Department of Electronics and Communication Engineering

Digital signal processing

Subject Code – **ECN-312**

Digital filter Design (IIR)

Frequency Selective Response

Aim

Find appropriate coefficients of FIR , IIR filters to approximate desired gain/phase characteristics within specified frequency range.

Some General Observations

Requirement	Preferred Filter
1. Linear Phase	FIR
2. Lower Side-bands	IIR
3.Less memory	IIR
4. Fewer coefficients	IIR
5. Lower complexity	IIR

Why

IIR Filter Design

- Better performance with less coefficients.
- Analog filter to digital conversion due to mature field of Analog filters.

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

***Can be given by its coefficients / its impulse response**

$$\boxed{H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt}$$

$$\boxed{\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}}$$

Stability condition while using these conversions

Analog linear invariant system is stable , if its all poles lie on left-half of the plane.

Methods should take care that the

The $j\Omega$ axis in the s-plane $\xrightarrow{\text{Map}}$ unit circle in the z-plane

(LHP) of the s-plane $\xrightarrow{\text{Map}}$ inside of the unit circle in the z-plane

IIR filters theoretically cannot have linear phase

- Condition of linear phase in Z-domain

$$H(z) = \pm z^{-N} H(z^{-1})$$

- All poles/zeros will have mirror image counterpart along the radius of that pole/zero.

Stability condition will not satisfy!

The main focus is on magnitude characteristics! Phase is related to the magnitude for a causal system.

IIR Filter Design by impulse invariance method

Sampling of analog filter impulse response:

$$h(n) \equiv h(nT), \qquad n = 0, 1, 2, \ldots$$

where T is the sampling interval.

Recalling the relation between spectrum of sampled signal and spectrum of analog signal:

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f-k)F_s]$$

where $f = F/F_s$ is the normalized frequency

Impulse invariance method

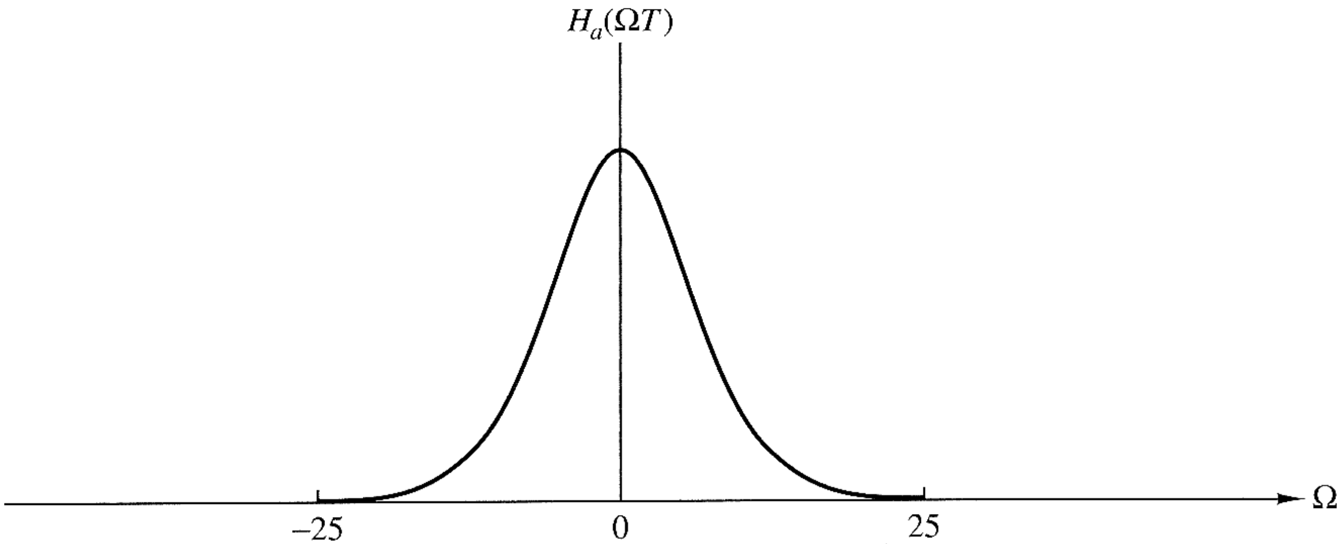
- Aliasing occurs if the Fs is less than twice of the highest frequency component of the signal.

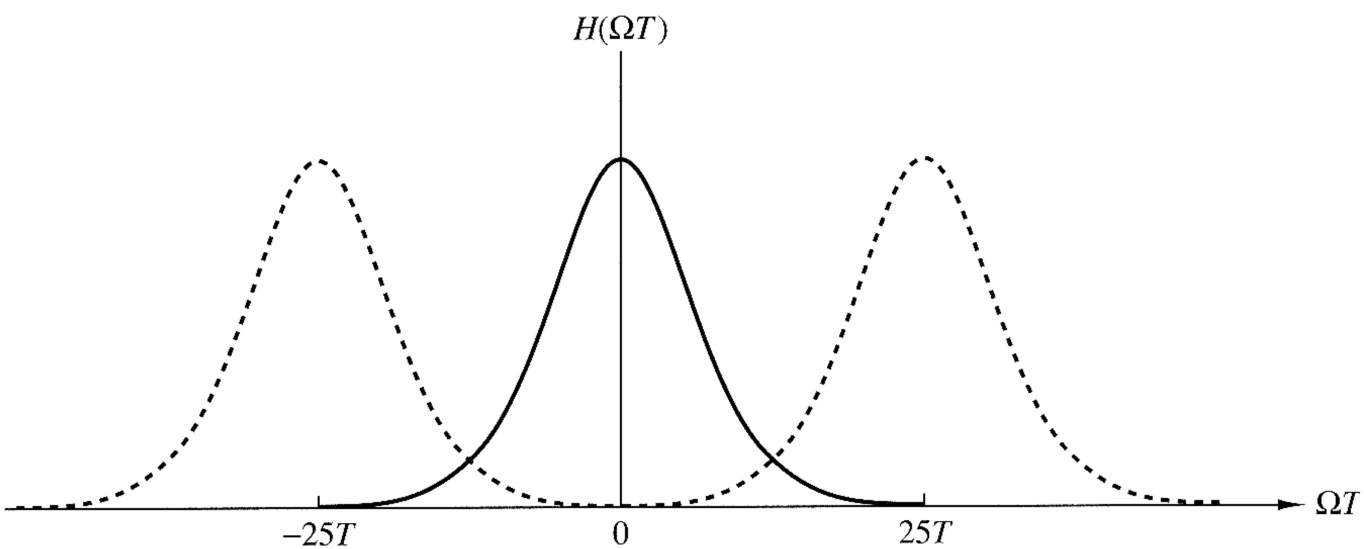
$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s]$$

or $$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

Filter response of analog and digital filter





Observations

- If time period is very small then analog and digital will be similar.
- Not appropriate for designing high-pass filters due to aliasing components.

We need to investigate the relation of z-transform of $h(n)$ to the Laplace transform of $h_a(t)$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \Rightarrow H(z)|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn}$$

$$\text{when } s = j\Omega,$$

$$H(z)\big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left[\left(s - j\frac{2\pi k}{T}\right)\boldsymbol{(-j)}\right]$$

Now let us consider the mapping given by $z = e^{sT}$

If we substitute $s = \sigma + j\Omega$ $\quad z = re^{j\omega}$

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T} \Rightarrow r = e^{\sigma T}$$

$$\omega = \Omega T$$

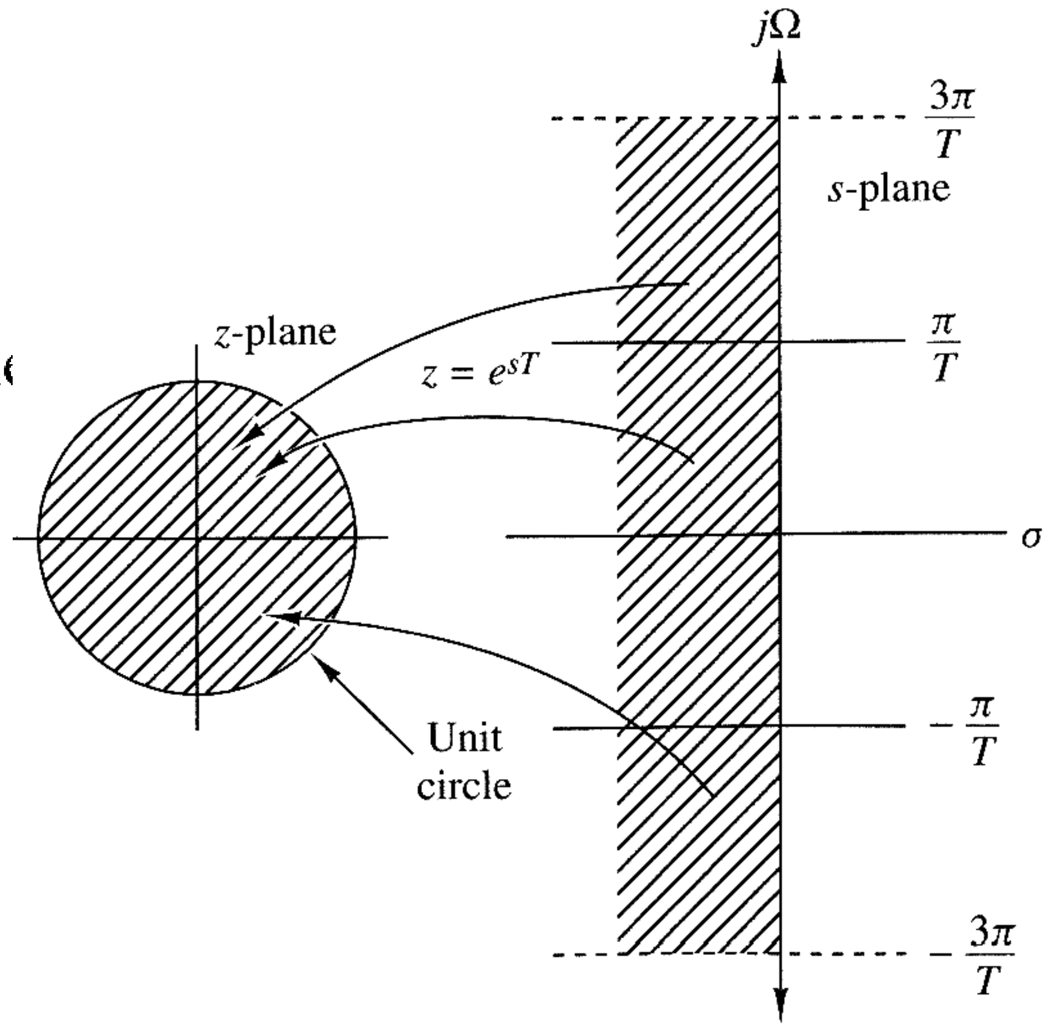
$\sigma < 0$ implies that $0 < r < 1$

$\sigma > 0$ implies that $r > 1$

$\sigma = 0$, we have $r = 1$

Mapping for stable case

The mapping of $z = e^{sT}$ maps strips of width $2\pi/T$ (for $\sigma < 0$) in the s-plane into points in the unit circl in the z-plane.



Mapping from analog to digital domain is many to one!

Digital domain frequency lies between $(-\pi, \pi)$

$$(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T, \Rightarrow -\pi < \omega < \pi$$

Rearrange analog filter system function into partial-fraction form

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \Rightarrow h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}, \qquad t \geq 0$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} c_k e^{p_k T n}$$

Use this $h(n)$ for deciding filter expression in digital domain:

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n}$$

$$= \sum_{k=1}^{N} c_k \underbrace{\sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n}$$

$$\sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n = \frac{1}{1 - e^{p_k T} z^{-1}}$$

Therefore, the system function of the digital filter is

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots, N$$

<u>This method will be successful for distinctive poles !</u>

Example:

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by means of the impulse invariance method.

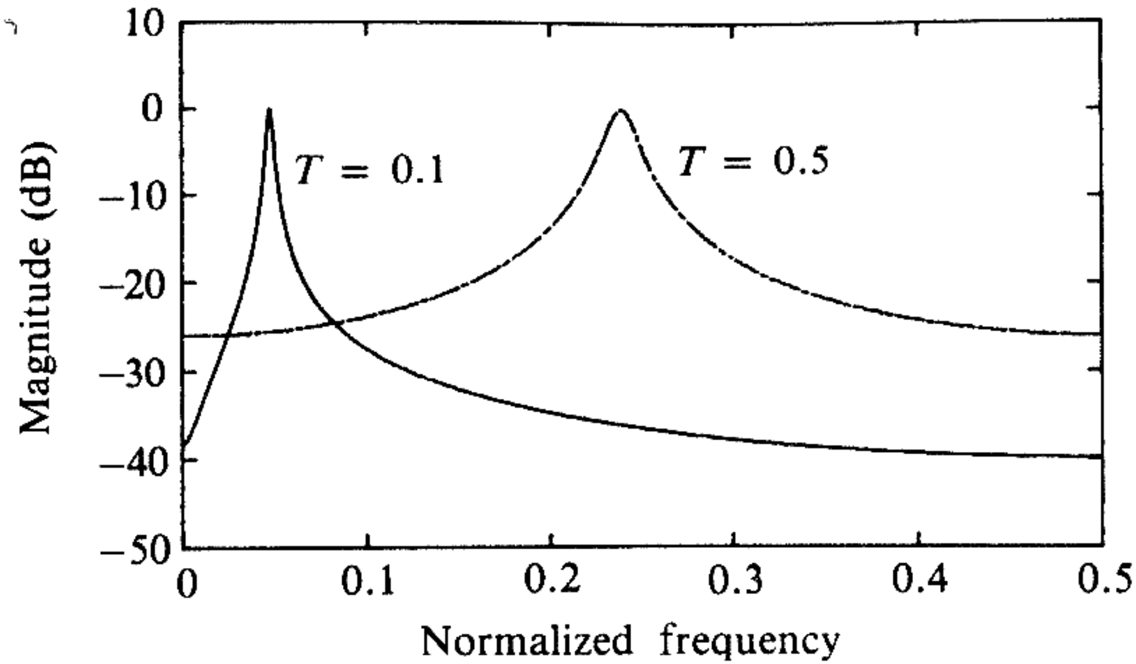
$$p_k = -0.1 \pm j3$$

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

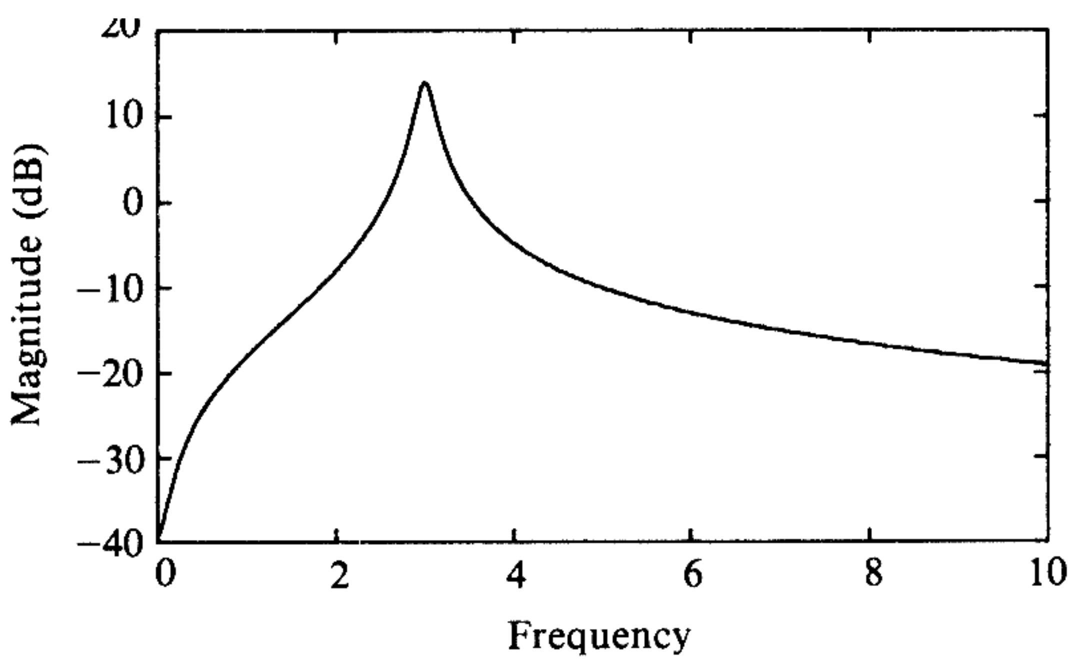
OR

$$H(z) = \frac{1 - (e^{-0.1T} \cos 3T) z^{-1}}{1 - (2e^{-0.1T} \cos 3T) z^{-1} + e^{-0.2T} z^{-1}}$$



Digital

Repeat this problem in MATLAB with different *T* values.



Analog

Bilinear Transformation

- Impulse invariance method is limited to low-pass and small cases of band-pass filters.
- Bilinear transformation maps the unit circle only once, avoiding aliasing of components.
- Derivation starts by considering trapezoidal rules for integration, lets start by considering

$$H(s) = \frac{b}{s + a}$$

This system is also characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$$

$$\Downarrow$$

Trapezoidal rule in discrete domain

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T)$$

$$y'(nT) = -ay(nT) + bx(nT)$$

$$\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}[x(n)+x(n-1)]$$

The z-transform of this difference equation is

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}(1+z^{-1})X(z)$$

the system function of the equivalent digital filter is

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(bT/2)(1+z^{-1})}{1+aT/2-(1-aT/2)z^{-1}}$$

$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

the mapping from the s-plane to the z-plane is

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

To investigate the characteristics of the bilinear transformation, let

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

$$s = \frac{2}{T}\frac{z-1}{z+1} \quad = \frac{2}{T}\frac{re^{j\omega}-1}{re^{j\omega}+1}$$

$$= \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}$$

$$\Omega = \frac{2}{T}\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}$$

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1 + \cos\omega}$$

$$= \frac{2}{T}\tan\frac{\omega}{2} \quad \Rightarrow \quad \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 16}$$

resonant frequency of $\omega_r = \pi/2$.

$\Omega_r = 4 \qquad \omega_r = \pi/2$

$T = \frac{1}{2} \qquad s = 4\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-1}}{1 + 0.0006z^{-1} + 0.975z^{-2}}$$

$$p_{1,2} = 0.987e^{\pm j\pi/2}$$

$$z_{1,2} = -1, 0.95$$

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.975z^{-2}}$$