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Assignment - 3 Probability & Statistics (CSL003P1M) - Rohit Tiwari - 2024PCS0036

Sol 1. $X_i = 0,1,2,3$ for $i = 1,2,3$

$N = 1,2,3$

Possible configuration with 3 boxes and 3 balls

B_1	B_2	B_3
3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0
1	0	2
0	1	2
0	2	1
~~1~~	~~0~~	~~2~~
1	1	1

$\longrightarrow$ each with prob = $\frac{1}{\binom{3+3-1}{3}} = \frac{1}{10}$

↓N	X_1	X_2	X_3
1	3	0	0
1	0	3	0
1	0	0	3
2	2	1	0
2	2	0	1
2	1	2	0
2	0	2	1
2	1	0	2
2	0	1	2
3	1	1	1

Joint pmf of (N, X_1)

$N \backslash X_1$	0	1	2	3
1	2/10	0	0	1/10
2	2/10	2/10	2/10	0
3	0	1/10	0	0
	4/10	3/10	2/10	1/10

Joint pmf of (X_1, X_2)

$X_1 \backslash X_2$	0	1	2	3
0	1/10	1/10	1/10	1/10
1	1/10	1/10	1/10	0
2	1/10	1/10	0	0
3	1/10	0	0	0
	4/10	3/10	2/10	1/10

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Q.2 Jt pmf of X, Y

$$p(x,y) = \begin{cases} cxy, & (x,y) \in \{(1,1),(2,1),(2,2),(3,1)\} \\ 0, & \text{o.w.} \end{cases}$$

find c, marginal pmf of X & Y and $P(X|Y=2)$

Sol) $\sum_{(x,y)} P(x,y) = c \sum_{(x,y)} (xy) = 1$

$\Rightarrow c[(1,1)+(2,1)+(2,2)+(3,1)] = 1$

$\Rightarrow c = 1/10$

J' pmf

X \ Y	1	2	
1	1/10	0	1/10
2	2/10	4/10	6/10
3	3/10	0	3/10
	6/10	4/10	

marginal of X

marginal of Y

Condition pmf of X given Y=2

$$\frac{P(x,2)}{P_Y(2)} = 1 \quad \text{if } x=2$$

$= 0$ otherwise

$4\alpha^2 - 9\alpha + 6 = 4$

$4\alpha^2 - 9\alpha + 2 = 0$

$4\alpha^2 - 8\alpha - \alpha + 2 = 0$

$4\alpha(\alpha-2) - 1(\alpha-2)$

$\alpha = \frac{1}{4}, 2$

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Sol 3. X_2: # of black balls. X_1: # of white balls.

W, 2 B, 1, R-7

$$P_{X_1,X_2}(x_1, x_2) = \frac{3!}{x_1!\,x_2!\,(3-x_1-x_2)!}\left(\frac{3}{8}\right)^{x_1}\left(\frac{2}{8}\right)^{x_2}\left(\frac{3}{8}\right)^{3-x_1-x_2} \quad ; x_i \geq 0, x_1+x_2 \leq 3$$

$$(X_1, X_2) \sim \text{Mult}\left(3, \frac{3}{8}, \frac{2}{8}\right)$$

$$P_{X_1}(x_1) = \binom{3}{x_1}\left(\frac{3}{8}\right)^{x_1}\left(\frac{5}{8}\right)^{3-x_1} \quad ; x_1 = 0, 1, 2, 3$$

$$P_{X_2}(x_2) = \binom{3}{x_2}\left(\frac{2}{8}\right)^{x_2}\left(\frac{6}{8}\right)^{3-x_2} \quad ; x_2 = 0, 1, 2, 3$$

i.e. $X_1 \sim B\left(3, \frac{3}{8}\right)$; $X_2 \sim B\left(3, \frac{2}{8}\right)$

$$P_{X_1}(x_1)\,P_{X_2}(x_2) \neq P_{X_1,X_2}(x_1, x_2)$$

$\Rightarrow$ X_1 & X_2 are not indep.

Sol 4. $P(X_1=0, X_2=0) = P(X_1=0, X_2=1) = P(X_1=1, X_2=1) = P(X_1=1, X_2=0) = \frac{1}{4}$

Further $(X_1, X_2) \equiv (X_1, X_3) \equiv (X_2, X_3)$

& $P(X_i=0) = \frac{1}{2} = P(X_i=1)$; $i = 1, 2, 3$

$\Rightarrow$ $X_1\ X_2\ X_3$ are Pairwise indep.

But

$$P(X_1=0, X_2=0, X_3=0) = \frac{1}{4} \neq P(X_1=0)\,P(X_2=0)\,P(X_3=0) = \frac{1}{8}$$

$\Rightarrow$ $X_1\ X_2\ X_3$ are not independent.

$1 - \frac{1}{16}$

$\frac{15}{16}$

~~sol~~. 5 ~~jt p.d.f. of (X, Y)~~

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0<x<1 \\ & 0<y<1 \\ 0 & ow \end{cases}$$

~~Marginal of~~ X:

$$f_X(x) = \int_0^1 4xy\,dy = 2x \quad , 0<x<1$$
$$= 0 \quad ow$$

~~similarly.~~ $f_Y(y) = \begin{cases} 2y & 0<y<1 \\ 0 & ow \end{cases}$

~~Now~~ $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$\Rightarrow$ X & Y are indep

$$P(0<X<1/2,\ 1/4<Y<1) = P(0<X<\tfrac{1}{2})\, P(1/4<Y<1)$$
$$= \left(\int_0^{1/2} 2x\,dx\right)\left(\int_{1/4}^{1} 2y\,dy\right) =$$
$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^{1/2} \cdot \left[y^2\right]_{1/4}^{1}$$
$$= \frac{1}{4} \cdot \frac{15}{16} = \frac{15}{64}$$

$$P(X+Y<1) = \int_0^1 P(X<1-y)\, f_Y(y)\,dy$$
$$= \int_0^1 \left[\int_0^{1-y} 2x\,dx\right] 2y\,dy$$
$$= \left[x^2\right]_0^{1-y} \quad \underline{(1-y)^2} \quad 1+y^2-2y$$

$$= \int_0^1 2y(1+y^2-2y)\,dy$$

$$= \int_0^1 2y + \int_0^1 2y^2 - \int_0^1 4y^2$$

$$= \left[y^2\right]_0^1 + \frac{2}{3}\left[y^3\right]_0^1 - \frac{4}{3}\left[y^3\right]_0^1$$

$$= 1 + 2/3 - 4/3 = 1 - 2/3 = 1/3$$

Sol 7

$$f(x,y) = \begin{cases} cx^2y, & 0<x,y<1 \\ 0, & ow \end{cases}$$

i) Find c.

b) marginal pdf of X, Y

(c) $P(X+Y \le 1)$

$$\int_0^1 \int_x^1 f(x,y)\,dy\,dx = 1.$$

i.e. $c\int_0^1 x^2 \int_x^1 y\,dy\,dx = 1$

$$\Rightarrow c\int_0^1 x^2 \frac{1}{2}(1-x^2)\,dx = 1$$

$$\Rightarrow \frac{c}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 1 \Rightarrow c = 15$$

b). $f_X(x) = 15x^2 \int_x^1 y\,dy = \begin{cases} \frac{15}{2}x^2(1-x^2), & 0<x<1 \\ 0 & , ow \end{cases}$

$f_Y(y) = 15y\int_0^y x^2\,dx = \begin{cases} 5y^4 & , 0<y<1 \\ 0 & , ow \end{cases}$

(c) $P(\cancel{X+Y \le 1}) = \phi$

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(c) $P(X+Y\le 1) = \iint_{\substack{x+y\le 1 \\ x<y}} 15x^2y\,dy\,dx$

$= 15\int_0^{1/2} x^2 \int_x^{1-x} y\,dy\,dx = 15\int_0^{1/2} x^2 \frac{y^2}{2}\Big|_x^{1-x} dx$

$\cdots = \cdots = \frac{15}{192}$

Alternatively.

$P(X+Y\le 1) = \iint_{x+y\le 1,\ x<y} 15x^2y\,dy\,dx$

$= 15\int_0^{1/2} y\int_0^{y} x^2\,dx\,dy + 15\int_{1/2}^{1} y\int_0^{1-y} x^2\,dx\,dy$

$= \frac{15}{15\times 32} + \frac{15}{10\times 32} = \frac{15}{192}$

Sol 6. $\int_x^{\infty} 2e^{-x}e^{-y}\,dy$

$= 2e^{-x}e^{-x} = 2e^{-2x} \quad x>0$

$= 0 \quad$ ow.

Similarly $f_Y(y) = 2\int_0^{y} e^{-y}e^{-x}\,dx = 2e^{-y}(1-e^{-y}) \quad y>0$

$= 0 \quad$ ow

$f(x,y) \neq f(x)f(y)$

$\Rightarrow X, Y$ are not indep.

sol Q9 Jt pdf. of (X,Y).

$$f(x,y) = \begin{cases} x+y, & 0<x,y<1 \\ 0, & ow \end{cases}$$

find

$P(Y \mid X=x)$, $0<x<1$. conditional mean and conditional variance of conditional distribution.

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dx$$

$$= \begin{cases} x+1/2 & 0<x<1 \\ 0 & ow. \end{cases}$$

$$f_{Y/X} = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{(2x+1)} \quad 0<y<1$$

$$E(Y|X) = \int_0^1 y\,\frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (xy+y^2)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{2}+\frac{1}{3}\right) = \frac{2(3x+2)}{6(2x+1)} = \frac{3x+2}{6x+3}$$

$$E(Y^2|X) = \int_0^1 y^2\,\frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (y^2x+y^3)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{3}+\frac{1}{4}\right) = \frac{2(4x+3)}{12(2x+1)} = \frac{4x+3}{6(2x+1)}$$

$$V(Y|X) = E(Y^2|X) - E^2(Y|X)$$

$$= \frac{4x+3}{6(2x+1)} - \left(\frac{3x+2}{3(2x+1)}\right)^2$$

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Sol.8.

$$f_{(x,y)} = \begin{cases} 6(1-x-y) & x>0, y>0, \; x+y<1 \\ 0. & \text{ow.} \end{cases}$$

Find marginal of X, Y.
and $P(2X+3Y<1)$

$$f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy = 6\int_0^{1-x}(1-x-y)\,dy = 6\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x}$$

$$= \begin{cases} 3(1-x)^2, & 0<x<1 \\ 0 & \text{, ow} \end{cases}$$

by symmetry

$$f_Y(y) = \begin{cases} 3(1-y)^2, & 0<y<1 \\ 0 & \text{ow} \end{cases}$$

$$P(2X+3Y<1) = 6\int_0^{1/2}\int_0^{\frac{1-2x}{3}} (1-x-y)\,dy\,dx$$

$$= 6\int_0^{1/2}\left[(1-x)y - \frac{y^2}{2}\right]_0^{\frac{1-2x}{3}} dx$$

$$= 6\int_0^{1/2}\left\{(1-x)\left(\frac{1-2x}{3}\right) - \frac{1}{2}\left(\frac{1-2x}{3}\right)^2\right\} dx$$

$$= 6\int_0^{1/2}\left(\frac{1+2x^2-3x}{3} - \frac{1+4x^2-4x}{18}\right) dx$$

$$= 6\int_0^{1/2} \frac{8x^2-14x+5}{18}\,dx$$

$$= \frac{6}{18}\left(\left. 8\frac{x^3}{8} - 14\frac{x^2}{2} + 5x \right|_0^{1/2}\right)$$

$$= \frac{6}{18}\left(\frac{8}{3}\cdot\frac{1}{8} - 7\cdot\frac{1}{4} + \frac{5}{2}\right) = \frac{13}{36}$$

Ans. $P(2X+3Y<1) = 6\int_0^{1/3}\int_0^{\frac{1-3y}{2}} (1-y-x)\,dx\,dy.$

$$= 6\int_0^{1/3}\left((1-y)x - \frac{x^2}{2}\Big|_0^{\frac{1-3y}{2}}\right)dy$$

$$= 6\int_0^{1/3}\left[(1-y)\left(\frac{1-3y}{2}\right) - \frac{1}{2}\left(\frac{1-3y}{2}\right)^2\right]dy$$

$$= \ldots = \frac{13}{36}$$

Q.10

$$f_X(x) = \begin{cases} 4x(1-x^2), & 0<x<1 \\ 0, & ow \end{cases}$$

and conditional density of Y given $X=x$ .n

$$f_{Y|X=x}(y|x) = \begin{cases} 2y/(1-x^2), & x<y<1,\ 0<x<1 \\ 0, & ow \end{cases}$$

Find conditional p.d.f of X given $Y=y$ $E(X|Y=\frac{1}{2})$ and and $Var(X|Y=1/2)$

$$f_{X,Y}(x,y) = f_{Y|X=x}(y|x)\, f_X(x)$$

$$= \begin{cases} 8xy, & 0<x<y<1 \\ 0 & ow \end{cases}$$

, marginal pdf of Y

$$f_Y(y) = \begin{cases} 8y\int_0^y x\,dx = 4y^3, & 0<y<1 \\ 0, & ow \end{cases}$$

conditional pdf. of X given Y

$$f_{X|Y=y}(x|y) = \begin{cases} \frac{8xy}{y^3} = \frac{2x}{y^3}, & 0<x<y;\ 0<y<1 \\ 0, & ow. \end{cases}$$

$$E(X|Y=y) = \frac{2}{y^2}\int_0^y x^2\,dx = \frac{2}{y^2}\cdot\frac{y^3}{3} = \frac{2y}{3}$$

$$\Rightarrow E(X|Y=1/2) = 1/3$$

$$E(X^2/Y=y) = \frac{2}{y^2}\int_0^y x^3\,dx = \frac{2}{y^2}\cdot\frac{y^4}{4} = \frac{y^2}{2} = \frac{1}{8}$$

$$V(X|Y=y) = E(X^2|Y=y) - E^2(X|Y=y) = \frac{1}{8} - \frac{1}{9} = [illegible]$$

$$= 1/72$$