Discrete Maths CODING THEORY chapter 4

Generator Matrix: Parity – Check Matrix

Example 1) An encoding function $E : Z_2^2 \to Z_2^5$ is given by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

a) Determine all the code words. What can be said about the error detection capability of this code? What about its error correction capability?

b) Find the associated parity - check matrix H.

c)Use H to decode the received words: 11101, 11011

Soln: We know that the given G is of the form $G = [I_2 / A]$, where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

a) We find that

$$[E(00)] = [0 \quad 0]G = [0 \quad 0]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$[E(01)] = [0 \quad 0]G = [0 \quad 1]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= [0 \quad 1 \quad 0 \quad 1 \quad 1]$$

$$[E(10)] = [1 \quad 0]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= [1 \quad 0 \quad 1 \quad 1 \quad 0]$$

$$[E(11)] = [1 \quad 1]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= [1 \quad 1 \quad 1 \quad 0 \quad 1]$$

These matrix equations show that the code words are

$E(00) = 00000,\ E(01) = 01001,\ E(10) = 10110, E(11) = 11101$

From these, we find that

$$\begin{aligned} &d(E(00), E(01)) = 3, && d(E(00), E(10)) = 3 \\ &d(E(00), E(11)) = 4 && d(E(01), E(10)) = 4 \\ &d(E(01), E(11)) = 3 && d(E(10), E(11)) = 3 \end{aligned}$$

Thus $\min(E) = 3$. Therefore the code can detect all errors of weight ≤ 2 and can correct all single errors.

b) The Parity-check matrix H associated with G is given by

$$H = [A^T / I_3] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

We observe that H does not contain a column of 0s and further no two columns of H are identical. Therefore, H corrects single errors in transmission.

c) For $r = 11101$, the syndrome of r is

$$H\begin{bmatrix}1 & 1 & 1 & 0 & 1\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since this is a zero matrix , the decoded message is got by retaining the first two components of r. The decoded message is therefore 11.

For $r = 11011$, the syndrome of r is

$$H\begin{bmatrix}1 & 1 & 0 & 1 & 1\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

We observe that the matrix is identical with the first column of H. Therefore, we change the first component of r (from 1 to 0) to get 01011. This is the code word . The first two components of this code word, namely 01, is the original message.

Example 2) The generating function of an encoding function $E : Z_2^3 \to Z_2^6$ is given by

$$G = \begin{bmatrix}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 0 & 0 & 1 & 1\\0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}$$

a) Find the code words assigned to 110 and 010.

b) Obtain associated parity - check matrix.

c)Hence decode the received words : 110110, 111101.

d)Show that the decoding of 111111 is not possible by using H.

Soln: We note that $G=[I_3 / A]$, where

$$I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \text{ and } A=\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}$$

a) We find that

$$[E(110)]=[1 \quad 1 \quad 0]\begin{bmatrix}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 0 & 0 & 1 & 1\\0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}=[1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1]$$

$$[E(010)]=[0 \quad 1 \quad 0]\begin{bmatrix}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 0 & 0 & 1 & 1\\0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}=[0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1]$$

Thus the required code words are

$E(110)=110101$ and $E(010)=010011$.

b) The parity – check matrix associated with G is

$$H=[A^T / I_3]=\begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\0 & 1 & 1 & 0 & 0 & 1\end{bmatrix}$$

c)For $r=110110$, the syndrome of r is

$$H[1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0]^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We observe that this matrix is identical with the second column of H. Therefore, we change the second component in r (from 1 to 0) to get the word $c = 100110$. The first three components of this code word gives the original message $w = 100$.

For $r = 111101$, the syndrome of r is

$$H[1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1]^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We observe that this matrix is identical with the third column of H. Therefore, we change the third component in r (from 1 to 0) to get the word $c = 110101$. The first three components of this code word gives the original message $w = 110$.

d)For $r = 111111$, the syndrome of r is

$$H[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We observe this matrix is not a zero matrix and is not equal to any column of H. Therefore, we cannot decode $r = 111111$ by using H.

Example 3) The parity – check matrix for an encoding function $E: Z_2^3 \to Z_2^6$ is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

a) Determine the associated generator matrix.

b)Does this code correct all single errors in transmission?

Soln:

a) we have

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Which is of the form $[A^T / I_3]$. Accordingly,

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ so that } A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Hence, the associated generator matrix is

$$G = [I_3 / A] \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

b)We observe that two columns of H (namely the 2nd and 5th) are identical. Therefore, H does not provide a decoding scheme that corrects single errors in transmission.