CSE 5301: 002 – Fall 2018
Exam 3 - Variant α, Thursday 12/06/2018

Name: SOLUTION

Student ID:

(Not providing this information: -10 Points)
(Name missing in Individual pages: -5 Points)

Name: ______________________________

Total Exam Points: 100

Score

Question	Points	Max Points
1		12
2		18
3		25
4		10
5		20
6		15
Survey		10*
Total		**100 (110)***

*: Optional Extra Credit Question

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Question 1 – 12 points

For each of these questions use the random numbers in Table A1 starting from z value = 21

(a) Generate 4 samples from Binomial(5,0.2)
(b) Generate 4 samples from Binomial(50, 0.2)
(c) Generate 4 samples from Binomial(50, 0.02)

Show all the calculations (if any) involved. Hint: For part b and c, consider if you can use a different distribution to represent the Binomial.

(a)

0.0579	0.4966	0.5828	0.6333	0.1210	
1	0	0	0	1	= 2 //
0.1370	0.1939	0.3603	0.6700	0.1879	
1	1	0	0	1	3 //
0.3529	0.8998	0.4235	0.2126	0.4508	
0	0	0	0	0	0 //
0.8188	0.9048	0.5485	0.2009	0.4906	
0	0	0	0	0	0 //

(b) Approx as Normal$(50 \times 0.2, \sqrt{50 \times 0.2 \times 0.8})$

Nomal$(10, 2.8284)$

$U_1 = 0.8132 \quad U_2 = 0.8216$

$Z_1 = \sqrt{-2\ln(0.8132)}\,\cos(2 \times \pi \times 0.8216) = 0.64047$

$Z_2 = \sqrt{-2\ln(0.8132)}\,\sin(2 \times \pi \times 0.8216) = 0.05786$

$X_1 = 0.64047 \times 2.8284 + 10 \qquad X_2 = 0.05786 \times 2.8284 + 10$

$= 11.8115 \,// \qquad = 10.16365 \,//$

$U_3 = 0.5155 \qquad U_4 = 0.8392$

$Z_3 = \sqrt{-2\ln(0.5155)}\,\cos(2\pi\, 0.8392) = 1.14631$

$Z_4 = \sqrt{-2\ln(0.5155)}\,\sin(2\pi\, 0.8392) = 0.010579$

$X_3 = 1.14631 \times 2.8284 + 10 \qquad X_4 = 0.010579 \times 2.8284 + 10$

$= 13.24222 \qquad\qquad = 10.02992$

(c) Approx as Poisson(np) = Poisson(1)

$U_1 = 0.7159 \Rightarrow U_1 \le F(1) \qquad X_1 = 1$

$U_2 = 0.4302 \Rightarrow U_2 \le F(1) \qquad X_2 = 1$

$U_3 = 0.5692 \Rightarrow U_3 \le F(1) \qquad X_3 = 1$

$U_4 = 0.2618 \Rightarrow U_4 \le F(0) \qquad X_4 = 0$

Other method for sampling Poisson also OK.

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Question 2 – 18 points

You are playing a game of chance that consists of tossing a coin twice and rolling a 6-sided dice twice. If you get heads on the first coin toss, you win or lose cash equal to the product of the two dice rolls depending on whether you got heads or tails on the second coin toss. If you get tails on the first coin toss, and heads on the second coin toss, you 'win' by how much first dice roll was bigger than the second (note you can 'win' negative numbers here too). If you had tails on the second coin toss, you 'won' by how much the second dice roll was bigger than the first.

Assume that the coin and dice are unbiased.

You want to find out your estimated winnings from such a game by doing long run sampling experiments. How would you go about generating samples? (Give the procedure for generating one sample)

Generate 5 samples for the amount won in a single game (For the random numbers required use Table A1 starting from z = 11). Show all the calculations if any involved

Let ~~Be~~ X be amont wone

Let Dice roll follow : $\{1: \frac{1}{6},\ 2: \frac{1}{6},\ 3: \frac{1}{6},\ 4: \frac{1}{6},\ 5: \frac{1}{6},\ 6: \frac{1}{6}\}$

To generate Sample, So $F(x) = \{1: 0.1666,\ 2: 0.3333,\ 3: 0.5,\ 4: 0.6666,\ 5: 0.83333,\ 6: 1\}$

Get U_1, U_2, U_3, U_4 from $U(0,1)$

if $U_1 \leq \frac{1}{2}$

if $U_2 \leq \frac{1}{2}$

$X = [i \text{ st: } F(i-1) \leq U_3 < F(i)] * [i \text{ st: } F(i-1) \leq U_4 < F(i)]$

else

$X = 0 - [i \text{ st: } F(i-1) \leq U_3 < F(i)] * [i \text{ st: } F(i-1) \leq U_4 < F(i)]$

end if

else,

if $u_2 \le 1/2$

$$X = [i \text{ st}: F(i-1) \le u_3 \le F(i)] - [i \text{ st}: F(i-1) \le u_4 < F(i)]$$

else

$$X = [i \text{ st}: F(i-1) \le u_4 < F(i)] - [i \text{ st}: F(i-1) \le u_3 < F(i)]$$

end if

end if

$u_1 = 0.6154$ $u_2 = 0.1934$ $u_3 = 0.1365$ $u_4 = 0.2140$

$X = 1 - 1 = 0$

$u_1 = 0.0841$ $u_2 = 0.9342$ $u_3 = 0.3759$ $u_4 = 0.1146.$

$X = -3 \times 1 = -3$

$u_1 = 0.3567$ $u_2 = 0.9566$ $u_3 = .7919$ $u_4 = 0.6822$

$X = -5 \times 5 = -25$

$u_1 = 0.0118$ $u_2 = 0.6435$ $u_3 = 0.4544$ $u_4 = 0.2644.$

$X = -3 \times 2 = -6$

$u_1 = 0.0099$ $u_2 = 0.6649$ $u_3 = 0.4983$ $u_4 = 0.1472$

$X = -3 \times 1 = -3$

Name: ____________________

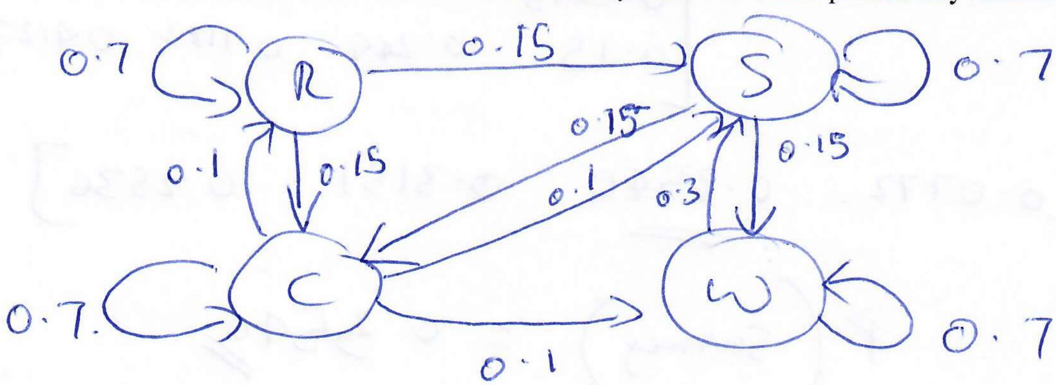
Question 3 – 25 points

Weather on a particular day can be rainy, sunny, windy or cloudy. There is a 0.7 probability that the weather remains unchanged the next day. Sunny weather can also change into either windy or cloudy (with equal chance). Rainy weather can change into sunny or cloudy (with equal chance). Windy weather can change to sunny. Cloudy weather can change to rainy, sunny or windy (with equal chance).

Draw the state transition diagram for this Markov chain. What is the transition probability matrix?

If today is windy or cloudy (with equal chance), what is the probability that tomorrow is sunny? What is probability that it is sunny 3 days from now?

Find the steady state distribution and steady state transition probability matrix.



$$P = \begin{bmatrix} 0.7 & 0.15 & 0 & 0.15 \\ 0 & 0.7 & 0.15 & 0.15 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}$$

Today is $[0 \quad 0 \quad 0.5 \quad 0.5]$

Tomorrow is $[0 \quad 0 \quad 0.5 \quad 0.5] \begin{bmatrix} 0.7 & 0.15 & 0 & 0.15 \\ 0 & 0.7 & 0.15 & 0.15 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}$

$= [0.05 \quad 0.2 \quad 0.4 \quad 0.35]$

$P(\text{Sunny}) = 0.2$

3 days from now

$$= \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.15 & 0 & 0.15 \\ 0 & 0.7 & 0.15 & 0.15 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3767 & 0.2677 & 0.0832 & 0.2722 \\ 0.0315 & 0.457 & 0.261 & 0.2317 \\ 0.0045 & 0.459 & 0.442 & 0.0945 \\ 0.15 & 0.249 & 0.1182 & 0.4127 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0772 & \underline{\underline{0.3540}} & 0.3151 & 0.2536 \end{bmatrix}$$

$$P(\text{sunny}) = 0.3540$$

To find Steady state

$$\pi P = \pi \qquad \begin{bmatrix} \pi_R & \pi_S & \pi_W & \pi_C \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.15 & 0 & 0.15 \\ 0 & 0.7 & 0.15 & 0.15 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} \pi_R & \pi_S & \pi_W & \pi_C \end{bmatrix}$$

$$\pi_R + \pi_S + \pi_W + \pi_C = 1.$$

$$\pi_R(0.7) + \pi_C(0.1) = \pi_R$$

$$\pi_R(0.15) + \pi_S(0.7) + \pi_W(0.3) + \pi_C(0.1) = \pi_S$$

$$\pi_R(0) + \pi_S(0.15) + \pi_W(0.7) + \pi_C(0.1) = \pi_W$$

$$\pi_R(0.15) + \pi_S(0.15) + \pi_C(0.1) = \pi_C.$$

Solving $\pi = \begin{bmatrix} 0.08 & 0.4 & 0.28 & 0.24 \end{bmatrix}$

From this you can also get Steady state Probability

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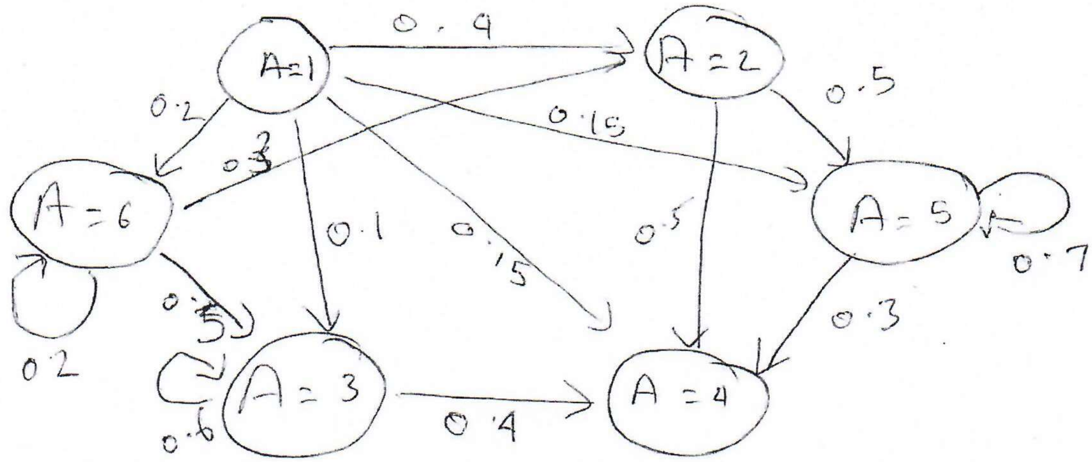
Question 4 – 10 points

Consider the following state transition diagram:



What is the state transition probability matrix? Is this a regular markov chain? Does it have a steady state distribution? If it does find it.

$$P = \begin{bmatrix} 0 & 0.4 & 0.1 & 0.15 & 0.15 & 0.2 \\ 0 & 0 & \cancel{0.5} & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0.3 & 0.5 & \cancel{0.2} & 0 & 0.2 \end{bmatrix}$$

This has $A=4$ as an absorbing state,

So it is not regular

And it has steady state distribution

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Name: ______________________

Question 5 – 20 points

The number of requests processed by a server is modeled as a binomial counting process. On average it receives 4 requests in a 5-minute period.

(a) If I want probability of request arrival in a frame to be equal to 0.16, what is frame length I should use.
(b) What is probability of the server getting less than 25 requests in half an hour.
(c) Find the expected time between two consecutive requests. Find the standard deviation of time between two consecutive requests
(d) Find the probability of waiting more than a half a minute between jobs

(a) $\lambda = 4/5 \text{ min}^{-1}$

$p = 0.16$

$\Delta = \frac{p}{\lambda} = \frac{0.16 \times 5}{4} = \underline{\underline{0.2}} \text{ mins.}$

(b) $t = 30 \text{ min.}$

$n = \frac{30}{0.2} = 150$

So $X(n) = \text{Binomial}(150, 0.16) = \text{Normal}(24, 4.4899)$

$P(X(n) < 25) = P(X(n) \leq 24.5)$

$= P\left(Z \leq \frac{24.5 - 24}{4.4899}\right)$

$= P(Z \leq 0.1113)$

$= 0.5478$

(c) $T = \Delta Y$ $\quad Y = \text{Geometric}(0.16)$

$= 0.2\, Y$

$$E(T) = 0.2\, E(Y) = \frac{0.2}{0.16} = 1.25 \text{ min.}$$

$$Var(T) = (0.2)^2\, Var(Y) = 0.04\, \frac{(1-0.16)}{(0.16)^2} = 1.3125 \text{ min}^2$$

$$Std(T) = 1.14564 \text{ min}$$

(d) $P(T \geq 0.5) = 1 - P(T < 0.5)$

$= 1 - P\left(Y < \frac{0.5}{0.2}\right) = 1 - P(Y < 2.5)$

$= 1 - P(Y \leq 2)$

$\approx 1 - (P(Y = 1) + P(Y = 2))$

$\approx 1 - \left((1-0.16)^0 (0.16) + (1-0.16)^1 (0.16)\right)$

$\approx 1 - 0.2944 = 0.7056$

or Treat it as Normal. (Also Acceptable)

$1 - P(T < 0.5) \approx 1 - P\left(Z < \frac{0.5 - 1.25}{1.14564}\right) = 1 - P(Z \leq -0.6546)$

$= 1 - P(Z \leq -0.6546)$

$= 1 - 0.2578 = 0.7422$

Name: ______________________

Question 6 – 15 points

If the previous scenario was modeled as a Poisson counting process, you no longer have to worry about frame sizes.

Answer the same questions as 5(b), 5(c) and 5(d). Show all the substitutions and calculations

(b) $X(t)$ follows Poisson(λt)

Poisson$\left(\frac{4}{5} \times 30\right)$ = Poisson(24)

$P(X(t) < 25) = P(X(t) \leq 24)$

$= F(24) = 0.554$

(c) T follow Exponential(λ) = Exponential$(4/5)$

$E(T) = \frac{1}{4/5} = \frac{5}{4} = 1.25$

$Var(X) = \frac{1}{(4/5)^2} = \frac{25}{16}$

$Std(X) = \frac{5}{4} = 1.25$

(c)

$$P(T < 0.5) = \int_{0}^{0.5} \lambda e^{-\lambda x}$$

$$= \left[- e^{-\lambda x} \right]_{0}^{0.5}$$

$$= - e^{-\frac{4}{5} 0.5} + 1$$

$$= 1 - e^{-0.4}$$

$$= 1 - \frac{1}{e^{0.4}} = 0.3296$$

OR Aprox as Normal. [Also Acceptable]

$$P(T < 0.5) \approx P\left(Z < \frac{0.5 - 1.25}{1.25}\right)$$

$$\approx P(Z < -0.6) = P(Z \leq -0.6)$$

$$= 0.2743$$

Name: ______________________

Question 7 – 10 points EC [Optional Survey Question]

The following table has the topics that were discussed in this class. In column 1 indicate whether you would have preferred **more** or **less** discussion about the theoretical material. In column 2 indicate whether you would have preferred **more** or **less** solved examples during class.

Student Preference Survey

Topic	Theory and Material	Solved Examples
Probability, Random Variables		
Joint and Conditional probability distributions		
Combinations of Random variables		
CLT and Chebyshev's Inequality		
Families of Distributions		
Parameter estimations		
Graphical Statistics		
Method of Moments/ Maximum Likelihood estimation		
Confidence Intervals		
Hypothesis Testing (Z and t)		
Hypothesis Testing of Variance (chi-square)		
Bayesian inference		
Simulation of Random Variables & Monte Carlo methods		
Stochastic and Markov Processes		
Markov Counting Processes		

Attempting This will give credit.

Name: ________________________

SCRATCH