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RECURRENCE RELATIONS
(Difference Equations)

The 'forwards' difference operator Δ is defined by

$$(\Delta f)(n) = f(n+1) - f(n)$$

The 'backwards' difference operator $\nabla = \Delta^{-}$ is defined by

$$(\nabla f)(n) = (\Delta^{-} f)(n) = f(n) - f(n-1)$$

Discrete numeric function or sequence
$f: N \rightarrow R$
$f(n) = \pi,\ \pi \in R$

$$(\Delta^2 f)(n) = \Delta(\Delta f)(n) = \Delta(f(n+1) - f(n)) = \Delta f(n+1) - \Delta f(n)$$
$$= [f(n+2) - f(n+1)] - [f(n+1) - f(n)]$$

$$\therefore (\Delta^2 f)(n) = f(n+2) - 2f(n+1) + f(n)$$

$$(\Delta^3 f)(n) = \Delta^2 f(n+1) - \Delta^2 f(n)$$
$$= [f(n+3) - 2f(n+2) + f(n+1)] - [f(n+2) - 2f(n+1) + f(n)]$$

$$\therefore (\Delta^3 f)(n) = f(n+3) - 3f(n+2) + 3f(n+1) - f(n)$$

etc.

Thus, the difference operator may be entirely eliminated in ~~favour~~ favour of function values at integer steps.

A DIFFERENCE EQUATION is an equation involving the difference operator Δ, an unknown function f, and known functions $a_1(n)$, $a_2(n)$, ---, $a_k(n)$, $b(n)$.

Order of difference equation = the highest order of a difference operator, occurring

When rewritten without the use of Δ, a k^{th}-order equation may involve $f(n+k)$, $f(n+k-1)$, $f(n+k-2)$, --, $f(n)$.

A difference equation which can be written in the form:

$$a_k f(n+k) + a_{k-1} f(n+k-1) + \cdots + a_0(n) f(n) = b(n)$$

where the a_i and b are known functions, is called LINEAR.

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Linear Equations with constant coefficients

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = b_n$$

where the a_n are constants and b_n is a given function of n

To find the complementary function, we try

$y_n = \lambda^n$, as a solution of the homogeneous equation.

$$\therefore \quad a_k \lambda^{n+k} + a_{k-1} \lambda^{n+k-1} + \cdots + a_0 \lambda^n = 0$$

$$\therefore \quad a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_0 = 0$$

This is called the auxiliary equation (characteristic equation) of the difference equation.

1. Distinct Roots

Ex: Solve : $y_{n+2} + 3y_{n+1} + 2y_n = n$

The auxiliary equation is $\lambda^2 + 3\lambda + 2 = 0$, which has solutions $\lambda = -1, -2$

Thus, $y_n = (-1)^n$ and $y_n = (-2)^n$ are solutions of the homogeneous equation.

CF: $\therefore \quad y_n^{(h)} = A(-1)^n + B(-2)^n$ is also a solution.

$\therefore$ the most general solution of the homogeneous equation. i.e

$\therefore$ it is the complementary function.

To find a particular integral, we use the method of 'undetermined coefficients'.

The R.H.S is linear (first degree polynomial), we try

$$y_n^{(p)} = kn + l \qquad \text{(most general linear poly.)}$$

Putting in the inhomogeneous equation

$$[k(n+2)+l] + 3[k(n+1)] + 2(kn+l) = n$$

Equation the coefficients of n and $n^0 = 1$:

$$k + 3k + 2k = 1 \qquad \Rightarrow (2k+l) + (3k+l) + 2l = 0$$

so, $k = \frac{1}{6}$ and $l = -5/36$ $\qquad \therefore y_n^{(p)} = \left(\frac{1}{6}\right)n - \frac{5}{36}$

$\therefore$ the general solution:

$$y_n = y_n^{(h)} + y_n^{(p)} \Rightarrow \boxed{y_n = A(-1)^n + B(-2)^n + \frac{n}{6} - \frac{5}{36}}$$

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Linear Equations with constant coefficients

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \dots + a_0 y_n = b_n$$

where the a_r are constants and b_n is a given function of n

To find the complementary function, we try

$\underline{\underline{y_n = \lambda^n}}$, as a solution of the homogeneous equation

$$\therefore \quad a_k \lambda^{n+k} + a_{k-1} \lambda^{n+k-1} + \dots + a_0 \lambda^n = 0$$

$$\therefore \quad \underline{a_k \lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0 = 0}$$

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Equating the coefficients of n and $n^0 = 1$:

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EX. (Distinct Complex Roots)

solve : $y_{n+2}+w^2 y_n = 2^n$; $y_0=\alpha$, $y_1=\beta$

| modulus-amplitude form:
$r=\sqrt{x^2+y^2}=\omega$
$=\omega$
$\theta=\tan^{-1}[y/x]=\frac{\pi}{2}$
$0\pm i\omega=\omega[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}]$

Sol. A.E. is $\lambda^2+w^2=0$, which has two solutions: $\lambda=\pm i\omega$

$\therefore$ CF is
$$y_n = A_1(i\omega)^n + B_1(-i\omega)^n$$
$$=\omega^n[A_1 i^n + B_1(-i)^n]$$
$$=w^n[A_1 e^{n\pi i/2}+B_1 e^{-n\pi i/2}]$$
$| \because i = e^{\pi i/2}$

Euler's identity:
$| e^{i\theta}=\cos\theta + i\sin\theta$

$$=\omega^n\{A_1[\cos(n\pi/2)+i\sin(n\pi/2)] + B_1[\cos(n\pi/2)-i\sin(n\pi/2)]\}$$
$$=\omega^n[(A_1+B_1)\cos(n\pi/2)+(i(A_1-B_1))\sin(n\pi/2)]$$
$$\therefore y_n^{(h)}=\omega^n[A\cos(n\pi/2)+B\sin(n\pi/2)]$$

For PI, we try $y_n^{(p)}=k2^n$. On substituting into the inhomogeneous equation:

$$k2^{n+2}+\omega^2 k2^n=2^n \Rightarrow k(4+\omega^2)=1 \Rightarrow k=\frac{1}{(4+\omega^2)}$$

$\therefore y_n^{(p)}=\left(\frac{1}{4+\omega^2}\right).2^n$

$\therefore$ The general solution of the inhomogeneous equation:

$$y_n = y_n^{(k)}+y_n^{(h)}=\frac{2^n}{4+\omega^2}+\omega^n[A\cos(n\pi/2)+B\sin(n\pi/2)]$$

Determining A and B in terms of α and β:

$$y_0=\frac{1}{4+\omega^2}+A=\alpha \quad , \quad y_1=\frac{2}{4+\omega^2}+\omega B=\beta$$

so that $A=\alpha-\frac{1}{4+\omega^2}$, $B=\frac{1}{\omega}\left[\beta-\frac{2}{4+\omega^2}\right]$

$\therefore$ The solution is

$$y_n=\frac{2^n}{4+\omega^2}+\omega^n\left[\left(\alpha-\frac{1}{4+\omega^2}\right)\cos(n\pi/2)+\frac{1}{\omega}\left(\beta-\frac{2}{4+\omega^2}\right)\sin(n\pi/2)\right]$$

Repeated Roots:

EX: Solve: $y_{n+2}-4y_{n+1}+4y_n=n2^n$

The auxiliary equation is $\lambda^2-4\lambda+4=0$ with repeated roots $\lambda=2,2$

$\therefore$ CF is $y_n^{(h)}=(A+Bn)2^n$

For a PI, we try $y_n^{(p)}=(kn+l).n^2.2^n=(kn^3+ln^2).2^n$

$$\therefore [k(n+2)^3+l(n+2)^2]2^{n+2}-4[k(n+1)^3+l(n+1)^2]2^{n+1}+4(kn^3+ln^2)2^n=n.2^n$$

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$n^3 2^n$: $4k-8k+4k=0$

$n^2 2^n$: $24k+4l-24k-8l+4l=0$

$n 2^n$: $48k+16l-24k-16l=1$

2^n : $32k+16l-8k-8l=0$

These give

$$\left.\begin{array}{l} 24k=1 \\ 24k+8l=0 \end{array}\right\} \Rightarrow \begin{array}{l} k=\frac{1}{24} \\ l=-\frac{1}{8} \end{array}$$

$\therefore$ the general solution is

$$y_n = y_n^{(h)} + y_n^{(p)} = (A+Bn-n^2/8+n^3/24)2^n$$

EX: Solve the recurrence relation: $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$

Ans: $a_n = (A+Bn)2^n + n^2(\frac{n}{6}+1)2^n$

EX: solve the recurrence relation: $a_n = 2(a_{n-1} - a_{n-2})$

Sol:

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

AE: $\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i$

The modulus-amplitude form of

$$1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right)$$

$\therefore$ The general solution of the R.R. is

$$a_n = (\sqrt{2})^n \left\{c_1 \cos\frac{n\pi}{4} \pm c_2 \sin\frac{n\pi}{4}\right\}.$$

EX: solve the recurrence relation:

$$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n), \ n \geq 0$$

Sol: $a_n^{(h)} = (c_1 + c_2 n)3^n$; Assume; $a_n^{(p)} = A_0 \cdot 2^n + A_1 \cdot n^2 \cdot 3^n$ | $\lambda = 3,3$

$A_0 = 1$, $A_1 = \frac{7}{18}$

$$a_n^{(p)} = 2^n + \frac{7}{18} n^2 \cdot 3^n$$

$$\therefore a_n = a_n^{(h)} + a_n^{(p)} = (c_1 + c_2 n)3^n + 2^n + \frac{7}{18} n^2 \cdot 3^n$$

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The method of Generating Functions

Let a_n : n^{th} term of a discrete numeric function/sequence. Its O.G.F

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots$$

$$A(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \quad \Rightarrow \sum_{n=1}^{\infty} a_n z^n = A(z) - a_0$$

$$A(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n \quad \Rightarrow \sum_{n=2}^{\infty} a_n z^n = A(z) - a_0 - a_1 z$$

If $a_n = \alpha^n$

$$G(z) = \sum_{n=0}^{\infty} \alpha^n z^n = 1 + \alpha z + \alpha^2 z^2 + \alpha^3 z^3 + \cdots = \frac{1}{1-\alpha z}$$

Also, note : $\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5 + \cdots$

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

$$\therefore \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + \cdots$$

Ex: Use generating functions to solve : $a_n - 3a_{n-1} = n$, $n \geq 1$, $a_0 = 1$

Sol.

$$\sum_{n=1}^{\infty} a_n z^n - 3\sum_{n=1}^{\infty} a_{n-1} z^n = \sum_{n=1}^{\infty} n z^n$$

$$\sum_{n=1}^{\infty} a_n z^n - 3z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} = \sum_{n=1}^{\infty} n z^n$$

$$(A(z) - a_0) - 3z A(z) = z + 2z^2 + 3z^3 + 4z^4 + \cdots \qquad (AP + GP)$$

$$(A(z) - 1) - 3z A(z) = \frac{z}{(1-z)^2} \Rightarrow A(z) = \frac{1}{1-3z} + \frac{z}{(1-z)^2(1-3z)}$$

Using a partial fraction decomposition,

$$\frac{z}{(1-z)^2(1-3z)} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{(1-3z)}$$

$$\therefore z = A(1-z)(1-3z) + B(1-3z) + C(1-z)^2$$

For the following assignments for z we get

$(z=1)$: $1 = B(-2)$, $B = -\frac{1}{2}$

$(z=\frac{1}{3})$: $\frac{1}{3} = C(\frac{2}{3})^2$, $C = \frac{3}{4}$

$(z=0)$: $0 = A + B + C$, $A = -(B+C) = -\frac{1}{4}$

$$\therefore A(z) = \frac{1}{(1-3z)} + \frac{(-1/4)}{(1-z)} + \frac{(-1/2)}{(1-z)^2} + \frac{(3/4)}{(1-3z)}$$

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Linear = -

$$A(z) = \frac{(7/4)}{(1-3z)} + \frac{(-1/4)}{(1-z)} + \frac{(-1/2)}{(1-z)^2}$$

Now,

$$\frac{(7/4)}{(1-3z)} = \frac{7}{4}\left[1/(1-3z)\right] = \left(\frac{7}{4}\right)\left[1+(3z)+(3z)^2+(3z^3)+\cdots+(3z)^n+\cdots\right]$$

$\therefore$ coefficient of $z^n = \frac{7}{4} \cdot 3^n$

$$\left(-\frac{1}{4}\right)\frac{1}{1-z} = \left(-\frac{1}{4}\right)\left[1+z+z^2+\cdots+z^n+\cdots\right]$$

$\therefore$ coefficient of $z^n = \left(-\frac{1}{4}\right) \cdot 1$

$$(-1/2)/(1-z)^2 = \left(-\frac{1}{2}\right)\left[\underline{1-z}\right]^{-2}$$
$$= \left(-\frac{1}{2}\right)\left[\binom{-2}{0} + \binom{-2}{1}(-z) + \binom{-2}{2}(-z)^2 + \binom{-2}{3}(-z)^3 + \cdots + \binom{-2}{n}(-z)^n + \cdots\right]$$

$\therefore$ coefficient of $z^n = \left(-\frac{1}{2}\right)\binom{-2}{n}(-1)^n = \left(-\frac{1}{2}\right)(-1)^n \cdot \binom{2+n-1}{n}(-1)^n$

$$= \left(-\frac{1}{2}\right)\underline{(n+1)} = \left(\frac{-1}{2}\right)n - \frac{1}{2}$$

$$a_n = \frac{7}{4}3^n - \frac{1}{4} + \left(\frac{-1}{2}\right)n - \frac{1}{2} = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}; \; n \geq 0$$

coefficient of z^n in $(1-\alpha x)^{-n} = \binom{n+n-1}{n}\alpha^n$

✓ __Ex__ Use generating functions to solve:

$$a_{n+2} - 5a_{n+1} + 6a_n = 2, \; n \geq 0, \; \underline{a_0 = 3}, \; \underline{a_1 = 7}$$

$$\sum_{n=0}^{\infty} a_{n+2}z^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}z^{n+2} + 6\sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=0}^{\infty} 2z^{n+2}$$

$$\sum_{n=0}^{\infty} a_{n+2}z^{n+2} - 5z\sum_{n=0}^{\infty} a_{n+1}z^{n+1} + 6z^2\sum_{n=0}^{\infty} a_n z^n = 2z^2\sum_{n=0}^{\infty} z^n$$

$$(A(z) - a_0 - a_1 z) - 5z(A(z) - a_0) + 6z^2 A(z) = 2z^2 \cdot \frac{1}{1-z}$$

$$(A(z) - 3 - 7z) - 5z(A(z) - 3) + 6z^2 A(z) = \frac{2z^2}{1-z}$$

$$(1 - 5z + 6z^2)A(z) = 3 - 8z + \frac{2z^2}{1-z} = \frac{3 - 11z + 10z^2}{1-z}$$

$$\therefore A(z) = \frac{3 - 11z + 10z^2}{(1-5z+6z^2)(1-z)} = \frac{(3-5z)(1-2z)}{(1-3z)(1-2z)(1-z)} = \frac{3-5z}{(1-3z)(1-z)}$$

By a partial fraction decomposition

$$A(z) = \frac{2}{1-3z} + \frac{1}{1-z} = 2\sum_{n=0}^{\infty}(3z)^n + \sum_{n=0}^{\infty} z^n$$

Consequently $a_n = 2(3^n) + 1 \; ; \; n \geq 0$

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RECURRE... Numeric Eq... ... is defined ... f(n) lined by ... f(n)=

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Q. Use the method of generating function to solve the recurrence relations :

(a) $a_n = 4a_{n-1} - 4a_{n-2} + 4^n \; ; n \geq 2 \; , \; a_0 = 2, \; a_1 = 8$

(b) $a_{n+1} - 8a_n + 16a_{n-1} = 4^n \; ; \; n \geq 1 \; , \; a_0 = 1, \; a_1 = 8$

(c) $a_{n+2} - 4a_n = 9n^2, \; n \geq 0$

(d) $a_n = 4a_{n-1} + 3n.2^n \; ; n \geq 1, \; a_0 = 4$

Solutions

(a) $(A(z) - a_0 - a_1 z) = 4z(A(z) - a_0) - 4z^2 A(z) + \frac{1}{1-4z} - 1 - 4z$

$$A(z) = \frac{1 + (1-4z)^2}{(1-2z)^2(1-4z)} = \frac{4}{1-4z} - \frac{2}{(1-2z)^2}$$

$$a_n = 4.4^n - 2.(n+1)2^n = 4^{n+1} - (n+1)2^{n+1}$$

(b) $\sum_{n=1}^{\infty} a_{n+1} z^n - 8\sum_{n=1}^{\infty} a_n z^n + 16\sum_{n=1}^{\infty} a_{n-1} z^n = \sum_{n=1}^{\infty} (4z)^n$

$$\frac{1}{z}(A(z) - a_0 - a_1 z) - 8(A(z) - a_0) + 16zA(z) = \frac{1}{1-4z} - 1$$

$$(1 - 8z + 16z^2)A(z) - a_0 - a_1 z + 8a_0 z = \frac{4z^2}{1-4z}$$

$$A(z) = \frac{1}{(1-4z)^2} - \frac{4z^2}{(1-4z)^3}.$$

$$= \frac{(1 - 4z + 4z^2)}{(1-4z)^3}$$

Coefft of z^n in $(1-4z)^{-3}$
$= 4^n \binom{3+n-1}{n}$
$= 4^n \binom{n+2}{n}$
$= 4^n \binom{n+2}{2}$
$= \frac{1}{2}(n+1)(n+2) 4^n$

$$\sum_{n=0}^{\infty} a_n z^n = (1 - 4z + 4z^2).\frac{1}{2}[1.2(4z)^0 + 2.3(4z) + 3.4(4z)^2 + \cdots + (n+1)(n+2)(4z^n) + \cdots]$$

$$a_n = \frac{1}{2}[(n+1)(n+2)4^n - n(n+1)4^n + (n-1)(n)4^{n-1}]$$

$$= \frac{1}{2}4^{n-1}[4(n^2+3n+2) - 4(n^2+n) + (n^2-n)]$$

$$= \frac{1}{2}(n^2 + 7n + 8).4^{n-1}$$

(c) $\sum_{n=0}^{\infty} a_{n+2} z^n - 4\sum_{n=0}^{\infty} a_n z^n = 9\sum_{n=0}^{\infty} n^2 z^n = 9\sum_{n=0}^{\infty}[n(n+1) - n]z^n$

$$\frac{1}{z^2}(A(z) - a_0 - a_1 z) - 4A(z) = 9[1.2z + 2.3z^2 + \cdots] - 9[z + 2z^2 + 3z^3 + \cdots]$$

$$= 18x\frac{1}{(1-z)^3} - \frac{9z}{(1-z)^2}$$

$$\left(\frac{1}{z^2} - 4\right)A(z) = \frac{a_0}{z^2} + \frac{a_1}{z} + \frac{18z}{(1-z)^3} - \frac{9z}{(1-z)^2}.$$

$$A(z) = \frac{a_0 + a_1 z}{(1-4z^2)} + \frac{18z^3}{(1-z)^3(1-4z^2)} - \frac{9z^3}{(1-z)^2(1-4z^2)}$$

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$$A(z) = \frac{a_0 + a_1 z}{(1-2z)(1+2z)} + \frac{9z^3 + 9z^4}{(1-z)^3(1-2z)(1+2z)}$$

$$= \frac{A}{1-2z} + \frac{B}{1+2z} - \frac{17/3}{1-z} + \frac{5}{(1-z)^2} - \frac{6}{(1-z)^3} - \frac{\frac{1}{12}}{1+2z} + \frac{\frac{27}{4}}{1-2z}$$

$$= \frac{1}{1-2z}\left(A + \frac{27}{4}\right) + \frac{1}{1+2z}\left(B - \frac{1}{12}\right) - \frac{17}{3}\cdot\frac{1}{(1-z)} + 5\cdot\frac{1}{(1-z)^2} - 6\cdot\frac{1}{(1-z)^3}$$

$$= \frac{c_1}{(1-2z)} + \frac{c_2}{1+2z} - \frac{17}{3}\cdot\frac{1}{1-z} + 5\cdot\frac{1}{(1-z)^2} - \frac{6}{(1-z)^3}$$

$$a_n = c_1 2^n + c_2(-2)^n - \frac{17}{3}\cdot 1^n + 5(n+1) - \cancel{6}^{3}\cdot\frac{1}{\cancel{2}}(n+1)(n+2)$$

$$a_n = c_1(2)^n + c_2(-2)^n - 3\left(n^2 + \frac{4}{3}n + \frac{20}{9}\right)$$

d) $$\sum_{n=1}^{\infty} a_n z^n - 4\sum_{n=1}^{\infty} a_{n-1} z^n = 3\sum_{n=1}^{\infty} n\cdot 2^n\cdot z^n = 3\sum_{n=1}^{\infty} n(2z)^n$$

$$(A(z) - a_0) - 4z\,A(z) = 3\times 2z\sum_{n=1}^{\infty} n(2z)^{n-1} = 6z\cdot\left[1 + 2(2z)^1 + 3(2z)^2 + 4(2z)^3 + \cdots\right]$$

$$= \frac{6z}{(1-2z)^2}$$

$$(1-4z)\,A(z) = \frac{6z}{(1-2z)^2} + 4$$

$$A(z) = \frac{6z}{(1-4z)(1-2z)^2} + \frac{4}{(1-4z)}$$

Resolve into partial fractions and simplify.

Q. solve the system of recurrence ~~la~~ relations:

$$a_n = 3a_{n-1} + 2b_{n-1}, \quad n \geq 1$$
$$b_n = a_{n-1} + b_{n-1}; \quad n \geq 1$$

BC: $a_0 = 1$, $b_0 = 0$

$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} 3a_{n-1} z^n + \sum_{n=1}^{\infty} 2b_{n-1} z^n$ i.e. $A(z) - a_0 = 3z\,A(z) + 2z\,B(z)$

$\sum_{n=1}^{\infty} b_n z^n = \sum_{n=1}^{\infty} a_{n-1} z^n + \sum_{n=1}^{\infty} b_{n-1} z^n$ i.e. $B(z) - b_0 = z\,A(z) + z\,B(z)$

$$A(z) = \frac{1-z}{1-4z+z^2} = \frac{(3+\sqrt{3})/6}{1-(2+\sqrt{3})z} + \frac{(3-\sqrt{3})/6}{1-(2-\sqrt{3})z}$$

$$B(z) = \frac{z}{1-4z+z^2} = \frac{\sqrt{3}/6}{1-(2+\sqrt{3})z} - \frac{\sqrt{3}/6}{1-(2-\sqrt{3})z}$$

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Modeling with Recurrence Relations:

Recurrence relations can be used a variety of problems, such as finding compound interest, ...

(1) Compound Interest:

Amount in the account after n years

= amount in the account after (n-1) years + Interest for the nth year

$P_n = P_{n-1} + \left(\frac{R}{100}\right) P_{n-1}$, $n \geq 1$, $P_0 = v$: Interest compounded annually

$P_n = P_{n-1} + \left(\frac{R}{2 \times 100}\right) P_{n-1}$, $n \geq 1$, $P_0 = v$ " " " half-yearly

$P_n = P_{n-1} + \left(\frac{R}{4 \times 100}\right) P_{n-1}$, $n \geq 1$, $P_0 = v$ " " " quarterly

$P_n = P_{n-1} + \left(\frac{R}{12 \times 100}\right) P_{n-1}$, $n \geq 1$, $P_0 = v$ " " " monthly.

(2) Population Growth:

Population after n years = Population after (n-1) years + population increase in nth year

$$P_n = P_{n-1} + \frac{R}{100} P_{n-1}, \; n \geq 1, \; P_0 = v$$

R% : rate of increase.

(3) Machine Evaluation:

Value after n years = Value after (n-1) years − depreciation in value during nth year

$$P_n = P_{n-1} - \frac{R}{100} P_{n-1}, \; n \geq 1, \; P_0 = v$$

R% = rate of depreciation

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(4) A Geometric Progression: $a_0, a_1, a_2, \cdots$ is described by a Recurrence Relation:

$a_{n+1} = d\,a_n$; $n \geq 0$, d (common ratio) is a constant and $a_0 = A$.

Its solution: $a_n = Ad^n$, $n \geq 0$ defines a discrete function whose domain is the set N of all nonnegative integers.
[(n+1)st term]

(5) An Arithmetic Progression: $a_0, a_1, a_2, \cdots$ is represented by a R.R.:

$a_{n+1} = a_n + d$, $n \geq 0$, d (common difference) is constant and $a_0 = A$.

Its solution is [: (n+1)st term of the sequence]

$a_n = A + nd$, $n \geq 0$

(6) Number of permutations of n objects, $n \geq 0$ is given by RR:

$a_n = n \cdot a_{n-1}$, $n \geq 1$ and $a_0 = 1$

Its solution is: $a_n = n!$

(7) If the number of bacteria in a colony doubles every hour and if the colony begins with 5 bacteria, then the number of bacteria at the end of n hours is given by a R.R.:

$a_n = 2a_{n-1}$, $n \geq 1$, $a_0 = 5$

(8) Codeword Enumeration:

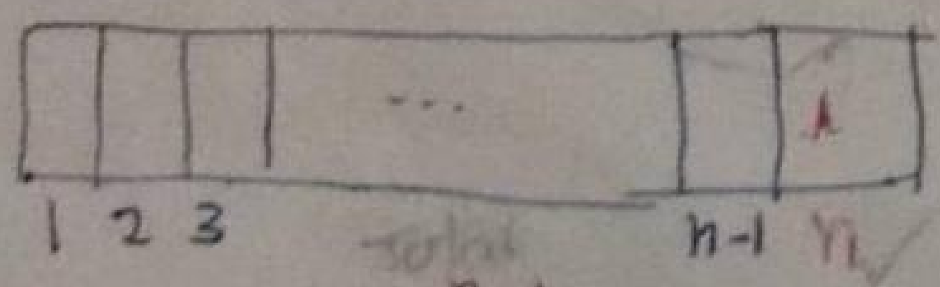
A string of decimal digits is a VALID codeword if it contains an even number of 0 digits.

a_n : number of valid n-digit codewords.

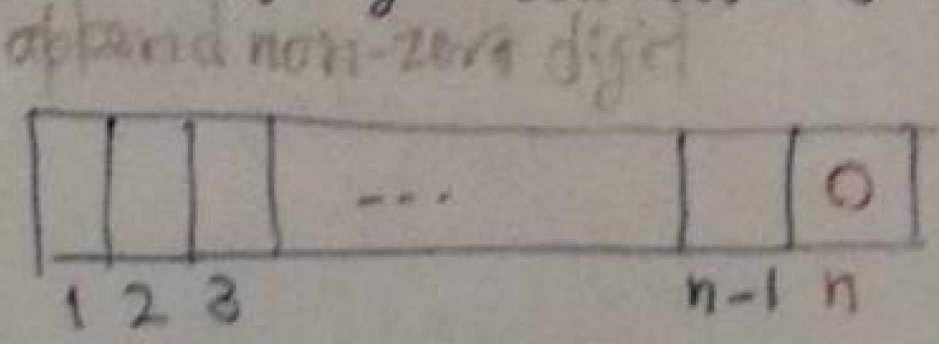
$a_1 = 9$ (∵ there are 10 one-digit strings and string 0 is not valid)

Construct a valid n-digit string from strings of $(n-1)$ digits

Strings of $(n-1)$ digits = strings of $(n-1)$ digits with even number of 0s
or + strings of $(n-1)$ digits with no ~~even~~ odd number of 0s
= valid $(n-1)$ digit codewords or + invalid $(n-1)$ digit codewords



or



Use OGF to solve:

$\boxed{a_n = 8a_{n-1} + 10^{n-1} \quad a_1 = 9 \text{ (or } a_0 = 1)}$

$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) \cdot 1 \Rightarrow a_n = 8a_{n-1} + 10^{n-1}$

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EX: Prove : If m is an even integer, then m+7 is odd. : $p \to q$
$\equiv \neg q \to \neg p$

1. **Direct Proof :**

m is even $\Rightarrow m = 2a$ for some integer a

$\therefore m+7 = 2a+7 = 2(a+3)+1 = 2k_1+1$, $k_1 = a+3$ is an integer

$\therefore m+7 = 2k+1$ is odd

2. **The Contrapositive Method:**

The given statement can be written as

For all m, if m is is an even integer, then m+7 is odd

$\forall m[p(m) \to q(m)]$

We know that $\forall m[p(m) \to q(m)] \equiv \forall m[\neg q(m) \to \neg p(m)]$

Suppose m+7 is not odd, for every m

$\therefore$ m+7 is even

$\therefore m+7 = 2b$, for some integer b

$\therefore m = 2b-7 = 2(b-4)+1 = 2k_2+1$, $k_2 = b-4$ is integer

$\therefore$ m is odd, for every integer m

$\therefore \forall m[\neg q(m) \to \neg p(m)]$

But $\forall m[p(m) \to q(m)] \equiv \forall m[\neg q(m) \to \neg p(m)]$ { contraposition Rule.

$\therefore \forall m[p(m) \to q(m)]$

$\therefore$ For every m, if m is an even integer, then m+7 is odd

3. **The method of Proof by contradiction:**

statement: If m is an even integer, then m+7 is odd

suppose that m is even and that (m+7) is also even $\to$ negation of what we want to prove.

m+7 is even implies that $m+7 = 2c$ for some integer c.

$m = 2c-7 = 2(c-4)+1$, with c-4 an integer

$\therefore$ m is odd

$\therefore$ we have a contraction to m is even

No integer can be both even and odd.

Some mistake made

Our assumption: (m+7) is even , is not possible

ie. our assumption is false

Its negation is true , $\therefore$ m+7 is odd.

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Q. Prove:

For all positive real ~~m~~ numbers x and y, if the product xy exceeds 25, then $x>5$ or $y>5$

Q. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

EX: Establish the validity of the argument:

$$
\begin{array}{l}
(\forall x)[p(x) \vee q(x)] \\
(\exists x)\neg p(x) \\
(\forall x)[\neg q(x) \vee r(x)] \\
(\forall x)[s(x) \rightarrow \neg r(x)] \\
\hline
\therefore (\exists x)\neg s(x)
\end{array}
$$

$\{1\}$	(1)	$\forall x[p(x) \vee q(x)]$	Rule P
$\{1\}$	(2)	$p(a) \vee q(a)$	Rule US, (1)
$\{1\}$	(3)	$\neg p \rightarrow q(a)$	Rule T, (2), $p \rightarrow q \equiv \neg p \vee q$, $\neg\neg p \equiv p$
$\{4\}$	(4)	$\exists x \neg p(x)$	Rule P
$\{4\}$	(5)	$\neg p(a)$	Rule ES, (5)
$\{1,4\}$	(6)	$q(a)$	Rule T, (3), (5), $p, p \rightarrow q \Rightarrow q$
$\{7\}$	(7)	$\forall x[\neg q(x) \vee r(x)]$	Rule P
$\{7\}$	(8)	$\neg q(a) \vee r(a)$	Rule US, (7)
$\{7\}$	(9)	$q(a) \rightarrow r(a)$	Rule T, (8), $p \rightarrow q \equiv \neg p \vee q$
$\{1,4,7\}$	(10)	$r(a)$	Rule T, (6), (9), $p, p \rightarrow q \Rightarrow q$
$\{11\}$	(11)	$\forall x[s(x) \rightarrow \neg r(x)]$	Rule P
$\{11\}$	(12)	$s(a) \rightarrow \neg r(a)$	Rule US, (11)
$\{11\}$	(13)	$r(a) \rightarrow \neg s(a)$	Rule T, (12), $p \rightarrow q \equiv \neg q \rightarrow \neg p$, $\neg\neg p \equiv p$
$\{1,4,7,11\}$	(14)	$\neg s(a)$	Rule T, (10), (13), $p, p \rightarrow q \Rightarrow q$
$\{1,4,7,11\}$	(15)	$(\exists x)\neg s(x)$	Rule EG, (14)

$$
\begin{array}{l}
(a \wedge q) \rightarrow m \\
(f \rightarrow q) \wedge (\neg p \rightarrow a) \\
\hline
\therefore \neg m \rightarrow (\neg f \vee p)
\end{array}
$$

Q. Examine the validity of the argument:

"The meeting can take place if all members have been informed in advance, and it is quorate. It is quorate provided that there are at least 15 members present, and members will have been informed in advance if there is not a postal strike. Therefore, if the meeting was cancelled, there were fewer than 15 members present, or there was a postal strike."

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11/01/2021

Exponential Generating Function

The exponential generating function for the sequence $\{a_n\}$ is the series

$$E(z) = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^n$$

(i) $a_n = n+1$,

$$E(z) = \sum_{n=0}^{\infty} \frac{(n+1)}{n!} z^n = \sum_{n=0}^{\infty} \frac{n}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{n}{n!} z^n + e^z$$

$$= z \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} + e^z = ze^z + e^z$$

(ii) $a_n = \frac{1}{(n+1)}$

$$E(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)} \cdot \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!}$$

$$= \frac{1}{z}\left[z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{z}(e^z - 1)$$

(iii) $a_n = \frac{1}{(n+1)(n+2)}$

$$E(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \cdot \frac{z^n}{n!} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!}$$

$$= \frac{1}{z^2}\left[\frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{z^2}\left[e^z - z - 1\right]$$

(iv) $a_n = n(n-1)$

$$E(z) = \sum_{n=0}^{\infty} \frac{n(n-1)}{n!} z^n \; \cancel{= \sum_{n=0}^{\infty} \frac{z^n}{(n-2)!} = \frac{1}{z^2} \sum_{n=0}^{\infty}}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{n!} z^n \; \cancel{= z^2 \sum_{n=0}^{\infty} \frac{z^{(n-2)}}{(n-2)!}}$$

$$= z^2 \sum_{n=2}^{\infty} \frac{z^{n-2}}{(n-2)!}$$

$$= z^2 e^z$$

Ex: Find the sequence for the ~~seque~~ function as its EGF.

$$f(z) = e^{z^3}$$

$$f(z) = e^{z^3} = 1 + \frac{z^3}{1!} + \frac{z^6}{2!} + \frac{z^9}{3!} + \cdots + \frac{z^n}{(n/3)!} + \cdots + \frac{z^{3n}}{n!} + \cdots$$

$$\therefore a_n = \text{coefft. of } \frac{z^n}{n!} = \begin{cases} \frac{n!}{(n/3)!}, & \text{if } n \text{ is a multiple of } 3. \\ 0, & \text{otherwise} \end{cases}$$

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12/01/2021

Ex. In how many ways can a police captain distribute 24 rifle shells to four police officers so that each officer gets at least three shells, but not more than eight?

SOL. The choices for the number of shells each officer receives are given by

$$(z^3+z^4+\cdots+z^8)$$

There are four officers, so the resulting G.F.:

$$f(z) = (z^3+z^4+\cdots+z^8)^4 = z^{12}(\underbrace{1+z+z^2+\cdots+z^5}_{\text{finite G.P}})$$

$$= z^{12}\left[\frac{1-z^6}{1-z}\right]^4 = z^{12}(1-z^6)^4(1-z)^{-4}$$

The required number of ways is given by the coefficient of z^{24} in $f(z)$.

$\therefore$ Coefficient of z^{24} in $f(z) = z^{12}(1-z^6)^4(1-z)^{-4}$

$= \text{coefficient of } z^{12} \text{ in } (1-z^6)^4(1-z)^{-4}$

$= \text{coefft. of } z^{12} \text{ in } \left[1-\binom{4}{1}z^6+\binom{4}{2}z^{12}-\binom{4}{3}z^{18}+\binom{4}{4}z^{24}\right] \times \left[\binom{-4}{0}+\binom{-4}{1}(-z)+\binom{-4}{2}(-z)^2+\cdots\right]$

$= \left[1\cdot\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}\right]$

$= \left[\binom{4+12-1}{12}(-1)^{12}-\binom{4}{1}\binom{4+6-1}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}\right]$

$= \left[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}\right] = \underline{125}$

$\binom{n}{0} = 1$ — any real number

$\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$

NOTE: $\binom{n}{r}$ = coefficient of z^r in $(1+z)^n = \binom{n}{0}+\binom{n}{1}z+\cdots+\binom{n}{r}z^r+\cdots+\binom{n}{n}z^n$

Ex: Verify that for all $n \in \mathbb{Z}^+$, $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$

Proof:

$$(1+z)^{2n} = [(1+z)^n]^2 = (1+z)^n(1+z)^n$$

$$(1+z)^{2n} = \left[\binom{n}{0}+\binom{n}{1}z+\binom{n}{2}z^2+\cdots+\binom{n}{n}z^n\right]\left[\binom{n}{0}+\binom{n}{1}z+\binom{n}{2}z^2+\cdots+\binom{n}{n}z^n\right]$$

$$= \left[\binom{n}{0}+\binom{n}{1}z+\binom{n}{2}z^2+\cdots+\binom{n}{n}z^n\right]\left[\binom{n}{n}+\binom{n}{n-1}z+\binom{n}{n-2}z^2+\cdots+\binom{n}{0}z^n\right]$$

$\binom{2n}{n}$ = coefficient of z^n in $(1+z)^{2n}$

$= \text{coefft. of } z^n \text{ in } \left\{\left[\binom{n}{0}+\binom{n}{1}z+\binom{n}{2}z^2+\cdots+\binom{n}{n-1}z^{n-1}+\binom{n}{n}z^n\right] \times \left[\binom{n}{n}+\binom{n}{n-1}z+\binom{n}{n-2}z^2+\cdots+\binom{n}{1}z^{n-1}+\binom{n}{0}z^n\right]\right\}$

$\therefore \binom{2n}{n} = \binom{n}{0}\binom{n}{0}+\binom{n}{1}\binom{n}{1}+\binom{n}{2}\binom{n}{2}+\cdots+\binom{n}{n}\binom{n}{n} = \underline{\sum_{i=0}^{n}\binom{n}{i}^2}$

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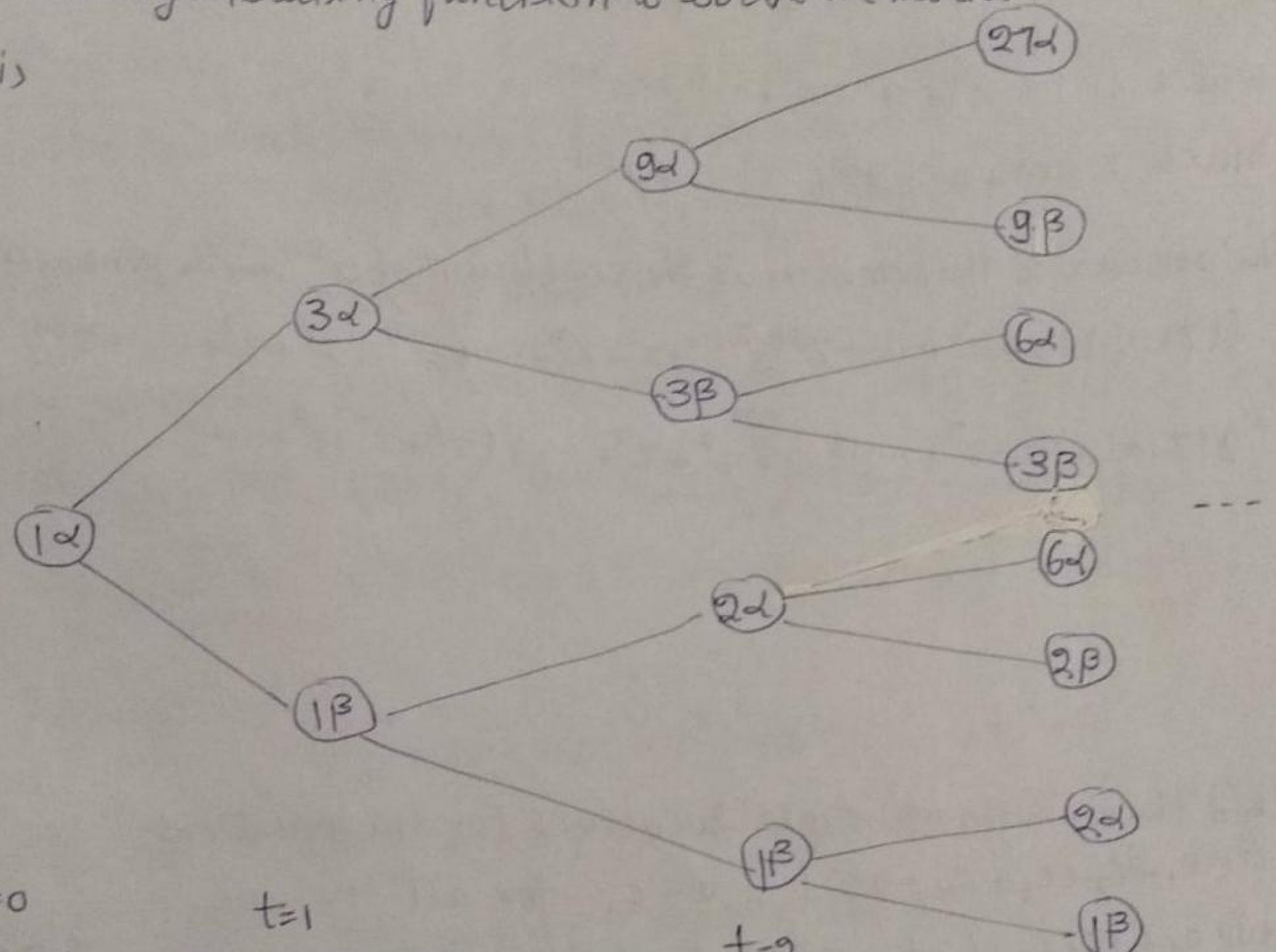
EX: There are two kinds of particles inside a nuclear reactor. In every second, an α particle will split into three α particles and one β particle, and a β particle will split into two α particles and one β particle. If there is a single α particle in the reactor at time $t=0$,

(i) Develop a recurrence relation model for the number of α particles and β particles in the reactor at time $t=n$ seconds.

(ii) Use generating function to solve the model.

SOL. (i)



Time: $t=0$ $\quad t=1$ $\quad t=2$ $\quad t=3$

$a_0 = 1$, $b_0 = 0$; $\quad a_1 = 3 = 3a_0 + 2b_0$, $b_1 = 1 = a_0 + b_0$; $\quad a_2 = 11 = 3a_1 + 2b_1$, $b_2 = 4 = a_1 + b_1$; $\quad a_3 = 41 = 3a_2 + 2b_2$, $b_3 = 15 = a_2 + b_2$

Thus, the recurrence relation model is

$$a_n = 3a_{n-1} + 2b_{n-1} \quad ; \; n \geq 1$$
$$b_n = a_{n-1} + b_{n-1} \quad ; \; n \geq 1$$
$$a_0 = 1 \text{ and } b_0 = 0$$

(ii) The solution to the model is: $a_n = \frac{(3+\sqrt{3})}{6}(2+\sqrt{3})^n + \frac{(3-\sqrt{3})}{6}(2-\sqrt{3})^n$

$A(z) = \frac{1-z}{1-4z+z^2} = \frac{1-z}{(1-\alpha z)(1-\beta z)}$;

$b_n = \frac{\sqrt{3}}{6}(2+\sqrt{3})^n - \frac{\sqrt{3}}{6}(2-\sqrt{3})^n$

$B(z) = \frac{z}{1-4z+z^2} = \frac{z}{(1-\alpha z)(1-\beta z)}$;

Q. Above exc. with condition: (i) α into three β particle
(ii) β into an α particle and two β particles

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Ex. A company hires 11 new ...

Counting Problems and Generating functions:

Ex: If there is an unlimited number (or at least 24 of each color) of red, green, white, and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even number of white beans and at least six black ones?

Sol: The polynomials associated with jelly bean colors are as follows:

- red : $1+x+x^2+x^3+\cdots+x^{24}$
- green : $1+x+x^2+x^3+\cdots+x^{24}$
- white : $1+x^2+x^4+x^6+\cdots+x^{24}$
- black : $x^6+x^7+x^8+\cdots+x^{24}$

The answer to the problem is the coefficient of x^{24} in the generating function:

$$f(z)=(1+z+z^2+\cdots+z^{24})^2(1+z^2+z^4+\cdots+z^{24})(z^6+z^7+\cdots+z^{24})$$

or

$$g(z)=(1+z+z^2+\cdots)^2(1+z^2+z^4+\cdots)(z^6+z^7+z^8+\cdots)$$

Ex. Find the number of integer solutions for the equations:

(i) $c_1+c_2+c_3+c_4=25$; $0\le c_i$ for all $1\le i\le 4$

(ii) $c_1+c_2+c_3+c_4+c_5=30$; $2\le c_1\le 4$ and $3\le c_i\le 8$ for all $2\le i\le 5$

(iii) $c_1+c_2+c_3+c_4+c_5=30$; $0\le c_i$ for all $1\le i\le 5$, with c_2 even and c_3 odd

Sol: (Hint)

(i) Coefficient of z^{25} in the generating function:

$$f(z)=(1+z+z^2+\cdots+z^{25})^4 \text{ or } g(z)=(1+z+z^2+\cdots+z^{25}+z^{26}+\cdots)^4$$

(ii) Coefficient of z^{30} in $(z^2+z^3+z^4)(z^3+z^4+\cdots+z^8)^4$

(iii) Coefficient of z^{30} in $(1+z+z^2+\cdots+z^{30})^3(1+z^2+z^4+\cdots+z^{30})(z+z^3+z^5+\cdots+z^{29})$

or

in $(1+z+z^2+\cdots)^3(1+z^2+z^4+\cdots)(z+z^3+z^5+\cdots)$

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EX. A company hires 11 new em...

Counting Problems and Generating functions:

EX: If there is an unlimited number (or at least 24 of each color) of red, green, white, and black jelly beans, in how many ways can Douglas select 24 of these candies so that he has an even number of white beans and at least six black ones?

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- black : $x^6+x^7+x^8+\cdots\;+x^{24}$

The answer to the problem is the coefficient of $\underline{x^{24}}$ in the generating function:

$$f(z)=(1+z+z^2+\cdots+z^{24})^2(1+z^2+z^4+\cdots+z^{24})(z^6+z^7+\cdots+z^{24})$$

or

$$g(z)=\underline{(1+z+z^2+\cdots)^2}\,\underline{(1+z^2+z^4+\cdots)}\,\underline{(z^6+z^7+z^8+\cdots)}$$

EX. Find the number of integer solutions for the equations:

(i) $c_1+c_2+c_3+c_4=\underline{25}$; $0\le c_i$ for all $1\le i\le 4$

(ii) $c_1+c_2+c_3+c_4+c_5=\underline{30}$; $\underline{2\le c_1\le 4}$ and $3\underline{\le c_i \le}8$ for all $2\le i\le 5$

(iii) $c_1+c_2+c_3+c_4+c_5=\underline{30}$; $0\le c_i$ for all $1\le i\le 5$, with c_2 even and c_3 odd

SOL: (Hint)

(i) Coefficient of $\underline{z^{25}}$ in the generating function:

$$f(z)=(1+z+z^2+\cdots+z^{25})^4 \text{ or } g(z)=\underline{(1+z+z^2+\cdots+z^{25}+z^{26}+\cdots)^4}$$

(ii) Coefficient of $\underline{z^{30}}$ in $(z^2+z^3+z^4)(z^3+z^4+\cdots+z^8)^4$

(iii) Coefficient of $\underline{z^{30}}$ in $(1+z+z^2+\cdots+z^{30})^3(1+z^2+z^4+\cdots+z^{30})(z+z^3+z^5+\cdots+z^{29})$

or

in $(1+z+z^2+\cdots)^3\underline{(1+z^2+z^4+\cdots)}\underline{(z+z^3+z^5+\cdots)}$

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13/01/21

Let $\{a_r\}$ and $\{b_r\}$ be two arbitrary discrete numeric functions/sequences.

$$C = a * b$$

$$c_r = \sum_{i=0}^{r} a_i b_{r-i}$$

(I) If $a_r = r$, $b_r = 1$, then $c_r = \sum_{i=0}^{r} r \cdot 1 = 0+1+2+3+\cdots+r$

(II) If $a_r = r^2$, $b_r = 1$, then $c_r = \sum_{i=0}^{r} r^2 \cdot 1 = 0^2+1^2+2^2+3^2+\cdots+r^2$

(III) If $a_r = r^3$, $b_r = 1$, then $c_r = \sum_{i=0}^{r} r^3 \cdot 1 = 0^3+1^3+2^3+3^3+\cdots+r^3$

<u>Convolution Theorem</u>: If $\{a_r\}$ and $\{b_r\}$ with respective OGFs $A(z)$, $B(z)$ are two discrete numeric functions and

$$c_r = a_r * b_r = \sum_{i=0}^{r} a_i b_{r-i}, \text{ then}$$

$$C(z) = A(z)B(z)$$

<u>Ex</u>: Use generating function to evaluate: $1+2+3+4+\cdots+r$

Let $a_r = r$, $b_r = 1$, then $c_r = \sum_{i=0}^{r} i \cdot 1 = 0+1+2+3+\cdots+r$.

$$A(z) = \sum_{r=0}^{\infty} r z^r = 0 + 1 \cdot z + 2z^2 + 3z^3 + 4z^4 + \cdots + rz^r + \cdots = \frac{z}{(1-z)^2}$$

Alternatively, $\frac{1}{1-z} = 1+z+z^2+z^3+z^4+\cdots+z^r+\cdots$

$$\frac{1}{(1-z)^2} = 1+2z+3z^2+4z^3+\cdots+rz^{r-1}+\cdots$$

$$\therefore \frac{z}{(1-z)^2} = z+2z^2+3z^3+4z^4+\cdots+rz^r+\cdots$$

$$\therefore A(z) = \frac{z}{(1-z)^2}; \text{ Also, } B(z) = \sum_{r=0}^{\infty} 1 \cdot z^r = \frac{1}{1-z}.$$

$\therefore$ By the convolution theorem

$$C(z) = A(z) \cdot B(z) = \frac{z}{(1-z)^3}$$

$$\therefore c_r = \text{coefficient of } z^r \text{ in } \frac{z}{(1-z)^3} = \text{coefficient of } z^{r-1} \text{ in } \frac{1}{(1-z)^3}$$

$$\therefore c_r =$$

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Ex. A company hires ..

Ex: Use generating functions to evaluate the sum : $1^2+2^2+3^2+4^2+\cdots+n^2$

let $a_n = n^2$, $b_n = 1$, then $c_n = \sum_{i=0}^{n} i^2 . 1 = 0^2+1^2+2^2+3^2+4^2+\cdots+n^2$

$B(z) = \frac{1}{1-z}$,

$A(z) = \sum_{n=0}^{\infty} n^2 . z^n = 0 + 1^2 z + 2^2 z^2 + 3^2 z^3 + 4^2 . z^4 + \cdots + n^2 . z^n + \cdots$

$A(z) = 0^2 + 1^2 . z + 2^2 . z^2 + 3^2 . z^3 + 4^2 . z^4 + \cdots + n^2 . z^n + \cdots =$

We know,

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1} + \cdots$$

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots + nz^n + \cdots$$

$$\frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = 1^2 + 2^2 . z + 3^2 z^2 + 4^2 . z^3 + \cdots + n^2 . z^{n-1} + \cdots$$

$$z \frac{d}{dz} \frac{z}{(1-z)^2} = 0^2 + 1^2 . z + 2^2 . z^2 + 3^2 . z^3 + 4^2 . z^4 + \cdots + n^2 . z^n + \cdots$$

$$\frac{z(1+z)}{(1-z)^3} = 0^2 + 1^2 z + 2^2 z^2 + 3^2 z^3 + 4^2 z^4 + \cdots + n^2 . z^n + \cdots$$

$$\therefore C(z) = A(z)B(z) = \frac{z(1+z)}{(1-z)^4} = \frac{z}{(1-z)^4} + \frac{z^2}{(1-z)^4}$$

$\therefore c_n$ = coefficient of z^n in $\frac{z(1+z)}{(1-z)^4}$ = coefft of z^n in $\frac{z}{(1-z)^4}$ + coefft of z^n in $\frac{z^2}{(1-z)^4}$

= coefficient of z^{n-1} in $\frac{1}{(1-z)^4}$ + coefficient of z^{n-2} in $\frac{1}{(1-z)^4}$

$$= \frac{n(n+1)(n+2)}{1.2.3} + \frac{(n-1)n(n+1)}{1.2.3} = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore 1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Q. Use generating function to evaluate the sum :

$$1^3+2^3+3^3+\cdots+n^3$$

Hint : Suppose, $a_n = n^3$, $b_n = 1$

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EX: A ship carries 48 flags, 12 each of the colors, red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

(a) How many of these signals use an even number of blue flags and an odd number of black flags?

(b) How many of the signals have at least three white flags or no white flags at all?

(c) How many signals use at least one flag of each color?

(d)

Sol: (a) In the situation mentioned, the exponential generating function:

$$f(z)=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)^2\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]$$

considers all such signals made up of n flags, where $n\geq 1$.

$$f(z)=(e^z)^2\left(\frac{e^z+e^{-z}}{2}\right)\left(\frac{e^z-e^{-z}}{2}\right)=\frac{1}{4}(e^{2z})(e^{2z}-e^{-2z})=\frac{1}{4}[e^{4z}-1]$$

$$=\frac{1}{4}\left[\sum_{i=0}^{\infty}\frac{(4z)^i}{i!}-1\right]=\left(\frac{1}{4}\right)\sum_{i=1}^{\infty}\frac{(4z)^i}{i!}$$

∴ Number of signals made up of 12 flags with an even number of blue flags and an odd number of black flags

$= \text{coefficient of } \frac{z^{12}}{12!} \text{ in } f(z)$

$=\left(\frac{1}{4}\right)(4^{12})=4^{11}.$

(b) The exponential generating function required

$$g(z)=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots\right)\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)^2$$

$$=e^z\left(e^z-z-\frac{z^2}{2!}\right)(e^z)^2=e^{3z}\left(e^z-z-\frac{z^2}{2!}\right)$$

$$=e^{4z}-ze^{3z}-\left(\frac{1}{2}\right)z^2\cdot e^{3z}$$

∴ Required number of signals = Coefficient of $\frac{z^{12}}{12!}$ in $g(z)$

$$=\sum_{i=0}^{\infty}\frac{(4z)^i}{i!}-z\sum_{i=0}^{\infty}\frac{(3z)^i}{i!}-\left(\frac{z^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3z)^i}{i!}\right)$$

$$=4^{11}-12(3^{11})-\left(\frac{1}{2}\right)(12)(11)(3^{10})=10{,}754{,}218$$

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EX: A ship carries 48 flags, 12 each of the colors red, white, blue, and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships.

(a) How many of these signals use an even number of blue flags and an odd number of black flags?

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$\therefore$ Number of signals made up of 12 flags with an even number of blue flags and an odd number of black flags

$$=\text{coefficient of } \frac{z^{12}}{12!} \text{ in } f(z)$$

$$=\left(\frac{1}{4}\right)(4^{12})=4^{11}.$$

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$$g(z)=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots\right)\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)^2$$

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$$=4^{11}-12(3^{11})-\left(\frac{1}{2}\right)(12)(11)(3^{10})=10{,}754{,}218$$

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Ex. A company hires 11 new employees, each of whom is to be assigned to one of four subdivisions. Each subdivision will get at least one new employee. In how many ways can these assignments be made?

Sol. Let the subdivisions be A, B, C, D, then the equivalent problem is to count the number of 11-letter sequences in which there is at least one occurrence of each of the letters A, B, C, and D.

The exponential generating function for these arrangements is

$$f(z) = \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)^4 = (e^z - 1)^4$$
$$= e^{4z} - 4e^{3z} + 6e^{2z} - 4e^z + 1$$

Required number of arrangements = coefficient of $\frac{z^{11}}{11!}$ in $f(z)$

$$= 4^{11} - 4(3^{11}) + 6.(2^{11}) - 4(1^{11})$$

Q. Determine the number of ways to assign each of 25 new employees to the four subdivisions so that each subdivision receives at least 3, but no more than 10, new people.

EGF for the number of ways to arrange n $(n \geq 0)$ letters selected from :

WORD	Arrangements of	EGF: $f(z)$
ENGINE	0, 1, or 2 E's, 0, 1, or 2 N's, 0 or 1 G, & I	$\left[1 + z + \left(\frac{z^2}{2!}\right)\right]^2 [1+z]^2$
HAWAII	0 or 1 H & W; 0, 1, or 2 A's; 0, 1, or 2 I's	$(1+z)^2 \left[1 + \frac{z}{1!} + \frac{z^2}{2!}\right]^2$
MISSISSIPPI	0 or 1 M; 0, 1, or 2 P's; 0, 1, 2, 3 or 4 I's; 0, 1, 2, 3 or 4 S's	$(1+z)\left(1 + \frac{z}{1!} + \frac{z^2}{2!}\right)\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}\right]^2$
ISOMORPHISM	0 or 1 R, P, H; 0, 1, or 2 Is, Ss.	$(1+z)^3\left[1 + \frac{z}{1!} + \frac{z^2}{2!}\right]^4$

Number of arrangements of four of the letters in the word = coeff. of $z^4/4!$ in $f(z)$: EGF.

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