these numbers in frequent situations involving permutations and combinations, with or without replacement.

Given occurrence of event B, one can compute conditional probability of event A. Unconditional probability of A can be computed from its conditional probabilities by the Law of Total Probability. The Bayes Rule, often used in testing and diagnostics, relates conditional probabilities of A given B and of B given A.

Exercises

2.1. Out of six computer chips, two are defective. If two chips are randomly chosen for testing (without replacement), compute the probability that both of them are defective. List all the outcomes in the sample space.

2.2. Suppose that after 10 years of service, 40% of computers have problems with motherboards (MB), 30% have problems with hard drives (HD), and 15% have problems with both MB and HD. What is the probability that a 10-year old computer still has fully functioning MB and HD?

2.3. A new computer virus can enter the system through e-mail or through the internet. There is a 30% chance of receiving this virus through e-mail. There is a 40% chance of receiving it through the internet. Also, the virus enters the system simultaneously through e-mail and the internet with probability 0.15. What is the probability that the virus does not enter the system at all?

2.4. Among employees of a certain firm, 70% know C/C++, 60% know Fortran, and 50% know both languages. What portion of programmers

(a) does not know Fortran?

(b) does not know Fortran and does not know C/C++?

(c) knows C/C++ but not Fortran?

(d) knows Fortran but not C/C++?

(e) If someone knows Fortran, what is the probability that he/she knows C/C++ too?

(f) If someone knows C/C++, what is the probability that he/she knows Fortran too?

2.5. A computer program is tested by 3 *independent* tests. When there is an error, these tests will discover it with probabilities 0.2, 0.3, and 0.5, respectively. Suppose that the program contains an error. What is the probability that it will be found by at least one test?

2.6. Under good weather conditions, 80% of flights arrive on time. During bad weather, only 30% of flights arrive on time. Tomorrow, the chance of good weather is 60%. What is the probability that your flight will arrive on time?

2.7. A system may become infected by some spyware through the internet or e-mail. Seventy percent of the time the spyware arrives via the internet, thirty percent of the time via e-mail. If it enters via the internet, the system detects it immediately with probability 0.6. If via e-mail, it is detected with probability 0.8. What percentage of times is this spyware detected?

2.8. A shuttle's launch depends on three key devices that may fail independently of each other with probabilities 0.01, 0.02, and 0.02, respectively. If any of the key devices fails, the launch will be postponed. Compute the probability for the shuttle to be launched on time, according to its schedule.

2.9. Successful implementation of a new system is based on three independent modules. Module 1 works properly with probability 0.96. For modules 2 and 3, these probabilities equal 0.95 and 0.90. Compute the probability that at least one of these three modules fails to work properly.

2.10. Three computer viruses arrived as an e-mail attachment. Virus A damages the system with probability 0.4. Independently of it, virus B damages the system with probability 0.5. Independently of A and B, virus C damages the system with probability 0.2. What is the probability that the system gets damaged?

2.11. A computer program is tested by 5 independent tests. If there is an error, these tests will discover it with probabilities 0.1, 0.2, 0.3, 0.4, and 0.5, respectively. Suppose that the program contains an error. What is the probability that it will be found

(a) by at least one test?

(b) by at least two tests?

(c) by all five tests?

2.12. A building is examined by policemen with four dogs that are trained to detect the scent of explosives. If there are explosives in a certain building, and each dog detects them with probability 0.6, independently of other dogs, what is the probability that the explosives will be detected by at least one dog?

2.13. An important module is tested by three independent teams of inspectors. Each team detects a problem in a defective module with probability 0.8. What is the probability that at least one team of inspectors detects a problem in a defective module?

2.14. A spyware is trying to break into a system by guessing its password. It does not give up until it tries 1 million different passwords. What is the probability that it will guess the password and break in if by rules, the password must consist of

(a) 6 different lower-case letters

(b) 6 different letters, some may be upper-case, and it is case-sensitive

(c) any 6 letters, upper- or lower-case, and it is case-sensitive

(d) any 6 characters including letters and digits

2.15. A computer program consists of two blocks written independently by two different programmers. The first block has an error with probability 0.2. The second block has an error with probability 0.3. If the program returns an error, what is the probability that there is an error in both blocks?

2.16. A computer maker receives parts from three suppliers, S1, S2, and S3. Fifty percent come from S1, twenty percent from S2, and thirty percent from S3. Among all the parts supplied by S1, 5% are defective. For S2 and S3, the portion of defective parts is 3% and 6%, respectively.

(a) What portion of all the parts is defective?

(b) A customer complains that a certain part in her recently purchased computer is defective. What is the probability that it was supplied by S1?

2.17. A computer assembling company receives 24% of parts from supplier X, 36% of parts from supplier Y, and the remaining 40% of parts from supplier Z. Five percent of parts supplied by X, ten percent of parts supplied by Y, and six percent of parts supplied by Z are defective. If an assembled computer has a defective part in it, what is the probability that this part was received from supplier Z?

2.18. A problem on a multiple-choice quiz is answered correctly with probability 0.9 if a student is prepared. An unprepared student guesses between 4 possible answers, so the probability of choosing the right answer is 1/4. Seventy-five percent of students prepare for the quiz. If Mr. X gives a correct answer to this problem, what is the chance that he did not prepare for the quiz?

2.19. At a plant, 20% of all the produced parts are subject to a special electronic inspection. It is known that any produced part which was inspected electronically has no defects with probability 0.95. For a part that was not inspected electronically this probability is only 0.7. A customer receives a part and find defects in it. What is the probability that this part went through an electronic inspection?

2.20. All athletes at the Olympic games are tested for performance-enhancing steroid drug use. The imperfect test gives positive results (indicating drug use) for 90% of all steroid-users but also (and incorrectly) for 2% of those who do not use steroids. Suppose that 5% of all registered athletes use steroids. If an athlete is tested negative, what is the probability that he/she uses steroids?

2.21. In the system in Figure 2.7, each component fails with probability 0.3 independently of other components. Compute the system's reliability.

2.22. Three highways connect city A with city B. Two highways connect city B with city C. During a rush hour, each highway is blocked by a traffic accident with probability 0.2, independently of other highways.

(a) Compute the probability that there is at least one open route from A to C.

(b) How will a new highway, also blocked with probability 0.2 independently of other highways, change the probability in (a) if it is built

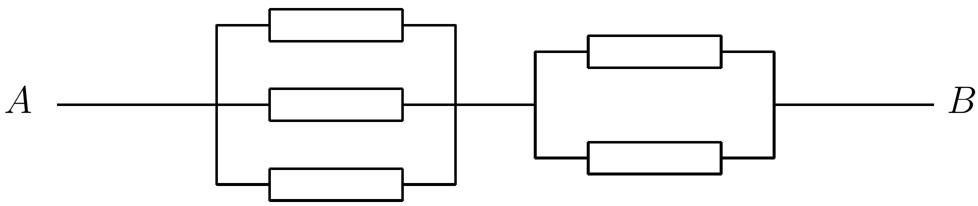

FIGURE 2.7: *Calculate reliability of this system (Exercise 2.21).*

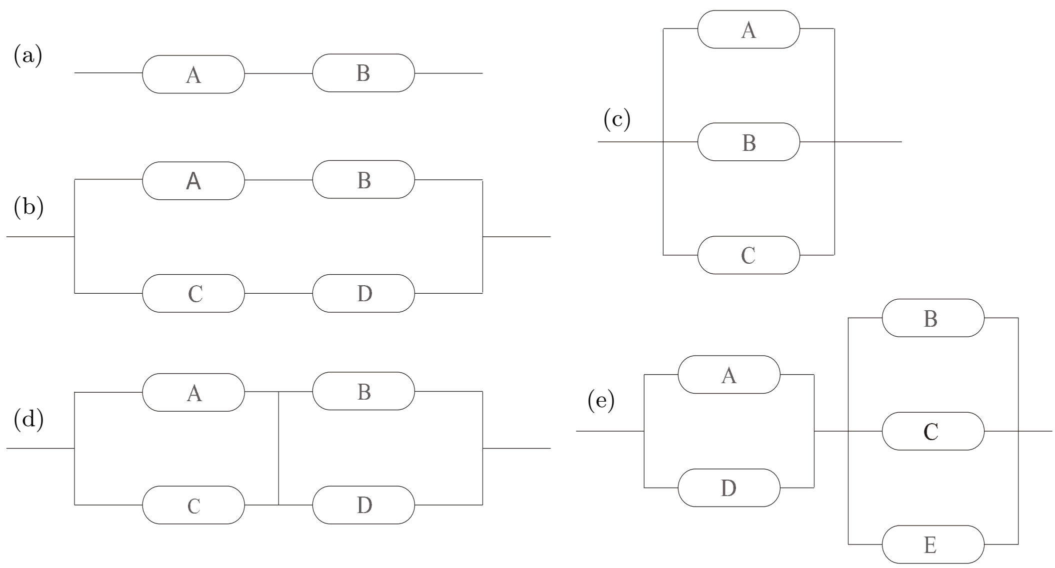

FIGURE 2.8: *Calculate reliability of each system (Exercise 2.23).*

(α) between A and B?
(β) between B and C?
(γ) between A and C?

2.23. Calculate the reliability of each system shown in Figure 2.8, if components A, B, C, D, and E function properly with probabilities 0.9, 0.8, 0.7, 0.6, and 0.5, respectively.

2.24. Among 10 laptop computers, five are good and five have defects. Unaware of this, a customer buys 6 laptops.

(a) What is the probability of exactly 2 defective laptops among them?

(b) Given that *at least* 2 purchased laptops are defective, what is the probability that *exactly* 2 are defective?

2.25. This is known as *the Birthday Problem*.

(a) Consider a class with 30 students. Compute the probability that at least two of them have their birthdays on the same day. (For simplicity, ignore the leap year).

(b) How many students should be in class in order to have this probability above 0.5?

2.26. Two out of six computers in a lab have problems with hard drives. If three computers are selected at random for inspection, what is the probability that none of them has hard drive problems?

2.27. Among eighteen computers in some store, six have defects. Five randomly selected computers are bought for the university lab. Compute the probability that all five computers have no defects.

2.28. A quiz consists of 6 multiple-choice questions. Each question has 4 possible answers. A student is unprepared, and he has no choice but guessing answers completely at random. He passes the quiz if he gets at least 3 questions correctly. What is the probability that he will pass?

2.29. An internet search engine looks for a keyword in 9 databases, searching them in a random order. Only 5 of these databases contain the given keyword. Find the probability that it will be found in at least 2 of the first 4 searched databases.

2.30. Consider the situation described in Example 2.24 on p. 22, but this time let us define the sample space clearly. Suppose that one child is older, and the other is younger, their gender is independent of their age, and the child you meet is one or the other with probabilities 1/2 and 1/2.

(a) List all the outcomes in this sample space. Each outcome should tell the children's gender, which child is older, and which child you have met.

(b) Show that *unconditional* probabilities of outcomes BB, BG, and GB are equal.

(c) Show that *conditional* probabilities of BB, BG, and GB, after you met Leo, are not equal.

(d) Show that the *conditional* probability that Leo has a brother is 1/2.

2.31. Show that events $A, B, C, \ldots$ are disjoint if and only if $\overline{A}, \overline{B}, \overline{C}, \ldots$ are exhaustive.

2.32. Events A and B are independent. Show, intuitively and mathematically, that:

(a) Their complements are also independent.

(b) If they are disjoint, then $\boldsymbol{P}\{A\} = 0$ or $\boldsymbol{P}\{B\} = 0$.

(c) If they are exhaustive, then $\boldsymbol{P}\{A\} = 1$ or $\boldsymbol{P}\{B\} = 1$.

2.33. Derive a computational formula for the probability of a union of N arbitrary events. Assume that probabilities of all individual events and their intersections are given.

2.34. Prove that

$$\overline{E_1 \cap \ldots \cap E_n} = \overline{E}_1 \cup \ldots \cup \overline{E}_n$$

for arbitrary events $E_1, \ldots, E_n$.

2.35. From the "addition" rule of probability, derive the "subtraction" rule:

$$\text{if } B \subset A, \text{ then } \boldsymbol{P}\{A \backslash B\} = \boldsymbol{P}(A) - \boldsymbol{P}(B).$$

2.36. Prove "subadditivity": $\boldsymbol{P}\{E_1 \cup E_2 \cup \ldots\} \leq \sum \boldsymbol{P}\{E_i\}$ for *any* events $E_1, E_2, \ldots \in \mathfrak{M}$.

Exercises

3.1. A computer virus is trying to corrupt two files. The first file will be corrupted with probability 0.4. Independently of it, the second file will be corrupted with probability 0.3.

(a) Compute the probability mass function (pmf) of X, the number of corrupted files.

(b) Draw a graph of its cumulative distribution function (cdf).

3.2. Every day, the number of network blackouts has a distribution (probability mass function)

x	0	1	2
$P(x)$	0.7	0.2	0.1

A small internet trading company estimates that each network blackout results in a \$500 loss. Compute expectation and variance of this company's daily loss due to blackouts.

3.3. There is one error in one of five blocks of a program. To find the error, we test three randomly selected blocks. Let X be the number of errors in these three blocks. Compute $\mathbf{E}(X)$ and $\mathrm{Var}(X)$.

3.4. Tossing a fair die is an experiment that can result in any integer number from 1 to 6 with equal probabilities. Let X be the number of dots on the top face of a die. Compute $\mathbf{E}(X)$ and $\mathrm{Var}(X)$.

3.5. A software package consists of 12 programs, five of which must be upgraded. If 4 programs are randomly chosen for testing,

(a) What is the probability that at least two of them must be upgraded?

(b) What is the expected number of programs, out of the chosen four, that must be upgraded?

3.6. A computer program contains one error. In order to find the error, we split the program into 6 blocks and test two of them, selected at random. Let X be the number of errors in these blocks. Compute $\mathbf{E}(X)$.

3.7. The number of home runs scored by a certain team in one baseball game is a random variable with the distribution

x	0	1	2
$P(x)$	0.4	0.4	0.2

The team plays 2 games. The number of home runs scored in one game is independent of the number of home runs in the other game. Let Y be the *total* number of home runs. Find $\mathbf{E}(Y)$ and $\mathrm{Var}(Y)$.

3.8. A computer user tries to recall her password. She knows it can be one of 4 possible passwords. She tries her passwords until she finds the right one. Let X be the number of wrong passwords she uses before she finds the right one. Find $\mathbf{E}(X)$ and $\mathrm{Var}(X)$.

3.9. It takes an average of 40 seconds to download a certain file, with a standard deviation of 5 seconds. The actual distribution of the download time is unknown. Using Chebyshev's inequality, what can be said about the probability of spending more than 1 minute for this download?

3.10. Every day, the number of traffic accidents has the probability mass function

x	0	1	2	more than 2
$P(x)$	0.6	0.2	0.2	0

independently of other days. What is the probability that there are more accidents on Friday than on Thursday?

3.11. Two dice are tossed. Let X be *the smaller* number of points. Let Y be *the larger* number of points. If both dice show the same number, say, z points, then $X = Y = z$.

(a) Find the joint probability mass function of (X, Y).

(b) Are X and Y independent? Explain.

(c) Find the probability mass function of X.

(d) If $X = 2$, what is the probability that $Y = 5$?

3.12. Two random variables, X and Y, have the joint distribution $P(x, y)$,

$P(x,y)$		x = 0	x = 1
y	0	0.5	0.2
	1	0.2	0.1

(a) Are X and Y independent? Explain.

(b) Are $(X + Y)$ and $(X - Y)$ independent? Explain.

3.13. Two random variables X and Y have the joint distribution, $P(0, 0) = 0.2$, $P(0, 2) = 0.3$, $P(1, 1) = 0.1$, $P(2, 0) = 0.3$, $P(2, 2) = 0.1$, and $P(x, y) = 0$ for all other pairs (x, y).

(a) Find the probability mass function of $Z = X + Y$.

(b) Find the probability mass function of $U = X - Y$.

(c) Find the probability mass function of $V = XY$.

3.14. An internet service provider charges its customers for the time of the internet use rounding it up to the nearest hour. The joint distribution of the used time (X, hours) and the charge per hour (Y, cents) is given in the table below.

$P(x,y)$		x = 1	2	3	4
y	1	0	0.06	0.06	0.10
	2	0.10	0.10	0.04	0.04
	3	0.40	0.10	0	0

Each customer is charged $Z = X \cdot Y$ cents, which is the number of hours multiplied by the price of each hour. Find the distribution of Z.

3.15. Let X and Y be the number of hardware failures in two computer labs in a given month. The joint distribution of X and Y is given in the table below.

$P(x,y)$		x = 0	1	2
y	0	0.52	0.20	0.04
	1	0.14	0.02	0.01
	2	0.06	0.01	0

(a) Compute the probability of at least one hardware failure.

(b) From the given distribution, are X and Y independent? Why or why not?

3.16. The number of hardware failures, X, and the number of software failures, Y, on any day in a small computer lab have the joint distribution $P(x,y)$, where $P(0,0) = 0.6$, $P(0,1) = 0.1$, $P(1,0) = 0.1$, $P(1,1) = 0.2$. Based on this information,

(a) Are X and Y (hardware and software failures) independent?

(b) Compute $\mathbf{E}(X+Y)$, i.e., the expected total number of failures during 1 day.

3.17. Shares of company A are sold at \$10 per share. Shares of company B are sold at \$50 per share. According to a market analyst, 1 share of each company can either gain \$1, with probability 0.5, or lose \$1, with probability 0.5, independently of the other company. Which of the following portfolios has the lowest risk:

(a) 100 shares of A

(b) 50 shares of A + 10 shares of B

(c) 40 shares of A + 12 shares of B

3.18. Shares of company A cost \$10 per share and give a profit of X%. Independently of A, shares of company B cost \$50 per share and give a profit of Y%. Deciding how to invest \$1,000, Mr. X chooses between 3 portfolios:

(a) 100 shares of A,

(b) 50 shares of A and 10 shares of B,

(c) 20 shares of B.

The distribution of X is given by probabilities:

$$P\{X = -3\} = 0.3, P\{X = 0\} = 0.2, P\{X = 3\} = 0.5.$$

The distribution of Y is given by probabilities:

$$P\{Y = -3\} = 0.4, P\{Y = 3\} = 0.6.$$

Compute expectations and variances of the total dollar profit generated by portfolios (a), (b), and (c). What is the least risky portfolio? What is the most risky portfolio?

3.19. A and B are two competing companies. An investor decides whether to buy

(a) 100 shares of A, or

(b) 100 shares of B, or

(c) 50 shares of A and 50 shares of B.

A profit made on 1 share of A is a random variable X with the distribution $P(X = 2) = P(X = -2) = 0.5$.

A profit made on 1 share of B is a random variable Y with the distribution $P(Y = 4) = 0.2, P(Y = -1) = 0.8$.

If X and Y are independent, compute the expected value and variance of the total profit for strategies (a), (b), and (c).

3.20. A quality control engineer tests the quality of produced computers. Suppose that 5% of computers have defects, and defects occur independently of each other.

(a) Find the probability of exactly 3 defective computers in a shipment of twenty.

(b) Find the probability that the engineer has to test at least 5 computers in order to find 2 defective ones.

3.21. A lab network consisting of 20 computers was attacked by a computer virus. This virus enters each computer with probability 0.4, independently of other computers. Find the probability that it entered at least 10 computers.

3.22. Five percent of computer parts produced by a certain supplier are defective. What is the probability that a sample of 16 parts contains more than 3 defective ones?

3.23. Every day, a lecture may be canceled due to inclement weather with probability 0.05. Class cancelations on different days are independent.

(a) There are 15 classes left this semester. Compute the probability that at least 4 of them get canceled.

(b) Compute the probability that the tenth class this semester is the third class that gets canceled.

3.24. An internet search engine looks for a certain keyword in a sequence of independent web sites. It is believed that 20% of the sites contain this keyword.

(a) Compute the probability that at least 5 of the first 10 sites contain the given keyword.

(b) Compute the probability that the search engine had to visit at least 5 sites in order to find the first occurrence of a keyword.

3.25. About ten percent of users do not close Windows properly. Suppose that Windows is installed in a public library that is used by random people in a random order.

(a) On the average, how many users of this computer *do not* close Windows properly before someone *does* close it properly?

(b) What is the probability that exactly 8 of the next 10 users will close Windows properly?

3.26. After a computer virus entered the system, a computer manager checks the condition of all important files. She knows that each file has probability 0.2 to be damaged by the virus, independently of other files.

(a) Compute the probability that at least 5 of the first 20 files are damaged.

(b) Compute the probability that the manager has to check at least 6 files in order to find 3 undamaged files.

3.27. Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

(a) What is the probability of receiving *at least* five messages during the next hour?

(b) What is the probability of receiving *exactly* five messages during the next hour?

3.28. The number of received electronic messages has Poisson distribution with some parameter λ. Using Chebyshev inequality, show that the probability of receiving more than 4λ messages does not exceed $1/(9\lambda)$.

3.29. An insurance company divides its customers into 2 groups. Twenty percent of customers are in the high-risk group, and eighty percent are in the low-risk group. The high-risk customers make an average of 1 accident per year while the low-risk customers make an average of 0.1 accidents per year. Eric had no accidents last year. What is the probability that he is a high-risk driver?

3.30. Eric from Exercise 3.29 continues driving. After three years, he still has no traffic accidents. Now, what is the conditional probability that he is a high-risk driver?

3.31. Before the computer is assembled, its vital component (motherboard) goes through a special inspection. Only 80% of components pass this inspection.

(a) What is the probability that at least 18 of the next 20 components pass inspection?

(b) On the average, how many components should be inspected until a component that passes inspection is found?

3.32. On the average, 1 computer in 800 crashes during a severe thunderstorm. A certain company had 4,000 working computers when the area was hit by a severe thunderstorm.

(a) Compute the probability that less than 10 computers crashed.

(b) Compute the probability that exactly 10 computers crashed.

You may want to use a suitable approximation.

3.33. The number of computer shutdowns during any month has a Poisson distribution, averaging 0.25 shutdowns per month.

(a) What is the probability of at least 3 computer shutdowns during the next year?

(b) During the next year, what is the probability of at least 3 months (out of 12) with exactly 1 computer shutdown in each?

3.34. A dangerous computer virus attacks a folder consisting of 250 files. Files are affected by the virus independently of one another. Each file is affected with the probability 0.032. What is the probability that more than 7 files are affected by this virus?

3.35. In some city, the probability of a thunderstorm on any day is 0.6. During a thunderstorm, the number of traffic accidents has Poisson distribution with parameter 10. Otherwise, the number of traffic accidents has Poisson distribution with parameter 4. If there were 7 accidents yesterday, what is the probability that there was a thunderstorm?

3.36. An interactive system consists of ten terminals that are connected to the central computer. At any time, each terminal is ready to transmit a message with probability 0.7, independently of other terminals. Find the probability that exactly 6 terminals are ready to transmit at 8 o'clock.

3.37. Network breakdowns are unexpected rare events that occur every 3 weeks, on the average. Compute the probability of more than 4 breakdowns during a 21-week period.

3.38. Simplifying expressions, derive from the definitions of variance and covariance that

(a) $\mathrm{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}^2(X)$;

(b) $\mathrm{Cov}(X, Y) = \mathbf{E}(XY) - \mathbf{E}(X)\,\mathbf{E}(Y)$.

3.39. Show that

$$\begin{aligned} &\mathrm{Cov}(aX + bY + c, dZ + eW + f) \\ &= \; ad\,\mathrm{Cov}(X, Z) + ae\,\mathrm{Cov}(X, W) + bd\,\mathrm{Cov}(Y, Z) + be\,\mathrm{Cov}(Y, W) \end{aligned}$$

for any random variables X, Y, Z, W, and any non-random numbers a, b, c, d, e, f.