Exercises

7.1. Customers arrive at the ATM at the rate of 10 customers per hour and spend 2 minutes, on average, on all the transactions. This system is modeled by the Bernoulli single-server queuing process with 10-second frames. Write the transition probability matrix for the number of customers at the ATM at the end of each frame.

7.2. Consider Bernoulli single-server queuing process with an arrival rate of 2 jobs per minute, a service rate of 4 jobs per minute, frames of 0.2 minutes, and a capacity limited by 2 jobs. Compute the steady-state distribution of the number of jobs in the system.

7.3. Performance of a car wash center is modeled by the Bernoulli single-server queuing process with 2-minute frames. The cars arrive every 10 minutes, on the average. The average service time is 6 minutes. Capacity is unlimited. If there are no cars at the center at 10 am, compute the probability that one car will be washed and another car will be waiting at 10:04 am.

7.4. Masha has a telephone with two lines that allows her to talk with a friend and have at most one other friend on hold. On the average, she gets 10 calls every hour, and an average conversation takes 2 minutes. Assuming a single-server limited-capacity Bernoulli queuing process with 1-minute frames, compute the fraction of time Masha spends using her telephone.

7.5. A customer service representative can work with one person at a time and have at most one other customer waiting. Compute the steady-state distribution of the number of customers in this queuing system at any time, assuming that customers arrive according to a Binomial counting process with 3-minute frames and the average interarrival time of 10 minutes, and the average service takes 15 minutes.

7.6. Performance of a telephone with 2 lines is modeled by the Bernoulli single-server queuing process with limited capacity ($C = 2$). If both lines of a telephone are busy, the new callers receive a busy signal and cannot enter the queue. On the average, there are 5 calls per hour, and the average call takes 20 minutes. Compute the steady-state probabilities using four-minute frames.

7.7. Jobs arrive at the server at the rate of 8 jobs per hour. The service takes 3 minutes, on the average. This system is modeled by the Bernoulli single-server queuing process with 5-second frames and capacity limited by 3 jobs. Write the transition probability matrix for the number of jobs in the system at the end of each frame.

7.8. For an M/M/1 queuing process with the arrival rate of 5 min^{-1} and the average service time of 4 seconds, compute

(a) the proportion of time when there are exactly 2 customers in the system;

(b) the expected response time (the expected time from the arrival till the departure).

7.9. Jobs are sent to a printer at random times, according to a Poisson process of arrivals, with a rate of 12 jobs per hour. The time it takes to print a job is an Exponential random variable, independent of the arrival time, with the average of 2 minutes per job.

(a) A job is sent to a printer at noon. When is it expected to be printed?

(b) How often does the total number of jobs in a queue and currently being printed exceed 2?

7.10. A vending machine is modeled as an M/M/1 queue with the arrival rate of 20 customers per hour and the average service time of 2 minutes.

(a) A customer arrives at 8 pm. What is the expected waiting time?

(b) What is the probability that nobody is using the vending machine at 3 pm?

7.11. Customers come to a teller's window according to a Poisson process with a rate of 10 customers every 25 minutes. Service times are Exponential. The average service takes 2 minutes. Compute

(a) the average number of customers in the system and the average number of customers waiting in a line;

(b) the fraction of time when the teller is busy with a customer;

(c) the fraction of time when the teller is busy and at least five other customers are waiting in a line.

7.12. For an M/M/1 queuing system with the average interarrival time of 5 minutes and the average service time of 3 minutes, compute

(a) the expected response time;

(b) the fraction of time when there are fewer than 2 jobs in the system;

(c) the fraction of customers who have to wait before their service starts.

7.13. Jobs arrive at the service facility according to a Poisson process with the average interarrival time of 4.5 minutes. A typical job spends a Gamma distributed time with parameters $\alpha = 12$, $\lambda = 5\ \text{min}^{-1}$ in the system and leaves.

(a) Compute the average number of jobs in the system at any time.

(b) Suppose that only 20 jobs arrived during the last three hours. Is this an evidence that the expected interarrival time has increased?

7.14. Trucks arrive at a weigh station according to a Poisson process with the average rate of 1 truck every 10 minutes. Inspection time is Exponential with the average of 3 minutes. When a truck is on scale, the other arrived trucks stay in a line waiting for their turn. Compute

(a) the expected number of trucks in the line at any time;

(b) the proportion of time when the weigh station is empty;

(c) the expected time each truck spends at the station, from arrival till departure.

7.15. Consider a hot-line telephone that has no second line. When the telephone is busy, the new callers get a busy signal. People call at the average rate of 2 calls per minute. The average duration of a telephone conversation is 1 minute. The system behaves like a Bernoulli single-server queuing process with a frame size of 1 second.

(a) Compute the steady-state distribution for the number of concurrent jobs.

(b) What is the probability that more than 150 people attempted to call this number between 2 pm and 3 pm?

7.16. On the average, every 6 minutes a customer arrives at an M/M/k queuing system, spends an average of 20 minutes there, and departs. What is the mean number of customers in the system at any given time?

7.17. Verify the Little's Law for the M/M/1 queuing system and for its components – waiting and service.

7.18. A metered parking lot with two parking spaces represents a Bernoulli two-server queuing system with capacity limited by two cars and 30-second frames. Cars arrive at the rate of one car every 4 minutes. Each car is parked for 5 minutes, on the average.

(a) Compute the transition probability matrix for the number of parked cars.

(b) Find the steady-state distribution for the number of parked cars.

(c) What fraction of the time are both parking spaces vacant?

(d) What fraction of arriving cars will not be able to park?

(e) Every 2 minutes of parking costs 25 cents. Assuming that the drivers use all the parking time they pay for, what revenue is the owner of this parking lot expected to get every 24 hours?

7.19. (*This exercise may require a computer or at least a calculator.*) A walk-in hairdressing saloon has two hairdressers and two extra chairs for waiting customers. We model this system as a Bernoulli queuing process with two servers, 1-minute frames, capacity limited by 4 customers, arrival rate of 3 customers per hour, and the average service time of 45 minutes, not including the waiting time.

(a) Compute the transition probability matrix for the number of customers in the saloon at any time and find the steady-state distribution.

(b) Use this steady-state distribution to compute the expected number of customers in the saloon at any time, the expected number of customers waiting for a hairdresser, and the fraction of customers who found all the available seats already occupied.

(c) Each hairdresser comes for an eight-hour working day. How does it split between their working time and their resting time?

(d) How will the performance characteristics in (b,c) change if they simply put two more chairs into the waiting area?

7.20. Two tellers are now on duty in a bank from Exercise 7.11, and they work as an M/M/2 queuing system with the arrival rate of 0.4 min^{-1} and the mean service time of 2 min.

(a) Compute the steady state probabilities $\pi_0, ..., \pi_{10}$ and use them to approximate the expected number of customers in the bank as

$$\mathbf{E}(X) \approx \sum_{x=0}^{10} x\pi_x.$$

(b) Use the Little's Law to find the expected response time.

(c) From this, derive the expected waiting time and the expected number of waiting customers.

(d*) Derive the exact expected number of customers $\mathbf{E}(X)$ without the approximation in (a) and use it to recalculate the answers in (b) and (c). *(Hint:* $\mathbf{E}(X)$ *is computed as a series; relate it to the expected number of jobs in the M/M/1 system).*

7.21. A toll area on a highway has three toll booths and works as an M/M/3 queuing system. On the average, cars arrive at the rate of one car every 5 seconds, and it takes 12 seconds to pay the toll, not including the waiting time. Compute the fraction of time when there are ten or more cars waiting in the line.

7.22. Sports fans tune to a local sports talk radio station according to a Poisson process with the rate of three fans every two minutes and listen it for an Exponential amount of time with the average of 20 minutes.

(a) What queuing system is the most appropriate for this situation?

(b) Compute the expected number of concurrent listeners at any time.

(b) Find the fraction of time when 40 or more fans are tuned to this station.

7.23. Internet users visit a certain web site according to an M/M/∞ queuing system with the arrival rate of 2 min^{-1} and the expected time of 5 minutes spent at the site. Compute

(a) the expected number of visitors of the web site at any time;

(b) the fraction of time when nobody is browsing this web site.

7.24. (MINI-PROJECT) Messages arrive at an electronic mail server according to a Poisson process with the average frequency of 5 messages per minute. The server can process only one message at a time, and messages are processed on a "first come – first serve" basis. It takes an Exponential amount of time U to process any text message, *plus* an Exponential amount of time V, independent of U, to process attachments (if there are any), with $\mathbf{E}(U) = 2$ seconds and $\mathbf{E}(V) = 7$ seconds. Forty percent of messages contain attachments. Estimate the expected response time of this server by the Monte Carlo method.

(*Notice that because of the attachments, the overall service time is not Exponential, so the system is not M/M/1.*).

7.25. (Mini-project) A doctor scheduled her patients to arrive at equal 15-minute intervals. Patients are then served in the order of their arrivals, and each of them needs a Gamma time with the doctor that has parameters $\alpha = 4$ and $\lambda = 0.3$ min^{-1}. Use Monte Carlo simulations to estimate

(a) the expected response time;

(b) the expected waiting time;

(c) the probability that a patient has to wait before seeing the doctor.

(*This system is not M/M/1 either because the interarrival times are not random.*).

7.26. (Computer project) A multiserver system (computer lab, customer service, telephone company) consists of $n = 4$ servers (computers, customer service representatives, telephone cables). Every server is able to process any job, but some of them work faster than the others. The service times are distributed according to the table.

Server	Distribution	Parameters
I	Gamma	$\alpha = 7$, $\lambda = 3$ min^{-1}
II	Gamma	$\alpha = 5$, $\lambda = 2$ min^{-1}
III	Exponential	$\lambda = 0.3$ min^{-1}
IV	Uniform	$a = 4$ min, $b = 9$ min

The jobs (customers, telephone calls) arrive to the system at random times, independently of each other, according to a Poisson process. The average interarrival time is 2 minutes. If a job arrives, and there are free servers available, then the job is *equally likely* to be processed by any of the available servers. If no servers are available at the time of arrival, the job enters a queue. After waiting for 6 minutes, if the service has not started, the job leaves the system. The system works 10 hours a day, from 8 am till 6 pm.

Run at least 1000 Monte Carlo simulations and estimate the following quantities:

(a) the expected waiting time for a randomly selected job;

(b) the expected response time;

(c) the expected length of a queue (excluding the jobs receiving service), when a new job arrives;

(d) the expected *maximum* waiting time during a 10-hour day;

(e) the expected *maximum* length of a queue during a 10-hour day;

(f) the probability that at least one server is available, when a job arrives;

(g) the probability that at least two servers are available, when a job arrives;

(h) the expected number of jobs processed by each server;

(i) the expected time each server is idle during the day;

(j) the expected number of jobs still remaining in the system at 6:03 pm;

(k) the expected percentage of jobs that left the queue prematurely.

7.27. (COMPUTER PROJECT) An IT support help desk represents a queuing system with five assistants taking calls from customers. The calls occur according to a Poisson process with the average rate of one call every 45 seconds. The service times for the 1st, 2nd, 3rd, 4th, and 5th assistants are all Exponential random variables with parameters $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 0.4$, and $\lambda_5 = 0.5$ min^{-1}, respectively (the jth help desk assistant has $\lambda_k = k/10$ min^{-1}). Besides the customers who are being assisted, up to ten other customers can be placed on hold. At times when this capacity is reached, the new callers receive a busy signal.

Use the Monte Carlo methods to estimate the following performance characteristics,

(a) the fraction of customers who receive a busy signal;

(b) the expected response time;

(c) the average waiting time;

(d) the portion of customers served by each help desk assistant;

(e) all of the above if the 6th help desk assistant has been hired to help the first five, and $\lambda_6 = 0.6$. How has this assistant, the fastest on the team, improved performance of the queuing system?

Questions (b-d) refer to those customers who got into the system and did not receive a busy signal.