PARTIAL ORDERINGS

to use relations to order some or all of the elements of sets. For instance:

- order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary.
- Schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins.
- order the set of integers using the relation containing the pairs (x, y), where x is less than y.
- Schedule the tasks needed to build a house using the relation consisting of the pairs of tasks, (x, y) such that x must be completed/performed before y begins/starts.
- order the tasks of a software project by specifying the order
- Schedule the tasks needed to cook a Chinese meal by specifying their order

When we add all of the pairs of the form (x, x) to these relations: $\{(x, y) \mid x, y \in S\}$, we obtain a relation that is reflexive, antisymmetric, and transitive.

A: set of tasks that must be performed in building a house. Define R on A by xRy if x, y denote the same task or if task x must be performed before the start of task y

These are properties that characterize relations used to order the elements of sets.

on $A = \{1, 2, 3\}$
$R = \{(1,2), (2,1), (2,3)\}$:
$R_1 = \{(1,1), (2,2)\}$:

A relation R on a set S is:

- ✓ Reflexive iff $\forall x (x \in S \rightarrow xRx)$
- Symmetric $\forall x \forall y (x \in S \land y \in S \land xRy \rightarrow yRx)$
- ✓ Antisymmetric if $\forall x \forall y (x \in S \land y \in S \land xRy \land yRx \rightarrow x = y)$
- ✓ Transitive if $\forall x \forall y \forall z (x \in S \land y \in S \land z \in S \land xRy \land yRz \rightarrow xRz)$

DEF. A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R, (S, R) is called a partially ordered set or poset. members of S are called elements of the poset. Examples: $(Z, \geq)$, $(Z^+, |)$, $(P(S), \subseteq)$ are posets.

$a < b$ if ...
$a > b$ if $a \geq b$ but $a \neq$ ...

DEF. The elements a and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.

Ex. In the poset $(Z^+, |)$, the integers 3 and 9 are comparable because $3|9$, the integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

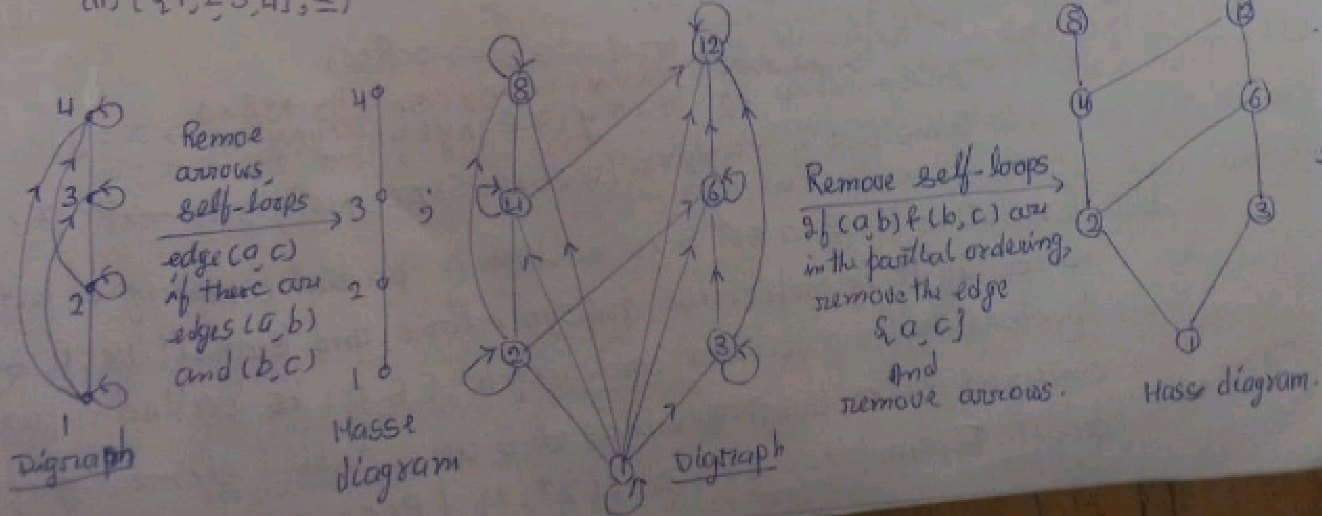
DEF. If $(S, \preceq)$ is a poset and every two elements of S are comparable, the poset $(S, \preceq)$ is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered is also called a chain.

EX: $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$
$(Z^+, |)$ is not totally ordered, because $5 \nmid 7$ or $7 \nmid 5$ etc.

DEF. $(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering ~~ang~~ every non-empty subset of S has a least element.

EX. $(Z^+, \leq)$ is well-ordered set ; $(N, \leq)$
$(Z, \leq)$ is not well-ordered because $Z^- \subseteq Z$ has no least element

EX: Construct the Hasse Diagram for (i) $(P\{a, b, c\}; \subseteq)$
(ii) $(\{1, 2, 3, 4\}, \leq)$
(iii) $(\{1, 2, 3, 4, 6, 8, 12\}, |)$



a is MAXIMAL in the poset $(S, \leq)$ if $\exists$ no $b \in S$ such that $a < b$

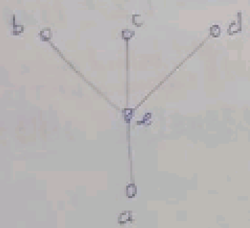
a is MINIMAL in the poset $(S, \leq)$ if $\exists$ no $b \in S$ such that $b < a$

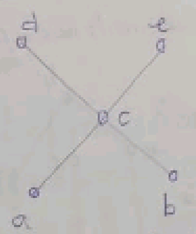
Maximal and minimal elements are the "top" and "bottom" elements in the Hasse diagram. They are not unique elements

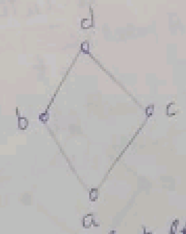
a is the GREATEST ELEMENT of the poset $(S, \leq)$ if $b \leq a$, $\forall b \in S$ (Unique, if exists)

a is the LEAST ELEMENT of the poset $(S, \leq)$ if $a \leq b$, $\forall b \in S$ (Unique if exists)

The poset with Hasse diagram has:





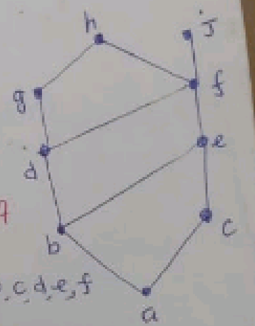


least element a
no greatest element
minimal elts: a
maximal elts: b, c, d

neither least elt. nor greatest elt.
Minimal : a, b
maximal : d, e

no least elt
greatest elt d
Minimal : a, b
maximal : d

least elt : a
greatest elt : d
minimal : a
maximal : d



Given a poset $(S, \leq)$ and $A \subseteq S$.

$u \in S$ is called an UPPER BOUND of A if $a \leq u$, $\forall a \in A$

$l \in S$ is called a LOWER BOUND of A if $l \leq a$, $\forall a \in A$

EX:

→ $\{a, b, c\}$ its upper bounds : e, f, j, + h and lower bound : a

→ $\{f, h\}$ " : no upper bounds f " : a, b, c, d, e, f

→ $\{a, c, d, f\}$ " : f, h, j f " : a

x is the least upper bound of A if $a \leq x$, whenever $a \in A$ and $x \leq z$, whenever z is an upper bound of A.

y is the greatest lower bound of A if $y \leq a$, whenever $a \in A$ and $z \leq y$, whenever z is a lower bound of A.

EX: upper bounds of $\{b, d, g\}$ are g and h. Because, $g < h$, $\therefore$ g is the lub of $\{b, d, g\}$

lower bounds of $\{b, d, g\}$ are a and b. because $a < b$, $\therefore$ b is the glb of $\{b, d, g\}$

EX. Find the greatest lower bound (glb) and the least upper bound (lub) of the set : $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(Z^+, |)$

Sol : Lower bounds for the set $\{3, 9, 12\}$ are 1, 3. Because, $1|3$, $glb\{3,9,12\} = 3$
Lower bound for $\{1,2,4,5,10\}$ w.r.t. '|' is the element 1. $glb\{1,2,4,5,10\} = 1$

$lub\{3,9,12\} = 36$

$lub\{1,2,4,5,10\} = 20$

Let $a, b \in Z^+$ in the poset $(Z^+, |)$.

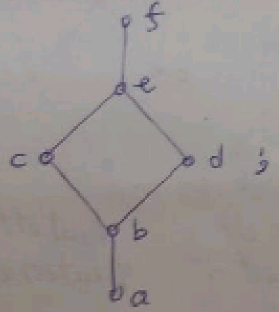
$lub\{a,b\} = a \vee b$ = the least common multiple of a and b

$glb\{a,b\} = a \wedge b$ = the greatest common divisor of a and b

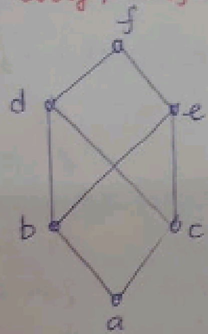
Thus, every pair of elements, $\{a,b\}$ in $(Z^+, |)$ has both a least upper bound and a greatest lower bound. Such a poset i.e. $(Z^+, |)$ is called a lattice.

DEF : A poset $(S, \leq)$ in which every pair of elements has both a least upper bound and a greatest lower bound is called a LATTICE.

Ex :

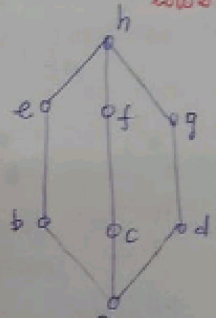


Lattice



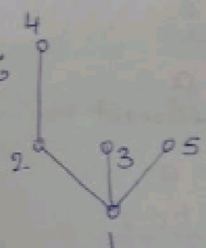
Not lattice

$lub\{b,c\} = b \vee c$ does not exist.



lattice

; $(\{1,2,3,4,5\}, |)$



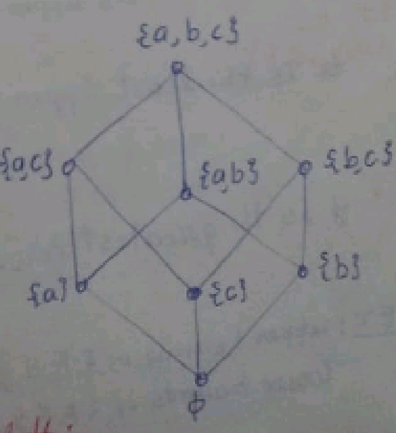
Not lattice

$lub\{2,3\} = 2 \vee 3$ & $lub\{3,5\} = 3 \vee 5$ do not exist

; $(\{1,2,4,8,16\}, |)$

Lattice

EX : Determine whether $(P(\{a,b,c\}), \subseteq)$ is a lattice.



Sol. Let $A, B \in P(\{a,b,c\})$

$lub\{A,B\} = A \vee B = A \cup B$

$glb\{A,B\} = A \wedge B = A \cap B$

$(P(\{a,b,c\}); \subseteq, \wedge, \vee)$ is a lattice.

DEF. A lattice L is BOUNDED if it has both a upper bound, denoted by 1, such that $x \leq 1$ for all $x \in L$ and a lower bound, denoted by 0, such that $0 \leq x$ for all $x \in L$.

(a) $x \vee 1 = 1$, (b) $x \wedge 1 = x$, (c) $x \vee 0 = x$, (d) $x \wedge 0 = 0$

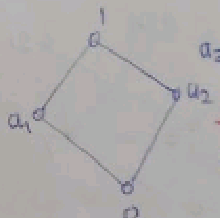
EX: For a finite set S, $(P(S), \subseteq)$ is bounded.

DEF. The complement of an element a of a bounded lattice L with upper bound 1 and lower bound 0 is an element b such that

$$a \vee b = 1 \quad \text{and} \quad a \wedge b = 0$$

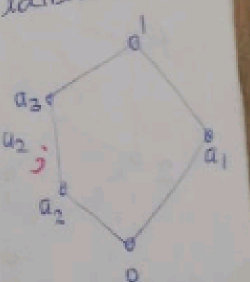
Such a lattice is complemented if every element of the lattice has a complement.

EX: For a finite set S, $(P(S), \subseteq)$ is complemented



$0' = 1$, $1' = 0$
$a_1' = a_2$
$a_2' = a_1$

;



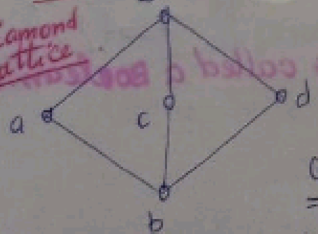
$0' = 1$, $1' = 0$
$a_1' = a_2$ or a_3

DEF. A lattice L is called distributive if (i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ $\forall x, y, z \in L$.

EX: For a finite set S, $(P(S), \subseteq)$ is distributive.

EX

'diamond lattice'



$a \wedge (c \vee d)$
$= a \wedge e$
$= a$

$(a \wedge c) \vee (a \wedge d)$
$= b \vee b = b$

$\therefore a \wedge (c \vee d) \neq (a \wedge c) \vee (a \wedge d)$

NOT distributive.

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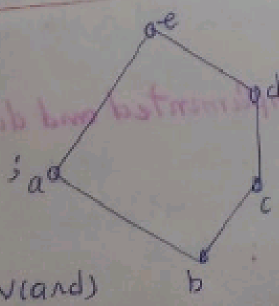
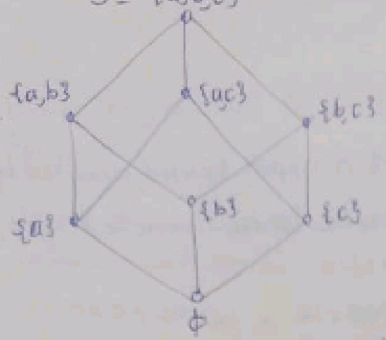


$d \wedge (a \vee c) = d \wedge e = d$
$(d \wedge a) \vee (d \wedge c) = b \vee c = e$

$\therefore d \wedge (a \vee c) \neq (d \wedge a) \vee (d \wedge c)$

NOT DISTRIBUTIVE lattice.

Ex. $S = \{a,b,c\}$



$\because \{a\} \cup \{b,c\} = S$, $\{a\} \cap \{b,c\} = \phi$

$\therefore \overline{\{a\}} = \{b,c\}$ and $\overline{\{b,c\}} = \{a\}$

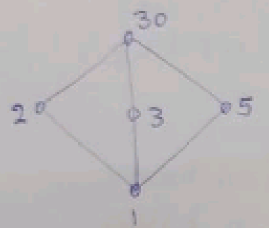
$\overline{\{b\}} = \{a,c\}$ and $\overline{\{a,c\}} = \{b\}$

$\overline{\{c\}} = \{a,b\}$ and $\overline{\{a,b\}} = \{c\}$

$\phi' = \{a,b,c\}$ and $\overline{\{a,b,c\}} = \phi$

least element or lower bound $= \phi$

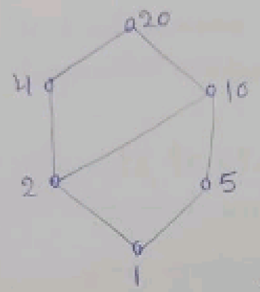
greatest element or upper bound $= \{a,b,c\}$



$1' = 30$ & $30' = 1$

$2' = 3$ & 5 and $5' = 2$ and 3

$3' = 2$ and 5



$(D_{20}, |)$

not complemented lattice

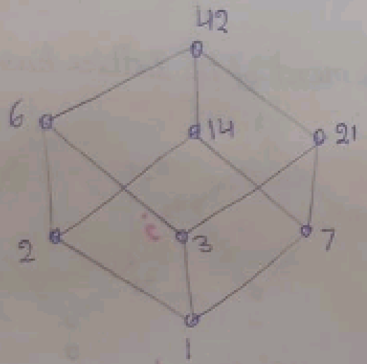
$1' = 20$ and $20' = 1$

$2 \wedge 5 = 1$ whereas $2 \vee 5 = 10$

2 has no complement and so 5.

$10 \wedge 4 = 2$ whereas $10 \vee 4 = 20$

10 has no complement and so 4.



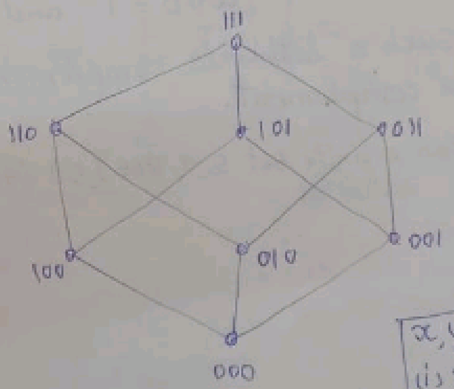
$(D_{42}, |)$

$\overline{1} = 42$ and $\overline{42} = 1$

$\overline{2} = 21$ and $\overline{21} = 2$

$\overline{3} = 14$ and $\overline{14} = 3$

$\overline{6} = 7$ and $\overline{7} = 6$



$(B_3, \leq)$

$\overline{000} = 111$ and $\overline{111} = 000$

$\overline{100} = 011$ and $\overline{011} = 100$

$\overline{010} = 101$ and $\overline{101} = 010$

$\overline{001} = 110$ and $\overline{110} = 001$

$x, y \in B_n$

(i) $x \leq y$ iff $x_k \leq y_k$

(ii) $x \wedge y = c_1 c_2 \cdots c_n$, $c_k = \min\{x_k, y_k\}$

(iii) $x \vee y = d_1 d_2 \cdots d_n$, $d_k = \max\{x_k, y_k\}$

(iv) $x' = z_1 z_2 \cdots z_n$, $z_k = \begin{cases} 1 & \text{if } x_k = 0 \\ 0 & \text{if } x_k = 1 \end{cases}$

DEF: A lattice which is complemented and distributive is called a Boolean Algebra.

Let (A, R_1), (B, R_2) be two posets.
A relation R on $A \times B$ is defined by
$(a,b) \, R \, (x,y)$ if $a R_1 x$ and $b R_2 y$

For $\forall a \in A$, $\forall b \in B$, we have $a R_1 a$ and $b R_2 b$, so $(a,b) R (a,b)$ and R is reflexive.

For (a,b), $(c,d) \in A \times B$
suppose $(a,b) R (c,d) \Rightarrow a R_1 c$ and $b R_2 d$
& $(c,d) R (a,b) \Rightarrow c R_1 a$ and $d R_2 b$
$\Rightarrow a = c$ and $b = d \Rightarrow (a,b) = (c,d)$
$\Rightarrow R$ is antisymmetric.
(antisymmetric) (antisymmetric)

For (a,b), (c,d), $(e,f) \in A \times B$
Suppose,
$(a,b) R (c,d) \Rightarrow a R_1 c$ and $b R_2 d$
$(c,d) R (e,f) \Rightarrow c R_1 e$ and $d R_2 f$
$\Rightarrow a R_1 e$ and $b R_2 f \Rightarrow (a,b) R (e,f)$
$\Rightarrow R$ is transitive.
(transitive) (transitive)

$\therefore$ R is a partial order on $A \times B$ and $(A \times B, R)$ is a poset.
This R is called a product partial order.

Lexicographic Order:

Usual on $(Z \times Z, \leq)$

Given two posets: $(A_1, \leq_1)$ and $(A_2, \leq_2)$.
The lexicographic ordering $\leq$ on $A_1 \times A_2$ is defined by
$(a_1, a_2) < (b_1, b_2)$ either if $a_1 <_1 b_1$ or if both $a_1 = b_1$ and $a_2 < b_2$

$(3,5) < (4,8)$
$(3,8) < (4,5)$
$(4,9) < (4,11)$

Note: a partial ordering $\leq$ can be obtained by adding equality to the ordering $<$ on $A_1 \times A_2$

Given n posets: $(A_1, \leq_1)$, $(A_2, \leq_2)$, $\ldots$, $(A_n, \leq_n)$
The partial ordering (lexicographic ordering) $\leq$ on $A_1 \times A_2 \times \cdots \times A_n$ is defined by
$(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ if $a_1 <_1 b_1$,
or if there is an integer $i > 0$ such that $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} <_{i+1} b_{i+1}$

$(1,2,3,5) < (1,2,4,3)$

Lexicographic Ordering of strings:

Given the strings $a_1a_2\cdots a_m$ and $b_1b_2\cdots b_n$ on a poset S.

The string $a_1a_2a_3\cdots a_m$ is less than $b_1b_2b_3\cdots b_n$ if and only if

$(a_1,a_2\cdots a_t) \prec (b_1,b_2\cdots b_t)$ ← truncated or

$(a_1,a_2\cdots a_t) = (b_1,b_2\cdots b_t)$ and $m<n$ where $t=\min(m,n)$.

Ex: discreet < discrete (e<t)

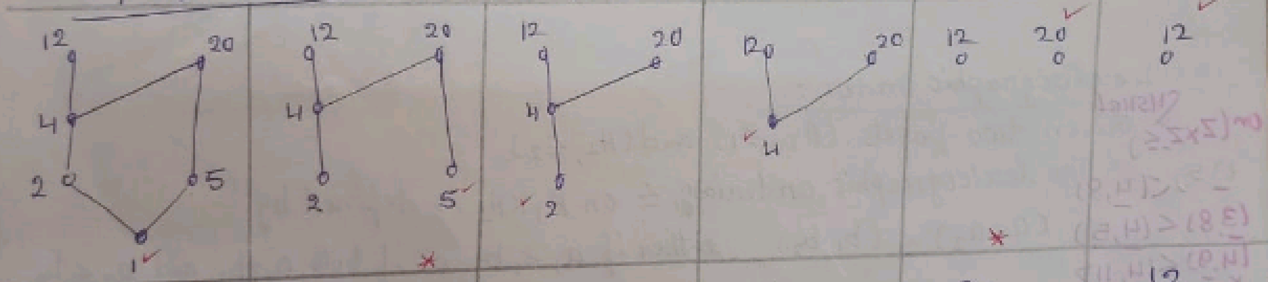
discreet < discreetness

discrete < discretion (e<i)

Topological Sorting:

— has an application to the scheduling of projects.

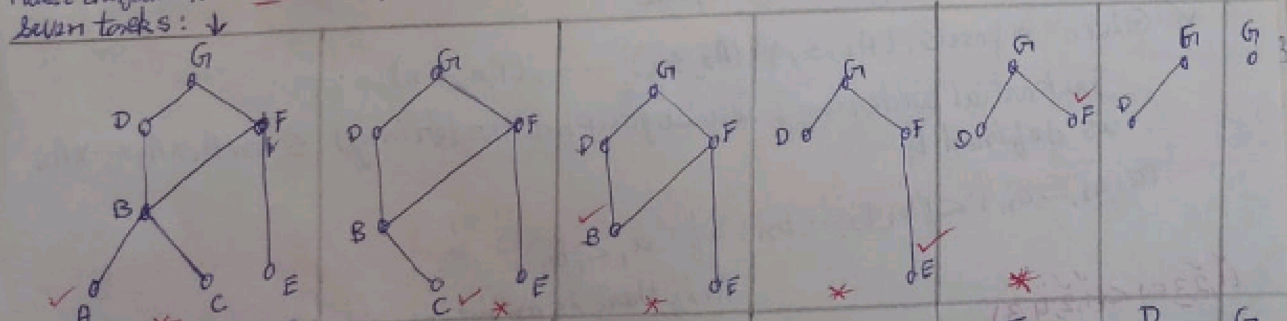
Find a compatible total ordering for the poset $(\{1,2,4,5,12,20\}, |)$



minimal element chosen	1	5	2	4	20	12

Ans. $1<5<2<4<20<12$

Hasse diagram for seven tasks: ↓



Minimal element chosen	A	C	B	E	F	D	G

$A<C<B<E<F<D<G$: one possible order for the tasks

MORPHISM

DEF: Let (S, R) and (S', R') be posets.

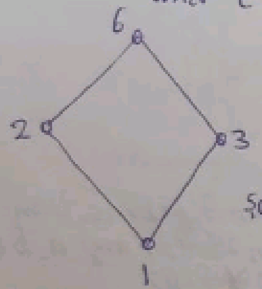
A function $f : S \longrightarrow S'$ is called a poset homomorphism from (S,R) to (S',R') if for any $a, b \in S$.

1. aRb, then $f(a)\, R'\, f(b)$, and
2. $a \| b$ (comparable), then $f(a) \| f(b)$

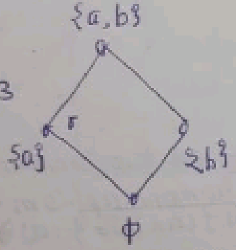
In addition, if $f : S \longrightarrow S'$ is

3. one-to-one
4. onto.

then f is called poset isomorphism or similarity mapping from (S,R) to (S',R') and (S,R) and (S',R') are isomorphic posets.



$(\{1,2,3,6\}, |)$



$(P(\{a,b\}), \subseteq)$

Define set function $f : \{1,2,3,6\} \longrightarrow P(\{a,b\})$ by , f is one-to-one onto.

$f(1) = \phi$
$f(2) = \{a\}$
$f(3) = \{b\}$
$f(6) = \{a,b\}$

$1|2$, $f(1) \subseteq f(2)$
$1|3$, $f(1) \subseteq f(3)$
$1|6$, $f(1) \subseteq f(6)$
etc.

Also $2 \nmid 3$, $\{a\} \not\subseteq \{b\}$ or a

$\therefore$ Order-relation is preserved.

Q. Examine whether $(\{1,2,3,5,6,10,15,30\}, |)$ and $(P(\{a,b,c\}), \subseteq)$ are isomorphic.

DEF. If $(L_1, \vee, \wedge)$ and $(L_2, \oplus, *)$ are two lattices, a mapping $f : L_1 \longrightarrow L_2$ is called a lattice homomorphism from $(L_1, \vee, \wedge)$ to $(L_2, \oplus, *)$, if for any $a, b \in L_1$

(1) $f(a \vee b) = f(a) \oplus f(b)$: Join-homomorphism
(2) $f(a \wedge b) = f(a) * f(b)$: meet-homomorphism
(3) $a \leq_1 b$, then $f(a) \leq_2 f(b)$: order-homomorphism

In addition, if $f : L_1 \longrightarrow L_2$ is

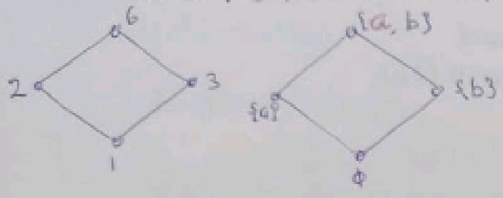
(4) one-to-one
(5) onto

then homomorphism $f : L_1 \rightarrow L_2$ is called lattice isomorphism.

EX. $(D_6, |, \vee, \wedge)$ and $(P(\{a,b\}), \subseteq, \oplus, *)$ are isomorphic lattices.

where $LUB\{a,b\} = a \vee b = LCM\{a,b\}$ $\qquad$ $LUB\{A,B\} = A \oplus B = A \cup B$

$GLB\{a,b\} = a \wedge b = GCD\{a,b\}$ $\qquad$ $GLB\{A,B\} = A * B = A \cap B$



$f : D_6 \longrightarrow P(\{a,b\})$

$f(1) = \phi, \ f(2) = \{a\}, \ f(3) = \{b\}, \ f(6) = \{a,b\}$

$f(2 \vee 3) = f(6) = \{a,b\} = \{a\} \cup \{b\} = \{a\} \oplus \{b\} = f(2) \oplus f(3)$

$f(2 \wedge 3) = f(1) = \phi = \{a\} \cap \{b\} = \{a\} * \{b\} = f(2) * f(3)$

etc. $\qquad$ Also $1 | 3$, $f(1) \subseteq f(3)$ $(\because \phi \subseteq \{b\})$

DEF. If $(B_1, \le_1, \vee, \wedge, ', 0, 1)$ and $(B_2, \le_2, \oplus, *, \bar{\ }, \alpha, \beta)$ are two Boolean Algebras, then a mapping $f : B_1 \longrightarrow B_2$ is called a Boolean homomorphism if for any $a, b \in B_1$,

(i) $f(a \vee b) = f(a) \oplus f(b)$

(ii) $f(a \wedge b) = f(a) * f(b)$

(iii) $f(a') = \overline{f(a)}$

(iv) $f(0) = \alpha$ and $f(1) = \beta$

(v) $a \le_1 b$ then $f(a) \le_2 f(b)$

DEF. Boolean Algebras $(B_1, \le_1, \vee, \wedge, \bar{\ }, 0, 1)$ and $(B_2, \le_2, \oplus, *, \bar{\ }, 0, 1)$ are called **ISOMORPHIC** if there is a function $f : B_1 \to B_2$ such that f is one-to-one and onto, and for all $x_1, y_1 \in B_1$.

(a) $f(x_1 \vee y_1) = f(x_1) \oplus f(y_1)$

(b) $f(x_1 \wedge y_1) = f(x_1) * f(y_1)$

(c) $f(\bar{x}_1) = \overline{f(x_1)}$

EX: $U = \{1,2,3\}$, $D_{30} = \{1,2,3,5,6,10,15,30\}$

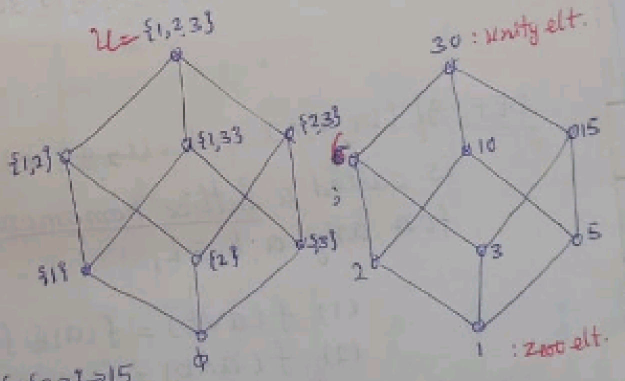
Consider two Boolean Algebras:

$(P(U), \subseteq, \vee, \wedge, \bar{\ }, \phi, U)$

$(P(U), \subseteq, \cup, \cap, \bar{\ }, \phi, U)$

and $(D_{30}, |, \oplus, *, \bar{\ }, 1, 30)$

$(D_{30}, |, lcm\{a,b\}, GCD\{a,b\}, \bar{\ }, 1, 30)$



$f : P(U) \longrightarrow D_{30}$ by

$f : \phi \to 1$, $f\{2\} \to 3$, $f : \{1,2\} \to 6$, $f : \{2,3\} \to 15$

$f : \{1\} \to 2$, $f : \{3\} \to 5$, $f : \{1,3\} \to 10$, $f : \{1,2,3\} \to 30$

Schedule the tasks needed to build a house, by specifying their, if the Hasse diagram representing these tasks is shown in Fig. 1

Q. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is shown in Fig. 2.

Q Schedule the tasks needed to cook a Chinese meal by specifying their order. if the Hasse diagram representing their tasks is shown in Fig. 3

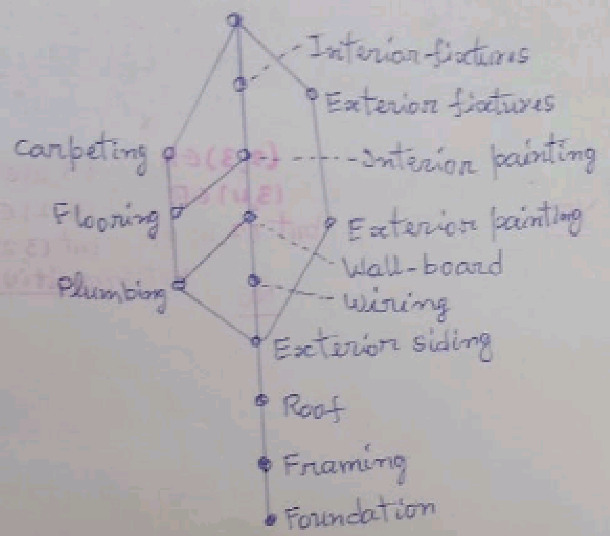


Fig. 1

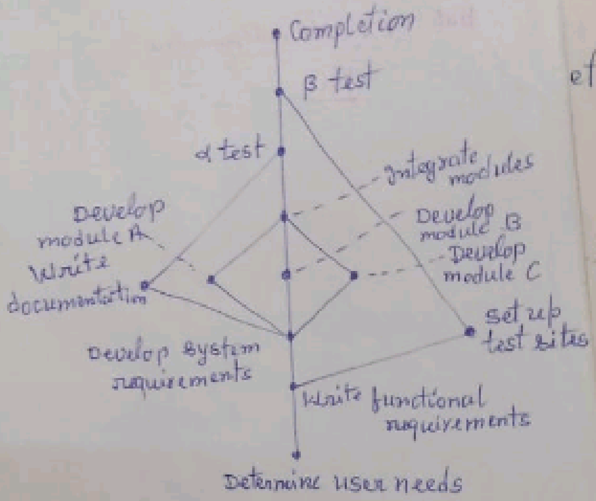


Fig. 2

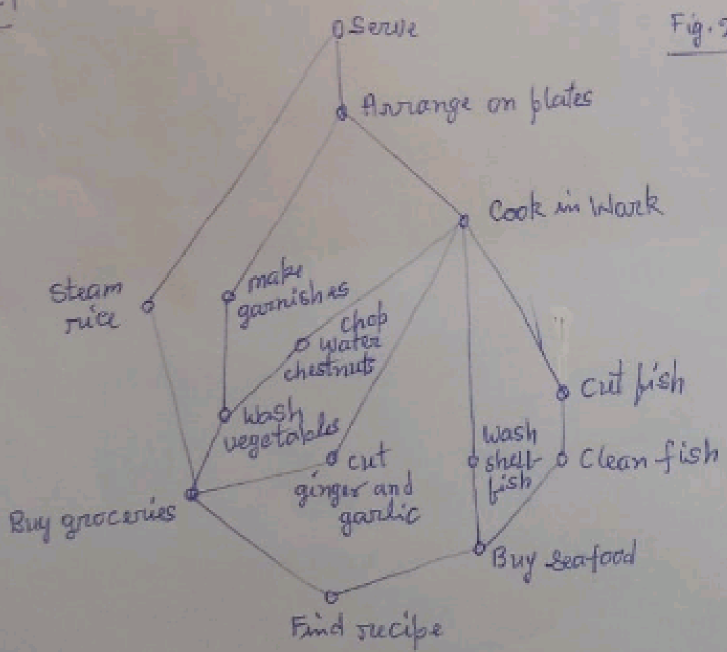


Fig. 3

EX. Determine whether the relations represented by these zero-one matrices are partial orders.

(a)
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b)
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c)
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Sol. (a) Let $S = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,3)\}$

$(1,2) \in R$

$(2,1) \in R$

but $1 \neq 2$, $\therefore$ Not antisymmetric.

NO

(b) $S = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

YES

(c) $S = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (3,3), (3,4), (4,1), (4,2), (4,4)\}$

$(2,3) \in R$; $(3,4) \in R$

$(3,4) \in R$; $(4,2) \in R$

but $(2,4) \notin R$; but $(3,2) \notin R$

Not transitive

NO

EX. Let B be the set of positive integer divisors of 30. The binary operations (+), (·) and unary operation (') are defined by

$$x + y = \text{lcm}(x, y), \quad x \cdot y = \text{gcd}(x, y), \text{ and } x' = \frac{30}{x}$$

Examine (B, +, ·, ') for

SOL. $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Since lcm and gcd of any two numbers $x, y \in B$, belong to B, the binary operations (+) and (·) are defined.

(1) Commutative Laws: Examine whether $x+y = y+x$ and $x \cdot y = y \cdot x$. $\forall x, y \in B$

(2) Associative laws: Examine whether: $(x+y)+z = x+(y+z)$
: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$\forall x, y, z \in B$.

(3) Distributive Laws: Examine whether: $x \cdot (y+z) = x \cdot y + x \cdot z$
$x + (y \cdot z) = (x+y) \cdot (x+z)$
$\forall x, y, z \in B$

(4) Identity Laws: Existence of identity elements.

$x + 1 = 1 + x = \text{lcm}\{1, x\} = x \quad , \forall x \in B$

and $x \cdot 30 = 30 \cdot x = \text{gcd}\{x, 30\} = x \quad , \forall x \in B$

∴ 1 is identity for '+' and 30 is identity for '·'.

(5) Complement Laws / complementation

Given; $x' = \frac{30}{x}$ $\quad \therefore 5' = \frac{30}{5} = 6$

Thus $x + x' = 30$ and $x \cdot x' = 1$, $\forall x \in B$

$5 + 5' = 5 + 6 = \text{lcm}\{5, 6\} = 30$
$5' + 5 = 6 + 5 = \text{lcm}\{6, 5\} = 30$
$5 \cdot 5' = 5' \cdot 5 = \text{gcd}(5, 6) = 1$

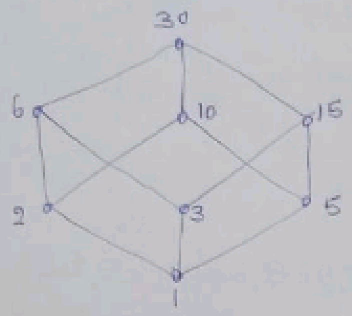
Q. Examine whether $(P(S), \subseteq, \cup, \cap, ', \phi, S)$ is a Boolean Algebra, S being a finite set.

$\boxed{X \cup (S-X) = S}$ $X \cap (S-X) = \phi$ $\boxed{X \cup \phi = X \quad X \cap S = X}$

DEF: A non-empty set consisting of at least two elements 0 and 1, two binary operations (+) and (·) called addition and multiplication, and a unary operator (') called complementation is called a Boolean Algebra provided that the above following axioms for all $x, y, z \in B$ are satisfied.

EX. Consider posets / lattices : $(D_{30}, |, \vee, \wedge)$ and $(B_3, \leq, \vee, \wedge)$

✓ Examine whether both are isomorphic lattices



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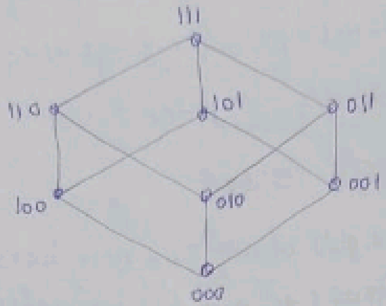


$(D_{30}, |, \vee, \wedge, ', 1, 30)$ $(B_3, \leq, \vee, \wedge, ', 000, 111)$

$x, y \in B_n$

(1) $x \leq y$ if $x_k \leq y_k$ for $k = 1, 2, 3, \ldots n$

(2) $x \wedge y = c_1 c_2 \cdots c_n$, where $c_k = \min\{x_k, y_k\}$

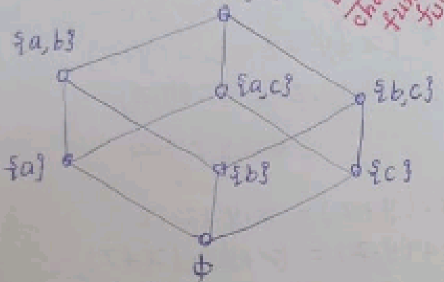
(3) $x \vee y = d_1 d_2 \cdots d_n$, where $d_k = \max\{x_k, y_k\}$

(4) $x' = z_1 z_2 \cdots z_n$
$z_k = \begin{cases} 1 & \text{if } x_k = 0 \\ 0 & \text{if } y_k = 1 \end{cases}$

✓ Examine $(B_n \leq, \quad)$ is a Boolean Algebra

$\mathcal{U} = \{a, b, c\}$

use characteristic function $f_U(x)$



$(P(\mathcal{U}), \leq, \vee, \wedge, ^-, \phi, \mathcal{U})$

DEF ✓ A finite lattice $(L, \leq)$ which is ISOMORPHIC to a lattice $(B_n, \leq)$ for some non-negative number n, is called a Boolean Algebra.

Each $(B_n, \leq)$ is a Boolean Algebra and so is each lattice $(P(S), \subseteq)$ where S is a finite set.

✓ If a finite lattice $(L, \leq)$ does not contain 2^n elements for some non-negative integer n, $(L, \leq)$ can not be a Boolean Algebra. If $|L| = 2^n$, $(L, \leq)$ may or may not be Boolean Algebra.

✓ Theorem : If $n = p_1 p_2 \cdots p_k$, where p_i are distinct primes, then $(D_n, |)$ is a Boolean algebra.

EX $(D_{385}, |)$
$(D_{646}, |)$

$385 = 5 \cdot 7 \cdot 11$
$646 = 1 \cdot 2 \cdot 17 \cdot 19$