In the world of finance, few concepts are as revered and yet enigmatic as the Black-Scholes formula. Developed by economists Fischer Black and Myron Scholes in the early 1970s, this groundbreaking formula revolutionized the way we think about option pricing. Despite its widespread use by financial institutions and investors around the globe, the Black-Scholes formula remains a topic of curiosity and fascination for many. In this blog post, we embark on a journey to unravel the intricate mathematics underlying this formula, demystifying its components and shedding light on the significance of this remarkable equation in the realm of financial derivatives. So, fasten your seatbelts as we dive into the depths of the Black-Scholes formula to gain a comprehensive understanding of its inner workings and implications.

Assumptions

Before we delve into the world of complex mathematics, it is important to understand the assumptions on which the Black-Scholes equation is based. These assumptions are a key aspect for the exact derivation of the model

- The underlying follows a lognormal random walk: This means that the price changes randomly over time, and the distribution of these changes is logarithmically normal with a constant drift.

- The short selling of securities with full use of proceeds is permitted: investors can sell securities they do not own with the intention of buying them back at a later time. The assumption of full use of proceeds means that the investor can invest the entire amount obtained from the short sale. This assumption allows for more flexibility in trading strategies.

- There are no transaction costs or taxes: This simplifies the model by excluding additional factors that would impact the profitability of trades.

- Security trading is continuous: Investors can buy or sell securities at any time within the market operating hours. This assumption allows for continuous price adjustments based on market demand and supply, enabling smoother modeling of the asset's price dynamics.

- The risk-free rate of interest, r, is constant and the same for all maturities: that applies to all maturities. This assumption simplifies the model by disregarding changes in interest rates over time and across different time periods. It allows for a single, constant rate to be used in discounting future cash flows and determining the present value of the derivative.

- There are no riskless arbitrage opportunities: This assumption implies that it is not possible to make a guaranteed profit without taking any risk. It ensures that the model is consistent and avoids situations where an investor could exploit discrepancies in prices to make risk-free profits.

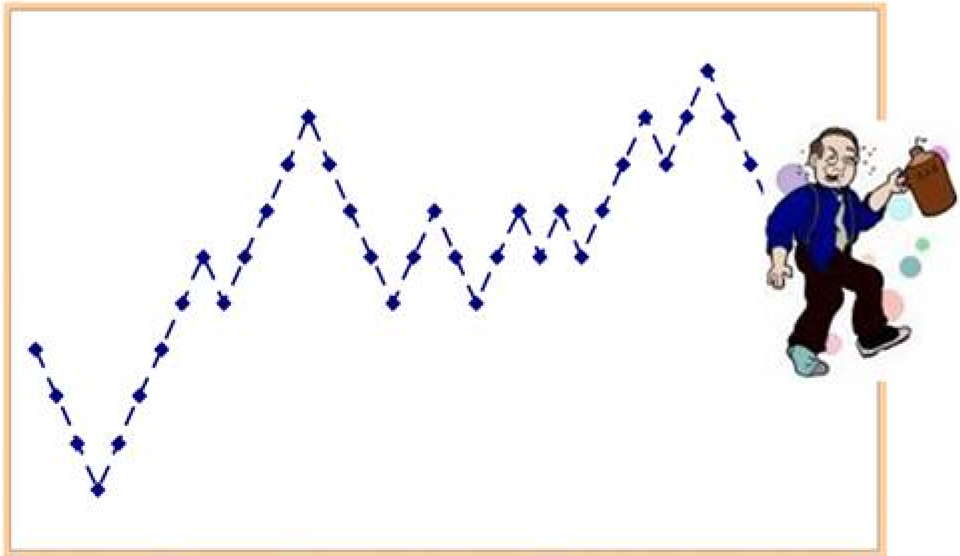
- There are no dividends during the life of the derivative: Dividends are periodic payments made by a company to its shareholders. By assuming no dividends, the model focuses solely on the price dynamics of the underlying asset, disregarding any additional cash flows from dividend payments.

- The contract cannot be exercised before the expiration date: Such option contracts are termed European Options. This is a key factor for the validity of the Black-Scholes Model.

Key aspects of the Black-Scholes formula:

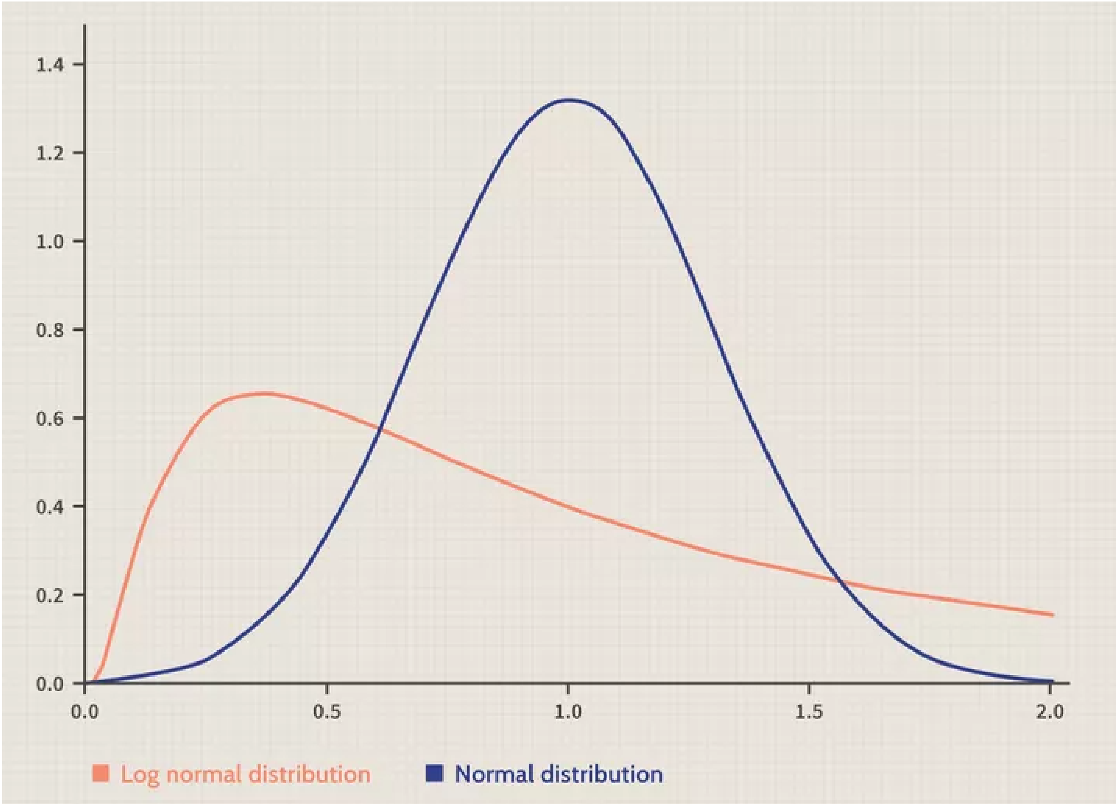
Following are some useful statistical and financial concepts that are used throughout the modeling of the equation:

Random Walk: It is a process for determining the probable location of a point subjected to random paths/motions. The probability at each step to move a certain distance in some direction is kept constant. It is a type of Markov Process where the future behavior is assumed to be independent of the past history.



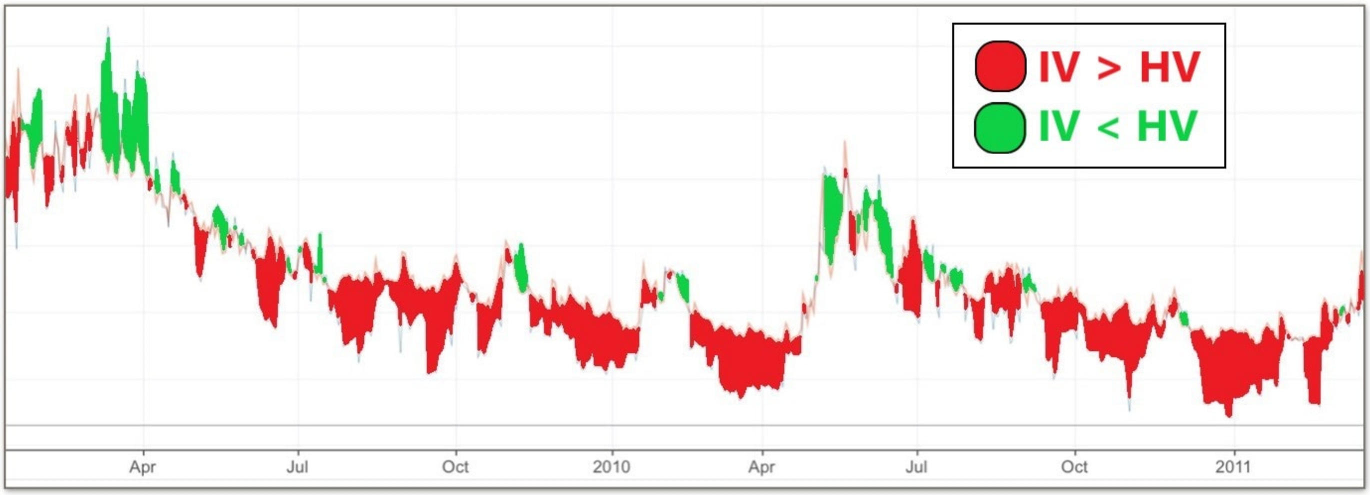
<u>Lognormal Distribution</u>: A continuous probability distribution of random numbers, where the logarithmic is normally distributed.

- This deviation of using a 'lognormal' distribution instead of the usual normal distribution is due to the fact that stock prices cannot be negative since they represent ownership in an asset.
- Secondly, Black-Scholes is based on Generic Brownian Motion whose logarithm follows a normal distribution.
- Thirdly, empirical evidence shows that asset returns often exhibit a skewness(asymmetry) and kurtosis(fat tails) which are not captured accurately by normal distribution. Whereas lognormal distribution inherently has positive skewness and fat tails, and thus reflects the observed characteristics better.



Risk-Free interest rate: The risk-free interest rate plays a crucial role in the Black-Scholes option pricing model. It represents the theoretical return an investor would earn from a risk-free investment with no possibility of default. Typically, a short-term government bond yield is used as a proxy for the risk-free rate. This ensures fair valuation based on the time value of money and the opportunity cost of investing in risk-free assets.

Implied Volatility: It represents the market's expectations of future price volatility for the underlying asset. Implied volatility is preferred over historical volatility in option pricing because it incorporates forward-looking information. The Black-Scholes model assumes that the underlying asset follows geometric Brownian motion, where this volatility plays a crucial role. One reason implied volatility is favored is that it captures the impact of anticipated events on the future. These events could include earnings announcements, economic reports, political developments, or other factors like demand and supply that might significantly affect the underlying asset's price. Historical volatility, on the other hand, only reflects past price movements and may not fully reflect the market's current expectations or anticipated future events. Implied volatility is derived from observed option prices in the market using various pricing models, including the Black-Scholes model.



The formula

Weiner process: It is a particular type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year. It is sometimes referred to as Brownian motion. Intuitively, we may think of Brownian motion as a limiting case of some random walk as its time increment goes to zero. A variable z follows a Wiener process if it has the following two properties:

1) $\Delta z = \epsilon\sqrt{\Delta t}$, where e has a standard normal distribution(mean=0, standard deviation=1)
2) The values of z for any two different short intervals of time, t, are independent.

From property 1 it is evident that z follows a normal distribution with

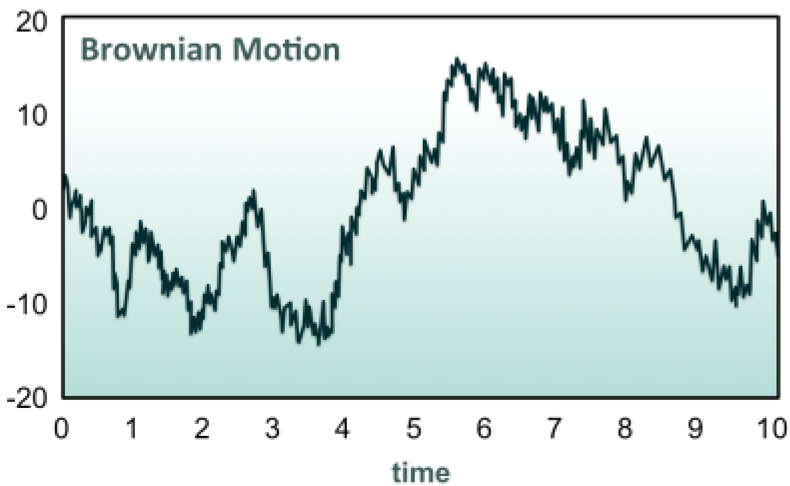
Mean = 0

Variance = Δt

Standard deviation = $\sqrt{\Delta t}$

Drift rate and Variance Rate: The mean change per unit time for a stochastic process is known as the drift rate and the variance per unit time is known as the variance rate.

The basic Wiener process, dz, that has been developed so far has a drift rate of zero and a variance rate of 1.0. The drift rate of zero means that the expected value of z at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in z in a time interval of length T equals T.

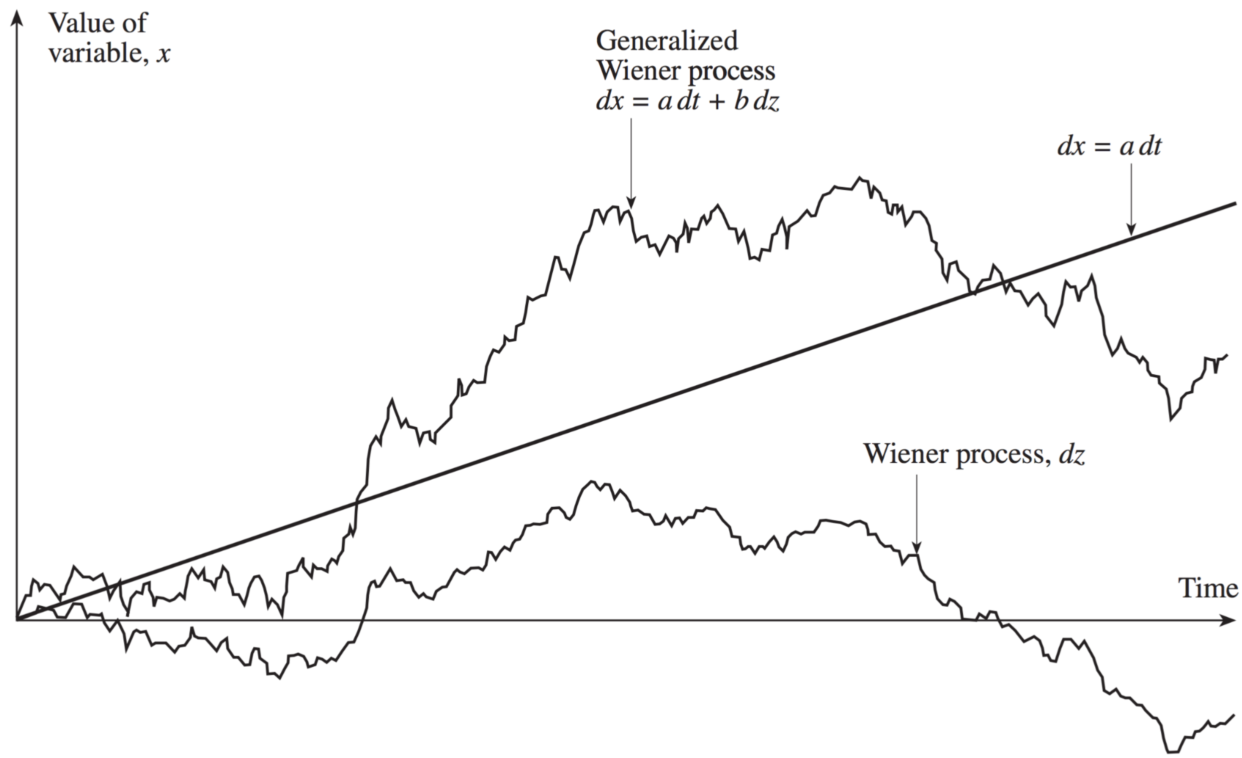


A generalized Wiener process for a variable x can be defined in terms of dz as

$$\mathbf{dz = adt + bdz}$$

It is useful to consider the two components on the right-hand side separately. The adt term implies that x has an expected drift rate of a per unit of time. The bdz term on the right-hand side of the equation can be regarded as adding noise or variability to the path followed by x. The amount of this noise or variability is b times a Wiener process. Hence the equation can be written as

$$\mathbf{\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}}$$



Process for a Stock price

Constant drift?

The idea of a generalized Wiener process, with a consistent expected drift rate and constant variance rate, may seem appealing for modeling stock price movements. However, this model fails to capture a crucial aspect: the expected percentage return demanded by investors remains independent of the stock's price. Whether the stock price is \$10 or \$50, investors would still expect the same annual return rate (e.g., 14%) assuming all other factors remain unchanged. It becomes clear that assuming a constant expected drift rate is not appropriate. Instead, the assumption should be that the expected return rate (i.e., expected drift divided by the stock price) remains constant. This way, we acknowledge that investors demand the same rate of return relative to the stock's price. Therefore, it is necessary to replace the assumption of a constant expected drift rate with the assumption of a constant expected return rate.

Expected return

If S is the stock price at time t, then the expected drift rate in S should be assumed to be μS for some constant parameter. This means that in a short interval of time, Δt, the expected increase in S is $\mu S \Delta t$. The parameter μ is the expected rate of return on the stock.

Brownian Motion (Stock as an equation)

Since there is uncertainty in the practical world, the coefficient of dz (i.e. noise factor/variability factor) cannot be assumed to be zero. A reason might be that the variability of the return in a short period of time, t, is the same regardless of the stock price. In other words, an investor is just as uncertain of the return when the stock price is \$50 as when it is \$10. This suggests that the standard deviation of the change in a short period of time t should be proportional to the stock price and leads to the model:

$$dS = \mu S\, dt + \sigma S\, dz$$

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz$$

This equation is the most widely used model of stock price behavior. The variable μ is the stock's expected rate of return. The variable σ is the volatility of the stock price. The model in the equation represents the stock price process in the real world. In a risk-neutral world, equals the risk-free rate r. The discrete-time version of the derived model is known as **Generic Brownian Motion** and is written as

$$\frac{\Delta S}{S} = \mu\, \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

$$\Delta S = \mu S\, \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

This equation shows that $\Delta S/S$ is approximately normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$.

$$\frac{\Delta S}{S} \sim \phi(\mu\, \Delta t,\ \sigma^2 \Delta t)$$

Ito's lemma: Ito's process is a type of stochastic process, and is a generalized version of the Wiener process where the parameters a and b are the functions of the value of the underlying variables x and t. Since the price of the option is a function of the underlying asset's price and the time until the contract expires, Ito's lemma fits best for modeling the value of the option. Thus Ito's process can be defined as

$$\mathbf{dx = a(x,t)dt + b(x,t)dz}$$

Ito's Lemma, therefore, shows that a function G of x and t follows a certain process (whose derivation is beyond our scope of the study) given by:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

Black-Scholes

Suppose f is the price of the call option contingent on S, then f must be a function of S and t. From Ito's lemma

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z$$

From Ito's lemma, the Wiener processes underlying f and S are the same. This means the Δz in both the equations are same. Suppose a portfolio is constructed to eliminate the Weiner process, such that

Derivatives: -1
Shares: +Δf/ΔS

The holder of this portfolio is short one derivative and long an amount +Δf/ΔS of shares. Hence, the value of the portfolio is

$$\Pi = -f + \frac{\partial f}{\partial S}S$$

Hence the change in the value of the portfolio can be given as:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

Substituting the 2 equations gives

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

Because this equation does not involve z, the portfolio should be riskless during time t.

No Arbitrage Principle: The assumptions mentioned earlier imply that the portfolio should earn the same rate of return as other short-term risk-free investments in an instant. If the portfolio were to earn a higher return, it would create an opportunity for arbitrageurs to make a risk-free profit by borrowing money to invest in the portfolio. Conversely, if the portfolio were to earn a lower return, arbitrageurs could profit by shorting the portfolio and investing in risk-free securities. In simpler terms, if there is a completely risk-free change (d∏) in the portfolio value (∏), it must be equivalent to the growth obtained by investing the same amount of money in a risk-free interest-bearing account. This ensures that there are no riskless opportunities for profit in the market and maintains the principle that the portfolio should earn the same return as risk-free investments. Hence,

$$\Delta\Pi = r\Pi\,\Delta t$$

Substituting the equations and rearranging gives,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

This is known as **Black-Scholes-Merton differential equation**.

The Black-Scholes equation has multiple solutions, each corresponding to different derivatives that can be defined with the underlying variable (S). The specific derivative obtained when solving the equation depends on the boundary conditions used, which determine the values of the derivative at the boundaries of possible values for S and t. This allows us to calculate the value of a particular financial contract by applying the appropriate boundary conditions in the Black-Scholes equation.
For a European call option,

$$f = \max(S - K,\ 0) \quad \text{when } t = T$$

For a European put option,

$$f = \max(K - S,\ 0) \quad \text{when } t = T$$

<u>Where did Drift Rate go?</u>

The Black–Scholes equation contains all the obvious variables and parameters such as the underlying, time, and volatility, but there is no mention of the drift rate μ. Why is this? Any dependence on the drift dropped out at the same time as we eliminated the dS component of the portfolio. The economic argument for this is that by perfectly hedging the option with the underlying asset, any dependence on the drift rate is eliminated. This means that investors should not be rewarded for taking unnecessary risks beyond the risk-free rate. Therefore, the drift rate is not considered in the equation. In practical terms, this implies that even if two individuals have different estimates of the drift rate for an asset, as long as they agree on the volatility, they will still arrive at the same value for its derivatives according to the Black-Scholes model. The agreement on volatility is sufficient to determine the value of the options, regardless of differing views on the drift rate.

Solution to PDE

The most famous solutions to the differential equation are the Black–Scholes– Merton formulas for the prices of European call-and-put options. These are:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

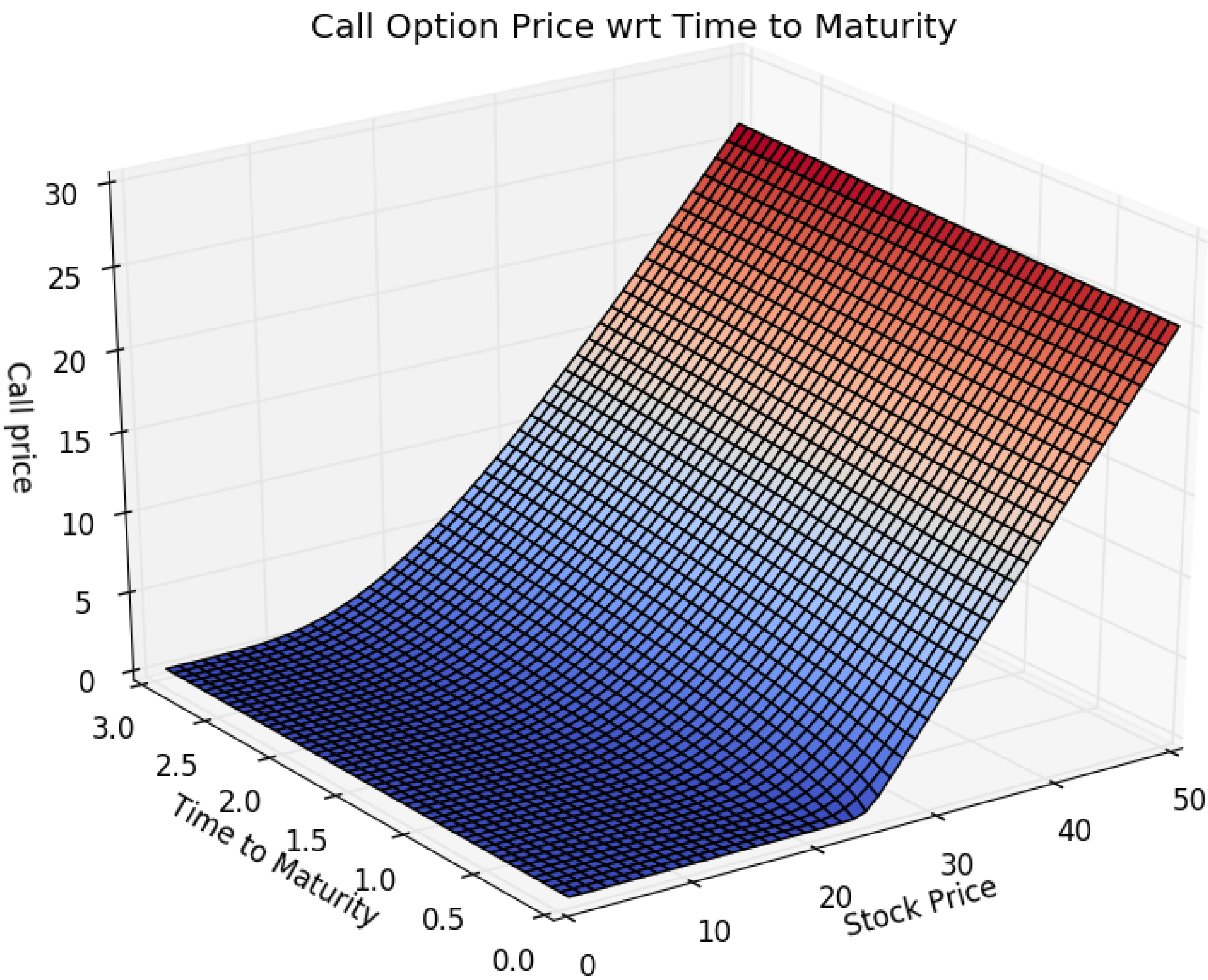
$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where,

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Where,
K = Strike Price of Option
r = Risk-free interest rate
S = Underlying asset's current price
σ = Implied Volatility
T = Number of years for a contract to expire from the current date



Why everything done so far is correct knowing the future is completely uncertain?

The Black-Scholes PDE is derived under specific assumptions, including efficient markets, constant volatility, continuous trading, no transaction costs, and risk-free interest rates. Recall that all these parameters weren't assumed but proved while deriving the differential equation. When these assumptions hold, the PDE accurately reflects the dynamics of the underlying asset's price and allows for the calculation of option prices. The correctness of Black Scholes can be elaborated from several perspectives:

- No arbitrage principle
- The equation was derived using a replication strategy, i.e. by considering a portfolio of the option and underlying asset.
- Continuous trading and no trading costs ensure accuracy in capturing the dynamics of the derivative market.

- The model assumes the rational and efficient behavior of the market and investors. This implies that the price of an option reflects all available information, and investors make decisions based on expected future returns.

Limitations

- Assumption of Efficient Markets: The Black-Scholes formula assumes that markets are efficient, meaning that prices instantly and accurately reflect all available information. However, this assumption may not hold true in real-world markets, where inefficiencies, market frictions, and delays in information dissemination can impact option pricing.

- Constant Volatility Assumption: The formula assumes that volatility remains constant over the life of the option. In reality, volatility can fluctuate, and this assumption may not accurately capture the dynamics of market volatility, especially during times of high uncertainty or market stress.

- No Transaction Costs: The formula assumes no transaction costs, such as brokerage fees or taxes. In practice, these costs can significantly impact the profitability of options trading and may need to be considered separately when applying the Black-Scholes model.

- Risk-Neutral Assumption: The formula assumes a risk-neutral world, where investors do not require a risk premium for holding the underlying asset. This assumption implies that the risk-free interest rate is used as the discount rate in the formula. However, in practice, investors might demand a premium for bearing risk, which can impact option prices and the accuracy of the model.

Ideally, the PDE for the Black Scholes can be derived using various other methods like Risk-Neutral valuation, Martingale approach, Binomial Tree approach, and many more. Here portfolio replicating method which is the most widely known method was used. But the basic approach remains the same irrespective of the approaches we use. Just the simulation differs from approach to approach. This was all about the Black-Scholes formula and the underlying mathematical and statistical concepts used to derive it.

Thank You!!