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2024PCS0036

Probability and Statistics

Assignment-5

[Q]

Convergence of Random Variables:

Q3 $X_1 \dots X_n \overset{iid}{\sim} U(0,1)$. $Y_n = \min(X_1 \dots X_n)$ $Z_n = \max(X_1 \dots X_n)$. Show

(a) $\sqrt{Y_n} \xrightarrow{P} 0$. (b) $Z_n^2 \xrightarrow{P} 1$ (c) $Y_n^2 Z_n^2 \xrightarrow{P} 0$

Sol. (a) $Y_n = \min(X_1 \dots X_n)$

$$F_{Y_n}(y) = 1 - (1 - F_X(y))^n$$
$$= 1 - (1-y)^n \text{ for } y \in [0,1].$$

for uniform dist. i.e. $U(0,1)$ cdf is $F_X(x) = \begin{cases} 0 & x<0 \\ x & 0<x<1 \\ 1 & 1>x \end{cases}$

let $Z_n = \sqrt{Y_n}$

$$F_{Z_n}(Z) = P(\sqrt{Y_n} \le Z)$$
$$= P(Y_n \le Z^2)$$
$$= F_{Y_n}(Z^2)$$
$$= 1 - (1 - Z^2)^n \text{ for } Z \in [0,1]$$

T.P:- $\sqrt{Y_n} \xrightarrow{P} 0$ as $n \to \infty$

i.e. $\forall \epsilon > 0, P(|\sqrt{Y_n} - 0| > \epsilon) \to 0$ as $n \to \infty$

Now.
$$\Rightarrow P(|\sqrt{Y_n}| > \epsilon) = 1 - P(\sqrt{Y_n} \le \epsilon)$$
$$= 1 - P(Z_n \le \epsilon)$$
$$= 1 - F_{Z_n}(\epsilon)$$
$$= 1 - [1 - (1-\epsilon^2)^n]$$
$$= (1-\epsilon^2)^n$$
$$\to 0 \text{ as } n \to \infty$$

$\{ \because 0 < (1-\epsilon^2) < 1$

$\therefore P(|\sqrt{Y_n}| > \epsilon) \to 0$ as $n \to \infty$

$\Rightarrow \sqrt{Y_n} \xrightarrow{P} 0$ as $n \to \infty$ Hence Proved

(b) $Z_n = \max(X_1 \ldots X_n)$

$F_{Z_n}(z) = (F_X(z))^n$

$= z^n$ for $z \in [0,1]$

$F_{Z_n^2}(z) = P(Z_n^2 \le z) = P(Z_n \le \sqrt{z})$

$= (\sqrt{z})^n = z^{n/2}$ for $z \in [0,1]$

TP $\quad Z_n^2 \xrightarrow{P} 1$ as $n \to \infty$

Now $\quad P(|Z_n^2 - 1| > \epsilon) = P(Z_n^2 < 1-\epsilon) + P(Z_n^2 > 1+\epsilon)$

$= P(Z_n < \sqrt{1-\epsilon}) + P(Z_n > \sqrt{1+\epsilon})$

$= (1-\epsilon)^{n/2} + 1 - (1+\epsilon)^{n/2}$

But for large n we expect Z_n to be close to 1, so we focus on first term.

$\Rightarrow \quad P(|Z_n^2 - 1| > \epsilon) \le (1-\epsilon)^{n/2}$

for fixed ϵ as $n \to \infty$ $\quad (1-\epsilon)^{n/2} \to 0$

$\Rightarrow \quad P(|Z_n^2 - 1| > \epsilon) \to 0$ as $n \to \infty$

$\Rightarrow \quad Z_n^2 \xrightarrow{P} 1$ as $n \to \infty$ $\qquad$ Hence proved

(c) As $\sqrt{Y_n} \xrightarrow{P} 0$ as $n \to \infty$ (proved in part a)

& $Z_n^2 \xrightarrow{D} 1$ as $n \to \infty$ (proved in part b)

$Y_n^2 \xrightarrow{D} 0$ as $n \to \infty$ $\quad \{\because$ it is continuous function$\}$

$\left\{ \begin{array}{l} \text{as} \\ \text{Let} \end{array} \right. \sqrt{Y_n} = A_n \;\; \& \;\; (\sqrt{Y_n})^4 = Y_n^2 = A_n^4 \}$

$\therefore \quad Y_n^2 Z_n^2 \xrightarrow{D} 0 \cdot 1$ as $n \to \infty$ $\qquad \left\{ \begin{array}{l} \because X_n \xrightarrow{P} X \\ Y_n \xrightarrow{D} Y \\ X_n Y_n \xrightarrow{D} XY. \end{array} \right.$

$\therefore \quad Y_n^2 Z_n^2 \xrightarrow{D} 0$ as $n \to \infty$

Hence proved

Q.4 Let $X_1 \dots X_n \overset{iid}{\sim} N(0,1)$ Show that

$\frac{\bar{X}_n}{S_n} \xrightarrow{P} 0$ when $\bar{X}_n = \frac{1}{n}\sum_1^n X_i$ & $S_n^2 = \frac{1}{n}\sum_1^n (X_i - \bar{X}_n)^2$

Sol. $X_1 \dots X_n$ are iid & $E(X_i) = 0 < \infty \quad \forall\ i = 1$ to n

they by KWLLN

$$\frac{1}{n}\sum_1^n X_i = \bar{X}_n \xrightarrow{p} \mu \ (= E(X_i)) \text{ as } n \to \infty$$

$$\Rightarrow \bar{X}_n \xrightarrow{P} 0 \text{ as } n \to \infty$$

Also $\frac{1}{n}\sum_1^n (X_i - \bar{X}_n)^2 \Rightarrow S_n^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$

$$\Rightarrow S_n \xrightarrow{p} \sigma \quad . \quad n \to \infty$$ {∵ it is a cont. func

$$\Rightarrow \frac{\bar{X}_n}{S_n} \xrightarrow{P} \frac{0}{\sigma} \text{ as } n \to \infty$$

$$\Rightarrow \frac{\bar{X}_n}{S_n} \xrightarrow{p} 0 \text{ as } n \to \infty$$

Q.5 $Y_n \sim Bin(n, p)$, Show $\left(1 - \frac{Y_n}{n}\right) \xrightarrow{P} 1 - p$.

As $Y_n \sim Bin(n, p)$

Y_n = sum of n-independent Bernoulli trials with prob of success p

i.e. $Y_n = \sum_1^n X_i$

$$\Rightarrow E(Y_n) = \sum_1^n E(X_i) = np$$

$$V(Y_n) = npq.$$

$$\Rightarrow E\left(\frac{Y_n}{n}\right) = p \quad \& \quad V\left(\frac{Y_n}{n}\right) = \frac{pq}{n}$$

By Chebyshev. inequality

$$P\left(\left|\frac{Y_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2}$$

$$\Rightarrow P\left(\left|\frac{Y_n}{n} - p\right| \geq \epsilon\right) \leq \frac{pq}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow P\left(\left|\frac{Y_n}{n} - p\right| \geq \epsilon\right) = 0$$

$$\Rightarrow \frac{Y_n}{n} \xrightarrow{P} p \text{ as } n \to \infty$$

$$\Rightarrow 1 - \frac{Y_n}{n} \xrightarrow{P} 1 - p \text{ as } n \to \infty$$

Q.20 $X_1 .. X_n \sim$ iid $U(0,2)$. Let $Y_n = \bar{X}_n$, show

$$\sqrt{n}(Y_n - 1) \xrightarrow{L} N(0, \tfrac{1}{3})$$

Sol. mean $\mu = \frac{a+b}{2} = 1$

var $= \frac{(2-0)^2}{12} = 1/3$

By CLT

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{L} Y \quad \text{where } Y \sim N(0, \sigma^2)$$

$$\sqrt{n}(\bar{X}_n - 1) \xrightarrow{L} Y \quad \text{where } Y \sim N(0, 1/3)$$

Unbiased Estimator, Sufficient Statistics:

Q.1 Let $X_1 ... X_n$ be random sample from exponential dist with pdf.

$$f_X(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right); \quad x>0$$

Show that $\bar{X} = \sum_1^n X_i/n$ is an unbiased estimator of β.

Sol. $E(X_i) = \beta$ for $i = 1$ to n

Now $E(\bar{X}) = E\frac{(\sum_1^n X_i)}{n} = \frac{1}{n}\sum_1^n E(X_i)$

$= \frac{1}{n} . n\beta = \beta$

{ $S(\underline{X})$ is said to be unbiased estimator of $g(\theta)$ if $E(S(\underline{X})) = g(\theta)$ }

Therefore $\bar{X}$ is unbiased estimator of β.

Q.5 Let $X_1 ... X_n$ Random Sample from $P(\theta)$; $\theta > 0$. Find unbiased estimator of $\theta e^{-2\theta}$.

Sol: $X_1 ... X_n \overset{iid}{\sim} P(\theta)$, $\theta > 0$

Let $S(\underline{X}) = \begin{cases} 1 & : X_1 = 0, X_2 = 1 \\ 0 & : \text{ow} \end{cases}$

$E(S(\underline{X})) = 1 \cdot P(X_1 = 0, X_2 = 1) + 0 \; P(X_1 \neq 0, X_2 \neq 1)$

$= 1 \cdot P(X_1 = 0) \cdot P(X_2 = 1)$ { $\because X_i$ are i.i.d $X \sim P(\lambda)$ }

$= 1 \cdot \frac{e^{-\theta}\theta^0}{0!} \cdot \frac{e^{-\theta}\theta^1}{1!}$ { $f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ }

$= \theta e^{-2\theta}$

Therefore $S(\underline{X})$ is req. unbiased estimator of $\theta e^{-2\theta}$.

Q.8: $X_1 \ldots X_n \overset{iid}{\sim} B(1,\theta); \ 0 \le \theta \le 1$, Find an unbiased estimator of $\theta^2(1-\theta)$

$X_1 \ldots X_n \overset{iid}{\sim} B(1,\theta)$

Let $\delta(\underset{\sim}{X}) = \begin{cases} 1 & X_1=1, X_2=1, X_3=0 \\ 0 & o.w \end{cases}$ $\qquad \begin{cases} X \sim B(1,\theta) \\ f_X(x) = \theta^x (1-\theta)^{1-x} \end{cases}$

$E(\delta(\underset{\sim}{X})) = 1 \cdot P(X_1=1, X_2=1, X_3=0) + 0 \, (P X_1 \ne 1, X_2 \ne 1, X_3 \ne 0)$

$E(\delta(\underset{\sim}{X})) = 1 \cdot P(X_1=1) \cdot P(X_2=1)\, P(X_3=0)$

$= 1\theta^1 (1-\theta)^0 \ \theta^1 (1-\theta)^0 \ \theta^0 (1-\theta)^1$

$= \theta^2 (1-\theta)$

$\therefore$ $\delta(\underset{\sim}{X})$ is the req. unbiased estimator of $\theta^2(1-\theta)$.

Q.9: Using NFFT (Neyman Fisher Factorization Theorem) find a sufficient statistic based on a random sample $X_1 \ldots X_n$ from each of the following distri.

(a) $f_\alpha(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & ; x > 0 \\ 0 & o.w \end{cases}$

Sol $f_\alpha(\underset{\sim}{x}) = \prod_{i=1}^{n} \left(\frac{1}{\alpha} e^{-x_i/\alpha} \right)$

$= \frac{1}{\alpha^n} \left(e^{-\Sigma_1^n x_i/\alpha} \right)$

i.e. $f_\alpha(\underset{\sim}{x}) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\Sigma_1^n x_i}$

$h(\underset{\sim}{x}) = 1, \quad g_\alpha(t(\underset{\sim}{x})) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\Sigma_1^n x_i}$ i.e. $g_\alpha(\Sigma_1^n x_i) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\Sigma_1^n x_i}$

By NFFT: $T(\underset{\sim}{x}) = \Sigma_1^n x_i$ is sufficient statistic for α.

(b) $f_\beta(x) = \begin{cases} e^{-(x-\beta)} & ; x > \beta \\ 0 & ; o.w \end{cases}$

$f_\beta(\underset{\sim}{x}) = \prod_{i=1}^{n} \left(e^{-(x_i-\beta)} \right)$ for each $x_i > \beta$

$= e^{-\Sigma_1^n (x_i - \beta)}$

$= e^{n\beta} \cdot e^{-\Sigma_1^n x_i} \Rightarrow f_\beta(\underset{\sim}{x}) = e^{n\beta} \cdot e^{-\Sigma_1^n x_i} \ I(\beta, x_{(1)})$

$\Rightarrow h(\underset{\sim}{x}) = e^{-\Sigma_1^n x_i}$

i.e. $g_\beta(x_{(1)}) = e^{n\beta} I_{(\beta, x_{(1)})}$

$\begin{cases} \text{as } x_i > \beta \\ \text{i.e. } x_1 \ldots x_n > \beta \\ \Rightarrow \min\{x_1 \ldots x_n\} > \beta \\ \Rightarrow x_{(1)} > \beta \\ I_{(\beta, x_{(1)})} = \begin{cases} 1 : \beta < x < x_{(1)} \\ 0 : o.w \end{cases} \end{cases}$

By NFFT:

$T(\underset{\sim}{x}) = X_{(1)}$ is sufficient sta for β

(c) $f_{\alpha,\beta}(\underset{\sim}{x}) = \prod_{i=1}^{n}\left(\frac{1}{\alpha}e^{-\frac{(x_i-\beta)}{\alpha}}\right) : x_i > \beta$ for each i

$$= \frac{1}{\alpha^n} e^{\sum_i^n\left(\frac{x_i-\beta}{\alpha}\right)}$$

$$= \frac{1}{\alpha^n} e^{\frac{n\beta}{\alpha}} e^{-\frac{1}{\alpha}\sum_i^n x_i} = \frac{1}{\alpha^n} e^{\frac{n\beta}{\alpha}} e^{-\frac{1}{\alpha}\sum_i^n x_i} I_{(\beta, x_{(1)})} \quad \left\{\begin{array}{l}\infty \text{ each } x_i > \beta \\ \Rightarrow x_{(1)} > \beta\end{array}\right.$$

$$\Rightarrow h(\underset{\sim}{x}) = 1 \text{ \& } g_{\alpha,\beta}(t(\underset{\sim}{x})) = \frac{1}{\alpha^n} e^{\frac{n\beta}{\alpha}} \cdot e^{-\frac{1}{\alpha}\sum_i^n x_i} I_{(\beta, x_{(1)})}$$

By NFFT: $T(\underset{\sim}{x}) = \left(\sum_i^n x_i, X_{(1)}\right)$ is jointly sufficient for α, β.

(d) $f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{\left(\frac{-(\log x_i - \mu)^2}{2\sigma^2}\right)} & : x > 0 \\ 0 & : ow \end{cases}$

Sol: $f_{\mu,\sigma}(\underset{\sim}{x}) = \prod_{i=1}^{n}\left[\frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-(\log x_i - \mu)^2}{2\sigma^2}}\right] : \forall i \; x_i > 0$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot \frac{1}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2\sigma^2}\sum_i^n(\log x_i - \mu)^2}$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} \frac{1}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2\sigma^2}\sum_i^n\left[(\log x_i)^2 + \mu^2 - 2\mu\log x_i\right]}$$

$$e^{-\frac{1}{2\sigma^2}\sum_i^n(\log x_i)^2} \cdot e^{-\frac{1}{2\sigma^2}n\mu^2} \cdot e^{\frac{\mu}{\sigma^2}\sum_i^n \log \mu}$$

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\prod_{i=1}^{n} x_i} \cdot \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2}\left[\sum_i^n(\log x_i)^2\right] + \frac{\mu}{\sigma^2}\sum_i^n \log x_i - \frac{n\mu^2}{2\sigma^2}}$$

$$= \underbrace{\frac{1}{(\sqrt{2\pi})^n \prod_{i=1}^{n} x_i}}_{h(\underset{\sim}{x})} \cdot \underbrace{\frac{1}{\sigma^n} e^{\dots}}_{g_{\mu,\sigma}\left(\sum_i^n \log x_i, \sum_i^n(\log x_i)^2\right)}$$

By NFFT $T(\underset{\sim}{x}) = \left(\sum_i^n \log x_i, \sum_i^n (\log x_i)^2\right)$ is jointly sufficient for μ & σ^2.

(e). $f_\theta(x) = \begin{cases} \frac{1}{\theta} : & -\frac{\theta}{2} \le x \le \frac{\theta}{2} \\ 0 : & ow. \end{cases}$

Sol: $f_\theta(\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n} : & -\frac{\theta}{2} \le x_1, \dots, x_n \le \frac{\theta}{2} \\ 0 : & ow \end{cases}$

$$= \begin{cases} 1/\theta^n & : |x_i| < \theta/2 \text{ for } i = 1 \text{ to } n \\ 0 & : ow. \end{cases}$$

$$= \begin{cases} 1/\theta^n & : \max_i |x_i| < \theta/2 \text{ for } i = 1 \text{ to } n \\ 0 & : ow \end{cases}$$

$\Rightarrow f_\theta(\underset{\sim}{x}) = \underbrace{\frac{1}{\theta^n} I(\max_i |x_i|, \theta/2)}_{g_\theta(\max|x_i|)}$

$h(\underset{\sim}{x}) = 1$

By NFFT $T(\underset{\sim}{x}) = \max(x_i)$ is sufficient statistic for θ

Minimal & complete Sufficient Statistic :

Q.1: Find minimal sufficient statistic based on a random sample $X_1 \dots X_n$ in each of following cases.

(a) $f_\alpha(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & : x > 0 \\ 0 & : ow \end{cases} \quad \alpha > 0$

Sol: $f_\alpha(\underset{\sim}{x}) = \prod_{i=1}^{n} \left(\frac{1}{\alpha} e^{-x_i/\alpha} \right) \quad x_i > 0$

$f_\alpha(\underset{\sim}{y}) = \prod_{i=1}^{n} \left(\frac{1}{\alpha} e^{-y_i/\alpha} \right) \quad y_i > 0$

$\frac{f_\alpha(\underset{\sim}{x})}{f_\alpha(\underset{\sim}{y})} = \prod_{i=1}^{n} \left(\frac{e^{-x_i/\alpha}}{e^{-y_i/\alpha}} \right) = \frac{e^{-\frac{1}{\alpha}\sum_1^n x_i}}{e^{-1/\alpha \sum_1^n y_i}} \Rightarrow e^{\frac{1}{\alpha}(\sum_1^n y_i - \sum_1^n x_i)}$

i.e. $\frac{f_\alpha(\underset{\sim}{x})}{f_\alpha(\underset{\sim}{y})}$ is independent of α iff $\sum_1^n x_i = \sum_1^n y_i$

$\Rightarrow T(\underset{\sim}{x}) = \sum_1^n X_i$ is minimal sufficient statistic for α.

(b) $f_\beta(\underset{\sim}{x}) = \begin{cases} e^{-(x-\beta)} & : x > \beta \\ 0 & : ow \end{cases} \quad \beta \in \mathbb{R}$

$\frac{f_\beta(\underset{\sim}{x})}{f_\beta(\underset{\sim}{y})} = \frac{e^{-\sum_1^n (x_i - \beta)} I(\beta, x_{(1)})}{e^{-\sum_1^n (y_i - \beta)} I(\beta, y_{(1)})} = \frac{e^{-\sum_1^n x_i} e^{n\beta} \cdot I(\beta, x_{(1)})}{e^{-\sum_1^n y_i} \cdot e^{n\beta} \cdot I(\beta, y_{(1)})}$

$= e^{\sum_1^n y_i - \sum_1^n x_i} \cdot \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$ is independent of β iff $x_{(1)} = y_{(1)}$

$\therefore T(\underset{\sim}{x}) = X_{(1)}$ is MSS for β

(c) $f_{\alpha,\beta}(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{(x-\beta)}{\alpha}} ; & x > \beta \\ 0 & ; \text{ ow} \end{cases}$ $\alpha > 0,\ \beta \in \mathbb{R}.$

$$\frac{f_{\alpha,\beta}(\underset{\sim}{x})}{f_{\alpha,\beta}(\underset{\sim}{y})} = \frac{\cancel{\frac{1}{\alpha^n}} e^{-\Sigma_1^n\left(\frac{x_i-\beta}{\alpha}\right)} \cdot I(\beta, x_{(1)})}{\cancel{\frac{1}{\alpha^n}} e^{-\Sigma_1^n\left(\frac{y_i-\beta}{\alpha}\right)} I(\beta, y_{(1)})}$$

$$= e^{\Sigma_1^n\left(\frac{y_i-\beta}{\alpha}\right)} - e^{\Sigma_1^n\left(\frac{x_i-\beta}{\alpha}\right)} \cdot \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$$

$$= e^{\frac{\Sigma_1^n y_i}{\alpha}} \cdot e^{-\frac{n\beta}{\alpha}} - e^{\Sigma_1^n x_i/\alpha} \cdot \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$$

$$= e^{-\frac{n\beta}{\alpha}}\left(e^{\frac{1}{\alpha}\Sigma_1^n y_i - \Sigma_1^n x_i}\right) \cdot \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$$

is ind of α & β iff $\Sigma_1^n x_i = \Sigma_1^n y_i$ & $x_{(1)} = y_{(1)}$

$\Rightarrow T(\underset{\sim}{X}) = \Sigma_1^n X_i,\ X_{(1)}$ is minimal sufficient statistic for α, β resp.

where $\quad x_{(1)} = \min\{x_1, x_2 \dots x_n\}$.

(d) $f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}} ; & x > 0 \\ 0 & ; \text{ ow} \end{cases}$ $\quad \mu \in \mathbb{R},\ \sigma > 0$

$$\frac{f_{\mu,\sigma}(\underset{\sim}{x})}{f_{\mu,\sigma}(\underset{\sim}{y})} = \frac{\frac{1}{\prod_{i=1}^n x_i} \cdot e^{-\frac{\Sigma_1^n(\log x_i - \mu)^2}{\sigma^2}}}{\frac{1}{\prod_{i=1}^n y_i} \cdot e^{-\frac{\Sigma_1^n(\log y_i - \mu)^2}{2\sigma^2}}}$$

$$= \frac{\prod_{i=1}^n y_i}{\prod_{i=1}^n x_i} \cdot \frac{e^{-\frac{1}{2\sigma^2}\Sigma_1^n(\log x_i)^2} \cdot \cancel{e^{-\frac{n\mu^2}{2\sigma^2}}} \cdot e^{\frac{\mu}{\sigma^2}\Sigma_1^n \log x_i}}{e^{-\frac{1}{2\sigma^2}\Sigma_1^n(\log y_i)^2} \cdot \cancel{e^{-\frac{n\mu^2}{2\sigma^2}}} \cdot e^{\frac{\mu}{\sigma^2}\Sigma_1^n \log y_i}}$$

$$= \frac{\prod_{i=1}^n y_i}{\prod_{i=1}^n x_i} \cdot e^{\frac{1}{2\sigma^2}\left[\Sigma_1^n(\log y_i)^2 - \Sigma_1^n(\log x_i)^2\right]} \cdot e^{\frac{\mu}{\sigma^2}\left[\Sigma_1^n \log x_i - \Sigma_1^n \log y_i\right]}$$

is ind of μ, & σ^2 iff $\Sigma_1^n(\log x_i)^2 = \Sigma_1^n(\log y_i)^2$ & $\Sigma_1^n \log x_i = \Sigma_1^n \log y_i$

$\therefore \Sigma_1^n \log X_i$ & $\Sigma_1^n(\log X_i)^2$ is minimal sufficient statistic for μ & σ^2 resp.

(e) $f_\theta(x) = \begin{cases} \frac{1}{\theta} & : -\frac{\theta}{2} \le x \le \theta/2 \\ 0 & : o\omega. \end{cases}$

$f_\theta(\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n} & : -\frac{\theta}{2} \le x_1 \ldots x_n \le \frac{\theta}{2} \\ 0 & : o\omega \end{cases}$

i.e. $\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \frac{\frac{1}{\theta^n} I(\max_i |x_i|, \frac{\theta}{2})}{\frac{1}{\theta^n} I(\max_i |y_i|, \frac{\theta}{2})}$

is ind of θ iff $\max_i |X_i| = \max_i |Y_i|$

$\Rightarrow T(\underset{\sim}{X}) = \max_i |X_i|$ is minimal sufficient statistic for θ.

(f) $f(x) = \begin{cases} \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\alpha} x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1 \quad \alpha>0,\ \beta>0 \\ 0 & o.\omega \end{cases}$

$\frac{f(\underset{\sim}{x})}{f(\underset{\sim}{y})} = \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \prod_{i=1}^{n}(1-x_i)^{\beta-1}}{\left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} \prod_{i=1}^{n}(1-y_i)^{\beta-1}} \quad 0<x_i<1,\ 0<y_i<1$

$= \left(\prod_{i=1}^{n} \frac{x_i}{y_i}\right)^{\alpha-1} \prod_{i=1}^{n}\left(\frac{1-x_i}{1-y_i}\right)^{\beta-1}$ is ind of α & β iff. $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$ &

$\prod_{i=1}^{n}(1-x_i) = \prod_{i=1}^{n}(1-y_i)$

Therefor $\prod_{i=1}^{n} X_i$ & $\prod_{i=1}^{n}(1-X_i)$ is minimal sufficient for α & β respectively

Q.2. (a) $g(\theta)=\theta$ (b) $g(\theta)=e^{-\theta}$ (c) $g(\theta) = e^{-\theta}(1+\theta)$

Sol: $X_1 \ldots X_n \sim P(\theta)$, $\theta \in (0,\infty)$

$f_X(x) = \frac{e^{-\theta}\theta^x}{x!}$ $x = 0, 1, 2 \ldots$ $= \frac{1}{x!} e^{x\log\theta - \theta}$

$h(x) = \frac{1}{x!}$, $T_1(x) = x$, $\eta_1(\theta) = \log(\theta)$, $\beta(\theta) = \theta$

This is 1 parameter exponential family with natural parameter space $\{\eta : \eta \in \mathbb{R}\}$

$\Rightarrow$ The above parameter is of full rank.

$\Rightarrow T(\underline{X}) = \Sigma_1^n x_i$ is complete sufficient statistic

(a) $g(\theta) = \theta$

$E(T) = n\theta$

$\Rightarrow E(\frac{T}{n}) = \theta \Rightarrow \frac{T}{n}$ is unbiased estimator as complete sufficient statistic

$\Rightarrow \frac{T}{n}$ is uniform minimum variance unbiased estimator for θ.

(b) $g(\theta) = e^{-\theta}$.

let $S(\underline{X}) = \begin{cases} 1; & x_1 = 0 \\ 0: & ow \end{cases}$ $\Rightarrow$ $S(\underline{X})$ is unbiased estimator of $g(\theta)$.

$E(S(\underline{X})) = 1 \cdot P(X_1 = 0) + 0 \cdot P(X_1 \neq 0)$

$= \frac{e^{-\theta} \cdot \theta^0}{0!} = e^{-\theta}$

Rao-Blackwellization of $S(\underline{X})$

$\eta(T) = E(S(\underline{X})|T) = 1 \cdot P(X_1 = 0|T = t) + 0 \cdot P(X_1 \neq 0|T)$

$= P(X_1 = 0|T = t) = \frac{P(X_1 = 0, \sum_{i=2}^n x_i = t)}{P(T = t)}$

$= \frac{e^{(-\theta)} e^{-(n-1)\theta} (n-1)\theta)^t / t!}{e^{-\theta n}(n\theta)^t / t!} \Rightarrow \left(\frac{n-1}{n}\right)^t$

$\Rightarrow \left(\frac{n-1}{n}\right)^T$ is unbiased estimator as complete sufficient statistic

$\Rightarrow \left(\frac{n-1}{n}\right)^T$ is UMVUE of $e^{-\theta}$

$g(\theta) = e^{-\theta}(1+\theta)$

let $S(\underline{X}) = \begin{cases} 1: & x_1 = 0, x_1 = 1 \\ 0: & ow \end{cases}$

$E(S(\underline{X})) = 1 \cdot P(X_1 = 0) + P(X_1 = 1) = e^{-\theta} + e^{-\theta}\theta = e^{-\theta}(1+\theta)$

$\eta(T) = E(S(\underline{X})|T) = P((X_1 = 0 \cup X_1 = 1)|T = t)$

$= \frac{P(X_1 = 0, T = t) + P(X_1 = 1, T = t)}{P(T = t)} = \frac{P(X_1 = 0, \Sigma_2^n x_i = t) + P(X_1 = 1, \Sigma_2^n x_i = t-1)}{P(T = t)}$

$= \frac{\frac{e^{-\theta} e^{-(n-1)\theta}((n-1)\theta)^t}{t!} + \frac{\theta e^{-\theta} e^{-(n-1)\theta}((n-1)\theta)^{t-1}}{(t-1)!}}{\frac{e^{-n\theta}(n\theta)^t}{t!}}$

$$= \left(\frac{n-1}{n}\right)^{t}\left(1+\frac{t}{n-1}\right)$$

$\Rightarrow \left(\frac{n-1}{n}\right)^{t}\left(1+\frac{T}{n-1}\right)$ is unbiased estimator on complete sufficient statistic

$\Rightarrow \left(\frac{n-1}{n}\right)^{T}\left(1+\frac{T}{n-1}\right)$ is UMVUE of $e^{-\theta}(1+\theta)$.

Q3 (a) $g(\theta)=\theta$

(b) $g(\theta)=\theta^4$

(c) $g(\theta)=\theta(1-\theta)^2$

$$f_\theta(x) = \theta^{x}(1-\theta)^{1-x}, \quad x=0,1$$

$$= e^{\left(x\log\left(\frac{\theta}{1-\theta}\right)+\log(1-\theta)\right)}$$

$h(x)=1$, $T(x)=x$, $\eta(\theta)=\log\left(\frac{\theta}{1-\theta}\right)$, $\beta(\theta)=-\log(1-\theta)$

this is 1-parameter exp. family

$$f_\eta(x) = e^{(x\eta - A(\eta))}$$

with natural parameter space $\{\eta : \eta \in \mathbb{R}\}$

The above 1-parameter family is a full rank.

$\Rightarrow T(\underline{X}) = \sum_1^n X_i$ is complete sufficient statistic (CSS)

(a) $g(\theta)=\theta$

$E(T)=n\theta$

$E\left(\frac{T}{n}\right)=\theta$ $\quad \frac{T}{n}$ is UE on CSS $\Rightarrow \frac{T}{n}$ is UMVUE of θ

(b) $g(\theta)=\theta^4$

let $\delta(\underline{X}) = \begin{cases} 1 : & X_1=1, X_2=1, X_3=1, X_4=1 \\ 0 : & \text{ow} \end{cases}$

$(\delta(\underline{X})) = P(X_1=1, X_2=1, X_3=1, X_4=1)$

$= \theta^4$

$\Rightarrow \delta(\underline{X})$ is UE of θ^4

Raw-Blackwellization of θ^4

$$\eta(T) = E(\delta(\underline{X}) \mid T)$$

$$= P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1 \mid T = t)$$

$$= \frac{P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, \sum_5^n X_i = t-4)}{P(T = t)}$$

$$\Rightarrow \frac{P(X_1 = 1) \cdot P(X_2 = 1) \cdot P(X_3 = 1) \cdot P(X_4 = 1) \cdot P(\sum_5^n X_i = t-4)}{P(T = t)}$$

$$= \frac{\theta^4 \, {}^{n-4}C_{t-4} \, \theta^{t-4} (1-\theta)^{n-4-t+4}}{{}^nC_t \, \theta^t (1-\theta)^{n-t}} = \frac{{}^{n-4}C_{t-4}}{{}^nC_t} = \frac{\frac{(n-4)!}{(t-4)!\,\cancel{(n-t)!}}}{\frac{n!}{t!\,\cancel{(n-t)!}}}$$

$$= \frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)}$$

$$\Rightarrow \frac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)}$$ is ue on C.SS.

is UMVUE of θ^4

(c) $g(\theta) = \theta(1-\theta)^2$

$$\delta(\underline{X}) = \begin{cases} 1 & X_1 = 1, X_2 = 0, X_3 = 0 \\ 0 & \text{ow} \end{cases}$$

$$E(\delta(\underline{X})) = P(X_1 = 1, X_2 = 0, X_3 = 0) = P(X_1 = 0)\,P(X_2 = 0)\,P(X_3 = 0)$$

$$= \theta(1-\theta)^2$$

$\Rightarrow \delta(\underline{X})$ is ue of $g(\theta)$

$$\eta(T) = E(\delta(\underline{X}) \mid T).$$

$$= \frac{P(X_1 = 1, X_2 = 0, X_3 = 0, \sum_4^n X_i = t-1)}{P(T = t)}$$

$$= \frac{\theta(1-\theta)(1-\theta) \, {}^{n-3}C_{t-1} \, \theta^{t-1} (1-\theta)^{n-3-t+1}}{{}^nC_t \, \theta^t (1-\theta)^{n-t}}$$

$$\Rightarrow \frac{{}^{n-3}C_{t-1}}{{}^nC_t} = \frac{t(n-t)(n-t-1)}{n(n-1)(n-2)}$$

$$\Rightarrow \frac{T(n-T)(n-T-1)}{n(n-1)(n-2)}$$ is UMVUE of $\theta(1-\theta)^2$

Sol: 4 $\quad f_\theta(\underset{\sim}{x}) = e^{-\sum_1^n (x_i - \theta)} I(x_{(1)}, \theta)$

By NFFT $\quad T = X_{(1)}$ is sufficient

(pdf) $f_T(t) = \begin{cases} n e^{-n(t-\theta)}; & t > \theta \\ 0 & ; \text{ow} \end{cases}$

Now $E(g(T)) = 0 \ \forall \ \theta \in \Theta$

$\Rightarrow \int_0^\infty g(t)\, n e^{-n(t-\theta)} dt = 0 \quad \forall \theta \in \Theta$

i.e $\int_0^\infty g(t)\, e^{-nt}\, dt = 0$

diff w.r.t θ.

$g(\theta) e^{-n\theta} = 0 \quad \forall \ \theta \in \Theta$

$\Rightarrow g(\theta) = 0 \ \forall \ 0 < \theta < t, \quad 0 < t < \infty$

$\Rightarrow g(t) = 0$ almost surely.

$\Rightarrow X_{(1)}$ is complete.

$g(\theta) = \theta^2$

$E X_{(1)} = \int_0^\infty t\, e^{-n(t-\theta)}\, n$

$= n \int_0^\infty (y+\theta) e^{-ny} dy = \theta + \frac{1}{n}$

$\Rightarrow E X_{(1)} - \frac{1}{n} = \theta$

$E x_{(1)}^2 = n \int_0^\infty t^2 \cdot e^{-n(t-\theta)} dt$

Put $\quad t - \theta = y$

$\Rightarrow n \int_0^\infty (y+\theta)^2 e^{-ny} dy = n \int_0^\infty (y^2 + \theta^2 + 2\theta y e^{-ny}) dy$

$= \frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}$

$\Rightarrow E x_{(1)}^2 = \frac{2}{n^2} + \theta^2 + \frac{2}{n} E\left(X_{(1)} - \frac{1}{n}\right)$

$\Rightarrow E\left(X_{(1)}^2 - \frac{2}{n^2} - \frac{2}{n}\left(X_{(1)} - \frac{1}{n}\right)\right) = \theta^2$

$E\left(X_{(1)}^2 - \frac{2}{n} X_{(1)}\right) = \theta^2$

$X_{(1)}^2 - \frac{2}{n} X_{(1)}$ is UE of θ^2 on CSS

$\rightarrow$ $X_{(1)}^2 - \frac{2}{n} X_{(1)}$ is UMVUE of θ^2.