Assignment-1: Probability and Statistics (CSL003P1M)
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1. Let $\Omega = \{1, 2, 3, 4\}$ Check whether any of the following is a σ-field of subsets of Ω.
   $F_1 = \{\phi, \{1, 2\}, \{3, 4\}\}$
   $F_2 = \{\phi, \Omega, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}\}$
   $F_3 = \{\phi, \Omega, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2\}\}$

2. Prove that if F_1 and F_2 are σ-fields of subsets of Ω, then $F_1 \cap F_2$ is also a σ-field. Give a counter example to show that similar results for union of σ-fields does not hold.

3. Let F be a σ-field of subsets of the sample space Ω and let $A \in F$ be fixed. Show that $F_A = \{C : C = A \cap B, B \in F\}$ is a σ-field of subsets of A.

4. Let $\Omega = \{0, 1, 2, ...\}$. If for an event A,
   (a) $P(A) = \sum_{x \in A} \frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0.$
   (b) $P(A) = \sum_{x \in A} p(1-p)^x, 0 < p < 1.$
   (c) $P(A) = \begin{cases} 1, & \text{if the number of elements in } A \text{ is finite;} \\ 0, & \text{Otherwise.} \end{cases}$
   Determine in each of the above cases whether $P(\cdot)$ is a probability measure.
   In cases where your answer is in the affirmative, determine
   $$P(E), P(F), P(G), P(E \cap F), P(E \cup F), P(F \cup G), P(E \cap G) \text{ and } P(F \cap G)$$
   where $E = \{x \in \Omega : x > 2\}, F = \{x \in \Omega : 0 < x < 3\}$ and $G = \{x \in \Omega : 3 < x < 6\}$,

5. Consider the sample space $\Omega = \{0, 1, 2, ...\}$ and F the σ-field of subsets of Ω. To the elementary event $\{j\}$ assign the probability
   $$P(\{j\}) = c\frac{2^j}{j!}, j = 0, 1, 2, ...$$
   (a) Determine the constant c.
   (b) Define the events A, B and C by
   $$A = \{j : 2 \leq j \leq 4\}, B = \{j : j \geq 3\} \text{ and } C = \{j : j \text{ is an odd integer}\}.$$
   Evaluate $P(A), P(B), P(C), P(A \cap B), , P(A \cap C)$, $P(B \cap C), P(A \cap B \cap C)$, and verify the formula for $P(A \cup B \cup C)$.

6. Each packet of a certain cereal contains a small plastic model of one of the five different dinosaurs; a given packet is equally likely to contain any one of the five dinosaurs. Find the probability that someone buying six packets of the cereal will acquire models of three favorite dinosaurs.

7. Suppose n cards numbered $1, 2, \ldots, n$ are laid out at random in a row. Let A_i denote the event that 'card i appears in the i^{th} position of the row', which is termed as a match. What is the probability of obtaining at least one match?

8. A man addresses n envelopes and writes n cheques for payment of n bills.
(a) If the n bills are placed at random in the n envelopes, what would be the probability that each bill would be placed in the wrong envelope?
(b) If the n bills and n cheques are placed at random in the n envelopes, one bill and one cheque in each envelope, what would be the probability that in no instance would the enclosures be completely correct?

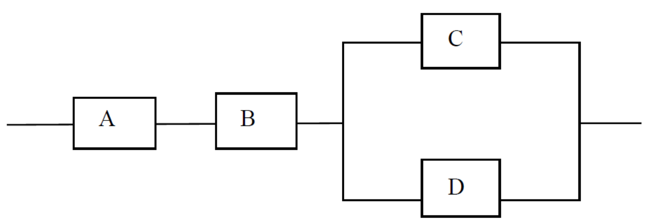
9. Consider an urn in which 4 balls have been placed by the following scheme. A fair coin is tossed, if the coin comes up heads, a white ball is placed in the urn otherwise a red ball is placed in the urn.
(a) What is the probability that the urn will contain exactly 3 white balls?
(b) What is the probability that the urn will contain exactly 3 white balls, given that the first ball placed in the urn was white?

10. A person has three coins in his pocket, two fair coins (heads and tails are equally likely) but the third one is biased with probability of heads $\frac{2}{3}$. One coin selected at random drops on the floor, landing heads up. How likely is it that it is one of the fair coins?

11. A slip of paper is given to A, who marks it with either $a+$ or $a-$ sign, with a probability of $\frac{1}{3}$ of writing $a+$ sign. A passes the slip to B, who may either leave it unchanged or change the sign before passing it to C. C in turn passes the slip to D after perhaps changing the sign; finally D passes it to a referee after perhaps changing the sign. It is further known that B, C and D each change the sign with probability $\frac{2}{3}$. Find the probability that A originally wrote $a+$ given that the referee sees $a+$ sign on the slip.

12. Give a counter example (different from class notes) to show that pairwise independence of a set of events $A_1, A_2, \ldots, A_n$ does not imply mutual independence.

13. An electrical system consists of four components as illustrated in the figure below. The



system works if components A and B work and either of the components C or D work. It is known that the components work independently and that

$$P(A \text{ works}) = P(B \text{ works}) = 0.9 \quad \text{and} \quad P(C \text{ works}) = P(D \text{ works}) = 0.8$$

Find the probability that (a) the entire system works, and (b) the component C does not work, given that the entire system works.

14. During the course of an experiment with a particular brand of a disinfectant on flies, it is found that 80% are killed in the first application. Those which survive develop a resistance, so that the percentage of survivors killed in any later application is half of that in the preceding application. Find the probability that (a) a fly will survive 4 applications; (b) it will survive 4 applications, given that it has survived the 1^{st} one.

15. Let X be a random variable defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Show that the following are also random variables;
(a) $|X|$, (b) X^2 and (c) $\sqrt{X}$, given that $\{X < 0\} = \phi$

16. Let $\Omega = [0, 1]$ and $\mathscr{F}$ be the Borel σ-field of subsets of Ω. Define X on Ω as follows:

$$X(\omega) = \begin{cases} \omega & \text{if } 0 \leq \omega \leq \dfrac{1}{2} \\ \omega - \dfrac{1}{2} & \text{if } \dfrac{1}{2} < \omega \leq 1 \end{cases}$$

Show that X defined above is a random variable.

17. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathscr{F} = \{\phi, \Omega, \{1\}, \{2, 3, 4\}, \}$ be a σ-field of subsets of Ω. Verify whether $X(\omega) = \omega + 1; \forall \omega \in \Omega$, is a random variable with respect to $\mathscr{F}$.

18. Let a card be selected from an ordinary pack of playing cards . The outcome ω is one of these 52 cards. Define X on Ω as

$$X(\omega) = \begin{cases} 4 & \text{if } \omega \text{ is an ace} \\ 3 & \text{if } \omega \text{ is a king} \\ 2 & \text{if } \omega \text{ is a queen} \\ 1 & \text{if } \omega \text{ is a jack} \\ 0 & \text{Otherwise} \end{cases}$$

Show that X is a random variable.

*** All the best ***