ECN – 347

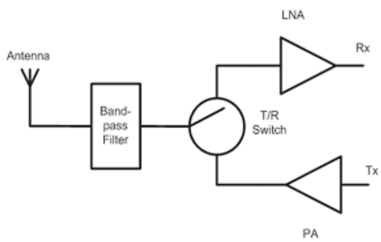
Introduction to Microwave Semiconductor Device Modelling Techniques

CWS: 25 MTE: 25 ETE: 50

Books:

1. R. H. Caverly, “Microwave and RF Semiconductor Control Device Modelling”, Artech House
2. K. Chang, I. Bahl and V. Nair, “RF and Microwave Circuit and Component Design for Wireless Systems”, John Wiley & Sons

Microwave and RF Semiconductor Control Devices



PIN Diode Control Elements
FET-Based Control Elements

Switch Concept:

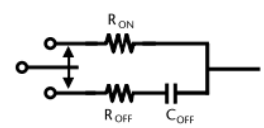
The basic concept of the switch depends on the location of the control element and its impedance Z_{CTL} in the control circuit.

For a series-connected switch element, low impedance Z_{low} is required for the low loss or on-state, whereas a high impedance Z_{high} is required for the high loss or off-state.

For a shunt-connected switch element, the opposite is true: a high impedance is required for the on-state and a low impedance for the off-state.

In an ideal series-connected mechanical switch, these two impedance states would correspond to the switch contacts touching with very small resistance $Z_{low} = R_{ON}$ between the contacts; the open switch would then have infinite resistance.

- However, while this may be true at dc, in the open-switch condition, the two air-separated contacts would create a capacitance C_{off} that is a function of the area and the contact separation distance.
- This capacitance exhibits a reactance of $\frac{1}{j\omega C_{off}}$ and would show a frequency dependent loss in the off state, with the loss decreasing with increasing frequency.
- In the off state, there will be a small resistance in series with this capacitance, R_{OFF}, because of the finite conductivity and dimensions of the contacts.
- The high impedance condition of the switch state Z_{high} is the series sum of these two components, $Z_{\mathrm{high}} = R_{\mathrm{OFF}} + \frac{1}{J\omega C_{\mathrm{OFF}}}$.
- The switch actuator then automatically controls which state is in play.



- In case of the semiconductor control devices, the impedance concepts are exactly the same: the control elements are required to exhibit low or high impedance values, depending on their circuit connections and the control state (on or off).
- The actuator that controls the impedance state is no longer mechanical but will be some electrical signal from one or more control points that governs the semiconductor device operating behavior as it toggles between the two impedance states.
- For FET-based circuits, the “actuator” is voltage-based and the application or removal of a voltage on the gate terminal controls the drain-source channel impedance state.

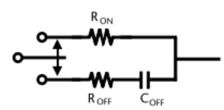
Switching Quality Factor (Q):

The two switch states (States 1 and 2) are defined by impedances based on the equivalent circuit of the switch element, $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$ when placed at the output of a lossless two-port network described by S-parameters S_{ij}. The switching quality factor is then defined in terms of these impedances, Z_i, or the corresponding reflection coefficients, Γ_i:

$$Q = \frac{\sqrt{(R_1 - R_2)^2 + (X_1 - X_2)^2}}{\sqrt{R_1 R_2}} = \frac{2|\Gamma_1 - \Gamma_2|}{\sqrt{\left(1 - |\Gamma_1|^2\right)\left(1 - |\Gamma_2|^2\right)}}$$

Using the general switch equivalent circuit shown in Figure where State 1 is the on-state and State 2 is the off-state, and further, if the difference between R_{ON} and R_{OFF} is significantly less than the reactance X_2, the switching Q can be written as

$$Q = \frac{X_2}{R_{ON}} = \frac{1}{\omega\sqrt{R_{ON}R_{OFF}}C_{OFF}}$$



where ω is the radian frequency.

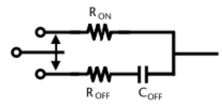
The *RC* product in the above eqn. can be used to define a cutoff frequency called the switch cutoff frequency, F_C, which allows the switching Q under this scenario to be written as:

$$F_C = \frac{1}{2\pi\sqrt{R_{ON}R_{OFF}}C_{OFF}}$$

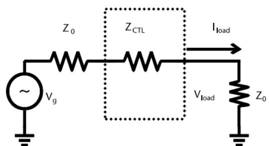
$$Q = \frac{F_C}{F}$$

where F is the frequency ($\omega = 2\pi F$)

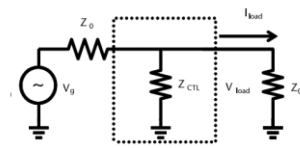
Circuit Analysis



[Simplified switch model]

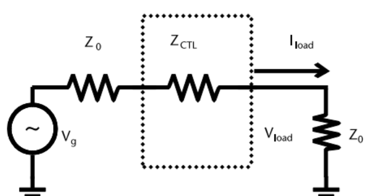


[Series-connected control element in a circuit]



[Shunt-connected control element in a circuit]

Series Connection



The control element (with impedance Z_{CTL}) is series connected in a Z_0 system with the generator matched to the load.

Using traditional circuit analysis, the peak voltage and current at the load is:

$$V_{\text{load}} = V_g \frac{Z_0}{2Z_0 + Z_{\text{CTL}}} \qquad I_{\text{load}} = V_g \frac{1}{2Z_0 + Z_{\text{CTL}}} = \frac{V_{\text{load}}}{Z_0}$$

The power available from the generator P_A is the load power in the absence of the control element and is defined as

$$P_A = \frac{V_g V_g^*}{8Z_0}$$

where the generator voltage is assumed to be a peak voltage and * denotes the complex conjugate.

The corresponding load power with in the circuit can then be computed, using the above equation, as

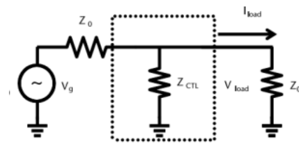
$$\begin{aligned} P_{\text{load}} &= \frac{1}{2} RE\left(V_{\text{load}} I_{\text{load}}^*\right) \\ &= \frac{1}{2} RE\left[\left(V_g \frac{Z_0}{2Z_0 + Z_{\text{CTL}}}\right)\left(\frac{V_g}{Z_0} \frac{Z_0}{2Z_0 + Z_{\text{CTL}}}\right)^*\right] \\ &= P_A \left|\frac{2Z_0}{2Z_0 + Z_{\text{CTL}}}\right|^2 \end{aligned}$$

RE(): Real part

$$P_{\text{load}} = P_A \left|\frac{2Z_0}{2Z_0 + Z_{\text{CTL}}}\right|^2$$

- For Z_{CTL} much less than Z_0, the load and the available power are approximately the same.
- A high impedance value for Z_{CTL} corresponds to a significant reduction in the load power compared with the available power from the generator.

Shunt Connection

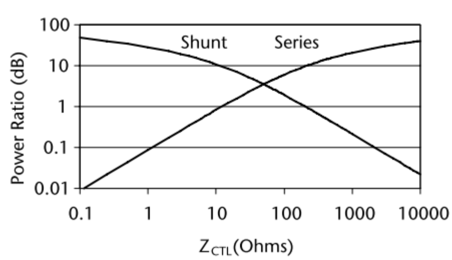


$$V_{load} = V_g \frac{Z_{CTL}}{2Z_{CTL} + Z_0}$$

$$I_{load} = \frac{V_g}{Z_0} \frac{Z_{CTL}}{2Z_{CTL} + Z_0} = \frac{V_{load}}{Z_0}$$

$$\begin{aligned} P_{load} &= \frac{1}{2} RE\left(V_{load} I_{load}^*\right) \\ &= \frac{1}{2} RE\left[\left(V_g \frac{Z_{CTL}}{2Z_{CTL} + Z_0}\right) \frac{V_g^*}{Z_0} \left(\frac{Z_{CTL}}{2Z_{CTL} + Z_0}\right)^*\right] \\ &= P_A \left|\frac{2Z_{CTL}}{2Z_{CTL} + Z_0}\right|^2 \end{aligned}$$

- For the shunt-connected case, the control behavior is complementary to that of the series-connected case.
- For Z_{CTL} much greater than Z_0, the load and the available power are approximately the same.
- A low impedance value for Z_{CTL} corresponds to a significant reduction in the load power compared with the available power from the generator.



Z_{CTL} is assumed purely resistive

Figure Power loss ratio P_A/P_{load} in dB as a function of the controlling impedance Z_{CTL} for series and shunt-connected circuits in a 50Ω system.

Control Circuit Power Handling

The semiconductor control elements are limited in the level of voltage across or current through them, and if either one is exceeded, device destruction will occur and the control module will fail.

Series:

$$V_{\text{load}} = V_g \frac{Z_0}{2Z_0 + R_{\text{CTL}}}; \qquad I_R = \frac{V_{\text{load}}}{Z_0}; \qquad V_R = R_{\text{CTL}} I_{\text{load}}$$

Shunt:

$$V_{\text{load}} = V_R = V_g \frac{R_{\text{CTL}}}{2R_{\text{CTL}} + Z_0}; \qquad I_R = \frac{V_{\text{load}}}{R_{\text{CTL}}}$$

The power handling capability of control devices was derived by observing the difference between the control module's ON-state and OFF-state reflection coefficients and the resulting voltages and currents in the circuit:

$$\left|\Gamma_{\text{OFF}} - \Gamma_{\text{ON}}\right| \leq \frac{I_{sc} V_{oc}}{8P_A}$$

I_{SC}: peak value of the current in the switch when closed
V_{OC}: peak value of the voltage across the switch when open

Assuming that the reflection coefficient in the ideal ON-state switch is zero and the reflection coefficient in the ideal OFF-state is unity, the above equation yields the factors limiting the power handling in the semiconductor control devices:

$$P_{\text{MAX}} \leq \frac{I_{sc} V_{oc}}{8}$$

Control Circuit Terms

$$IL = 10\log_{10}\frac{P_A}{P_{\text{load}}}\ \text{dB}$$

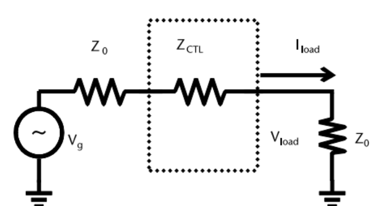
P_A: Power available from the source
P_{load}: Power delivered to the load

$$IL = -10\log_{10}|S_{21}|^2\ \text{dB}$$

Reflective Switches

The single-pole, single-throw (SPST) switches are often called series or shunt reflective switches, since they operate by creating mismatch between the matched source and load impedance, Z_0.

Series Reflective Control Circuit



The control impedance Z_{CTL} of Figure creates an impedance mismatch between the source and load and, in a Z_0 system, the associated reflection coefficient as seen by the source is defined as:

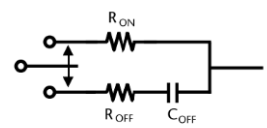
$$\Gamma = \frac{(Z_{\text{CTL}} + Z_0) - Z_0}{(Z_{\text{CTL}} + Z_0) + Z_0} = \frac{Z_{\text{CTL}}}{Z_{\text{CTL}} + 2Z_0}$$

A certain portion of the power available from the source, P_A, will be reflected due to the impedance mismatch and is defined as $P_{REF} = P_A \Gamma \Gamma^* = P_A |\Gamma|^2$. The total power delivered to the input of the switching module is the difference between the power available from the source and the reflected power, $P_{IN} = P_A(1 - |\Gamma|^2)$. The power dissipated by the load is then a fraction of the power available, P_A:

$$P_{load} = P_A \left| \frac{2Z_0}{2Z_0 + Z_{CTL}} \right|^2$$

Using the equivalent circuit for the control element as shown and the eqn. for Γ, the reflection coefficient in the two switch states for the series-connected control element can be written as:

$$\Gamma = \begin{cases} \dfrac{R_{ON}}{R_{ON} + 2Z_0} & \text{ON} \\[2ex] \dfrac{(R_{OFF} + 1/j\omega C_{OFF})}{(R_{OFF} + 1/j\omega C_{OFF}) + 2Z_0} & \text{OFF} \end{cases}$$



In the switch ON-state, application of above eqn. defines the insertion loss of the switch as:

$$IL = 20\log_{10}\left(1 + \frac{R_{ON}}{2Z_0}\right) \text{ dB}$$

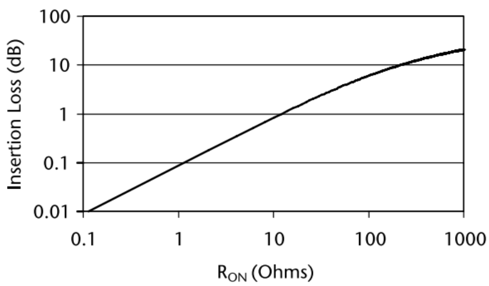
In a similar manner, the OFF-state isolation can be defined, again using Γ, as

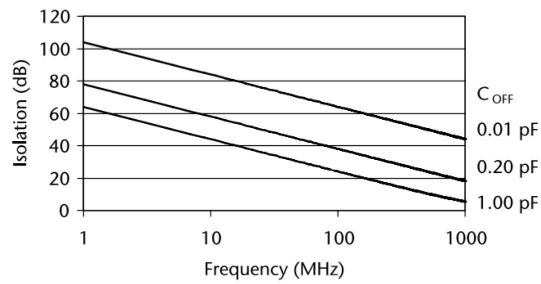
$$ISO = 10\log_{10}\left|1 + \frac{(R_{OFF} + 1/j\omega C_{OFF})}{2Z_0}\right|^2 \text{ dB}$$

For operation well below the switch cutoff frequency F_C, the reactance of the off-state capacitance is much greater than R_{OFF}, and so the isolation expression can be simplified:

$$ISO = 10\log_{10}\left|1 + \frac{1}{2j\omega C_{OFF} Z_0}\right|^2 = 10\log_{10}\left(1 + \frac{1}{(2\omega C_{OFF} Z_0)^2}\right) \text{ dB}$$

Ideal operation of the series-connected control device shows no frequency dependence in the low insertion loss on-state, but the isolation is frequency dependent because of the reactive nature of Z_{CTL} in this state.





The peak RF current through the control component in its on-state (low switch insertion loss) is $I_{R_{ON}} = V_g/(2Z_0 + R_{ON})$, and so the power dissipated in the control component can be computed as

$$P = \frac{1}{2} I_{R_{ON}}^2 R_{ON} = \frac{4R_{ON}Z_0}{\left(2Z_0 + R_{ON}\right)^2} P_A \cong \frac{R_{ON}}{Z_0} P_A$$

with an alternate form of the control component current in terms of P_A:

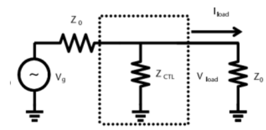
$$I_{R_{ON}} = \frac{\sqrt{8Z_0P_A}}{\left(2Z_0 + R_{ON}\right)} \cong \sqrt{2P_A/Z_0}$$

In the off-state with the control component at a high impedance for good isolation (reactance much greater than Z_0), the majority of the generator voltage V_g is dropped across the control component:

$$V = V_g = \sqrt{8Z_0P_A}$$

Shunt Reflective Control Circuit

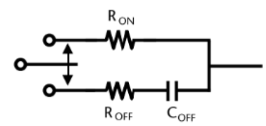
An approach similar to that used in the series-reflective switch case can be used for analysis of the shunt-connected or shunt-reflective SPST switch. The control impedance Z_{CTL} creates an impedance mismatch (Figure) between the source and load, and in a Z_0 system, the associated reflection coefficient as seen by the source can be written as



$$\Gamma = \frac{-Z_0}{2Z_{CTL} + Z_0}$$

Using the equivalent circuit for the control element as shown and the eqn. for Γ, the reflection coefficient in the two switch states for the shunt-connected control element can be written as:

$$\Gamma = \begin{cases} \dfrac{-Z_0}{2\left(R_{OFF} + 1/j\omega C_{OFF}\right) + Z_0} & \text{ON} \\ \dfrac{-Z_0}{2R_{ON} + Z_0} & \text{OFF} \end{cases}$$



In the switch ON-state, the insertion loss of the switch can be given by:

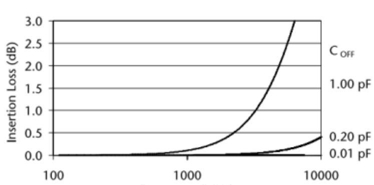
$$IL = 10\log_{10}\left|1 + \frac{Z_0}{2\left(R_{OFF} + 1/j\omega C_{OFF}\right)}\right|^2 \text{ dB}$$

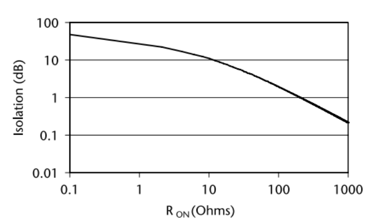
For normal operation, well below the switch cutoff frequency F_C, the reactance of the off-state capacitance is much greater than R_{OFF} , and so the insertion loss can be simplified:

$$IL = 10\log_{10}\left|1 + \frac{j\omega C_{OFF} Z_0}{2}\right|^2 = 10\log_{10}\left(1 + \left(\frac{\omega C_{OFF} Z_0}{2}\right)^2\right) \text{ dB}$$

In a similar manner, the off-state isolation (ISO) can be defined,

$$ISO = 10\log_{10}\left(1 + \frac{Z_0}{2R_{ON}}\right)^2 \text{ dB}$$





The peak RF current through the control component in its on-state (high switch isolation) is $I_{R_{ON}} = V_g/(2R_{ON} + Z_0)$, and so the power dissipated in the control component can be computed:

$$P_{PIN} = \frac{1}{2} I_{R_{ON}}^2 R_{ON} = \frac{4R_{ON}Z_0}{(2R_{ON} + Z_0)^2} P_A \cong \frac{4R_{ON}}{Z_0} P_A$$

with an alternative form of the control component current in terms of P_A:

$$I_{R_{ON}} = \frac{\sqrt{8Z_0P_A}}{(2R_{ON} + Z_0)} \cong \sqrt{8P_A/Z_0}$$

In the control component off-state with a high shunt impedance providing low switch insertion loss, half of the generator voltage V_g is dropped across the device:

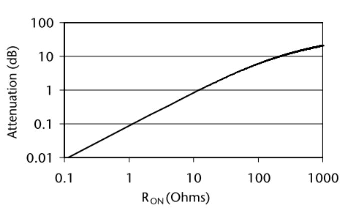
$$V_{PIN} = V_g/2 = \sqrt{2Z_0P_A}$$

Reflective Attenuators

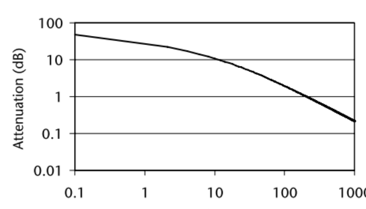
The insertion loss and isolation expressions for the series and shunt-connected device controlled by purely resistive control devices show that the control devices may be used as reflective attenuators since the on-state resistance in semiconductor control devices is often controllable.

$$ATT = 10\log_{10}\left(1 + \frac{R_{ON}}{2Z_0}\right)^2 \text{ dB} \quad \text{Series}$$

$$ATT = 10\log_{10}\left(1 + \frac{Z_0}{2R_{ON}}\right)^2 \text{ dB} \quad \text{Shunt}$$



[Series case]



[Shunt case]

Matched Attenuators

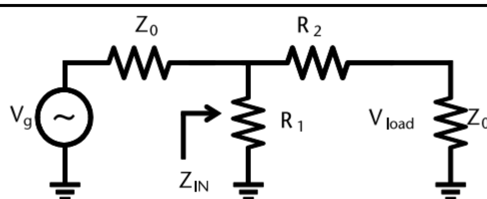
One of the major drawbacks of the reflective attenuator is that the management of this RF energy is governed by reflection of RF energy back toward the generator.

Depending on the power levels involved, the reflective energy would have to be dissipated by the generator and could exceed safe operating limits and cause undesired circuit operations.

This reflected energy is minimized if the generator looks into a matched impedance.

For matched attenuators, there are two degrees of freedom that must be addressed in the design: the desired level of attenuation and the conditions for impedance matching.

These two degrees of freedom indicate that, from the generator perspective, an attenuator circuit with two control resistances would be the simplest form.



The attenuator input impedance seen by the generator of impedance Z_0 is the parallel combination of R_1 and $R_2 + Z_0$. For the generator to see a matched impedance of Z_0, this parallel combination Z_{IN} should be equal to Z_0 and provides a relationship between R_1, R_2, and Z_0:

$$Z_{IN} = Z_0 = \frac{R_1(R_2 + Z_0)}{R_1 + R_2 + Z_0} \Rightarrow R_1 = Z_0\left(1 + \frac{Z_0}{R_2}\right)$$

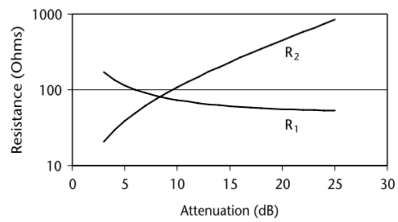
Since the input impedance of the matched attenuator Z_0 is the same as the generator impedance, half of the generator voltage $V_g/2$ is dropped across the attenuator input, and voltage division shows that the voltage drop across the Z_0 load is

$$V_{load} = \frac{V_g}{2}\frac{Z_0}{R_2 + Z_0}$$

The resulting attenuation ATT computed from the power loss ratio P_A/P_{load} can be derived as

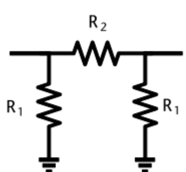
$$ATT = 10\log_{10}\frac{P_A}{P_{\text{load}}} = 20\log_{10}\left(1+\frac{R_2}{Z_0}\right)\text{ dB} \Rightarrow R_2 = Z_0\left(10^{\text{ATT}/20}-1\right)$$

Therefore, for a given level of attenuation ATT, specific values of R_2 and R_1 can be determined.



[Simple two-resistor matched attenuator showing the two resistance values as a function of desired attenuation]

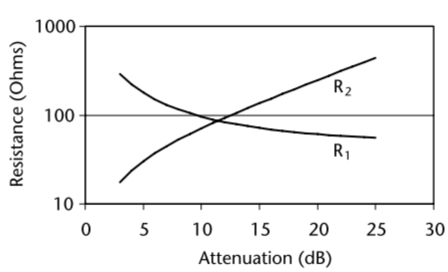
This simple two-resistor matched attenuator of the previous example is not a reciprocal network and presents a match only to the generator; the load sees a nonmatched condition that may not be desired in some applications.



Using a similar analysis as before, the two resistor values for a given level of attenuation and a match for the Π attenuator are:

$$ATT = 20\log_{10}\left(\frac{R_1+Z_0}{R_1+Z_0}\right)\text{ dB} \Rightarrow R_1 = Z_0\frac{10^{\text{ATT}/20}+1}{10^{\text{ATT}/20}-1}$$

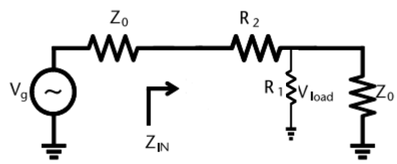
$$R_2 = \frac{2R_1Z_0^2}{R_1^2-Z_0^2}$$



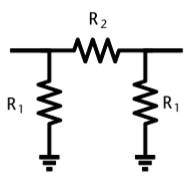
[Matched Π-connected attenuator showing the two resistance values as a function of the desired attenuation]

Assignments:

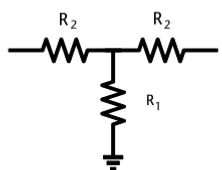
1. For the following purely resistive matched attenuator, matched to the generator, find the design equations:



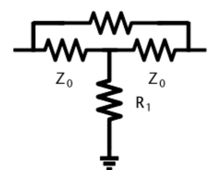
2. For the following purely resistive matched attenuators, find the design equations:



[Π-connected attenuator]



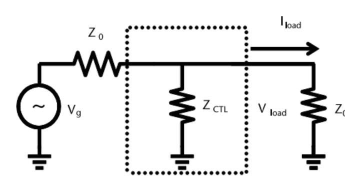
[T-connected attenuator]



[Bridged-T attenuator]

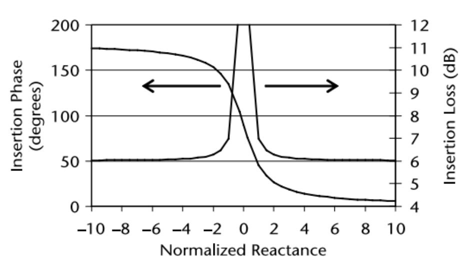
Phase Shifters

Consider that Z_{CTL} is a pure reactance jX.
The ratio of load to generator voltage can then be written as:

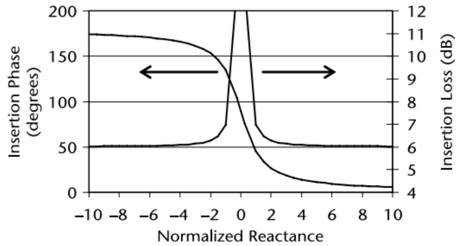


$$\frac{V_{\mathrm{load}}}{V_g} = \frac{Z_{\mathrm{CTL}}}{2Z_{\mathrm{CTL}} + Z_0} = \frac{jX}{2jX + Z_0} = \sqrt{\frac{X^2}{Z_0^2 + 4X^2}}\, e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2X}{Z_0}\right)\right)}$$

The voltage ratio shows both a reduction in amplitude and the introduction of phase, with a total phase shift (or insertion phase) of $\Delta\Theta = \pi/2 - \tan^{-1}(2X/Z_0)$.



[Insertion loss and phase as a function of normalized reactance X/Z_0 in a shunt-connected circuit]



[Insertion loss and phase as a function of normalized reactance X/Z_0 in a shunt-connected circuit]

For small normalized reactance X/Z_0, a phase shift between the generator and load of 90° plus or minus a few degrees is observed but at the expense of a significantly attenuated signal at the load. For a large normalized reactance X/Z_0, insertion phases near zero or 180° are possible with minimal signal loss (the 6 dB insertion loss figure indicates $V_{\text{load}} = 1/2V_g$ under no loading).

Noise

Resistive Noise Model:

The random motion of charge carriers in any circuit causes minute instantaneous fluctuations in the current flow in any current-carrying structure.

In a resistor of value R, these current fluctuations Δi occur around the mean current I.

The current fluctuations, in turn, create voltage fluctuations across R and give rise to so-called thermal resistive noise or just resistive noise.

The noise power has been shown to be linearly dependent on the physical temperature T of the resistor.

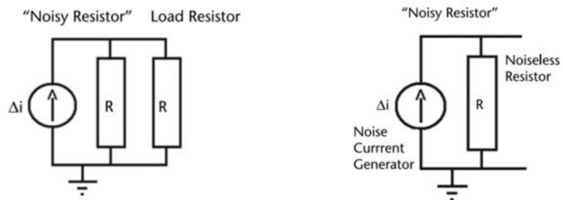
$$P = \alpha T = \Delta \bar{i}^2 R; \quad \Delta \bar{i}^2 = \alpha T/R = \alpha G T$$

where Δi is the rms value of the current fluctuations, and α is a function of the noise measurement.

The mean square noise current can be given by:

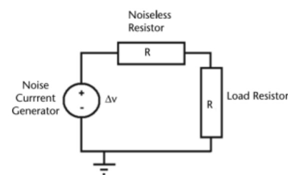
$$\Delta \bar{i}^2 = 4kTGB = 4kTB/R$$

B: Bandwidth
k: Boltzmann constant



Assuming that the noise source is driving a matched load of noiseless resistor *R*, the available noise power (in Watts) delivered to noiseless load *R* is:

$$P_N = \left(\frac{\Delta \bar{i}}{2}\right)^2 R = kTB$$



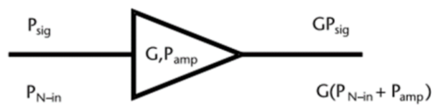
A Thévenin equivalent circuit for the noise source can also be drawn as shown in Figure where $\Delta \bar{v}^2 = 4kTRB$ and the available noise power $P_N = kTB$ is the same as before.

The noise power is independent of the value of resistance and so can be used to describe any element or system. In the general case, the noise temperature T is termed an equivalent noise temperature T_e of the element or system, and the IEEE standard noise reference temperature of 290K is defined with the symbol T_{ref}:

$$P_N = kT_eB; \;\; P_{\text{ref}} = kT_{\text{ref}}B$$

Noise Figure Model:

Any electronic circuit, active or passive, will generate its own noise, and so a signal with noise at the input of the circuit will find its way to the output, but with additional noise due to that generated by the circuit itself.



With the addition of the amplifier noise contribution, two signal-to-noise (SNR) values can be computed, one at the input (SNR_{in}) and the other at the output (SNR_{out}):

$$SNR_{in} = \frac{P_{sig}}{P_{N-in}}$$

$$SNR_{out} = \frac{GP_{sig}}{G\left[P_{N-in} + P_{amp}\right]} = \frac{P_{sig}}{\left[P_{N-in} + P_{amp}\right]}$$

SNR_{out} can be written in terms of SNR_{in} by rearranging terms

$$SNR_{out} = \frac{P_{sig}}{\left[P_{N-in} + P_{amp}\right]} = \frac{P_{sig}/P_{N-in}}{1 + P_{amp}/P_{N-in}} = \frac{SNR_{in}}{NF}$$

Noise Figure: $$NF = 1 + \frac{T_{amp}}{T_{N-in}}$$

where the noise powers have been replaced by their equivalent noise temperatures, T_{amp} and $T_{N\text{-}in}$, the amplifier equivalent noise temperature and input noise temperature, respectively.

Assuming that the input noise conforms to the IEEE standard allows NF to be written as:

$$NF = 1 + \frac{T_{amp}}{T_{ref}}$$

where T_{ref} is the input or standard noise temperature.

$$NF_{dB} = 10\log(NF); \quad SNR_{out\text{-}dB} = SNR_{in\text{-}db} - NF_{dB}$$

The control circuits of interest here are of course different than amplifiers and have losses associated with them, so instead of a gain G in the previous amplifier example, the circuit will exhibit a loss $L = 1/G$.

$$P_{\text{out}} = G\left(kTB + P_{N-\text{loss}}\right)$$

where $P_{N\text{-loss}}$ is the input-referred noise power of the lossy element. P_{out}, however, is kTB, and so $P_{N\text{-loss}}$ and its associated equivalent noise temperature can be written as

$$P_{N-\text{loss}} = \frac{1-G}{G} kTB = (L-1)kTB$$

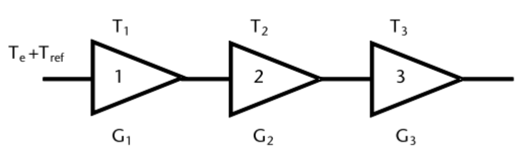
$$T_e = \frac{1-G}{G} T = (L-1)T$$

with a corresponding noise figure NF of

$$NF = 1 + (L-1)\frac{T}{T_{\text{ref}}}$$

If the system is at temperature T_{ref}, then $NF = L$, and the noise figure NF is identical to the circuit loss L.

Cascade System Noise:



A thermal noise source is placed at the input, set to the reference noise temperature T_{ref}. By tracking the input noise contributions at each input and multiplying them by the gain of each amplifier stage, the noise power output at the last state can be written as

$$P_{N-3} = kBG_1G_2G_3\left(T_1 + T_{\text{ref}}\right) + kBG_2G_3T_2 + kBG_3T_3$$

The output noise power P_{N-3} can be written in terms of an equivalent noise temperature T_e at the input of a single stage amplifier with gain $G_1G_2G_3$, so that P_{N-3} in this single amplifier-equivalent can be written as

$$P_{N-3} = kBG_1G_2G_3\left(T_e + T_{\text{ref}}\right)$$

Equating the above two equations yields the equivalent noise temperature and NF for the entire system:

$$T_e = T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1G_2}; \quad NF = 1 + \frac{T_e}{T_{\text{ref}}} = 1 + \frac{T_1}{T_{\text{ref}}} + \frac{T_2}{T_{\text{ref}}}\frac{1}{G_1} + \frac{T_3}{T_{\text{ref}}}\frac{1}{G_1G_2}$$

For a two-stage system with control module as Stage 1:

$$T_e = (L_1 - 1)T + L_1 T_2$$

$$NF = 1 + \frac{(L_1 - 1)T + L_1 T_2}{T_{\text{ref}}} = 1 + \frac{(L_1 - 1)T}{T_{\text{ref}}} + L_1 \frac{T_2}{T_{\text{ref}}}$$