1. A chicken lays n eggs. Each egg independently does or doesn't hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn't survive (independently of the other eggs), with probability s of survival. Let N ← Bin(n, p) be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don't survive (so X + Y = N). Find the marginal PMF of X, and the joint PMF of X and Y . Are they independent?

ANS:

A10

Intuitively, the probability that a egg hatches & chick survives is ps. We can consider each egg as a Bernoulli trial each with a success (hatching and surviving). There are n independent trials, so $X \sim Bin(n, ps)$

$$P(X=i) = \sum_{j=0}^{n} P(X=i \mid N=j)\,P(N=j)$$

$$= \sum_{j=1}^{n} P(X=i \mid N=j)\,P(N=j)$$

$$= \sum_{j=1}^{n} \binom{j}{i} s^i (1-s)^{j-i} \binom{n}{j} p^j (1-p)^{n-j}$$

$$= \sum_{j=1}^{n} \frac{j!}{i!(j-i)!}\,\frac{n!}{j!(n-j)!}\, s^i (1-s)^{j-i} p^j (1-p)^{n-j}$$

$$= \sum_{j=1}^{n} \frac{n!}{i!(j-i)!(n-j)!}\, s^i (1-s)^{j-i} p^j (1-p)^{n-j}$$

$$= \sum_{r=0}^{n-i} \frac{n!}{i!\,r!\,(n-i-r)!}\, s^i (1-s)^r p^{r+i} (1-p)^{n-i-r}$$

$$= \frac{n!}{i!(n-i)!} (ps)^i (1-p)^{n-i} \sum_{r=0}^{n-i} \frac{(n-i)!}{r!(n-i-r)!} (1-s)^r p^r (1-p)^{-r}$$

$$= \binom{n}{i} (ps)^i (1-p)^{n-i} \left(1 + \frac{(1-s)p}{1-p}\right)^{n-i}$$

$$= \binom{n}{i} (ps)^i (1-p)^{n-i} \left(\frac{1-ps}{1-p}\right)^{n-i}$$

$$= \binom{n}{i} (ps)^i (1-ps)^{n-i}$$

$$= Bin(i \mid n, ps)$$