PROBABILITY & STATISTICS
Convergence of random variable

[1] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) \to c$ and $V(X_n) \to 0$ as $n \to \infty$. Show that $X_n \xrightarrow{p} c.$

[2] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) = \mu_n$ and finite variance such that $\frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) \to 0$ as $n \to \infty$. Show that WLLN holds and $\bar{X}_n \xrightarrow{p} \bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$.

[3] Let $X_1, X_2, \ldots X_n$ be a random sample from $U(0,1)$. Let $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$. Show that (a) $\sqrt{Y_n} \xrightarrow{p} 0$, (b) $Z_n^2 \xrightarrow{p} 1$ and (c) $Y_n^2 Z_n^2 \xrightarrow{p} 0$.

[4] Let $X_1, X_2, \ldots X_n$ be i.i.d. $N(0,1)$. Show that $\bar{X}_n / S_n \xrightarrow{p} 0$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$.

[5] Suppose $Y_n \sim Bin(n, p)$, show that $(1 - Y_n/n) \xrightarrow{p} 1 - p$.

[6] Let $\{X_n\}$ be a sequence of independent random variables with

$$P(X_n = x) = \begin{cases} 1/2, & x = -n^{1/4},\ n^{1/4} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\bar{X}_n \xrightarrow{p} 0$.

[7] Let $\{X_n\}$ be a sequence of i.i.d. random variables with mean μ and finite variance. Show that

(a) $\frac{2}{n(n+1)} \sum_{i=1}^{n} i\, X_i \xrightarrow{p} \mu$

(b) $\frac{6}{n(n+1)(2n+1)} \sum_{i=1}^{n} i^2\, X_i \xrightarrow{p} \mu$

[8] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution and $Z_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$. Show that $Z_n \xrightarrow{p} e^{-1}$.

[9] Let $\{X_n\}$ be a sequence of uncorrelated random variables with $E(X_n) = \mu_n$ and $V(X_n) = \sigma_n^2$. Show that if $\sum_{i=1}^{n} \sigma_i^2 \to \infty$ as $n \to \infty$, then WLLN holds for $\{X_n\}$.

[10] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution. Find c such that $\bar{X}_n \xrightarrow{p} c$.

[11] Let $\{X_n\}$ be a sequence of $N\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Show that $X_n \xrightarrow{L} Z$, where $Z \sim N(0,1)$.

[12] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E(X_i)=\mu,\ Var(X_i)=\sigma^2$ and $E(X_i-\mu)^4=\sigma^4+1$. Find $\lim_{n\to\infty} P\left[\sigma^2-\frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2+\ldots+(X_n-\mu)^2}{n} \le \sigma^2+\frac{1}{\sqrt{n}}\right]$.

[13] Let $X_1, X_2, \ldots X_n$ be i.i.d. $B(1,p)$, $S_n=\sum_{i=1}^{n} X_i$. Find n which would guarantee $P\left(\left|\frac{S_n}{n}-p\right| \ge 0.01\right) \le 0.01$, no matter whatever the unknown p may be.

[14] Let $X_1,\ldots,X_n$ be i.i.d. from a distribution with mean μ and finite variance σ^2. Prove that $\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{S_n} \xrightarrow{\text{L}} Z$, where $Z \sim N(0,1)$.

[15] The p.d.f. of a random variable X is

$$f(x)=\begin{cases} 1/x^2 & x \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider a random sample of size 72 from the distribution having the above p.d.f. Compute, approximately, the probability that more than 50 of these observations are less than 3.

[16] Let $X_1,\ldots,X_{100}$ be i.i.d. from $\text{Poisson}(3)$ distribution and let $Y=\sum_{i=1}^{100} X_i$. Using CLT, find an approximate value of $P(100 \le Y \le 200)$.

[17] Let $X \sim Bin(100, 0.6)$. Find an approximate value of $P(10 \le X \le 16)$.

[18] The p.d.f. of X_n is given by

$$f_n(x)=\begin{cases} \frac{1}{\Gamma n} e^{-x} x^{n-1} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the limiting distribution of $Y_n = X_n / n$.

[19] Let $\bar{X}$ denote the mean of a random sample of size 64 from the Gamma distribution with density

$$f_n(x)=\begin{cases} \frac{1}{\Gamma p\, \alpha^p} e^{-x/\alpha} x^{p-1} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

With $\alpha=2, p=4$. Compute the approximate value of $P(7<\bar{X}<9)$.

[20] $X_1,\ldots,X_n$ is a random sample from $U(0,2)$. Let $Y_n=\bar{X}_n$, show that $\sqrt{n}\left(Y_n-1\right) \xrightarrow{\mathcal{L}} N(0, 1/3)$.

PROBABILITY & STATISTICS
Unbiased Estimator, Sufficient Statistic

[1] Let $X_1, X_2, \ldots X_n$ be a random sample from an exponential distribution with p.d.f.

$$f_X(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right); x > 0$$

Show that $\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$ is an unbiased estimator of β.

[2] Let $X_1, X_2, \ldots X_n$ be a random sample from $U(0, \theta)$; $\theta > 0$. Show that $\frac{n+1}{n} X_{(n)}$ and $2\,\bar{X}$ are both unbiased estimators of θ.

[3] Let $X_1, X_2, \ldots X_n$ be a random sample from an exponential distribution with p.d.f.

$$f(x) = \beta \exp(-\beta\, x); x > 0$$

Show that $\bar{X}$ is an unbiased estimator of $1/\beta$.

[4] Let $X_1, X_2, \ldots X_n$ be a random sample from $N(\theta, \theta^2)$, $\theta > 0$. Show that $\left(\sum_{i=1}^{n} X_i\right)^2 \Big/ n(n+1)$ and $\sum_{i=1}^{n} X_i^2 \Big/ 2n$ are both unbiased estimators of θ^2.

[5] Let $X_1, X_2, \ldots X_n$ be a random sample from $P(\theta)$; $\theta > 0$. Find an unbiased estimator of $\theta e^{-2\theta}$.

[6] Let $X_1, X_2, \ldots X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$.

(a) Show that the estimator $T(\underset{\sim}{X}) = \dfrac{\frac{1}{2}\sqrt{n} + \sum_{i=1}^{n} X_i}{n + \sqrt{n}}$ is not unbiased θ?

(b) Show that $\lim_{n\to\infty} E\left(T(\underset{\sim}{X})\right) = \theta$.

(An estimator satisfying the condition in (b) is said to be unbiased in the limit)

[7] $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2), \mu \in \Re, \sigma \in \Re^+$. Find unbiased estimators of μ/σ^2 and μ/σ.

[8] Let $X_1, X_2, \ldots X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$. Find an unbiased estimator of $\theta^2(1-\theta)$.

[9] Using Neyman Fisher Factorization Theorem, find a sufficient based on a random sample $X_1, X_2, \ldots X_n$ from each of the following distributions

(a) $f_\alpha(x)=\begin{cases}\dfrac{1}{\alpha}\exp\left(-\dfrac{x}{\alpha}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}$

(b) $f_\beta(x)=\begin{cases}\exp\left(-(x-\beta)\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}$

(c) $f_{\alpha,\beta}(x)=\begin{cases}\dfrac{1}{\alpha}\exp\left(-\dfrac{(x-\beta)}{\alpha}\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}$

(d) $f_{\mu,\sigma}(x)=\begin{cases}\dfrac{1}{x\,\sigma\,\sqrt{2\pi}}\ \exp\left(-\dfrac{(\log x_i-\mu)^2}{2\sigma^2}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}$

(e) $f_\theta(x)=\begin{cases}\dfrac{1}{\theta} & -\theta/2\le x\le\theta/2\\ 0 & \text{otherwise}\end{cases}$

[10] Let X_1 and X_2 be independent random samples with densities $f_1(x_1)=\theta\,e^{-\theta x_1}$ and $f_2(x_2)=2\,\theta\,e^{-2\theta x_2}$ as the respective p.d.f.s where $\theta>0$ is an unknown parameter and $0<x_1,x_2<\infty$. Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[11] Let $X_1,\dots,X_n$ be a random sample with densities

$$f_{X_i}(x)=\begin{cases}\exp(i\theta-x) & \text{if } x\ge i\theta\\ 0 & \text{otherwise.}\end{cases}$$

Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[12] Let $X_1, X_2,\dots X_n$ be a random sample from a $Beta(\alpha,\beta)$ distribution $(\alpha>0,\beta>0)$ with p.d.f.

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1\\ 0 & \text{otherwise.}\end{cases}$$

Show that

(a) $\prod_{i=1}^{n}X_i$ is sufficient for α if β is known to be a given constant.

(b) $\prod_{i=1}^{n}(1-X_i)$ is sufficient for β if α is known to be a given constant.

(c) $\left(\prod_{i=1}^{n}X_i,\prod_{i=1}^{n}(1-X_i)\right)$ is jointly sufficient for (α,β) if both the parameters are unknown.

[13] Let T and T^* be two statistic such that $T=\psi\left(T^*\right)$. Show that if T is sufficient then T^* is also sufficient.

[14] $X_1,\ldots,X_n$ be a random sample from $U\left(\theta-1/2,\theta+1/2\right),\theta\in\Re$. Find a sufficient statistic for θ.

[15] Let $X_1,\ldots,X_n$ be independent random variables with $X_i\left(i=1,2,\ldots,n\right)$ having the probability density function

$$f_i\left(x_i\right)=\begin{cases} i\theta e^{-i\theta x_i} & x_i>0 \\ 0 & \text{otherwise} \end{cases}$$

Find a sufficient statistic for θ.

PROBABILITY & STATISTICS
Minimal and Complete Sufficient Statistic

[1] Find minimal sufficient statistic based on a random sample $X_1,...,X_n$ in each of the following cases

(a) $f_\alpha(x) = \begin{cases} \frac{1}{\alpha}\exp\left(-\frac{x}{\alpha}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \quad \alpha > 0.$

(b) $f_\beta(x) = \begin{cases} \exp\left(-(x-\beta)\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases} \quad \beta \in \mathbb{R}.$

(c) $f_{\alpha,\beta}(x) = \begin{cases} \frac{1}{\alpha}\exp\left(-\frac{(x-\beta)}{\alpha}\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases} \quad \alpha > 0, \beta \in \mathbb{R}.$

(d) $f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{x\,\sigma\,\sqrt{2\pi}}\exp\left(-\frac{(\log x_i - \mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \quad \mu \in \mathbb{R}; \sigma > 0.$

(e) $f_\theta(x) = \begin{cases} \frac{1}{\theta} & -\theta/2 \le x \le \theta/2 \\ 0 & \text{otherwise} \end{cases} \quad \theta > 0.$

(f) $f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \quad \alpha > 0, \beta > 0.$

[2] Let $X_1,...,X_n$ be a random sample from $P(\theta), \theta \in (0,\infty)$. Show that $T = \sum_{i=1}^{n} X_i$ is complete sufficient statistic. Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of the following parametric functions: (a) $g(\theta) = \theta$, (b) $g(\theta) = e^{-\theta}$ and (c) $g(\theta) = e^{-\theta}(1+\theta)$.

[3] Suppose $X_1,...,X_n$ be a random sample from $B(1,\theta), \theta \in (0,1)$. Show that $T = \sum_{i=1}^{n} X_i$ is complete sufficient statistic and hence find the UMVUE for each of the following parametric functions: (a) $g(\theta) = \theta$, (b) $g(\theta) = \theta^4$ and (c) $g(\theta) = \theta(1-\theta)^2$.

[4] Let $X_1,...,X_n$ be a random sample from $Exp(\theta,1)$, i.e.

$$f(x\mid\theta)=\begin{cases} e^{-(x-\theta)} & \text{if } x>\theta \\ 0 & \text{otherwise} \end{cases}$$

Show that $T=X_{(1)}=\min\{X_1,...,X_n\}$ is a complete sufficient statistic and hence find the UMVUE of $g(\theta)=\theta.$

[5] $X_1,....,X_n$ is a random sample from $U(0,\theta),\theta>0.$ Show that $T=X_{(n)}=\max\{X_1,...,X_n\}$ is a complete sufficient statistic and find the UMVUE of $g(\theta)=\theta^2$.

[6] $X_1,....,X_n$ is a random sample from $Gamma(2,\theta),\theta>0,$ i.e.

$$f(x\mid\theta)=\begin{cases} \dfrac{1}{\Gamma 2\,\theta^2}e^{-x/\theta}x & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $T=\sum_{i=1}^{n}X_i$ is complete sufficient statistic and find the UMVUE of $\theta.$

[7] Let $X_1,...,X_n$ be a random sample from $U(\theta-1/2,\,\theta+1/2).$ Show that the minimal sufficient statistic is not complete.

[8] Let $X_1,...,X_n$ be a random sample from $N(0,\,\theta)$. Find the UMVUE of $\theta^2.$

[9] Let $X_1,...,X_n$ be a random sample from $N(\mu,\,\theta)$. Find the UMVUE of (a) θ when μ is known, (b) θ when μ is not known and (c) δ such that $P(X\le\delta)=p;\, p$ is a known fixed constant, both μ and θ are unknown parameters.

[10] Let $X_1,...,X_n$ be a random sample from $U(0,\,\theta),\theta>0.$ Of the following three estimators given below, which one would you prefer and why?

$$T_1(\underset{\sim}{X})=\frac{n+1}{n}X_{(n)},\; T_2(\underset{\sim}{X})=2\bar{X} \text{ and } T_3(\underset{\sim}{X})=X_{(1)}+X_{(n)}.$$

[11] $X_1,...,X_n$ be a random sample from $N(\mu,\sigma^2),\mu\in\Re,\sigma\in\Re^+.$ Assuming completeness of the associated minimal sufficient statistic find the UMVUE of μ^2 and $\mu+\sigma.$