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Probability and Statistics (CSL003P1S)

Assignment - 2

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Attempted Question Sequence
→ 2, 3, 5, 6, 9, 10, 11, 15, 12, 4, 7, 8, 13, 14, 1

Sol (2)

$E(X) = 3 = \mu$

$E(X^2) = 13 = \sigma^2$

(2). Let a random variable X, with mean, (μ), $E(X) = 3$ and ~~variance (σ^2), $E(X^2) = 13$~~. $E[X^2] = 13$,

$$Var(X) = E[X^2] - (E[X])^2 = 13 - 3^2 = 13 - 9 = 4$$

determine, lower bound for $P(-2 < X < 8)$

so, the interval can be rewritten as

$$-2 < X < 8 \Rightarrow -2-3 < X-3 < 8-3 \Rightarrow -5 < X < 5.$$

value of K (the distance from mean) is 5

Apply Chebyshev's inequality.

$$K = 5, \quad Var(X) = 4.$$

$$P(|X-3| < 5) \geq 1 - \frac{4}{5^2} = 1 - \frac{4}{25} = 1 - 0.16 = 0.84$$

The lower bound $P(-2 < X < 8)$ is 0.84

$$\boxed{P(|X - E[X]| < K) \geq 1 - \frac{Var(X)}{K^2}}$$

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(3) p.m.f $P(X=x) = \begin{cases} 1/8, & x=-1,1 \\ 6/8, & x=0 \\ 0 & o.w. \end{cases}$

show that bound for chebychevs inequality cannot be improved

$$\mu = \sum_x x \cdot P(X=x) \Rightarrow -1\cdot\frac{1}{8} + 0\cdot\frac{6}{8} + 1\cdot\frac{1}{8} = -\frac{1}{8}+\frac{1}{8}$$
$$=0$$

$$\sigma^2 = \sum_x (x-\mu)^2 \cdot P(X=x) \quad \text{. since } \mu=0$$

$$\Rightarrow \sum_x x^2 \cdot P(x) = (-1)^2\cdot\frac{1}{8} + (1)^2\frac{1}{8} = \frac{2}{8} = 1/4$$

Chebyshev's inequality states

$$P(|X-\mu| \geq K\sigma) \leq \frac{1}{K^2}$$

Here $\mu=0$, $\sigma = 1/2$

Let's compute bound for $K=2$, which chebyshev's ineq

$$P(|X| \geq 2\cdot\tfrac{1}{2}) \leq \frac{1}{2^2} = \frac{1}{4}$$

$$\Rightarrow \quad P(|X| \geq 1) \leq \frac{1}{4}$$

Compute Actual Prob

$$P(|X| \geq 1) = P(X=-1) + P(X=1) = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

Therefore, chebyshev inequality cannot be improve as $P(|X|\geq 1) = \frac{1}{4}$, which in exactly some as bound provided by chebyshev's ineq.

(5). Let X be normal r.v. with $\mu = 10$ $\sigma^2 = 36$. Compute

(a) $P(X > 5)$

$\sigma^2 = 36, \ \sigma = 6.$

To compute prob for normal distr. we standardize X to Z using formula

$$Z = \frac{X - \mu}{\sigma}$$

This transforms X to $Z \sim N(0,1)$, and we can use standard normal table (z-table)

$$Z = \frac{5-10}{6} = -5/6$$

$$\cancel{P(Z > -0.833) = 1 - P(Z < -0.833)}$$

$$P(Z > -5/6) = 1 - \phi\left(-\frac{5}{6}\right) = 1 - \left(1 - \phi\left(\frac{5}{6}\right)\right)$$

$$= \phi\left(\frac{5}{6}\right) = 0.7967$$

(b) $P(4 < X < 16) = \frac{4-10}{6} < Z < \frac{16-10}{6}$

$$= P(-1 < Z < 1) = \phi(1) - \phi(-1) = \phi(1) - (1 - \phi(1))$$

$$= 2\phi(1) - 1$$

$$= 2 \times 0.8413 - 1$$

$$= 1.6826 - 1$$

$$= 0.6826$$

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(5) $P(X>8)$, $Z = \frac{8-10}{6} = \frac{-1}{3}$

$\phi\left(-\frac{1}{3}\right) = 1-\phi\left(\frac{1}{3}\right) = 1-0.6293$
$= 0.3707$

(6). Let $X \sim N(\mu, \sigma^2)$. if $P(X \leq 0) = 1/2$ & $P(-1.96 \leq X \leq 1.96) = 0.95$. find μ, σ^2.

$P(X \leq 0) = \frac{1}{2} = P(X \geq 0) \Rightarrow \mu = 0$

$P(-1.96 \leq X \leq 1.96) = 0.95$

$P\left(-\frac{1.96}{\sigma} \leq \frac{X}{\sigma} \leq \frac{1.96}{\sigma}\right) = 0.95$

$P\left(-\frac{1.96}{\sigma} \leq Z \leq \frac{1.96}{\sigma}\right) = 0.95$; $Z \sim N(0,1)$

$2\phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.5$

$\phi\left(\frac{1.96}{\sigma}\right) = 0.975$

$\frac{1.96}{\sigma} = \phi^{-1}(0.975) = 1.96 \Rightarrow \sigma = 1$

(9) for a dis. r.v X with pmf.

$P(X=-2) = \frac{1}{5}$, $P(X=-1) = \frac{1}{6}$, $P(X=0) = \frac{1}{5}$

$P(X=1) = \frac{1}{15}$, $P(X=2) = \frac{10}{30}$, $P(X=3) = \frac{1}{30}$

Find PMF for $Y = X^2$

$X = \{-2, -1, 0, 1, 2, 3\}$
$X^2 = \{0, 1, 4, 9\}$

$$P(Y=y) = \begin{cases} 1/5, & y=0 \\ 1/6 + 1/15 = \frac{7}{30} & y=1 \\ 1/5 + 1/3 = \frac{8}{15} & y=4 \\ 1/30 & y=9 \end{cases}$$

(10) find p.m.f of $Y = \frac{X}{X+1}$, $P(X=x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x, & x = 0, 1, 2 \ldots \\ 0 & o\omega \end{cases}$

range of $Y = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots\}$

$P(Y=y) = P\left(\frac{X}{X+1} = y\right) = P\left(X = \frac{y}{1-y}\right)$

$$= \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{y/y-1}, & y = 0, \frac{1}{2}, \frac{2}{3} \ldots \\ 0 & o\omega \end{cases}$$

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(11) Let $X \sim U(0,1)$, find dist. of following

(a) $Y = \sqrt{X}$

$$f_X(x) = \begin{cases} 0 & \text{o.w.} \\ x & 0 \le x \le 1 \\ 0 & \end{cases}$$

$$F_Y(y) = P(\sqrt{X} \le y) = P(X \le y^2) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \le y < 1 \\ 1 & y > 1 \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ 2y & 0 \le y \le 1 \\ 0 & y > 1 \end{cases} = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(b) $y = X^2$

$$F_Y(y) = P(X^2 \le y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y < 1 \end{cases}$$

$\because P(X^2 \le y) = P(0 \le X \le \sqrt{y}) = \sqrt{y}$ for $0 \le y \le 1$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(c) $y = 2x + 3 \Rightarrow F_Y(y) = P(2X + 3 \le y) = P\left(X \le \frac{y-3}{2}\right)$

$$= \begin{cases} 0 & y < 0 \\ \frac{y-3}{2} & 3 \le y \le 5 \\ 1 & y > 5 \end{cases}$$

$$f_X(y) = \begin{cases} 1/2 & 3 \le y \le 5 \\ 0 & 0 \end{cases} \Rightarrow Y \sim U(3,5)$$

(d) $y = -\lambda \log X \Rightarrow F_X(y) = P(-\lambda \log X \le y) = P(X > e^{-y/\lambda})$

$\Rightarrow y \to (0, \infty)$ $\quad = 1 - P(X \le e^{-y/\lambda})$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y/\lambda} & y \ge 0 \end{cases}$$

$$f_X(y) = \begin{cases} \frac{1}{\lambda} e^{-y/\lambda} & y \ge 0 \\ 0 & \text{o.w.} \end{cases} \Rightarrow Y \sim Exp(\lambda)$$

(15) $Y = 2X - 6$, $y \in (-\infty, \infty)$

$F_Y(y) = P(Y \le y)$

$P(Y \le y) = P(2X - 6 \le y)$ $\because Y = 2X - 6$

$= P\left(X \le \frac{Y+6}{2}\right)$

$= P\left(\frac{X-\mu}{\sigma} \le \frac{\frac{Y+6}{2} - \mu}{\sigma}\right) = \phi\left(\frac{y+6-2\lambda}{2\sigma}\right)$

$f_Y(y) = \phi\left(\frac{y+6-2\lambda}{2\sigma}\right) \cdot \frac{1}{2\sigma}$ $y \in (-\infty, \infty)$

$= \frac{1}{\sqrt{2\pi}} e^{1/2\left(\frac{y-(2\mu-6)}{2\sigma}\right)^2} \cdot \frac{1}{2\sigma} = \frac{1}{\sqrt{2\pi}(2\sigma)} e^{-\left(-\frac{1}{8\sigma^2}(y-(2\mu-6))^2\right)}$

$Y \sim N(2\mu-6, 4\sigma^2)$

(12). $X \sim U(0, \theta)$, $\theta > 0$. Find dist of $Y = \min(X, \theta/2)$

$Y = \min(X, \theta/2) \rightarrow (0, \theta/2) \leftarrow$ range of Y

$F_Y(Y) = P(Y \le y) = P(\min(X, \theta/2) \le y) = 1 - P(\min(X, \theta/2) > y)$

$= 1 - P(X > y \wedge \theta/2 > y)$

Now. $P(X > y \wedge \theta/2 > y) = 1$ if $y = 0$

$P(X > y \wedge \theta/2 > y) = 0$ if $y \ge \theta/2$

for $0 \le y \le \theta/2$, $P(X > y \wedge \theta/2 > y) = P(X > y) = \frac{1}{\theta}\int_y^\theta dx = \frac{\theta - y}{\theta}$

$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ y/\theta & 0 \le y < \theta/2 \\ 1 & y \ge \theta/2 \end{cases}$

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(4). Let, Bell 1 be $X \sim Exp$ with mean $\alpha \Rightarrow f_x(x) = \frac{1}{\alpha} e^{-x/\alpha}, x > 0$
& Bell 2 be $Y \sim Exp$ with mean $2\alpha \Rightarrow f_y(y) = \frac{1}{2\alpha} e^{-y/2\alpha}, y > 0$

($\because$) Exp distribution $\Rightarrow f(x) = \frac{1}{\beta} e^{-x/\beta}$ & mean $= E(x) = \beta$

Now $P(X > \alpha \wedge Y > \alpha) = P(X > \alpha) P(Y > \alpha)$ {Independence

$$= \left(\int_{\alpha}^{\infty} \frac{1}{\alpha} e^{-x/\alpha} dx\right)\left(\int_{\alpha}^{\infty} \frac{1}{2\alpha} e^{-y/2\alpha} dy\right) = e^{-1} \times e^{-1/2} = e^{-3/2}$$

(7) Let X :- lifetime of compte chips & $X \sim N(\mu, \sigma^2)$
where $\mu = 1.4 \times 10^6 h$ & $\sigma = 3 \times 10^5 h \Rightarrow \sigma^2 = 9 \times 10^{10}$ hrs

$$P(X < 1.8 \times 10^6) = \left(\frac{X - 1.4 \times 10^6}{3 \times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5}\right) = P\left(Z < \frac{4}{3}\right)$$

$$= \Phi\left(4/3\right) = 0.918$$

Now let Y :- # chips having lifetime $< 1.8 \times 10^6$ hrs
& Acc to quest^n $Y \sim Bin(10, 0.918)$

$$= P(Y \geq 2) = 1 - P(Y < 2) = 1 - (P(Y=0) + P(Y=1))$$

$$= 1 - \left[{}^{10}C_0 (0.918)^0 (1-0.918)^{10} + {}^{10}C_1 (0.918)^1 (1-0.918)^9\right]$$

$$= 1 - \left[(0.082)^{10} + 10 \times 0.918 \times 0.082^9\right] = 0.999$$

(8) (i) $E((2+x)^{-1}) = \sum_{x=0}^{\infty} (2+x)^{-1} \frac{\lambda^x e^{\lambda}}{x!}$

(ii) As $X \sim P(\lambda)$ = pmf of $P(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2 \ldots \\ 0 & o\omega \end{cases}$

& $Y = X^2 - 5$

Now $\mathcal{X} = \{0, 1, 2 \ldots\}$ & $\mathcal{Y} = \{-5, -4, -1, 4 \ldots\}$

pmf . $P(Y=y) = P(X^2 - 5 = y) = P(X = \sqrt{y+5}) =$

$$\begin{cases} \frac{e^{\lambda}\lambda^{\sqrt{y+5}}}{\sqrt{y+5}\,!}, & y \in \mathcal{Y} \\ 0 & o\omega \end{cases}$$

(13) pdf of X, $f_X(x) = \begin{cases} 1/2 & -1/2 < x < 3/3 \\ 0 & o\omega \end{cases}$ As, $X \sim U(-1/2, 3/2)$

$Y = X^2 \Rightarrow y$ is $(0, 9/4)$

$f_y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

for $y < 0$, $f_Y(y) = 0$. for $0 \le y \le \frac{1}{4}$, $f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx = \sqrt{y}$.

for $y > 9/4$, $f_Y(y) = 1$

for $1/4 < y < 9/4$, $f_y(y) = \int_{-\sqrt{y}}^{-1/2} 0 . dx + \int_{-1/2}^{\sqrt{y}} \frac{1}{2} dx = \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right)$

$= \frac{1}{4} + \sqrt{y}/2$

$\Rightarrow f_X(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le 1/4 \\ \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right) & 1/4 \le y \le 9/4 \\ 1 & y \ge 9/4 \end{cases}$ & $f_y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 \le y \le \frac{1}{4} \\ \frac{1}{4\sqrt{y}}, & \frac{1}{4} \le y \le \frac{9}{4} \end{cases}$

(14) $X \sim U(0,1)$, $Y = \min(X, 1-X)$. $y \to (0, 1/2)$

$F_Y(y) = P(Y \le y) = P(\min(X,1-X) \le y) = 1 - P(\min(X,1-X) > y)$

$= 1 - P(X > y \wedge 1-X > y) = 1 - P(y < X < 1-y)$

$P(y<X<1-y) = \begin{cases} 1 & y \le 0 \\ \int_y^{1-y} dx & \text{if } 0 < y < 1/2 \\ 0 & y \ge 1/2 \end{cases} \Rightarrow F_Y(y) = \begin{cases} 0 & y \le 0 \\ 2y & 0<y<1/2 \\ 1 & y \ge 2. \end{cases}$

&

$f_Y(y) = \begin{cases} 2 & 0<y<1/2 \\ 0 & \text{o.w.} \end{cases}$

$Z = \frac{1-Y}{Y} \Rightarrow z \to (1, \infty)$

$F_Z(z) = P(Z \le z)$

if $z \le 1$, $F_Z(z) = 0$

if $z > 1$, $P(Z \le z) = P\left(\frac{1}{Y} - 1 \le z\right) = P\left(\frac{1}{y} \le z+1\right)$

$= P\left(Y \ge \frac{1}{z+1}\right) = 1 - P\left(y < \frac{1}{z+1}\right) = 1 - \frac{2}{z+1}$

$F_Z(z) = \begin{cases} 0 & z \le 1 \\ 1 - \frac{2}{z+1} & \text{if } z > 1 \end{cases}$ & $f_Z(z) = \begin{cases} \frac{2}{(1+z)^2} & z > 1 \\ 0 & \text{o.w.} \end{cases}$

(1). As $M_X(t) = E(e^{tx}) = \sum e^{tx} f(x)\,dx$ — (I)

given, $M_X(t) = \frac{1}{2}e^{-5t} + \frac{1}{6}e^{4t} + \frac{1}{8}e^{5t} + \frac{5}{24}e^{25t}$ — (II)

comparing (I) & (II): $\Rightarrow$

$X = x$	-5	4	5	25
$P(X=x)$	$1/2$	$1/6$	$1/8$	$5/24$

$\Rightarrow F_X(x) = \begin{cases} 0 & x < -5 \\ 1/2 & -5 \le x < 4 \\ 1/2 + 1/6 = 4/6 & 4 \le x < 5 \\ 1/2 + 1/6 + 1/8 = 38/48 & 5 \le x < 25 \\ 1/2 + 1/6 + 1/8 + 5/24 = 1 & x \ge 25 \end{cases}$