Digital Signal Processing (ECN-312)

Lecture 5 (Discrete Fourier Transform)

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January 25, 2022

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Fourier series of periodic sequences

- Consider a periodic sequence $\tilde{x}[n]$ with period N
  - $\tilde{x}[n] = \tilde{x}[n + rN]$ for any $r, n \in I$
- Fourier series representation:
  - Represent $\tilde{x}[n]$ as sum of harmonically related complex exponential sequences with fundamental frequency $\frac{2\pi}{N}$
  - General form of complex exponential: $e_k[n] = e^{j\frac{2\pi}{N}kn} = e_k[n + rN]$
    - where k is an integer
- Fourier series representation: $\tilde{x}[n] = \frac{1}{N}\sum_k \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$
  - N harmonically related complex exponentials
  - $e_{k+lN}[n] = e^{j\frac{2\pi}{N}(k+lN)n} = e^{j\frac{2\pi}{N}kn}e^{j2\pi ln} = e^{j\frac{2\pi}{N}kn} = e_k[n]$, for $l \in I$
    - i.e $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, ...
  - All the distinct periodic complex exponentials with frequencies that are integer multiples of $\frac{2\pi}{N}$
    - Set of N periodic complex exponentials $\{e_0[n], e_1[n], ..., e_{N-1}[n]\}$
- Thus, the Fourier series representation of a periodic sequence $\tilde{x}[n]$ need contain only N of these complex exponentials
  - Synthesis equation: $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$

Analysis equation

- Calculate Fourier series coefficients, $\tilde{X}[k]$ using orthogonality of complex exponential $e_k[n]$, for $k \in \{0, 1, ..., N-1\}$
  - $\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & k-r = mN, m \in I \\ 0, & otherwise \end{cases}$
- Multiply synthesis equation by $e^{-j\frac{2\pi}{N}rn}$ and summing from $n = 0$ to $N-1$
  -

$$\begin{aligned}
\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}rn} &= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}(k-r)n} \\
&= \sum_{k=0}^{N-1} \tilde{X}[k] \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n}\right] \\
&= \tilde{X}[r]
\end{aligned}$$

- Analysis equation: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$

Analysis equation

- Sequence $\tilde{X}[k]$ is periodic with period N
  - $$\begin{aligned}\tilde{X}[k+N] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}(k+N)n} \\ &= \left(\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}\right) e^{-j2\pi n} \\ &= \tilde{X}[k]\end{aligned}$$
  - For any integer k

Discrete Fourier series (DFS)

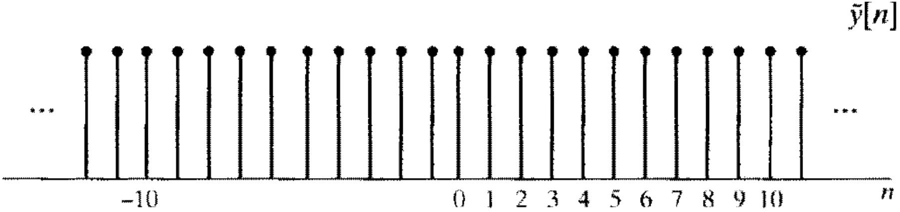
- Analysis equation: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$
- Synthesis equation: $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$
  - $W_N = e^{-j\frac{2\pi}{N}}$
- $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences
- Notation: $\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k]$

Example: DFS of a periodic impulse train

- Consider the periodic impulse train:
  - $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, r \in I \\ 0, & otherwise \end{cases}$
  - For $0 \leq n < N$, $\tilde{x}[n] = \delta[n]$
- $\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1$ (same for all k)
- Substituting in synthesis equation:
  - $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}$
  - A periodic impulse train can be expressed as a sum of complex exponentials having equal magnitude and phase
    - They add to unity at integer multiples of N and to zero for all other integers

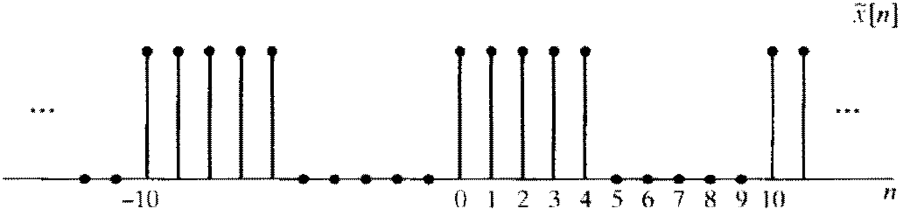
Example: Find a sequence whose DFS is a periodic impulse train

- Let $\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN]$
- Synthesis equation: $\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} N\delta[k]W_N^{-kn} = W_N^0 = 1$ for all n



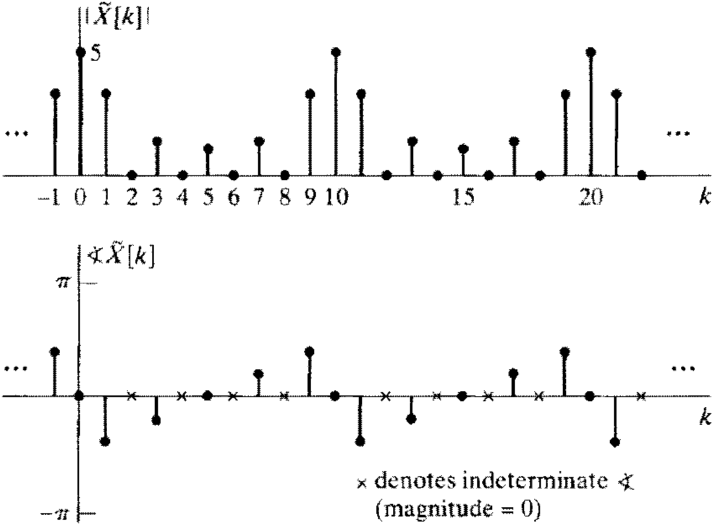
Example: DFS of a periodic rectangular pulse train

- Consider a periodic sequence $\tilde{x}[n]$ with $N = 10$ as shown below



- $\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j\frac{2\pi}{10}kn} = e^{-j\frac{4\pi k}{10}} \frac{sin(\frac{\pi k}{2})}{sin(\frac{\pi k}{10})}$

Example: DFS of a periodic rectangular pulse train



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Linearity

- Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N such that
  - $\tilde{x}_1[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]$
  - $\tilde{x}_2[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k]$
- Then, $a\tilde{x}_1[n] + b\tilde{x}_2[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]$
- Proof: Apply synthesis and analysis equations

Shift of a Sequence

- If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$
- DFS of a shifted version of $\tilde{x}[n]$, $\tilde{x}[n-m]$ is given by:
    - $\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]$
- Any shift that is greater than or equal to the period (i.e., $m \geq N$)
    - Cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2 N$
    - $m_1, m_2 \in I$ and $0 \leq m_1 \leq N-1$
    - Hence, $\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k] = W_N^{km_1}\tilde{X}[k] \overset{\mathcal{IDFS}}{\longleftrightarrow} \tilde{x}[n-m_1]$
- Similar result applies to a shift in the Fourier coefficients by an integer l
    - $W_N^{-nl}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-l]$

Duality

- DFS analysis and synthesis equations differ only in a factor of $\frac{1}{N}$ and in the sign of the exponent of W_N
- Periodic sequence and its DFS coefficients are the same kinds of functions
  - Both are periodic sequences
- $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \rightarrow N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{kn}$
  - By replacing n by $-n$
- Interchange the variables n and k $\rightarrow N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n] W_N^{kn}$
  - Takes the form of analysis equation
  - DFS coefficients of the periodic sequence $\tilde{X}[n]$ is $N\tilde{x}[-k]$
- $\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \rightarrow \tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]$

Periodic convolution

- Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N
  - $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$ be their DFS coefficients
- $\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k] = \tilde{X}_3[k]$
  - $\tilde{x}_3[n]$ is the periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}_3[n] W_N^{kn} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] W_N^{kn}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right)$$

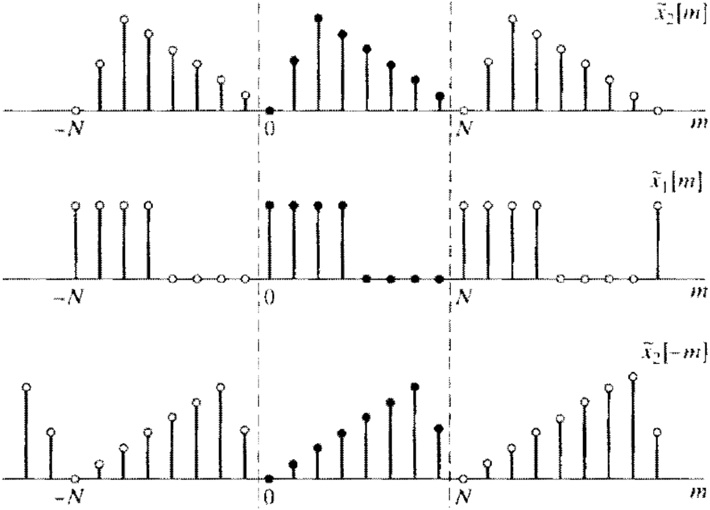
Interchange the order of summations

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \tilde{X}_2[k]$$

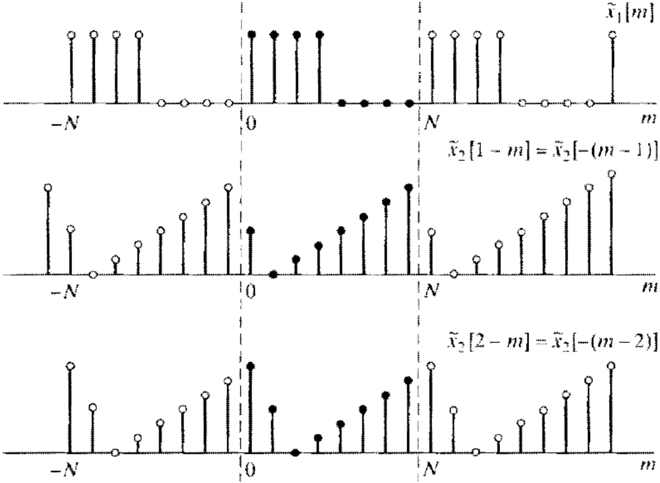
Using the shifting property of DFS

$$= \tilde{X}_1[k]\tilde{X}_2[k]$$

Periodic convolution example



Periodic convolution example



Duality property on periodic convolution

- $\tilde{x}_3[n] = \tilde{x}_1[k]\tilde{x}_2[k] \overset{\mathcal{DFS}}{\longleftrightarrow} \frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l] = \tilde{X}_3[k]$

Summary of properties of DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

Summary of properties of DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{J}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{J}m\{\tilde{X}[k]\} = -\mathcal{J}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$

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From DFS to Fourier transform

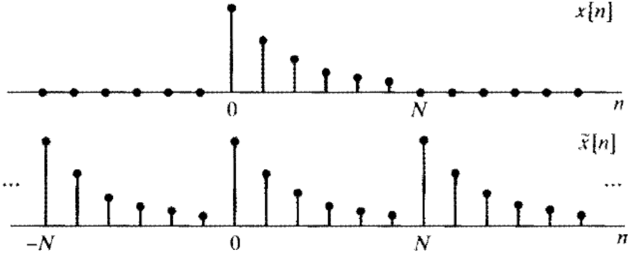
- DFS representation of periodic signals can be incorporated within the framework of the Fourier transform
  - Interpreting the Fourier transform of a periodic signal to be an impulse train in the frequency domain
  - Impulse values proportional to the DFS coefficients for the sequence
- Let $\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k]$ (Both periodic with period N)
- Fourier transform of $\tilde{x}[n]$ is defined as:
  - $\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$
  - $\tilde{X}(e^{j\omega})$ is periodic with period 2π

Example: Fourier transform of a periodic impulse train

- A periodic impulse train: $\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$
- Its DFS coefficients are: $\tilde{P}[k] = 1$ for all k
- Fourier transform: $\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi k}{N}\right)$

Relation between a periodic signal and a finite-length signal

- Consider a finite-length signal $x[n]$, such that $x[n] = 0$ except in the interval $0 \le n \le N-1$
- Consider the convolution of $x[n]$ with the periodic impulse train $\tilde{p}[n]$
  - $\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN]$
  - $\tilde{x}[n]$ consists of a set of periodically repeated copies of the finite-length sequence $x[n]$



Relation between a periodic signal and a finite-length signal

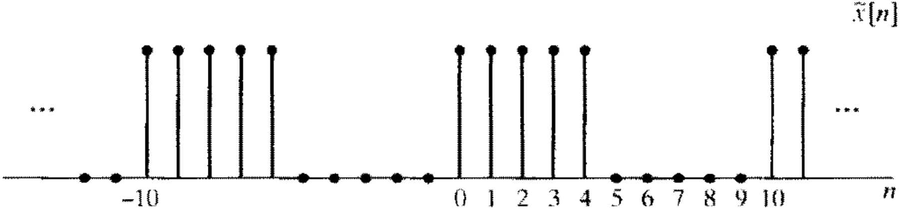
- Let Fourier transform of $x[n]$ is $X(e^{j\omega})$
- Fourier transform of $\tilde{x}[n]$:

$$\begin{aligned}\tilde{X}(e^{j\omega}) &= X(e^{j\omega})\tilde{P}(e^{j\omega}) \\ &= X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j\frac{2\pi k}{N}})\delta\left(\omega - \frac{2\pi k}{N}\right)\end{aligned}$$

- Comparing this with the expression of Fourier transform of a periodic sequence:
  - $\tilde{X}[k] = X(e^{j\frac{2\pi k}{N}}) = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}}$
  - DFS coefficients $\tilde{X}[k]$ of $\tilde{x}[n]$ are equally spaced samples of the Fourier transform of the finite-length sequence $x[n]$
    - Obtained by extracting one period of $\tilde{x}[n]$

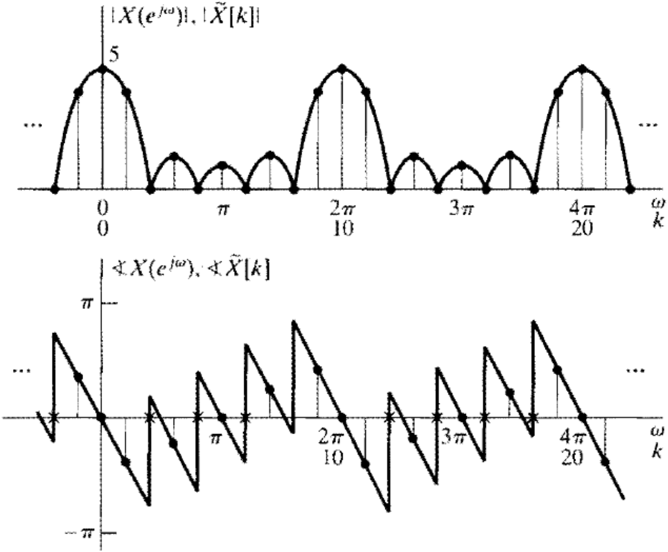
Example: Fourier transform of a rectangular pu

- Consider the periodic sequence $\tilde{x}[n]$ with $N = 10$ as shown below



- One period of $\tilde{x}[n]$ is $x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & otherwise \end{cases}$
- Fourier transform of $x[n]$: $X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega} \frac{sin(\frac{5\omega}{2})}{sin(\frac{\omega}{2})}$
- We can find DFS representation of periodic rectangular pulse train from Fourier transform of rectangular pulse
  - $\tilde{X}[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{10}} = e^{-j\frac{4\pi k}{10}} \frac{sin(\frac{\pi k}{2})}{sin(\frac{\pi k}{10})}$

Example: Fourier transform of a rectangular pu



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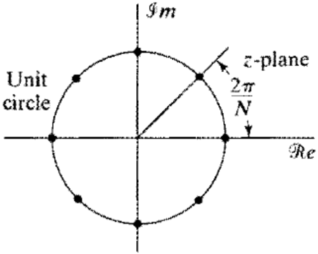
Sampling the Fourier transform

- Let an aperiodic sequence $x[n]$ has Fourier transform $X(e^{j\omega})$
  - $X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}$
- $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = \frac{2\pi k}{N}$
  - $\tilde{X}[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = X(e^{j\frac{2\pi k}{N}})$
- $X(e^{j\omega})$ is periodic in ω with period $2\pi \rightarrow \tilde{X}[k]$ is periodic in k with period N

- Fourier transform $=$ z-transform evaluated on the unit circle
- $\tilde{X}[k]$ can be obtained by sampling $X(z)$ at N equally spaced points on the unit circle



Sequence whose DFS coefficients are $\tilde{X}[k]$

- $\tilde{X}[k]$, being periodic with period N, could be the DFS coefficients of a sequence $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j\frac{2\pi k}{N}}) W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi k}{N}m}\right] W_N^{-kn}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right]$$

Interchange the order of summations and $W_N = e^{-j\frac{2\pi}{N}}$

$$= \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]$$

DFS representation of shifted periodic impulse train

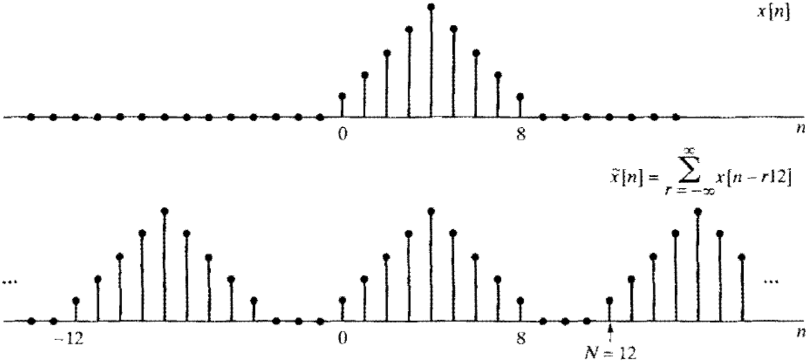
$$\tilde{p}[n-m] = \sum_{r=-\infty}^{\infty} \delta[n-m-rN] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}$$

Sequence whose DFS coefficients are $\tilde{X}[k]$

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - m - rN]$$
$$= \sum_{r=-\infty}^{\infty} x[n - m - rN]$$

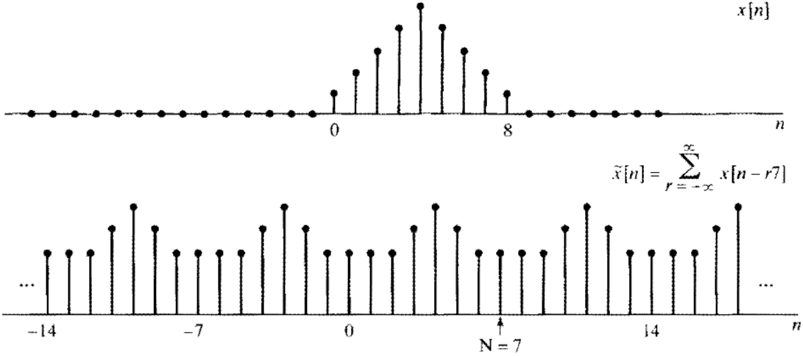
- $\tilde{x}[n]$ is the periodic sequence that results from the aperiodic convolution of $x[n]$ with a periodic unit-impulse train
- Infinite number of shifted replicas of $x[n]$
  - Shifted by integer multiples of N (Number of samples of $X(e^{j\omega})$ taken on the unit circle)

Case 1: $N >$ length of sequence $x[n]$



- Length of sequence $x[n] = 9$, and $N = 12$
  - Delayed replications of $x[n]$ do not overlap
  - One period of $\tilde{x}[n]$ is recognizable as $x[n]$
- Consistent with the observation:
  - "Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period"

Case 2: $N <$ length of sequence $x[n]$



- Length of sequence $x[n] = 9$, and $N = 7$
  - Delayed replications of $x[n]$ DO overlap
  - One period of $\tilde{x}[n]$ is no longer identical to $x[n]$
- DFS coefficients of $\tilde{x}[n]$ are still samples of the Fourier transform of $x[n]$ spaced in frequency at integer multiples of $\frac{2\pi}{N}$

Discussion

- Looks a bit similar to the concept of “Sampling” discussed earlier
  - However, here we are sampling in the frequency domain rather than in the time domain
- For $N >$ length of sequence $x[n]$
  - $x[n]$ can be recovered from $\tilde{x}[n]$ by extracting one period
  - Fourier transform $X(e^{j\omega})$ can be recovered from the samples spaced in frequency by $\frac{2\pi}{N}$
    - Fourier transform of $x[n]$ has been sampled at a sufficiently small spacing (in frequency) to be able to recover it from these samples

Discussion

- For $N <$ length of sequence $x[n]$
  - $x[n]$ cannot be recovered by extracting one period of $\tilde{x}[n]$
    - A form of aliasing in the time domain
    - Identical to the frequency-domain aliasing that results from undersampling in the time domain
  - Fourier transform $X(e^{j\omega})$ cannot be recovered from its samples
    - Fourier transform has been undersampled
- Just as frequency-domain aliasing can be avoided only for signals that have band limited Fourier transforms
  - Time-domain aliasing can be avoided only if $x[n]$ has finite length

Discussion

- If $x[n]$ has finite length and we take a sufficient number (greater than or equal to the length of $x[n]$) of equally spaced samples of its Fourier transform
  - Then Fourier transform is recoverable from these samples
  - $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$ using $x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \textit{otherwise} \end{cases}$
- To recover $x[n]$ it is not necessary to know $X(e^{j\omega})$ at all frequencies if $x[n]$ has finite length
  - We can form a periodic sequence $\tilde{x}[n]$
  - Which can be represented by DFS coefficients $\tilde{X}[k]$
- $\rightarrow$ *Discrete Fourier transform* (DFT) analysis

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Associate a periodic sequence to a finite-duration sequence

- Consider a finite-length sequence $x[n]$ of length N
  - $x[n] = 0$ outside the range $0 \leq n \leq N-1$
- We can always associate following periodic sequence to $x[n]$
  - $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN] = x[((n))_N]$
    - $((n))_N$ denote n modulo N
    - $\tilde{x}[n]$ can be visualized as wrapping a plot of $x[n]$ around a cylinder with a circumference equal to N
    - As we repeatedly traverse the circumference of the cylinder, we see the finite-length sequence periodically repeated
  - $x[n]$ is recoverable from $\tilde{x}[n]$ using $x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & otherwise \end{cases}$
    - Can be visualized as unwrapping the cylinder and laying it flat
    - Sequence is displayed on a linear time axis rather than a circular (modulo N) time axis

Associate a periodic sequence to a finite-duration sequence

- DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $\frac{2\pi}{N}$) of the Fourier transform of $x[n]$, i.e., $X(e^{j\omega})$
- $\tilde{X}[k]$ is itself a periodic sequence with period N
- We choose the Fourier coefficients that we associate with $x[n]$ to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$
  - This finite-duration sequence, $X[k]$, will be referred to as the *discrete Fourier transform* (DFT) of $x[n]$
  - $X[k] = \begin{cases} \tilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \textit{otherwise} \end{cases}$
  - $\tilde{X}[k] = X[((k))_N]$

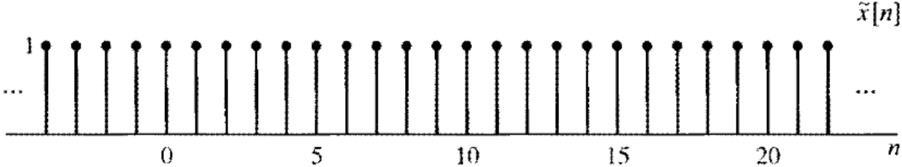
DFT analysis and synthesis equations

- DFS analysis and synthesis equations are given by:
  - Analysis equation: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$
  - Synthesis equation: $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$
    - $W_N = e^{-j\frac{2\pi}{N}}$
- Since the summations in these equations involve only the interval between zero and $(N-1)$
  - Where $\tilde{x}[n] = x[n]$ and $\tilde{X}[k] = X[k]$
- DFT analysis and synthesis equations are given by:
  - Analysis equation: $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$
  - Synthesis equation: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$
  - $X[k] = 0$ for k outside the interval $0 \leq k \leq N-1$ and that $x[n] = 0$ for n outside the interval $0 \leq n \leq N-1$
- Notation: $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$

Example: DFT of a rectangular pulse

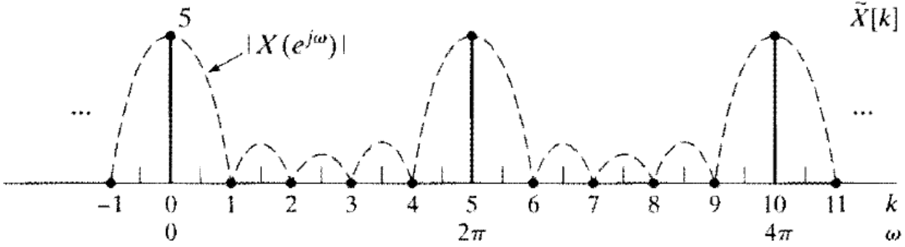
- Consider $x[n] = \begin{cases} 1, & 0 \leq k \leq 4 \\ 0, & otherwise \end{cases}$

- For calculating DFT, we can consider $x[n]$ as a finite-duration sequence of any length greater than or equal to 5
  - Let's consider $N = 5$



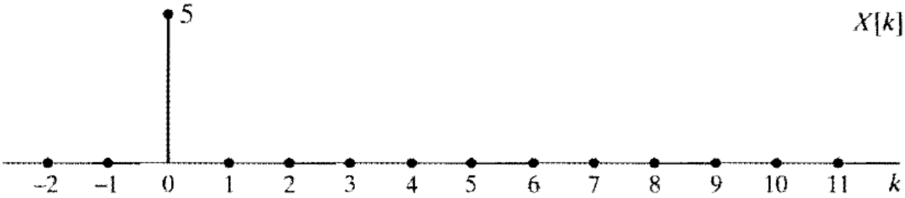
Example: DFT of a rectangular pulse

- $\tilde{X}[k] = \sum_{n=0}^{4} e^{-j\frac{2\pi k}{5}n} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi k}{5}}} = \begin{cases} 5, & k = 0, \pm 5, \pm 10, ... \\ 0, & otherwise \end{cases}$
- $\tilde{X}[k]$ is a sequence of samples of $X(e^{j\omega})$ at frequencies $\omega_k = \frac{2\pi k}{5}$



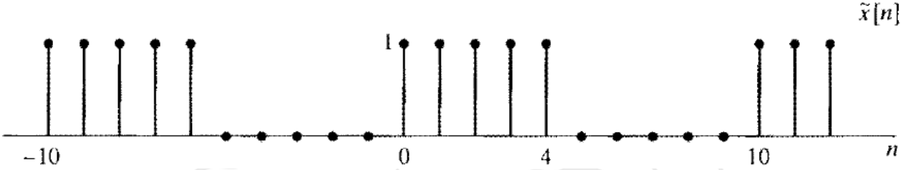
Example: DFT of a rectangular pulse

- Five-point DFT of $x[n]$ corresponds to one period of $\tilde{X}[k]$ as shown below



Example: DFT of a rectangular pulse

- It we consider $x[n]$ to be of length $N = 10$, underlying periodic sequence $\tilde{x}[n]$ is:



Example: DFT of a rectangular pulse

- Ten-point DFT of $x[n]$ is:

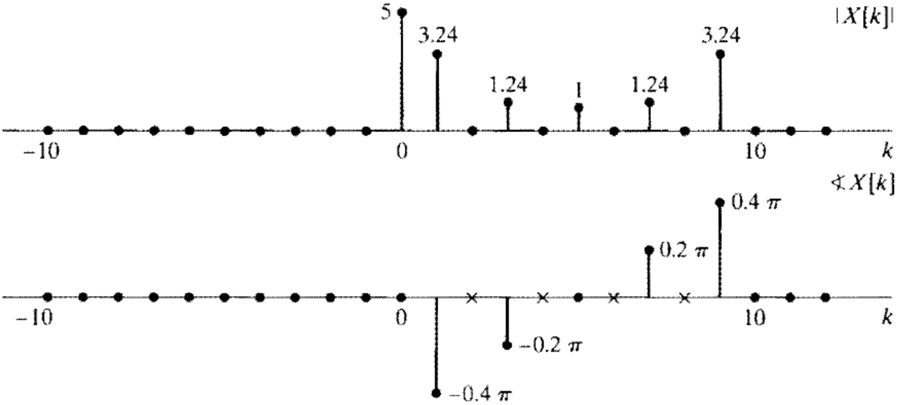


Table of Contents

Linearity

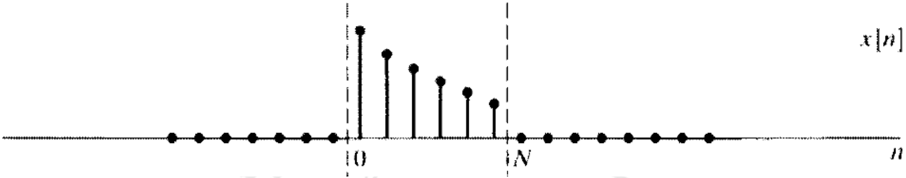
- If two finite-duration sequences $x_1[n]$ and $x_2[n]$ are linearly combined
  - $x_3[n] = ax_1[n] + bx_2[n]$
  - If $x_1[n]$ has length N_1 and $x_2[n]$ has length N_2, then the length of $x_3[n]$ will be $N_3 = max(N_1, N_2)$
- Then, the DFT of $x_3[n]$ is: $X_3[k] = aX_1[k] + bX_2[k]$
  - Both DFTs ($X_1[k]$ and $X_2[k]$) must be computed with the same length $N \geq N_3$
  - $ax_1[n] + bx_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} aX_1[k] + bX_2[k]$

Circular shift of a sequence

- Observations:
  - $x[n] \overset{\mathcal{FT}}{\longleftrightarrow} X(e^{j\omega}) \rightarrow x[n-m] \overset{\mathcal{FT}}{\longleftrightarrow} e^{-j\omega m} X(e^{j\omega})$
  - $\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \rightarrow \tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} e^{-j\frac{2\pi k}{N} m} \tilde{X}[k]$
- $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k] \rightarrow x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = e^{-j\frac{2\pi k}{N} m} X[k]$
  - $x[n]$ and $x_1[n]$ are both zero outside the interval $0 \leq n \leq N-1$
  - $\rightarrow x_1[n]$ cannot result from a simple time shift of $x[n]$
- Based on the interpretation of the DFT as the DFS coefficients of the periodic sequence:
  - $\tilde{x}_1[n] = x_1[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] = X_1[((k))_N]$
  - $\rightarrow \tilde{X}_1[k] = e^{-j\frac{2\pi((k))_N}{N} m} X[((k))_N] = e^{-j\frac{2\pi k}{N} m} \tilde{X}[k]$
  - $\rightarrow \tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N]$
  - $\rightarrow x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-m))_N], & 0 \leq n \leq N-1 \\ 0, & otherwise \end{cases}$

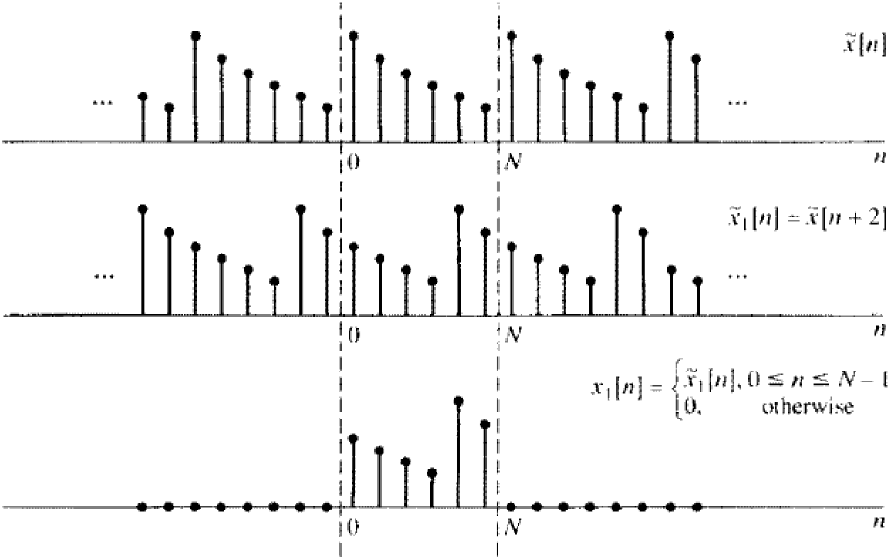
Circular shift example

- Consider finite-duration sequence $x[n]$ as shown below



- We want to determine $x_1[n] = x[((n+2))_N]$ for $N = 6$
  - Step 1: From $x[n]$, construct the periodic sequence $\tilde{x}[n] = x[((n))_6]$
  - Step 2: Shift $\tilde{x}[n]$ by 2 to the left obtaining $\tilde{x}_1[n] = \tilde{x}[n+2]$
  - Step 3: Extract one period of $\tilde{x}_1[n]$ to obtain $x_1[n]$
- $x_1[n]$ will have DFT $X_1[k] = W_6^{-2k} X[k]$

Circular shift example



Circular shift example

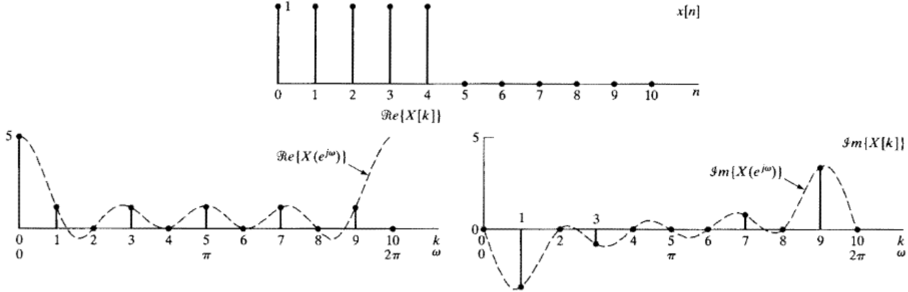
- $x_1[n]$ can be formed by *circularly* shifting $x[n]$
  - Sequence value leaves the interval 0 to $N-1$ at one end and enters at the other end
- Consider another sequence $x_2[n] = x[((n-4))_6]$ formed by circularly shifting the sequence $x[n]$ by 4 to the right
  - $x_2[n] = x_1[n]$
  - Also, DFTs $X_1[k] = X_2[k]$ since $W_6^{-2k} = W_6^{4k}$
- More generally, $W_N^{mk} = W_N^{-(N-m)k}$
  - N-point circular shift in one direction by m is the same as a circular shift in the opposite direction by $N-m$

Duality

- Let $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$
- $\tilde{x}[n] = x[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] = X[((k))_N]$
- From Duality of DFS: $\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]$
- Consider periodic sequence $\tilde{x}_1[n] = \tilde{X}[n]$, one period of which is the finite-length sequence $x_1[n] = X[n]$
  - $\tilde{x}_1[n] = \tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] = N\tilde{x}[-k]$
- $x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = \begin{cases} N\tilde{x}[-k] = Nx[((-k))_N], & 0 \le k \le N-1 \\ 0, & \textit{otherwise} \end{cases}$
- $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k] \rightarrow X[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Nx[((-k))_N], 0 \le k \le N-1$
  - sequence $Nx[((-k))_N]$ is $Nx[k]$ index reversed, modulo N

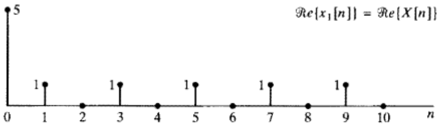
Duality example

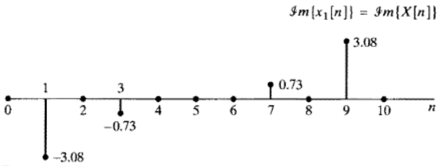
- Finite-length sequence $x[n]$ and its 10-point DFT $X[k]$



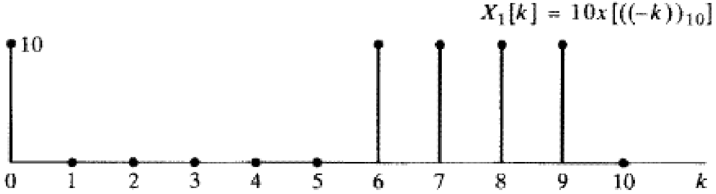
Duality example

- Obtain the complex sequence $x_1[n] = X[n]$ by relabeling the horizontal axis





- Its 10-point DFT (using duality property) is:



Summary of properties of DFT

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k] X_2[k]$
8.	$x_1[n] x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$

Summary of properties of DFT

10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{J}m\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

Properties 15–17 apply only when $x[n]$ is real.

15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

Thanks.