variables observed at time $t$, and $N_t$ is a noise series. If $N_t$ is white noise, then (1.1) is the familiar linear or nonlinear regression model. However, when one deals with time series, $N_t$ will generally be autocorrelated and frequently nonstationary. Numerous authors have warned against the consequences of using standard regression theory when $N_t$ is autocorrelated, the problem being well established as long ago as Anderson (1954).

In this article we are concerned with the converse problem—that of the effects of ignoring $f(\mathbf{X}_t;\ \boldsymbol{\xi})$ when analyzing a time series. In the particular case we consider, $f(\mathbf{X}_t;\ \boldsymbol{\xi})$ represents trading day and holiday effects. For this case we illustrate the important points that (a) pure ARIMA models should not be applied blindly to all time series, (b) to ignore known, relevant independent variables is to invite difficulties, and (c) substantial improvements in models can be obtained when relevant independent variables are incorporated in the model.

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## 2. MODEL-BUILDING PROCEDURES

In developing models of the form (1.1) for a specific set of data we follow the three-stage model-building procedure of identification, estimation, and diagnostic checking presented in Box and Jenkins (1976). In (1.1) we assume that $N_t$ follows the ARIMA model

$$\phi(B)\delta(B)N_t = \theta(B)a_t, \tag{2.1}$$

where $B$ is the backshift operator ($BN_t = N_{t-1}$), $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ have all their zeros outside the unit circle, $\phi(B)$ and $\theta(B)$ have no common zeroes, $\delta(B)$ is a differencing operator (all zeroes on the unit circle) such as $(1 - B)$ or $(1 - B)(1 - B^{12})$, and $\{a_t\}$ is a sequence of independent, identically distributed (iid) random variables with mean 0 and variance $\sigma^2$. Some of the $\phi$'s and $\theta$'s may be 0 or otherwise constrained, so that (2.1) could be a multiplicative seasonal model.

### 2.1 Model Identification

The regression portion of the model, $f(\mathbf{X}_t;\ \boldsymbol{\xi})$, can be identified by consideration of the nature of the independent variables, which in our case are describing the trading day or holiday variation. To identify the noise model (2.1) we first examine the sample autocorrelation function (SACF) of the time series $Z_t$. In our experience with series containing trading day or holiday variation, examination of the SACF of $Z_t$ is useful for determining the degree of differencing, $\delta(B)$, in $N_t$. We believe this is so because the effect of the nonstationary $N_t$ on the computed sample autocorrelations dominates the effect of the trading day or holiday variation. In contrast, after $Z_t$ (and thus $N_t$) has been appropriately differenced, the effect of the differenced $N_t$ on the computed sample autocorrelations no longer dominates the effect of the differenced $f(\mathbf{X}_t;\ \boldsymbol{\xi})$. The SACF and sample partial autocorrelation function (SPACF) of the differenced $Z_t$ series are usually confused. At this stage we must at least approximately remove the effects of $f(\mathbf{X}_t;\ \boldsymbol{\xi})$ from $Z_t$. To do this we fit the model

$$\delta(B)Z_t = \delta(B)f(\mathbf{X}_t;\ \boldsymbol{\xi}) + e_t \tag{2.2}$$

by least squares regression (linear or nonlinear) and examine the SACF and SPACF of the residuals from this regression in order to tentatively identify the noise model. A justification for this procedure is that the sample au-


September 1983, Volume 78, Number 383
Applications Section

tocorrelations and hence the sample partial autocorrelations of the residuals from the least squares fit of (2.2) differ from those of $\delta(B)N_t$ by an amount that converges in probability to zero (see Fuller 1976, p. 399). This procedure is illustrated by two examples later in this article.

## 2.2 Model Estimation

Combining (1.1) and (2.1), we can write our model as

$$\delta(B)Z_t = \delta(B)f(\mathbf{X}_t;\, \boldsymbol{\xi}) + \frac{\theta(B)}{\phi(B)}\, a_t. \qquad (2.3)$$

We can then estimate $\boldsymbol{\xi}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, in (2.3) by maximum likelihood methods assuming normality of the $a_t$'s We estimate $\sigma^2$ by $\hat{\sigma}^2 = (n - r)^{-1}\sum \hat{a}_t^2$ where $n$ is the number of observations less the degree of $\delta(B)\phi(B)$, $r$ is the number of parameters in (2.3), and

$$\hat{a}_t = \hat{\theta}(B)^{-1}\hat{\phi}(B)\delta(B)[Z_t - f(\mathbf{X}_t,\, \hat{\boldsymbol{\xi}})].$$

Since the model for $N_t$ is invertible this is asymptotically equivalent to nonlinear least squares.

Pierce (1971) discusses inference for the model (1.1) for the case in which $f(\mathbf{X}_t;\, \boldsymbol{\xi})$ is linear in $\boldsymbol{\xi}$. He shows that under some conditions on the $a_t$'s and the $\mathbf{X}_t$'s that the least squares estimates $\hat{\boldsymbol{\nu}} = (\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}})$ are consistent and asymptotically normal, $\hat{\boldsymbol{\xi}}$ is asymptotically independent of $(\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}})$, and $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2$. Also, the $(i, j)$th element of the inverse of the asymptotic covariance matrix of $\hat{\boldsymbol{\nu}}$ can be approximated numerically by $-(\partial^2 L/\partial\nu_i\partial\nu_j) \mid \hat{\boldsymbol{\nu}}$, where $L$ is the log-likelihood. Hannan (1971) and Gallant and Goebel (1976) obtain results analogous to those of Pierce for the case in which $f(\mathbf{X}_t;\, \boldsymbol{\xi})$ is nonlinear in $\boldsymbol{\xi}$, although they do not explicitly consider the asymptotic properties of $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$. They require the additional assumptions of continuity of $f(\mathbf{X}_t;\, \boldsymbol{\xi})$ for the consistency of $\hat{\boldsymbol{\xi}}$ (Hannan 1971) and twice differentiability for the asymptotic normality.

## 2.3 Diagnostic Checking

In general, the adequacy of both the assumed formulation of $f(\mathbf{X}_t;\boldsymbol{\xi})$ and the assumed noise model $\phi(B)\delta(B)N_t = \theta(B)a_t$ should be checked. To check the form of $f(\mathbf{X}_t;\, \boldsymbol{\xi})$ the residuals, $\hat{a}_t$, can be plotted against the $X_{it}$ and any other possible independent variables. The $\hat{a}_t$ should be plotted against time to check for outliers, constancy of variance, and trends. The SACF of the residuals should be examined for any large autocorrelations. Ljung and Box (1978) show that under the hypothesis that the model is correct, for large $n$ the statistic

$$Q = n(n + 2) \sum_{k=1}^{L} r_k(\hat{a})^2/(n - k)$$

has approximately a $\chi^2(L - s)$ distribution, where $r_k(\hat{a})$ is the lag $k$ sample autocorrelation of $\hat{a}_t$, and $s$ equals the number of parameters in the noise model. The noise model is judged inadequate if $Q$ exceeds $\chi^2_{\gamma}(L - s)$ for some suitable $\gamma$.

## 3. TRADING DAY AND HOLIDAY VARIATION

The variation in a monthly time series that is due to the changing number of times each day of the week occurs in a month is called *trading day variation*. Trading day variation occurs when the activity of a business or industry varies with the days of the week so that the activity for a particular month partially depends on which days of the week occur five times. In addition, Young (1965) notes that accounting and reporting practices can create trading day effects in a time series. For example, stores that perform their bookkeeping activities on Fridays tend to report higher sales in months with five Fridays than in months with four Fridays. *Holiday variation* refers to fluctuations in economic activity due to changes from year to year in the composition of the calendar with respect to holidays. The primary example of this for U.S. economic series is the increased buying that takes place in some retail sales series just before Easter. This is a holiday effect since Easter falls on various dates in March and April. Holiday effects must be distinguished from seasonal effects, which are attributable to the same month every year. For instance, the increase in retail sales in December prior to Christmas each year is a seasonal effect and not a holiday effect.

Almost all of the previous research on trading day and holiday effects has dealt with their relation to seasonal adjustment. Young (1965) describes the procedures that are used in the Census X-11 seasonal adjustment program to adjust time series for trading day variation, and briefly discusses the adjustments made for holiday variation. Cleveland and Devlin (1980,1982) have reported on methods to identify times in which trading day effects are present in a time series and on methods to remove these effects. Pfefferman and Fisher (1980) discuss adjustments for both trading day and holiday variation. All of these authors use a two-stage approach in which a regression model is fitted to data that have been preprocessed to remove the trend and seasonality. We prefer to postulate a model of the form (1.1) and ARIMA noise structure and simultaneously estimate the regression and ARIMA parameters. Once a model of the form (1.1) has been developed, it can be used for a variety of purposes including forecasting and seasonal adjustment.

## 4. MODELING TRADING DAY VARIATION IN TIME SERIES

Trading day variation arises in part because the activity for a monthly time series varies with the days of the week. We assume that trading day effects can be approximated by a deterministic model. We deal only with flow series for which the data are the accumulation of the daily values (flows) over the calendar months. (Cleveland and Grupe (1982) discuss modeling of trading day effects for other types of series such as stock series, e.g., inventories.) If $\xi_i$, $i = 1, \ldots, 7$, represent the average rates of activity on Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday for the series being modeled (i.e., the daily effects), then the effect attributable to the number of times each day of the week occurs in month $t$ is

$$TD_t = \sum_{i=1}^{7} \xi_i X_{it}, \tag{4.1}$$

where $X_{it}$, $i = 1, \ldots, 7$, are, respectively, the number of Mondays, Tuesdays, and so on in month $t$. A similar model was used by Young (1965), Cleveland and Devlin (1982), and Pfefferman and Fisher (1980). The model (4.1) accounts for variations in level due to differing month lengths, and allows for variations in level due to differing day of the week compositions for months of the same length. A model for the time series that incorporates trading day effects is

$$Z_t = TD_t + N_t, \tag{4.2}$$

where $TD_t$ is as defined in (4.1) and $N_t$ as in (2.1).

We obtain a useful reparameterization of (4.1) as follows. Let $\bar{\xi} = 1/7 \sum_{i=1}^{7} \xi_i$, $T_{it} = X_{it} - X_{7t}$, $i = 1, \ldots, 6$, and let $T_{7t} = \sum_{i=1}^{7} X_{it}$ denote the length of month $t$. Then we can write (4.1) as

$$\begin{aligned} TD_t &= \sum_{i=1}^{7} (\xi_i - \bar{\xi})(X_{it} - X_{7t}) \\ &\quad + \sum_{i=1}^{7} (\xi_i - \bar{\xi})X_{7t} + \bar{\xi} \sum_{i=1}^{7} X_{it} \\ &= \sum_{i=1}^{7} \beta_i T_{it}, \end{aligned} \tag{4.3}$$

where $\beta_i = \xi_i - \bar{\xi}$, $i = 1, \ldots, 6$, and $\beta_7 = \bar{\xi}$. Our model then becomes

$$Z_t = \sum_{i=1}^{7} \beta_i T_{it} + \frac{\theta(B)}{\phi(B)\delta(B)} a_t. \tag{4.4}$$

We get the same estimate for $\mathrm{TD}_t$ whether we use the parameterization (4.1) or (4.3); however, we have observed that estimates of the $\xi_i$'s tend to be highly correlated while estimates of $\beta_1, \ldots, \beta_6$ are less so and are not highly correlated with the estimate of $\beta_7$. The parameters $\beta_i = \xi_i - \bar{\xi}$, $i = 1, \ldots, 6$, measure the differences between the Monday, Tuesday, . . . , Saturday effects and the average of the daily effects, $\beta_7 = \bar{\xi}$. The difference between the Sunday effect and the average of the daily effects is then

$$\begin{aligned} \xi_7 - \bar{\xi} &= \sum_{1}^{7} \xi_i - \bar{\xi} - \sum_{1}^{6} \xi_i \\ &= 6\bar{\xi} - \sum_{1}^{6} (\beta_i + \bar{\xi}) = -\sum_{1}^{6} \beta_i, \end{aligned}$$

and one may solve for the Sunday effect, $\xi_7$, using $\beta_7 - \sum_1^6 \beta_i$.

### 4.1 An Example

As an example, consider the series retail sales of lumber and building materials from January 1967 to September 1979, (the data for which may be obtained from the U.S. Census Bureau). Examination of a plot of the series reveals that the amplitude of the seasonality increases with the level. Therefore, we have determined that it is appropriate to model the natural logarithms, which we denote by $Z_t$. Examination of the SACF of the logged data and the SACF of the first differenced logged data indicated that first and twelfth differences are needed to achieve stationarity. The SACF of $(1 - B)(1 - B^{12})Z_t$ together with plus and minus two standard error limits are reported in Figure 1. Figure 1 does not exhibit a recognizable pattern. In order to identify the noise model we note that (2.2) for this example can be written

$$(1 - B)(1 - B^{12})Z_t = \sum_{1}^{7} \beta_i(1 - B)(1 - B^{12})T_{it} + e_t,$$

so we examine the SACF of the residuals from the regression of $(1 - B)(1 - B^{12})Z_t$ on $(1 - B)(1 - B^{12})T_{it}$ for $i = 1, \ldots, 7$. From this SACF, Figure 2, the presence of the large negative value at lag 12 suggests the noise

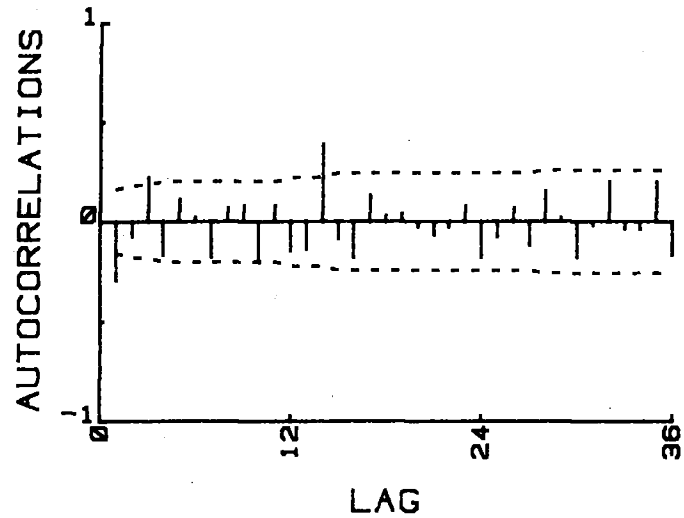


*Figure 1. SACF of $(1 - B)(1 - B^{12})Z_t$.*

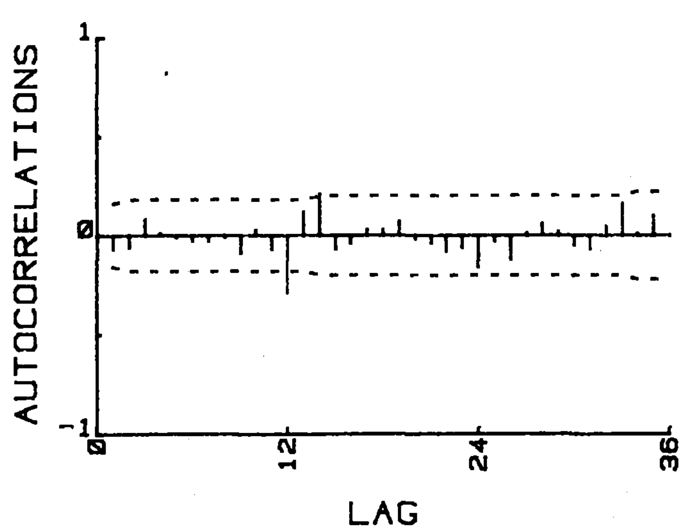


*Figure 2. SACF of Regression Residuals.*

model $(1 - B)(1 - B^{12})N_t = (1 - \theta_{12}B^{12})a_t$. Therefore, a tentatively entertained model for this series is

$$Z_t = \sum_{i=1}^{7} \beta_i T_{it} + \frac{(1 - \theta_{12}B^{12})}{(1 - B)(1 - B^{12})} a_t. \quad (4.5)$$

The BMDQ2T program (Liu 1979) was used to estimate the parameters in the model (4.5). The parameter estimates and corresponding standard errors are as follows:

$$\hat\beta_1 = \underset{(.0042)}{.0055} \quad \hat\beta_2 = \underset{(.0041)}{.0068} \quad \hat\beta_3 = \underset{(.0042)}{.0017}$$

$$\hat\beta_4 = \underset{(.0041)}{.0103} \quad \hat\beta_5 = \underset{(.0041)}{.0061} \quad \hat\beta_6 = \underset{(.0041)}{-.0098}$$

$$\hat\beta_7 = \underset{(.014)}{.037} \quad \hat\theta_{12} = \underset{(.028)}{.87} \quad \hat\sigma^2 = .00101.$$

The sample autocorrelations of the residuals from this model are all within plus or minus two standard errors of zero, and other diagnostic checks reveal no inadequacies with this model. The correlation matrix for the parameter estimates $\hat\beta_i$ $i = 1, \ldots, 7$ are reported in Table 1. The parameter estimates $\hat\beta_1, \ldots, \hat\beta_6$ are correlated so that individual inferences about these parameters must be made with caution. In contrast, $\hat\beta_7$ appears to be nearly uncorrelated with $\hat\beta_1, \ldots, \hat\beta_6$. This correlation pattern is typical of others that we have observed.

Inferences about the parameters in (4.5) can be made based on the asympotic theory referenced in Section 2. We first examine whether the daily effects ($\xi_i$) are different for the different days of the week by testing

$$H_0: \ \xi_1 = \cdots = \xi_7 \text{ vs.}$$

$$H_1: \ \text{not all } \xi_i \text{ are equal.}$$

This is equivalent to testing

$$H_0: \ \beta_1 = \cdots = \beta_6 = 0 \text{ vs.}$$

$$H_1: \ \text{not all } \beta_i = 0 \ i = 1, \ldots, 6. \quad (4.6)$$

If $\mathbf{A}$ is the estimated covariance matrix of $\hat{\boldsymbol\beta} = (\hat\beta_1, \ldots, \hat\beta_6)'$, then under $H_0$ in (4.6) the asymptotic distribution of $\hat{\boldsymbol\beta}' \mathbf{A}^{-1}\hat{\boldsymbol\beta}$ is chi-squared with 6 degrees of freedom. Because $\hat{\boldsymbol\beta}' \mathbf{A}^{-1}\hat{\boldsymbol\beta} = 147.63$ is larger than 12.6, which is the .05 critical value of a chi-squared distribution with 6 degrees of freedom, we reject $H_0$ in (4.6) and conclude that the different days of the week have significantly different effects.

*Table 1. Correlation Matrix of $\hat\beta$*

| | $\hat\beta_1$ | $\hat\beta_2$ | $\hat\beta_3$ | $\hat\beta_4$ | $\hat\beta_5$ | $\hat\beta_6$ | $\hat\beta_7$ |
|---|---|---|---|---|---|---|---|
| $\hat\beta_1$ | 1. | | | | | | |
| $\hat\beta_2$ | −.54 | 1. | | | | | |
| $\hat\beta_3$ | −.12 | −.50 | 1. | | | | |
| $\hat\beta_4$ | .14 | −.09 | −.53 | 1. | | | |
| $\hat\beta_5$ | .07 | .14 | −.12 | −.51 | 1. | | |
| $\hat\beta_6$ | −.04 | .05 | .15 | −.07 | −.55 | 1. | |
| $\hat\beta_7$ | .12 | −.11 | .13 | −.14 | .05 | .09 | 1. |

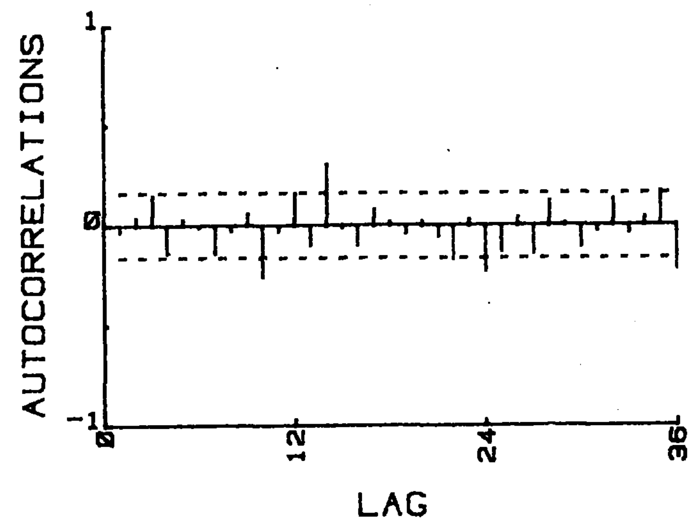


*Figure 3. SACF of Residuals From (4.9).*

It is also of interest to test

$$H_0: \ \beta_7 = 0 \text{ vs.}$$

$$H_1: \ \beta_7 \neq 0. \quad (4.7)$$

Since $\hat\beta_7$ divided by its standard error equals 2.6, we reject the null hypothesis in (4.7). When $\beta_7 \neq 0$, the term $\beta_7 T_{7t}$ in (4.5) accounts for an effect due to leap-year Februaries. To see this, notice from (4.5) that when we apply $1 - B^{12}$ to the data $Z_t$, we obtain

$$(1 - B^{12})Z_t = \sum_{i=1}^{7} \beta_i (1 - B^{12})T_{it} + \frac{(1 - \theta_{12}B^{12})}{1 - B} a_t.$$

Since $T_{7t}$ equals the length of month $t$, $(1 - B^{12})T_{7t} = 0$ except in a leap-year February and the February of the following year.

### 4.2 Ignoring Trading Day Effects

From the preceding analysis it is clear that the model (4.5) is an adequate description of this time series. It is of interest to get an idea of the effect of ignoring the trading day variables in this particular example. With this idea in mind we estimated the model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_{12}B^{12})a_t. \quad (4.8)$$

Examination of the residual autocorrelations from (4.8) revealed a number of significant values, including a significant autocorrelation at lag one. We therefore tried the model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12}B^{12})a_t. \quad (4.9)$$

The parameter estimates for (4.9) are $\hat\theta_1 = .40$, $\hat\theta_{12} = .88$, and $\hat\sigma^2 = .0017$. The residual autocorrelations are plotted in Figure 3. By comparing the results of the fit for (4.5) with those of the fit for (4.9), we can judge the impact of the trading day effects upon this data set. While the residual autocorrelations from model (4.9) did not reveal any specific pattern, there are a number of moderately large sample autocorrelations. Futhermore, the value of the Ljung-Box $Q$ based on 36 lags is 96.9, which greatly exceeds $\chi^2_{.01}(34) = 56.1$. We conclude that the residuals from (4.9) are not random. In contrast the model (4.5) passes the diagnostic checks and there is about a 40 percent reduction in the residual sum of squares from model (4.9) to (4.5). For this particular example the trading day effects are substantial and ignoring these effects is inappropriate.

## 5. MODELING HOLIDAY (EASTER) EFFECTS IN TIME SERIES

The Census Bureau adjusts certain retail sales series for holiday effects due to Easter, Labor Day, and Thanksgiving-Christmas (Young 1965). However, the Labor Day and Thanksgiving-Christmas adjustments are rather negligible, so we deal here only with modeling the effects of changing Easter dates. Techniques similar to those discussed here could be used to model other holiday effects, if necessary. For example, Liu (1980) discussed the problems involved with modeling a time series affected by the varying placement of the Chinese New Year.

The earliest and latest dates on which Easter can fall are March 22 and April 25. Thus, for series in which increased buying takes place before Easter we expect the March and April values in any particular year to depend on the date of Easter.

Specifying a functional form for the effect of Easter is not as simple as doing so for trading day effects. To be rather general, let $\bar{\alpha}_i$ denote the effect on the series being modeled on the $i$th day before Easter; let $h(i, t)$ be 1 when the $i$th day before Easter falls in the month corresponding to time point $t$, and 0 otherwise. Then the Easter effect at $t$, $E_t$, is

$$E_t = \sum_{i=1}^{K} \bar{\alpha}_i h(i, t), \tag{5.1}$$

where $K$ denotes some suitable upper bound on the length of the effect in days. Since many time series that contain Easter variation also contain trading day variation, we consider the model

$$Z_t = TD_t + E_t + N_t, \tag{5.2}$$

where $TD_t$ is given by (4.3), $N_t$ by (2.1), and $E_t$ by (5.1), although we will need to simplify $E_t$.

The relationship (5.1) was derived by consideration of the daily impact of Easter on the level of the series. Unfortunately, in most situations the only data available are monthly values of the series; as a consequence, in practice we cannot estimate effects as general as (5.1). To illustrate, consider the placement of Easter for the years 1967 to 1979. We chose these particular years because they correspond to the time frame of an actual set of data that is considered later; however, conclusions similar to those that we draw for these years are relevant for other time periods. Figure 4 shows the Easter dates for these years and constitutes the experimental design for determining the effect of Easter. From the diagram it is evident

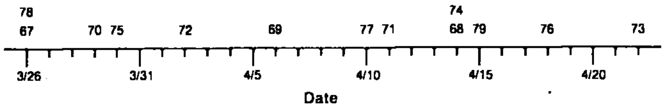


*Figure 4. The Placement of Easter for 1967 to 1979.*

that not all of the $\bar{\alpha}_i$ in (5.1) can be estimated. For example, in these years whenever the fourth day before Easter fell in March so did the fifth day before Easter; otherwise, they both fell in April. Thus we cannot distinguish the effect of $\bar{\alpha}_4 h(4, t)$ from that of $\bar{\alpha}_5 h(5, t)$, using the data from 1967–1979. Since we cannot estimate all of the $\bar{\alpha}_i$ in (5.1), we must look for special patterns in the $\bar{\alpha}_i$.

We initially use the simple pattern $\bar{\alpha}_1 = \cdots = \bar{\alpha}_\tau = \bar{\alpha}$, $\bar{\alpha}_{\tau+1} = \cdots = \bar{\alpha}_K = 0$ for some value $\tau$. This implies

$$E_t = \alpha \cdot H(\tau, t), \tag{5.3}$$

where $\alpha = \bar{\alpha}\tau$ and $H(\tau, t) = 1/\tau \sum_{i=1}^{\tau} h(i, t)$. Given $\tau$, $H(\tau, t)$ can be defined as the proportion of the time period $\tau$ days before Easter that falls in the month corresponding to time point $t$. With this definition $H(\tau, t)$ can be defined for any $\tau > 0$. For fixed $t$, $H(\tau, t)$ is in general a continuous but nondifferentiable function of $\tau$. Figure 5 shows $H(\tau, t)$ for $t$ corresponding to March 1969 and April 1969, Easter having been on April 6 that year.

Patterns other than that leading to (5.3) are possible. However, because Easter seldom occurred in early April from 1967 to 1979 (see Figure 4), it is unlikely that complex patterns can be detected from the data. This situation may change as additional data covering different Easter dates become available. We illustrate an approach to checking the adequacy of our assumed pattern in Section 5.3.

### 5.1 Noise Model Identification

It is of interest to consider the effect of $E_t$ on the ACF of the original series and its differences. Figure 6 shows the SACF of $(1 - B)(1 - B^{12})H(14, t)$ (using January 1967 through September 1975 data), its most unusual features being the spikes at and near lags 36 and 48. Patterns in the SACF's for $H(\tau, t)$ for other $\tau$ and other time periods are similar. The degree to which these characteristics are transmitted to the original series depends on the magnitude of the Easter effect relative to $TD_t$ and $N_t$. However, spikes at these lags can be taken as a possible indication of Easter effects in a series, especially when they show up in the SACF of a residual series from a model that has no terms to account for Easter effects.

To illustrate noise model identification, we consider the example of monthly retail sales of shoe stores (U.S.) from January 1967 through September 1979, which is available from the Census Bureau. (The observation for January 1970 ($t = 37$) was found to be an outlier and was modified from 243 to 270.3 (millions of dollars). The effect of the outlier was estimated by fitting the model with an indicator variable at $t = 37$.) We found it appropriate to analyze natural logarithms (denoted by $Z_t$) and to take $(1 - B)(1 - B^{12})Z_t$. Figure 7 gives the SACF of the differenced series, which exhibits behavior very similar to that in Figure 6, reflecting the Easter effect. To approximately remove $E_t$ we choose a preliminary value of $\tau$ in (5.3), such as $\tau = 14$, and regress $(1 - B)(1 - B^{12})Z_t$ on $(1 - B)(1 - B^{12})H(14, t)$ and $(1 - B)(1 - B^{12})T_{it}$ $i = 1, \ldots, 7$. The SACF of the residuals from this regression, shown in Figure 8, does not show any influence of Easter or trading day effects. From this we identify a tentative noise model:

$$(1 - B)(1 - B^{12})N_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t. \quad (5.4)$$

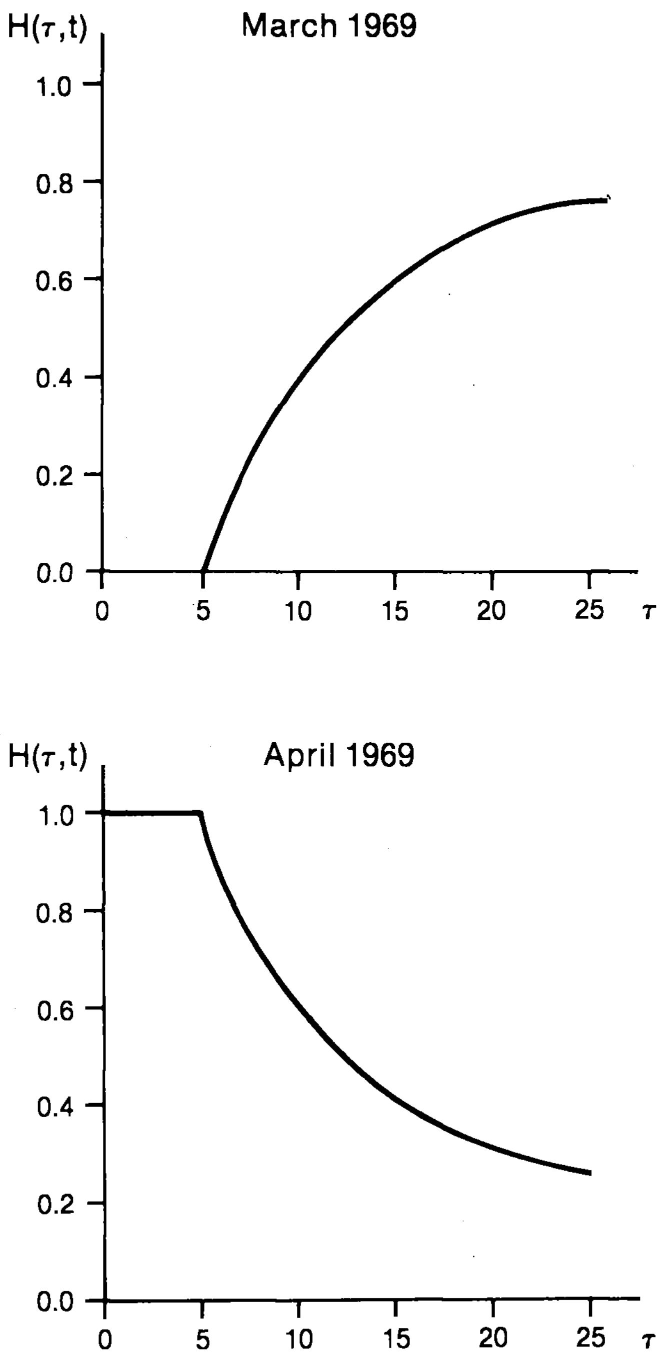

Figure 5. $H(\tau, t)$.

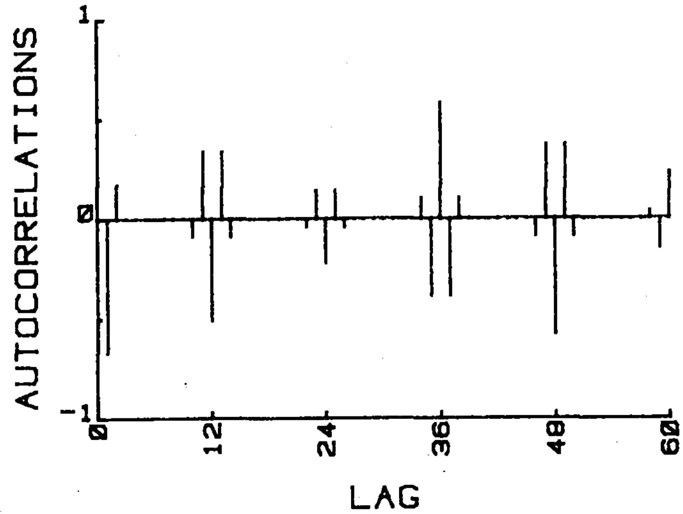

Figure 6. SACF of $(1 - B)(1 - B^{12})H(14, t)$.

## 5.2 Estimation of the Holiday Model

We demonstrate estimation of the model

$$Z_t = \sum_{i=1}^{7} \beta_i T_{it} + \alpha H(\tau, t) + \frac{\theta(B)}{\phi(B)\delta(B)} a_t \quad (5.5)$$

with the shoe stores example begun in Section (5.1). Notice that (5.5) is linear in $\beta_1, \ldots, \beta_7$, and $\alpha$ for fixed $\tau$, so for fixed $\tau$ estimation may proceed in a manner analogous to that for the trading day model (4.4). We can obtain maximum likelihood estimators for the parameters of (5.5), including $\tau$, by defining the asymptotic log-likelihood

$$L_{\max}(\tau) = \max_{\boldsymbol{\beta},\alpha,\boldsymbol{\phi},\boldsymbol{\theta},\sigma^2} L(\boldsymbol{\beta},\alpha,\tau,\boldsymbol{\phi},\boldsymbol{\theta},\sigma^2)$$

$$= -\frac{n}{2}\ln\hat{\sigma}^2(\tau) + \text{constant}$$

(where $\hat{\sigma}^2(\tau)$ is the estimate of $\sigma^2$ for fixed $\tau$) and maximizing this over $\tau$. Table 2 gives $\hat{\sigma}^2(\tau)$ for the shoe store

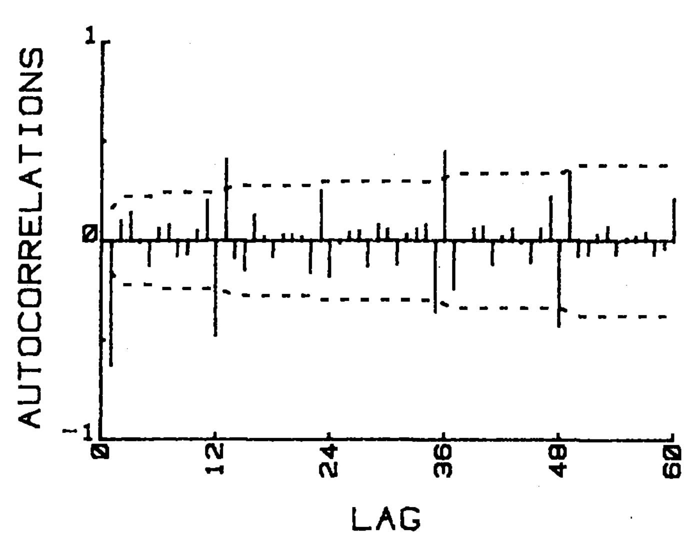

Figure 7. SACF of $(1 - B)(1 - B^{12})Z_t$.

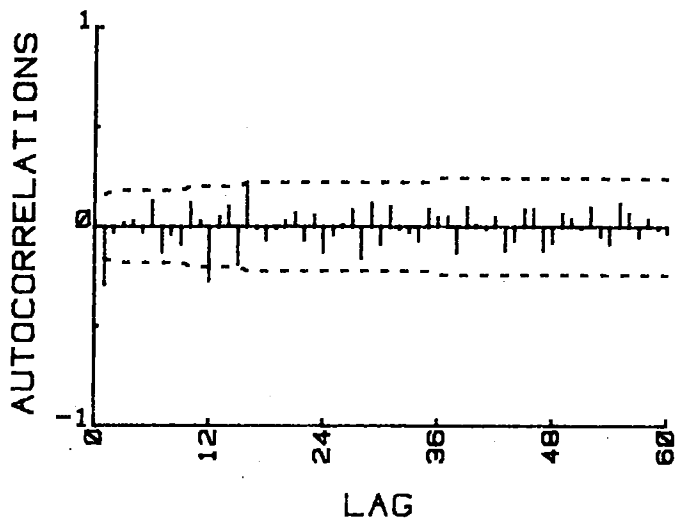


*Figure 8. SACF of Regression Residuals.*

series for $\tau = 1, \ldots, 25$. The MLE of $\tau$ is $\tau = 10$ (approximately) and the estimates of the other parameters together with their standard errors are (from the fit with $H(10, t)$ in the model) as follows:

$$\hat{\beta}_1 = \underset{(.0066)}{.0036} \qquad \hat{\beta}_2 = \underset{(.0062)}{-.0024} \qquad \hat{\beta}_3 = \underset{(.0063)}{-.0035}$$

$$\hat{\beta}_4 = \underset{(.0062)}{-.0087} \qquad \hat{\beta}_5 = \underset{(.0063)}{.0213} \qquad \hat{\beta}_6 = \underset{(.0065)}{.0131}$$

$$\hat{\beta}_7 = \underset{(.021)}{.045} \qquad \hat{\theta}_1 = \underset{(.079)}{.32} \qquad \hat{\theta}_{12} = \underset{(.029)}{.86}$$

$$\hat{\alpha} = \underset{(.012)}{.166} \qquad \hat{\sigma}^2 = .00160.$$

It is of interest to note that for this series there is not much difference in the values of $\hat{\sigma}^2(\tau)$ for values of $\tau$ near 10.

Since $H(\tau, t)$ is continuous, $\hat{\tau}$ should be a consistent estimator of $\tau$; however, since $H(\tau, t)$ is not differentiable for all $t$, the asymptotic normality need not hold. For fixed $\tau$ the results cited in Section 2.2 apply to the estimators of the other parameters, so that we can make inferences conditional on $\tau$. If $\hat{\tau}$ were approximately independent of the estimators of the other parameters, then we could fix $\tau$ at 10 to make inferences. This can be checked by examining the parameter estimates and their standard errors for various $\tau$. (in computing the standard errors, the parameter $\sigma$ was estimated using the residual standard error for $\tau = 10$ because we considered this value to be a better estimator of $\sigma$ than the residual standard error for other values of $\tau$.) For this example the standard errors of $\hat{\beta}_1(\tau), \ldots, \hat{\beta}_7(\tau)$ are quite nearly constant for $\tau = 1, \ldots, 25$. Also, $\hat{\beta}_3(\tau), \ldots, \hat{\beta}_7(\tau)$ vary little for $\tau = 2, \ldots, 25$ and $\hat{\beta}_1(\tau)$ and $\hat{\beta}_2(\tau)$ vary little for $\tau = 7, \ldots, 25$. The estimates at the lower values of $\tau$ differ more from the others, although the differences are not large relative to the standard errors. The standard errors for $\hat{\theta}_1(\tau)$ and $\hat{\theta}_{12}(\tau)$ show little variation (no more than 10 percent) for $\tau = 2, \ldots, 25$ and $\tau = 2, \ldots, 16$ respectively, and are slightly lower outside these ranges. The estimates $\hat{\theta}_1(\tau)$ and $\hat{\theta}_{12}(\tau)$ vary little with $\tau$, except possibly for $\tau = 1$. It seems for this series that $\hat{\beta}_1, \ldots, \hat{\beta}_7, \hat{\theta}_1, \hat{\theta}_{12}$, and their standard errors are relatively independent of $\hat{\tau}$, at least for a large part of the range of $\tau$ considered; thus we can make inferences on $\beta_1, \ldots, \beta_7, \theta_1$, and $\theta_{12}$ conditional on $\tau = 10$.

Table 2 shows that $\hat{\alpha}(\tau)$ and its estimated standard error depends more or $\tau$. Still, we note that for this example there appears to be a fairly wide range of values of $\tau$ for which the estimates of $\alpha$ and their standard errors are fairly constant. These results are partially due to the experimental design given in Figure 4. Thus for data covering approximately the same years as this particular example, it may be reasonable to choose an approximate value for $\tau$ (for example, $\tau = 10$) and proceed with the inference conditional upon the value of $\tau$ chosen.

### 5.3 Checking the Easter Model

One way that the model (5.5) can be inadequate is if the Easter effect is more complex than that described by the simple pattern $\tilde{\alpha}_1 = \cdots = \tilde{\alpha}_\tau, \tilde{\alpha}_{\tau+1} = \cdots = \tilde{\alpha}_k = 0$. We cannot estimate all the $\tilde{\alpha}_i$ in (5.1), but can estimate a somewhat general pattern by grouping some of the terms in (5.1) together. We used the grouping

$$\begin{aligned} E_t = {} & \alpha_1[h(1, t) + h(2, t)] \\ & + \alpha_2[h(3, t) + \cdots + h(6, t)] \\ & + \alpha_3[h(7, t) + \cdots + h(10, t)] \\ & + \alpha_4[h(11, t) + \cdots + h(14, t)] \\ & + \alpha_5[h(15, t) + \cdots + h(18, t)] \\ & + \alpha_6[h(19, t) + \cdots + h(22, t)]. \end{aligned} \tag{5.6}$$

*Table 2. Estimation of* $\tau$

| $\tau$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $100\hat{\sigma}^2(\tau)$ | .212 | .182 | .177 | .176 | .175 | .167 | .163 | .162 | .161 | .160 | .161 | .164 | .167 |
| $\hat{\alpha}(\tau)$ | .13 | .15 | .15 | .15 | .15 | .16 | .16 | .16 | .16 | .17 | .17 | .17 | .17 |
| $\hat{\sigma}(\hat{\alpha}(\tau))$ | .0129 | .0127 | .0125 | .0124 | .0123 | .0123 | .0123 | .0122 | .0122 | .0123 | .0126 | .0128 | .0130 |
| | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | |
| $100\hat{\sigma}^2(\tau)$ | .171 | .171 | .173 | .175 | .176 | .178 | .180 | .183 | .183 | .183 | .183 | .183 | |
| $\hat{\alpha}(\tau)$ | .17 | .18 | .18 | .19 | .19 | .20 | .21 | .21 | .22 | .23 | .24 | .25 | |
| $\hat{\sigma}(\hat{\alpha}(\tau))$ | .0135 | .0141 | .0146 | .0151 | .0157 | .0164 | .0169 | .0176 | .0183 | .0192 | .0200 | .0209 | |

Any grouping of the $h(i, t)$ can be used as long as it produces explanatory variables that are linearly independent over the span of the data. For our example we have 26 March and April observations for estimating the Easter effect. So as not to spread the observations too thin, we decided to use six groups, and chose the grouping in (5.6) to yield groups of equal length, except for a first group of length two to allow for a possibly important effect immediately before Easter.

Our general model at this point is (5.2) with $TD_t$ given by (4.3), $E_t$ by (5.6), and $N_t$ by (2.1). We investigate how complex an Easter effect is needed by sequentially testing

$$H_0: \quad \alpha_j = \alpha_{j+1} = \cdots = \alpha_6 = 0 \text{ vs.}$$

$$H_1: \quad \alpha_j \neq 0, \alpha_{j+1} = \cdots = \alpha_6 = 0$$

for $j = 1, \ldots, 6$. When $H_0$ is rejected we can investigate whether a simple pattern of the form $\bar{\alpha}_1 = \cdots = \bar{\alpha}_\tau = \bar{\alpha}$, $\bar{\alpha}_{\tau+1} = \cdots = \bar{\alpha}_k = 0$ is adequate by testing (for $j > 1$)

$$H_0': \quad \alpha_1 = \cdots = \alpha_j, \alpha_{j+1} = \cdots = \alpha_6 = 0$$

against $H_1$. Table 3 presents (asymptotic) likelihood ratio test statistics for the shoe stores example computed as

$$\frac{[\text{RSS}(H_0) - \text{RSS}(H_1)]/\nu_1}{\text{RSS}(H_1)/\nu_2}$$

and similarly for $H_0'$, where RSS denotes the residual sum of squares. The numerator degrees of freedom, $\nu_1$, is 1 for testing $H_0$ and $j - 1$ for testing $H_0'$. The denominator degrees of freedom, $\nu_2$, is 153 − 13 (for differencing) −1 (outlier) − 7 ($TD$ parameters) −2 ($\theta_1$ and $\theta_{12}$) $-j = 130 - j$. The 5% and 1% critical values for the $F(\nu_1, \nu_2)$ distribution are also reported. The test statistics do not indicate that $\alpha_j \neq 0$ for $j > 3$. Also, there is no reason to reject the assumption that $\alpha_1 = \alpha_2 = \alpha_3$. We conclude that for this example the data give no evidence that the simplified Easter effect given by (5.3) is inadequate.

In addition to checking the Easter effect, we also should check the adequacy of the noise model (5.4). The sample autocorrelations of the residuals for the shoe stores series (using the model (5.5) with $\tau = \hat{\tau} = 10$) are all within plus or minus two standard errors of zero with the exception of $r_8(\hat{a})$, which is 2.7 standard errors below zero. The Ljung-Box $Q$ statistic for 36 lags is 46.3. Since this is less than 48.6, the $\chi^2_{.05}(34)$ critical value, we conclude that the residuals appear to be white noise.

*Table 3. Investigating $\alpha_1 = \cdots = \alpha_j, \alpha_{j+1} = \cdots \alpha_6 = 0$*

| $j$ | F-statistic for $H_0$ | F-statistic for $H_0'$ | $F_{.05}(j-1, 130)$ | $F_{.01}(j-1, 130)$ |
|---|---|---|---|---|
| 1 | 90.3 | — | — | — |
| 2 | 12.1 | .9 | 3.9 | 6.8 |
| 3 | 5.7 | .6 | 3.0 | 4.8 |
| 4 | .0 | — | — | — |
| 5 | 2.7 | — | — | — |
| 6 | .3 | — | — | — |

$F_{.05}(1,130) = 3.9 \quad F_{.01}(1,130) = 6.8$

NOTE: The $F(\nu_1, 130-j)$ critical values are very close to the $F(\nu_1, 130)$ critical values.

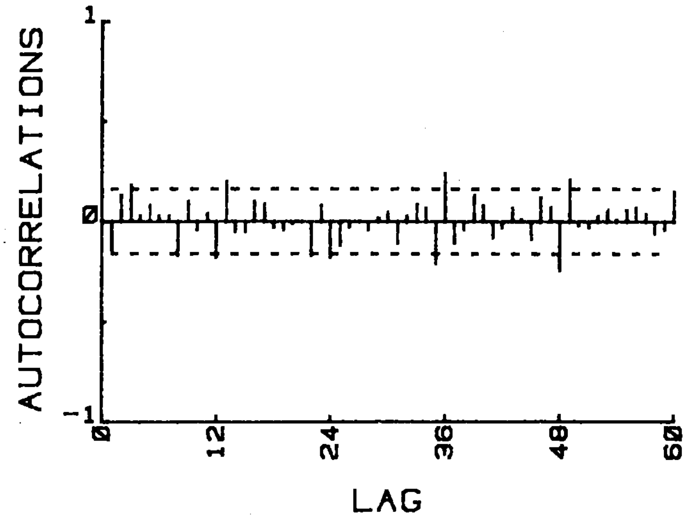

*Figure 9. SACF of Residuals From (5.7).*

## 5.4 Ignoring Trading Day and Easter Effects

As in the example of Section 4 it is of interest to investigate the influence of the trading day and Easter holiday terms in model (5.5) by fitting the model without these terms, which is

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t. \quad (5.7)$$

The parameter estimates for model (5.7) are $\hat{\theta}_1 = .68$, $\hat{\theta}_{12} = .93$, and $\hat{\sigma}^2 = .00333$. The residual autocorrelations are plotted in Figure 9. From Figure 9 there appear to be a number of moderately large $r_k(\hat{a})$'s at low lags, but there is not a recognizable pattern that would suggest a modification if a pure ARIMA model is to be used. Also, the behavior of the $r_k(\hat{a})$'s near lags 36 and 48 resembles that in Figure 6, indicating the presence of the Easter effect. The Ljung-Box $Q$ statistic based upon 36 lags is 79.5, which is larger than $\chi^2_{.01}(34) = 56.1$. Thus, we would reject the hypothesis that the residuals from model (5.7) were white noise. The model (5.7) has obvious inadequacies, and there is about a 50 percent reduction in the residual sum of squares when the trading day and Easter influences are appropriately modeled.

## 6. CONCLUSIONS

In the time series literature the model (1.1) has been considered from a theoretical viewpoint; however, in many applications there has been an apparent tendency either to consider pure regression models or to consider pure ARIMA time series models. We have argued that there are situations in which a combination of these two models is superior. As particular examples we considered in detail the cases of time series that include trading day variation and Easter holiday variation. These two particular examples are important because there are many time series that contain one or both of these effects. The actual time series we considered indicate that substantial improvements over pure ARIMA models can be achieved if trading day and Easter effects are appropriately modeled. We hope that from this research more model builders will become aware of trading day and Easter variations and, as a result, will be in a better position to handle them.

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## REFERENCES


ANDERSON, R.L. (1954), "The Problem of Autocorrelation in Regression Analysis," *Journal of the American Statistical Association*, 49, 113–129.

BOX, G.E.P., and JENKINS, G.M. (1976), *Time Series Analysis: Forecasting and Control*, San Francisco: Holden Day.

CLEVELAND, W.P., and GRUPE, M.R. (1982), "Modeling Times Series When Calendar Effects are Present," to appear in *Proceedings of the Conference on Applied Time Series Analysis of Economic Data*, ed. Arnold Zellner.

CLEVELAND, W.S., and DEVLIN, S.J. (1980), "Calendar Effects in Monthly Time Series: Detection by Spectrum Analysis and Graphical Methods," *Journal of the American Statistical Association*, 75, 487–496.

——— (1982), "Calendar Effects in Monthly Time Series: Modeling and Adjustment," *Journal of the American Statistical Association*, 77, 520–528.

FULLER, W. (1976), *Introduction to Statistical Time Series*, New York: John Wiley and Sons.

GALLANT, A.R., and GOEBEL, J.J. (1976), "Non-linear Regression with Autocorrelation Errors," *Journal of the American Statistical Association*, 71, 961–967.

HANNAN, E.J. (1971), "Non-linear Time Series Regression," *Journal of Applied Probability*, 8, 767–780.

LIU, L.M. (1979), "User's Manual for BMDQ2T Time Series Analysis," BMDP Technical Report.

——— (1980), "Analysis of Time Series with Calendar Effects," *Management Science*, 26, 106–112.

LJUNG, G.M., and BOX, G.E.P. (1978), "On a Measure of Lack of Fit in Time Series Models," *Biometrika*, 65, 297–303.

PFEFFERMAN, D., and FISHER, J. (1980), "Festival and Working Days Prior Adjustments in Economic Time Series," in *Time Series*, ed. O.D. Anderson, New York: North-Holland.

PIERCE, D.A. (1971), "Least Squares Estimation in the Regression Model With Autoregressive-moving Average Errors," *Biometrika*, 58, 299–312.

YOUNG, A.H. (1965), "Estimating Trading-Day Variation in Monthly Economic Time Series," Technical Paper 12, Bureau of the Census.