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ASSIGNMENT - 5

EXERCISE 6.4:

Given:
$P(\text{Green} \mid \text{green}) = 0.6$
$P(\text{Red} \mid \text{green}) = 0.4$
$P(\text{Green} \mid \text{red}) = 0.3$
$P(\text{Red} \mid \text{red}) = 0.7$

(a) Transition prob. matrix:

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

(b) To find prob. of third light we need 2-step transition matrix,

$$P^2 = P \times P$$
$$= \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \times \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$
$$= \begin{pmatrix} 0.48 & 0.52 \\ 0.39 & 0.61 \end{pmatrix}$$

$P(\text{third light is red}) = 0.52$

(c)

Steady state diag:

here,
π_1 = Last Light is green
π_2 = " " " red.

$$\pi = \pi P$$
$$\Sigma \pi_x = 1$$
$$0.6\pi_1 + 0.3\pi_2 = \pi_1$$
$$0.4\pi_1 + 0.7\pi_2 = \pi_2$$
$$0.3\pi_2 = 0.4\pi_1 \quad \text{—— (1)}$$

$$0.4\pi_1 = 0.3\pi_2 \quad \text{——} \quad (2)$$

$$\pi_1 = \frac{3\pi_2}{4}$$

from (2),

$$\frac{3\pi_2}{4} + \pi_2 = 1$$

$$\pi_2 = \frac{4}{7}$$

So,

$$\pi_1 = \frac{3}{7}.$$

So, prob. of last light to be red is,

$$\pi_2 = \frac{4}{7}.$$

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Exercise 6.7 :

Given : Markov chain's prob. matrix,

$$P = \begin{pmatrix} 0.3 & \_ & 0 \\ 0 & 0 & \_ \\ 1 & \_ & \_ \end{pmatrix}$$

(a)

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) For transition prob matrix to be regular there should be no zero entry in matrix, so start powering the matrix,

(i) $$P^2 = \begin{pmatrix} 0.3 & 0.7 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0.3 & 0.7 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.09 & 0.21 & 0.7 \\ 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

It has zero entries.

(ii) $$P^3 = \begin{pmatrix} 0.09 & 0.21 & 0.7 \\ 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \end{pmatrix} * \begin{pmatrix} 0.3 & 0.7 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.727 & 0.063 & 0.21 \\ 0.3 & 0.7 & 0 \\ 0.09 & 0.21 & 0.7 \end{pmatrix}$$ This also has zero entry.

(iii) $P^4 = \begin{pmatrix} 0.727 & 0.063 & 0.21 \\ 0.3 & 0.7 & 0 \\ 0.09 & 0.21 & 0.7 \end{pmatrix} \times \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0.4281 & 0.5089 & 0.063 \\ 0.09 & 0.21 & 0.7 \\ 0.727 & 0.063 & 0.21 \end{pmatrix}$

It has no zero entries, so this is regular matrix.

(C) Steady State Distribution,

$\pi = \pi P$

$\sum \pi_x = 1$

$0.3\pi_1 + \pi_3 = \pi_1$

$0.7\pi_1 = \pi_2$

$\pi_2 = \pi_3$

$\pi_2 = 0.7\pi_1$

$\pi_3 = 0.7\pi_1$

So,

$\pi_1 + \pi_2 + \pi_3 = 1$

$\pi_1 + 0.7\pi_1 + 0.7\pi_1 = 1$

$\pi_1 = 0.416$

$\pi_2 = 0.29167$

$\pi_3 = 0.29167$

So, steady state distribution is,

$\pi = (\pi_1, \pi_2, \pi_3) = (0.416, 0.29167, 0.29167)$

Exercise 6.8 :

Transition prob matrix,

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

Steady State Diag,

$$\pi = \pi P.$$

$$\Sigma \pi_x = 1$$

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) * \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

$$0.5\pi_1 + \pi_3 = \pi_1$$

$$0.5\pi_1 = \pi_2$$

$$0.5\pi_1 + 0.5\pi_2 = \pi_3.$$

$$\pi_2 = 0.5\pi_1$$

$$\pi_3 = 0.75\pi_1$$

So, $\pi_1 + \pi_2 + \pi_3 = 1$

$$\pi_1 + 0.5\pi_1 + 0.75\pi_1 = 1$$

$$\pi_1 = 0.444$$

$$\pi_2 = 0.222$$

$$\pi_3 = 0.333$$

$$\pi = (0.444, 0.222, 0.333)$$

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Exercise 6.15 :

Avg. Rate : 12/min (λ)
Modeled as binomial process.

(a)

$$\Delta = \frac{p}{\lambda} = \frac{0.15}{12} \Rightarrow 0.0125 \text{ min} \approx 0.75 \text{ sec.}$$

(b)

$$p = 0.15$$

$$E(T) = \frac{1}{\lambda} = \frac{1}{12} \text{ min}$$

$$Var(T) = \frac{1-p}{\lambda^2} = \frac{1-0.15}{(12)^2}$$

$$= 0.005903$$

SD., $s = \sqrt{Var(T)} = \sqrt{0.005903}$
$= 0.07683$

Exercise 6.18 :-

Given : $\lambda = \frac{1}{15}$,

$\Delta = 5$ sec.

So,

$$p = \lambda\Delta$$

$$= \frac{1}{15} * 5 = \frac{1}{3}.$$

$T = 200$ min.
$= 12000$ sec.

Sample size,

$$n = \frac{T}{\Delta}$$

$$= \frac{12000}{5} = 2400$$

As, n is large & $p < 0.5$. Convert binomial dist to normal dist,

$$\mu = np$$
$$= 2400 * \frac{1}{3} = 800.$$

$$\sigma = \sqrt{np(1-p)}$$

$$= \sqrt{2400 * \frac{1}{3}\left(1 - \frac{1}{3}\right)} \Rightarrow \sqrt{533.33}$$

$$= 23.094$$

Now,

$$P(Z \leq 750) = P(X(n) \leq 750.50)$$

$$= P\left(\frac{X(n) - \mu}{\sigma} \leq \frac{750.50 - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{750.50 - 800}{23.094}\right)$$

$$= P(Z \leq -2.14)$$

$$= \underline{\underline{0.0160}}$$

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Exercise 6.20:

$\lambda = 3 \text{ month}^{-1}$

$X(t) = \text{Poisson}(\lambda t)$

$P(X(3) > 5)$

$X(3) = \text{Poisson}(9)$

$P(X(3) > 5) = 1 - P(X(3) \le 5)$
$= 1 - 0.116$
$= 0.884$

Exercise 6.22:

$\lambda = 5$ per month.

(a)
$P(X(1) > 3) = 1 - P(X(1) \le 3)$
$= 1 - 0.265$
$= 0.735$

(b) Poisson dist,
$E(X) = \mu = \lambda t = 5 * 1 = 5$

Cost,
$1500 * E(X) = 1500 * 5 = 7500$.

$\sigma = \sqrt{\mu} = 2.236$

Cost,
$\sigma * 1500 = 3354$.

Exercise 7.1 :-

Given, $\lambda_A = 10 / hrs$ $\qquad \Delta = \frac{1}{6} min$

$= \frac{1}{6} min^{-1}$

$\lambda_S = \frac{1}{\mu_S} = \frac{1}{2} = 0.5 \, min^{-1}$

$P_A = \lambda_A \Delta = \frac{1}{36}$

$P_S = \lambda_S \Delta = \frac{1}{12}$

Transition prob,

$P_{00} = 1 - P_A = \frac{35}{36} = 0.972$

$P_{01} = P_A = \frac{1}{36} = 0.028$

For $i \geq 1$,

$P_{i,j-1} = (1 - P_A) P_S = 0.081$

$P_{i,j+1} = (1 - P_S) P_A = 0.025$

$P_{i,j} = 1 - 0.081 - 0.025 = 0.894$

Transition prob. matrix at end of each frame,

$$\Rightarrow \begin{bmatrix} 0.972 & 0.028 & 0 & \cdots \\ 0.081 & 0.894 & 0.025 & \cdots \\ 0 & 0.081 & 0.894 & 0.025 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Exercise 7.5:

Given:

$\mu_A = 10$ min

$\lambda_A = \frac{1}{10}$ $\min^{-1}$

$\mu_S = 15$ min

$\lambda_S = \frac{1}{15}$ $\min^{-1}$

$\Delta = 3$ min

Now,

$P_A = \lambda_A \Delta = 0.3$

$P_S = \lambda_S \Delta = 0.2$

Transition prob,

$P_{00} = 1 - P_A = 0.7$

$P_{01} = P_A = 0.3$

$P_{10} = (1-P_A) P_S = 0.14$

$P_{11} = 1 - 0.14 - 0.24 = 0.62$

$P_{12} = (1-P_S) P_A = 0.24$

$P_{20} = 0$

$P_{21} = (1-P_A) P_S = 0.14$

$P_{22} = 1 - 0.14 = 0.86$

Transition prob. matrix,

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.14 & 0.62 & 0.24 \\ 0 & 0.14 & 0.86 \end{bmatrix}$$

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Steady State Dist,

$\bar{\pi} = \bar{\pi} P$

$$\Rightarrow \begin{cases} 0.7\bar{\pi}_0 + 0.14\bar{\pi}_1 = \bar{\pi}_0 \\ 0.3\bar{\pi}_0 + 0.62\bar{\pi}_1 + 0.14\bar{\pi}_2 = \bar{\pi}_1 \\ 0.24\bar{\pi}_1 + 0.86\bar{\pi}_2 = \bar{\pi}_2 \end{cases}$$

So, using above equations.

$\bar{\pi}_0 + \bar{\pi}_1 + \bar{\pi}_2 = 1$

$\bar{\pi} = [0.1467, \; 0.3144, \; 0.5389]$

Exercise 7.11:-

Given, $\lambda_A = 10/25 \text{ min}^{-1} = 0.4 \text{ min}^{-1}$

$\mu_S = 2 \text{min}$

Utilization, $r = \lambda_A \mu_S = 0.4 * 2 = 0.8$.

(a)

No. of customer in system,

$$E(X) = \frac{r}{1-r} \Rightarrow \frac{0.8}{1-0.8} = 4.$$

No. of customer in waiting line,

$$E(X_w) = \frac{r^2}{1-r} = 3.2$$

(b)

As, $r = 0.8$

i.e, when teller is busy.

$r = \lambda_A * \mu_s = 0.4 * 2 = \underline{\underline{0.8}}$

(c)

$P(X \geq 5) = 1 - P(X \leq 4)$

$= 1 - 0.738 = \underline{0.262}$

Exercise 7.12:

Given, $\lambda_A = 1/5 \text{ min}^{-1} = 0.2 \text{ min}^{-1}$

$\mu_s = 3 \text{ min}$

Utilization, $r = \lambda_A \mu_s = 0.6$

(a) Response time,

$$E(R) = \frac{\mu_s}{1-r} = \frac{3}{1-0.6} = 7.5 \text{ min}$$

(b) $P(X<2) = P(X \leq 1)$

$= 0.64.$

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