CSL003P1M: Probability and Statistics
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Assignment-3

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1. 3 balls are placed randomly in 3 boxes B_1, B_2 and B_3. Let N be the total number of boxes which are occupied and X_i be the total number of balls in the box $B_i, i = 1, 2, 3$. Find the joint p.m.f. of (N, X_1) and (X_1, X_2). Obtain the marginal distributions of N, X_1 and, X_2 from the joint p.m.f.s.

2. The joint p.m.f. of X and Y is given by $p(x,y) = \begin{cases} cxy, & (x,y) \in \{(1,1),(2,1),(2,2),(3,1)\} \\ 0, & \text{otherwise} \end{cases}$
   Find the constant c, the marginal p.m.f. of X and Y, and the conditional p.m.f. of X given $Y = 2$.

3. Consider a sample of size 3 drawn with replacement from an urn containing 3 white, 2 black and 3 red balls. Let the random variables X_1 and X_2 denote the number of white balls and number of black balls in the sample, respectively. Determine whether the two random variables are independent.

4. Let $\underline{X} = (X_1, X_2, X_3)^T$, be a random vector with joint p.m.f.
$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} 1/4, & (x_1,x_2,x_3) \in \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\} \\ 0, & \text{otherwise.} \end{cases}$$
   Show that X_1, X_2, X_3 are pairwise independent but are not mutually independent.

5. The joint p.d.f. of (X, Y) is given by $f(x,y) = \begin{cases} 4xy, & 0 < x, y < 1 \\ 0, & \text{otherwise.} \end{cases}$. Find the marginal p.d.f.s and verify whether the random variables X and Y are independent.
   Also find $P(0 < X < 1/2, 1/4 < Y < 1)$, $P(X + Y < 1)$.

6. If the joint p.d.f. of (X, Y) is $f(x,y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$, show that X and Y are not independent.

7. Suppose the joint p.d.f. of (X, Y) is $f(x,y) = \begin{cases} cx^2y, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$. Find
   (a) the value of the constant c, (b) the marginal p.d.f.s of X and Y, and
   (c) $P(X + Y \leq 1)$.

8. The joint p.d.f. of (X, Y) is given by $f(x,y) = \begin{cases} 6(1-x-y), & x > 0, y > 0, x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$.
   Find the marginal p.d.f.s of X and Y, and $P(2X + 3Y < 1)$.

9. The joint p.d.f. of (X, Y) is given by $f(x,y) = \begin{cases} x + y, & 0 < x, y < 1, \\ 0, & \text{otherwise.} \end{cases}$. Find the conditional distribution of Y given $X = x, 0 < x < 1$; the conditional mean and conditional variance of the conditional distribution.

10. Suppose the marginal density of the random variable X is

$$f_X(x) = \begin{cases} 4x(1-x^2), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

and the conditional density of the random variable Y given $X = x$ is

$$f_{Y|X=x}(y|x) = \begin{cases} 2y/(1-x^2), & x < y < 1, 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional p.d.f. of X given $Y = y$, $E(X|Y = 1/2)$ and $Var(X|Y = 1/2)$.

11. Let X_1, X_2 and X_3 be three independent random variables each with a variance σ^2. Define the new random variables

$$W_1 = X_1,\ W_2 = \frac{\sqrt{3}-1}{2}X_1 + \frac{3-\sqrt{3}}{2}X_2,\ W_3 = (\sqrt{2}-1)X_2 + (2-\sqrt{2})X_3.$$

Find $\rho(W_1, W_2)$, $\rho(W_1, W_3)$ and $\rho(W_2, W_3)$.

12. Let $(X, Y) \sim N(3, 1, 16, 25, 0.6)$. Find
(a) $P(3 < Y < 8)$; (b) $P(3 < Y < 8|X = 7)$;
(c) $P(-3 < X < 3)$; (d) $P(-3 < X < 3|Y = 4)$.

13. Suppose that the lifetime of light bulbs of a certain kind follows exponential distribution with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{50}e^{-x/50}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability that among 8 such bulbs, 2 will last less that 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours. Find the expected number of bulbs in a lot of 8 bulbs with lifetime between 60 and 80 hours and also the expected number of bulbs in a lot of 8 with lifetime between 60 and 80 hours, given that the number of bulbs with lifetime anywhere between 40 and 60 hours is 2.

14. Let the random variables X and Y have the following joint p.m.f.s
(a) $P(X = x, Y = y) = 1/3$, if $(x, y) \in \{(0, 0), (1, 1), (2, 2)\}$ and 0 otherwise.
(b) $P(X = x, Y = y) = 1/3$, if $(x, y) \in \{(0, 2), (1, 1), (2, 0)\}$ and 0 otherwise.
(c) $P(X = x, Y = y) = 1/3$, if $(x, y) \in \{(0, 0), (1, 1), (2, 0)\}$ and 0 otherwise.
In each of the above cases find the coefficient of correlation between X and Y.

15. Let the joint p.d.f. of X and Y be given by

$$f_{X,Y}(x, y) = \begin{cases} k, & -x < y < x; 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant k and obtain the conditional expectations $E(X|Y = y)$ and $E(Y|X = x)$. Verify whether the 2 random variables are independent and/or uncorrelated.

16. Let $\underline{X} = (X_1, X_2, X_3)' \sim N_3(\mathbf{0}, \Sigma)$; $\Sigma = \begin{pmatrix} 1 & -0.5 & 0 \\ & 1 & -0.5 \\ & & 1 \end{pmatrix}$.
(a) Verify whether $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ are independent.
(b) Find the distribution of $(X_1 - X_2 - X_3)^2$.

17. Let $\underline{X} = (X_1, X_2, X_3)' \sim N_3(\mu, \Sigma)$; $\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ & 1 & \rho \\ & & 1 \end{pmatrix}$, $-1/2 < \rho < 1$.
Find the joint distribution of $(X_1 + X_2, X_1 - X_2)'$.

18. Let the joint p.d.f. of $\underline{X} = (X_1, X_2, X_3, X_4)'$ be

$$f_{\underline{X}}(\underline{x}) = \frac{exp(-\underline{x}^T \Sigma_1^{-1} \underline{x}/2)}{8\pi^2 |\Sigma_1|^{1/2}} + \frac{exp(-\underline{x}^T \Sigma_2^{-1} \underline{x}/2)}{8\pi^2 |\Sigma_2|^{1/2}}; \qquad \underline{x} \in \Re^4,$$

where,

$$\Sigma_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \Sigma_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}; \ |\rho| < 1.$$

(a) Find the joint p.d.f. of $(X_1, X_2)'$.
(b) Find the joint p.d.f. of $(X_3, X_4)'$.
(c) Prove or disprove "X_3 and X_4 are uncorrelated but not independent".

19. The joint probability mass function of the random variable X_1 and X_2 is given by

$$P(X_1 = x_1, X_2 = x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2} \left(\frac{1}{3}\right)^{2-x_1-x_2} & \text{if } (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\ 0 & \text{Otherwise} \end{cases}$$

(a) Find the joint probability mass function of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.
(b) Find the marginal probability mass functions of Y_1 and Y_2.
(c) Verify whether Y_1 and Y_2 are independent.

20. Let the joint probability mass function of X_1 and X_2 be

$$P(X_1 = X_1, X_2 = x_2) = \begin{cases} \frac{x_1 x_2}{36} & \text{if } x_1 = 1, 2, 3; x_2 = 1, 2, 3 \\ 0 & \text{Otherwise} \end{cases}$$

(a) Find the joint probability mass function of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.
(b) Find the marginal probability mass functions of Y_1.
(c) Find the probability mass function of $Z = X_1 + X_2$.

21. Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ be independent random variables. Find the conditional distribution of X given $X + Y = t$, $t \in \{0, 1, 2...\}$.

22. Let X_1, X_2, X_3 and X_4 be four mutually independent random variables each having probability density function

$$f(x) = \begin{cases} 3(1-x)^2 & \text{if } 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find the p.d.f. of $Y = \min(X_1, X_2, X_3, X_4)$ and $Z = \max(X_1, X_2, X_3, X_4)$.

23. Let X_1 and X_2 be i.i.d $U(0,1)$. Define two new random variables as $Y_1 = X_1 + X_2$ and $Y_2 = X_2 - X_1$. Find the joint probability density function of Y_1 and Y_2 and also marginal probability density of Y_1 and Y_2.

24. Let X and Y be i.i.d $N(0,1)$. Find the probability density function of $Z = \dfrac{X}{Y}$.

25. Let X and Y be i.i.d random variables with common probability density function

$$f(x) = \begin{cases} \dfrac{c}{1+x^4} & \text{if } -\infty < x < \infty \\ 0 & \text{Otherwise} \end{cases}$$

where, c is a normalizing constant. Find the probability density function of $Z = \dfrac{X}{Y}$

26. Let X and Y be i.i.d $N(0,1)$. Define the random variables R and θ by $X = Rcos(\theta)$, $Y = Rsin(\theta)$

(a) Show that R and θ are independent with $\dfrac{R^2}{2} \sim Exp(1)$ and $\theta \sim U(0, 2\pi)$.

(b) Show that $X^2 + Y^2$ and $\dfrac{X}{Y}$ are independently distributed.

27. Let U_1 and U_2 be i.i.d $U(0,1)$ random variables. Show that

$$X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2) \quad \text{and} \quad X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

are i.i.d $N(0,1)$ random variables.

28. Let X_1, X_2 and X_3 be i.i.d with probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

Find the probability density function of Y_1, Y_2, Y_3; where

$$Y_1 = \frac{X_1}{X_1 + X_2}, Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, Y_3 = X_1 + X_2 + X_3.$$

29. X and Y are i.i.d. random variables each having geometric distribution with the following p.m.f

$$P(X = x) = \begin{cases} (1-p)^x p & \text{if } x = 0, 1, .. \\ 0 & \textit{Otherwise} \end{cases}$$

Identify the distribution of $\dfrac{X}{X+Y}$. Further find the p.m.f of $Z = \min(X, Y)$.

*** All the best ***