Practice Set-2: Probability and Statistics (CSL003P1M): ©IIT Jammu

1. Let
$$F(x) := \begin{cases} 0 & \text{if } x < -1 \\ \dfrac{x+2}{4} & \text{if } -1 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$
Show that $F(\cdot)$ is a distribution function. Sketch the graph of $F(x)$ and compute the probabilities
$$P(-1/2 < X \le 1/2), P(X = 0), P(X = 1) \text{ and } P(-1 \le X < 1).$$

2. Which of the following functions is(are) distribution functions?
(a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1/2 \\ 1, & x > 1/2. \end{cases}$; (b) $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \ge 0. \end{cases}$

3. Let $F(x) := \begin{cases} 0 & \text{if } x \le 0 \\ 1 - \frac{2}{3}e^{-\frac{x}{3}} - \frac{1}{3}e^{-[\frac{x}{3}]} & \text{if } x > 0 \end{cases}$ where $[x]$ is the largest integer $\le x$. Show that $F(\cdot)$ is a distribution function and compute $P(X > 6), P(X = 5)$, and $P(5 \le X \le 8)$.

4. Find the value of α and k so that F given by
$$F(x) := \begin{cases} 0 & \text{if } x \le 0 \\ \alpha + ke^{-\frac{x^2}{2}} & \text{if } x > 0 \end{cases}$$
is distribution function of a continuous random variable.

5. Suppose F_X is the distribution function of a random variable X. Determine the distribution function of (a) X^+ and (b) $|X|$, where
$$X^+ := \begin{cases} X \text{ if } X \ge 0 \\ 0 \text{ if } X < 0 \end{cases}$$

6. Which of the following functions are probability mass functions?
(a) $f(x) := \begin{cases} \frac{x-2}{2}, & \text{if } x = 1, 2, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$
(b) $f(x) := \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x = 0, 1, 2, 3, 4, \ldots \\ 0, & \text{otherwise.} \end{cases}$ where $\lambda > 0$.
(c) $f(x) := \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x = 1, 2, 3, 4, \ldots \\ 0, & \text{otherwise.} \end{cases}$ where $\lambda > 0$.

7. Find the value of the constant c such that $f(x) = (1 - c)c^x; x = 0, 1, 2, 3, \ldots$ defines a probability mass function.

8. Let X be a discrete random variable taking value in $X = \{-3, -2, -1, 0, 1, 2, 3\}$ such that $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$ and $P(X < 0) = P(X = 0) = P(X > 0)$. Find the distributional function of X.

9. Show that
$$f(x) := \begin{cases} \frac{x^2}{18}, & \text{if } -3 < x < 3; \\ 0, & \text{otherwise.} \end{cases}$$
defines a probability density function. Find the corresponding distribution function and hence find $P(|X| < 1)$ and $P(X^2 < 9)$.

10. Find the expected number of throws of a fair die required to obtain a 6.

11. Let X be a continuous, nonnegative random variable with d.f. $F(x)$. Show that
$$E(X) = \int_0^\infty (1 - F(x))dx.$$

12. Find $E(X)$, if it exists, in the following cases:
(a) X has the p.m.f. $P(X = x) = \begin{cases} (x(x+1))^{-1}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$
(b) X has the p.d.f. $f(x) = \begin{cases} (2x^2)^{-1}, & \text{if } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$
(c) X has the p.d.f. $f(x) = \dfrac{1}{\pi}\dfrac{1}{1+x^2}, -\infty < x < \infty$

13. Find the mean and variance of the distributions having the following p.d.f. / p.m.f.
(a) $f(x) = ax^{a-1};\ 0 < x < 1, a > 0$, and $f(x) = 0$ otherwise.
(b) $f(x) = \dfrac{1}{n}$; for $x = 1, 2, \ldots, n$; where $n > 0$ is an integer, and $f(x) = 0$ otherwise.
(c) $f(x) = \frac{3}{2}(x-1)^2;\ \ 0 < x < 2$; and $f(x) = 0$ otherwise.

14. A target is made of three concentric circles of radii $1/\sqrt{3}, 1, \sqrt{3}$ feet. Shots within the inner circle give 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target give 0. Let X be the distance of the hit from the centre (in feet) and let the p.d.f. of X be
$$f(x) := \begin{cases} \frac{2}{\pi(1+x^2)}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$
What is the expected value of the score in a single shot?

*** All the best ***