**DEF.** A set $S$ is finite with cardinality $n \in N$ if there is a bijection from the set $\{0, 1, 2, \ldots, n-1\}$ to $S$. A set is infinite if it is not finite.

**DEF.** A set $S$ is infinite if there exists an injection $f : S \to S$ such that $f(S)$ is a proper subset of $S$.

**Theorem:** The set of natural numbers is an infinite set.

Proof: Consider the injection $f : N \to N$ defined by

$$f(x) = 3x$$

The range of $f$ is a subset of the domain of $f$.

**DEF.** A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable.

When an infinite set $S$ is countable, $|S| = \aleph_0$ (aleph null)

**EX.** Show that the set of odd positive integers is a countable set.

**Sol.** Let $A$ = set of odd positive integers.

$f : Z^+ \to A$ by

$f(n) = 2n - 1$ ; suppose, $f(n) = f(m)$

$\Rightarrow 2n - 1 = 2m - 1$. So $n = m$

$\therefore$ If $f(n) = f(m) \Rightarrow n = m$, $f$ is 1-1.

To see that $f$ is onto, suppose $t \in A$. Then $t$ is less than an even integer $2k$, where $k$ is a natural number.

$\therefore t = 2k - 1 = f(k)$, $\therefore$ $f$ is onto

An infinite set is countable if and only if it is possible to list the elements of the in a sequence (indexed by the positive integers)

| $n$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | |
| $f(n)$: | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | ... |

Fig. A One-to-One Correspondence bet $Z^+$ and the set of odd Positive integers.

Q. Prove that the set of positive rational numbers is countable.

DEF. An infinite set X is said to be countable if it can be put into 1-1 correspondence with the set N of natural numbers, and uncountable otherwise.

DEF. A 1-1 function from a set X onto a set Y is often called a 1-1 correspondence (bijection) between X and Y.

$|X| = |Y| \iff$ there is a 1-1 correspondence between X and Y.

$|X| \le |Y| \iff$ there is a 1-1 function from X into Y

EX. $N = \{0, 1, 2, \dots\}$

$f(n) = n+1, \ \forall n \in N$

$m \neq n$ implies $m+1 \neq n+1$

implies $f(m) \neq f(n)$

f is 1-1 function.

The range of f is the set: $\{1, 2, 3, \dots\}$ of positive integers, which is a proper subset of N.

Since N and $\{1, 2, 3, \dots\}$ are in 1-1 correspondence,

$|\{1, 2, 3, \dots\}| = |N| = \aleph_0$

In view of $f: N \to \{1, 2, 3, \dots\}$ being a bijection / 1-1 correspondence, set of positive integers is countable.

EX: $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$: set of all integers

$f: N \to Z$ by

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n+1)/2, & \text{if } n \text{ is odd} \end{cases}$$

f maps N onto Z (verify)

The correspondence.

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & \dots \end{array}$$

$\therefore$ Z is countable

EX. The set Q of rational numbers is countable.

$$Q = \left\{ \frac{a}{b} \,\middle|\, a, b \in Z \text{ and } b > 0 \wedge \gcd\{a, b\} = 1 \right\}$$

List as

$0, -1, 1, -2, -\frac{1}{2}, \frac{1}{2}, 2, -3, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3, \dots$

The position of $\frac{a}{b}$ is determined according to $\max(|a|, b)$ and elements are listed first in order of its value and for two rationals having the same value of $\max(|a|, b)$, we put them in the usual order.

$0, -1, 1$ : 1

$-2, -\frac{1}{2}, \frac{1}{2}, 2$ : 2

$-3, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3$ : 3

$-4, -\frac{4}{3}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, 4, \dots$ etc. : 4

**Ex.** The $Z^+ \times Z^+$ is countable.

**Proof**: We define

$\psi : Z^+ \times Z^+ \longrightarrow Z^+$ by

$f(a,b) = 2^a 3^b$

For $(m,n), (u,v) \in Z^+ \times Z^+$

Suppose

$f(m,n) = f(u,v)$

$\Rightarrow 2^m 3^n = 2^u 3^v$

$\Rightarrow m = u$ and $n = v$

$\Rightarrow (m,n) = (u,v)$

$\therefore f$ is one-to-one

$(Z^+ \times Z^+)$ is countable.

$Z^+ \times Z^+ = \{1, 2, 3, \dots\} \times \{1, 2, 3, \dots\}$

$$\begin{array}{llll}
(1,1), & (1,2) \rightarrow & (1,3), & (1,4), \rightarrow \dots \\
(2,1), & (2,2), & (2,3), & (2,4), \dots \\
(3,1), & (3,2), & (3,3), & (3,4), \dots \\
(4,1), & (4,2), & (4,3), & (4,4), \dots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Arrange the listing of elements $(m,n) \in Z^+ \times Z^+$ as per value of $(m+n)$.

$(1,1)$,

$(2,1), (1,2)$,

$(1,3), (2,2), (3,1)$,

$(4,1), (3,2), (2,3), (1,4)$

$\vdots$ etc

**OR**

$$\begin{array}{llll}
(1,1) & (1,2) & (1,3) & (1,4) \dots \\
(2,1) & (2,2) & (2,3) & (2,4), \dots \\
(3,1) & (3,2) & (3,3) & (3,4) \dots \\
(4,1) & (4,2) & (4,3) & (4,4) \dots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

**Q.** Examine whether $Z^+ \times Z^+ \times Z^+$ is a countable set.

**Theorem**: The union of a countable collection of countable sets is countable.

**Proof**: Let $S_0, S_1, S_2, \dots$ be a countable collection of countable sets and

$S_i = \langle a_{i0}, a_{i1}, a_{i2}, \dots \rangle$

Let $b_0, b_1, \dots$ be the sequence of elements.

$b_0 = a_{00}$

$b_1 = a_{01}$

$b_2 = a_{10}$

$b_3 = a_{02}$

$b_4 = a_{11}$

$b_5 = a_{20}$ and so on.

$$\begin{array}{lllll}
S_0 : (a_{00} & a_{01} & a_{02} & a_{03} & \dots) \\
S_1 : (a_{10} & a_{11} & a_{12} & a_{13} & \dots) \\
S_2 : (a_{20} & a_{21} & a_{22} & a_{23} & \dots) \\
S_3 : (a_{30} & a_{31} & a_{32} & a_{33} & \dots) \\
\vdots & & & &
\end{array}$$

This is an enumeration of the elements of $\bigcup_{i=0}^{\infty} S_i$ and hence $\bigcup_{i=0}^{\infty} S_i$ is countable.

Enumerate the elements as shown in the diagram.

**EX.** The set of positive rational numbers is countable — $\{p/q \mid p, q \in \mathbb{Z}^+\}$.

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

**Sol.** Let us list the positive rational numbers as a sequence:

$$r_1, r_2, r_3, \dots, r_n, \dots$$

We can arrange the positive rational numbers $p/q$ by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on.

$$\begin{array}{l|cccccccc}
\frac{p}{1}: & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \dots \\
\frac{p}{2}: & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \dots \\
\frac{p}{3}: & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \dots \\
\frac{p}{4}: & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \dots \\
\frac{p}{5}: & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \dots \\
\frac{p}{6}: & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & \frac{6}{6} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

First we list the positive rational numbers $\frac{p}{q}$ with $p+q=2$, followed by those with $p+q=3$, followed by those with $p+q=4$, and so on. We do not list a number $\frac{p}{q}$ again that is already listed. Thus we have a sequence:

$$1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, 5, \frac{1}{5}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, 6, \dots$$

Thus all positive rational numbers are listed.

$\therefore$ The set of positive rational numbers is countable.

**EX.** The set of integers that are multiples of 7 is countable.

Sol. Let

$$S = \{\dots, -21, -14, -7, 0, 7, 14, 21, \dots\} = A \cup B$$

We have a sequence:

$$0, 7, -7, +14, -14, 28, -28, \dots$$

Therefore, the set $S$ is countable.

*: $f: \mathbb{Z}^+ \to A$ by $f(n) = 7n$ ✓
& $f: \mathbb{Z}^+ \to B$, $f(n) = -7n$ ✓
A & B both countable
$\therefore S = A \cup B$ countable

Arrange the listing of elements $x \in S$ in order of increasing value of $|x|$.

0
7, −7
14, −14
21, −21
28, −28
⋮ ⋮

# RELATIONS

Let $A$ : set of courses offered by the mathematics department at HBTU.
$B$ : set of mathematics professors at HBTU.

$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ : all possible courses offered by mathematics professors at HBTU

$S$ : the set of students at HBTU.
$T$ : the set of teachers at HBTU.

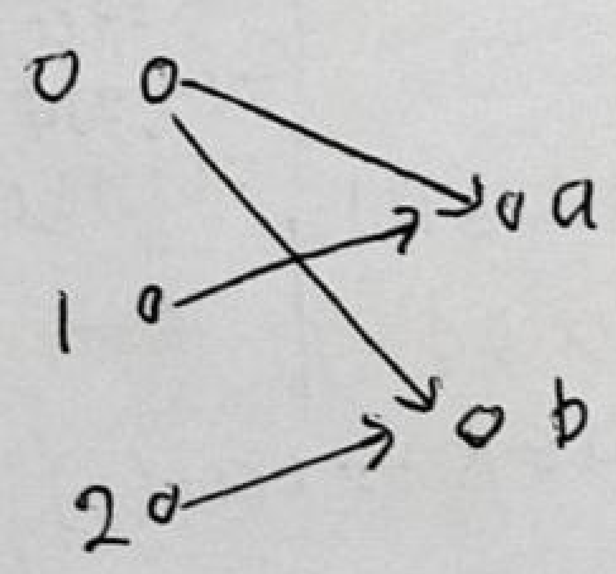
$S \times T = \{(x,y) \mid x \in S \wedge y \in T\} = \{x \text{ is a student of teacher } y\}$

Let $A = \{0,1,2\}$ and $B = \{a,b\}$

$A \times B = \{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b)\}$

$R = \{(0,a),(0,b),(1,a),(2,b)\} \subseteq A \times B$

| $A \times B$ | a | b |
|---|---|---|
| 0 | (0,a) ✓ | (0,b) ✓ |
| 1 | (1,a) ✓ | (1,b) |
| 2 | (2,a) | (2,b) ✓ |



Arrow diagram / Graphical form

| R | a | b |
|---|---|---|
| 0 | x | x |
| 1 | x | |
| 2 | | x |

Tabular form

$$m_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

zero-one matrix

**DEF.** Let $A$ and $B$ be sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.

Any subset $R \subseteq A \times B$ is called a binary relation from $A$ to $B$
$R = \{(a,b) \mid a \in A,\ b \in B\} \subseteq A \times B$.

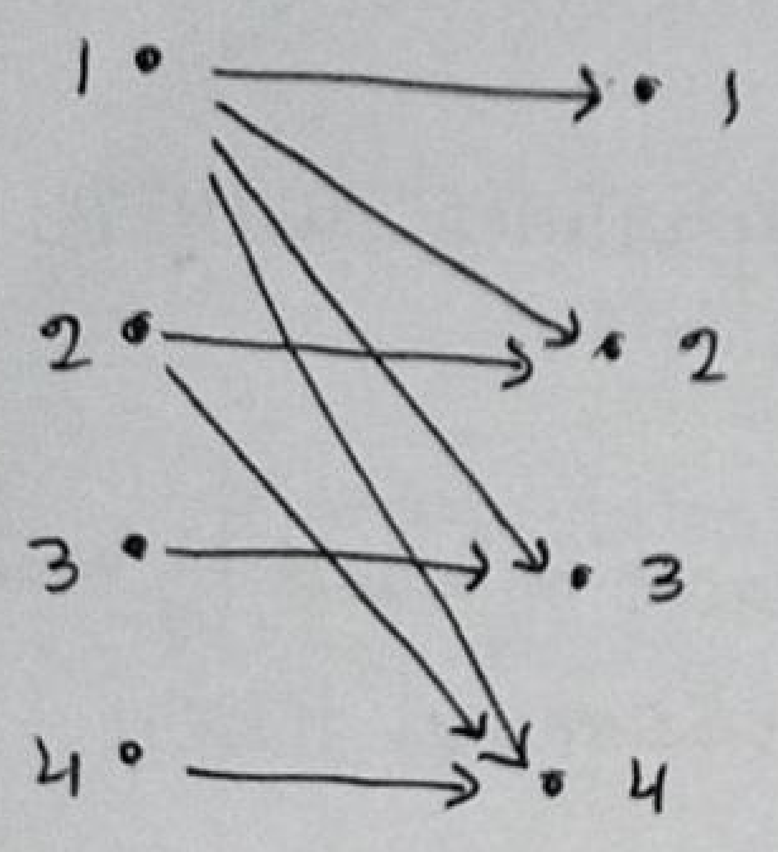
✓ (Inverse relation) $R^{-1}$ from $B$ to $A$
$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Complementary Relation $\overline{R}$
$\overline{R} = \{(a,b) \mid (a,b) \notin R\}$

**DEF.** A relation from $A$ to $A$ i.e. any subset $R \subseteq A \times A$ is called a relation on the set $A$.

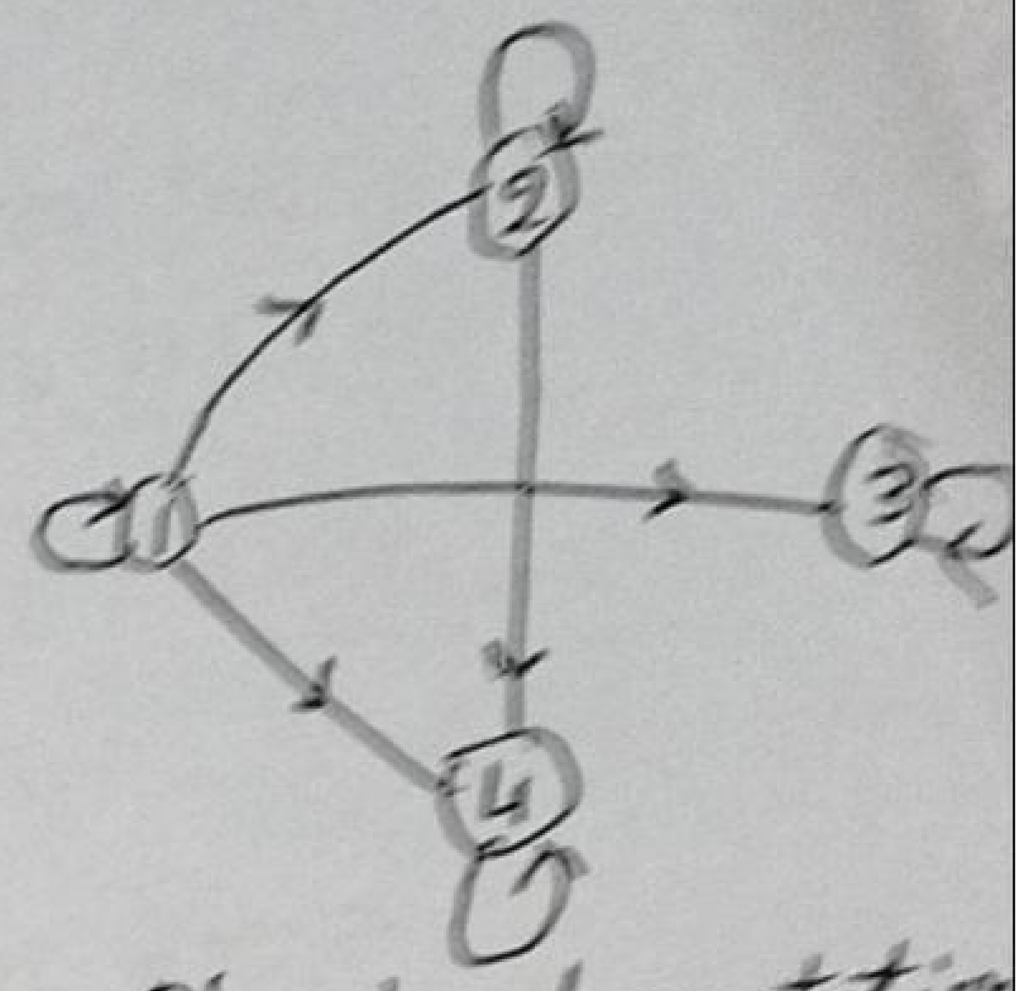
✓ Diagonal relation on $A$
$\Delta = \{(a,a) \mid a \in A\}$

**EX**: $A = \{1,2,3,4\}$
$R = \{(a,b) \mid a \in A \wedge b \in A \wedge a \text{ divides } b\}$
$= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),\cancel{(3,4)},(4,4)\} \subseteq A \times A$




| R | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | x | x | x | x |
| 2 |   | x |   | x |
| 3 |   |   | x |   |
| 4 |   |   |   | x |

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




Digraph representation

REMARK: There are $2^{|A|.|B|}$ binary relations from a set A to set B. and $2^{|A|^2}$ relations on a set A.

A relation R on the set A is

REFLEXIVE if $\forall a ((a,a) \in R)$
Irreflexive if $\forall a ((a,a) \notin R)$
SYMMETRIC if $\forall a \forall b [(a,b) \in R \rightarrow (b,a) \in R]$
Asymmetric if $\forall a \forall b [(a,b) \in R \rightarrow (b,a) \notin R]$
ANTISYMMETRIC if $\forall a \forall b ((a,b) \in R \wedge (b,a) \in R \rightarrow (a=b))$
TRANSITIVE if $\forall a \forall b \forall c ((a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R)$

On $A = \{1,2,3\}$
$R = \{(1,1), (1,2), (3,3), (2,3)\}$
$\because (2,2) \notin R$, not reflexive
$\because (1,1), (3,3) \in R$, not irreflexive

$R = \{(1,2), (2,1), (2,3), (1,1)\}$
$(2,3) \in R$ but $(3,2) \notin R$
not symmetric
$(1,2) \in R, (2,1) \in R$ but $1 \neq 2$
not antisymmetric

Given a finite set A with $|A| = n$. We have $|A \times A| = n^2$
$\therefore A = \{a_1, a_2, a_3, \ldots, a_n\}$
$R \subseteq A \times A$ is defined as $R = \{(a_i, a_j) \mid a_i, a_j \in A\}$

While defining/constructing a relation R on A, every $(a_i, a_j) \in A \times A$ may be included or excluded in R. Thus, there are two choices for each $(a_i, a_j)$ out of $n^2$ elements. By the multiplication rule, there are $\underbrace{2 \times 2 \times \cdots \times 2}_{(n^2 \text{ times})} = 2^{n^2}$ relations on A.

$S_1 = \{(a_i, a_i) \mid 1 \leq i \leq n\} \subseteq R$
$|S_1| = n$

$S_2 = \{(a_i, a_j) \mid i \neq j \text{ for } 1 \leq i, j \leq n\}$
$|S_2| = n^2 - n = n(n-1)$

$A = \{a_1, a_2, \cdots, a_n\}$, we write $A\times A = A_1 \cup A_2$ , $|A\times A| = n^2$

collection of diagonal elts (in Table of $A\times A$): where $A_1 = \{(a_i, a_i) \mid 1 \le i \le n\}$ , $|A_1| = n$

collection of off-diagonal elts: $A_2 = \{(a_i, a_j) \mid 1 \le i, j \le n, i \ne j\}$, $|A_2| = |A\times A| - |A_1|$

$\therefore |A_2| = n^2 - n$

The set $A_2$ contains $(\frac{1}{2})(n^2-1)$ subsets, $S_{ij} = \{(a_i, a_j), (a_j, a_i) \mid 1 \le i, j \le n, i \ne j\}$ $n(n-1)$, an even integer

- While we construct a reflexive relation $R \subseteq A\times A$ on $A$, we include each element of $A_1$ and either inclute or exclude each of the elements in $A_2$, so by the rule of product, there are

$$\underbrace{1\times 1\times \cdots \times 1}_{(n\text{ times})} \times \underbrace{2\times 2\times \cdots \times 2}_{(n^2-n)\text{ times}} = 2^{(n^2-n)} \text{ reflexive relations on } A$$

- In constructing a symmetric relation $R \subseteq A\times A$ on $A$, we have our usual choice of exclusion or inclusion

  for each ordered pair in $A_1$

  for each of the $(\frac{1}{2})(n^2-n)$ subsets $S_{ij}$ $(1 \le i, j \le n)$ taken from $A_2$.

  So by the rule of product there are

$$\underbrace{(2\times 2\times \cdots \times 2)}_{(n\text{ times})} \times \underbrace{(2\times 2\times \cdots \times 2)}_{((\frac{1}{2})(n^2-n)\text{ times})} = 2^n \cdot 2^{\frac{1}{2}(n^2-n)} = 2^{(1/2)(n^2+n)} \text{ symmetric relation on } A$$

- The number of relation $R \subseteq A\times A$ on $A$ that are both reflexive and symmetric

$$\underbrace{(1\times 1\times \cdots \times 1)}_{(n\text{ times})} \times \underbrace{(2\times 2\times \cdots \times 2)}_{(\frac{1}{2}(n^2-n)\text{ times})} = 2^{(1/2)(n^2-n)}$$

- While constructing an antisymmetric relation $R \subseteq A\times A$ on $A$,

  1. Each $(a_i, a_i) \in A_1$ can be either included or excluded with no concern whether or not $R$ is antisymmetric.
  2. For $R$ to remain antisymmetric there are three alternatives for each element $S_{ij}$ from $A_2$

     (a) to include $(a_i, a_j)$ in $R$ and exclude $(a_j, a_i)$

     (b) to include $(a_j, a_i)$ in $R$ and exclude $(a_i, a_j)$

     (c) to include neither $(a_i, a_j)$ nor $(a_j, a_i)$ in $R$

  So by the rule of product, the number of antisymmetric relations on $A$

$$= (2^n)(3^{(n^2-n)/2}), \; n>0.$$

$$A \times A = \begin{array}{c|cccc} & a_1 & a_2 & a_3 & \cdots \\ \hline a_1 & (a_1,a_1) & (a_1,a_2) & (a_1,a_3) & \cdots \\ a_2 & (a_2,a_1) & (a_2,a_2) & (a_2,a_3) & \cdots (a_2, \\ a_3 & (a_3,a_1) & (a_3,a_2) & (a_3,a_3) & \cdots (a_3, \\ \vdots & & & & \\ a_{n-1} & & & & \\ a_n & (a_n,a_1) & (a_n,a_2) & (a_n,a_3) & \cdots (a_n,a \end{array}$$

**EX**: A relation $R \subseteq Z \times Z$ on $Z$ is defined by

$$R = \{(x,y) \mid x \equiv y \pmod 4\} = \{(x,y) \mid (x-y) \text{ is divisible by } 4\}$$

For $\forall x \in Z$, $x - x = 0$ is divisible by $4 \Rightarrow x \equiv x \pmod 4 \Rightarrow xRx$

$\therefore R$ is Reflexive.

For $x, y \in Z$,

Suppose $xRy \Rightarrow x \equiv y \pmod 4 \Rightarrow (x-y) = 4k_1$

$\Rightarrow (y-x) = 4k$ where $k = -k_1 \in Z$

$\Rightarrow y \equiv x \pmod 4$

$\Rightarrow yRx$

$\therefore R$ is symmetric.

For $x, y, z \in Z$, suppose

$xRy \Rightarrow x \equiv y \pmod 4 \Rightarrow x - y = 4m \quad m \in Z$

& $yRz \Rightarrow y \equiv z \pmod 4 \Rightarrow y - z = 4n \quad , n \in Z$

$\Rightarrow x - z = 4(m+n) = 4l, \ l \in Z$

$\Rightarrow x \equiv z \pmod 4$

$\Rightarrow xRz$

$\therefore R$ is transitive

Verify
$[6] = [2] = [-2]$
$[5] = [3]$, $[1] = [1]$
etc

Thus, $R$ is an equivalence relation on $Z$.

$[x]_R = \{y \in Z \mid xRy\} = \{y \in Z \mid x \equiv y \pmod 4\} =$
$= \{y \in Z \mid x - y = 4l\} = \{y \in Z \mid y = x + 4k\}$

$[0]_R = \{4k \mid k \in Z\}$
$[1]_R = \{4k+1 \mid k \in Z\}$
$[2]_R = \{4k+2 \mid k \in Z\}$
$[3] = \{4k+3 \mid k \in Z\}$
etc.

NOTE: $Z = [0] \cup [1] \cup [2] \cup [3]$

# COMPOSITION OF RELATIONS

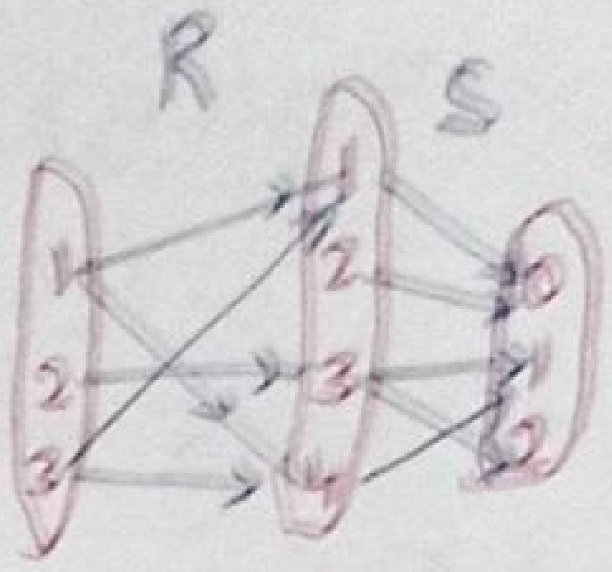
If $R \subseteq A \times B$, $S \subseteq B \times C$, then $S \circ R$ is defined

Consider a relation $R$ from $\{1,2,3\}$ to $\{1,2,3,4\}$ defined by

$$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$$

and a relation $S$ from $\{1,2,3,4\}$ to $\{0,1,2\}$ given by

$$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$$




✓ $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

DEF: If $A$, $B$, and $C$ are set with $R \subseteq A \times B$ and $S \subseteq B \times C$, then the composite relation $R \circ S$ is a relation from $A$ to $C$ defined by

✓ $R \circ S = \{(x, z) \mid x \in A, z \in C$, and there exits $y \in B$ with $(x,y) \in R$, $(y,z) \in S\}$.

## Composing the Parent Relation with Itself:

DEF: Let $R$ be a relation on the set $A$. The powers $R^n$, $n = 1, 2, 3 \cdots$ are defined recursively by

(1) $R^1 = R$

(2) $R^{n+1} = R^n \circ R$, $\forall n \in \mathbb{Z}^+$

| $A \times A$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | (1,1) | (1,2) | (1,3) | (1,4) |
| 2 | (2,1) | (2,2) | (2,3) | (2,4) |
| 3 | (3,1) | (3,2) | (3,3) | (3,4) |
| 4 | (4,1) | (4,2) | (4,3) | (4,4) |

EX: $R = \{(1,1), (2,1), (3,2), (4,3)\}$ on $A = \{1,2,3,4\}$

$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R = \{(1,1), (1,1), (3,1), (4,1)\} = R^3$

etc.

Verify $R^n = R^3$ for $n = 5, 6, 7, \cdots$

$\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

$\overline{R} = A \times A - R$

$R^{-1} = \{(1,1), (1,2), (2,3), (3,4)\}$

EX: On $A = \{1,2,3,4\}$, $R = \{(1,2), (1,3), (2,4), (3,2)\}$, then

$R^2 = \{(1,4), (1,2), (3,4)\}$

$R^3 = \{(1,4)\}$ : $R^n = \phi$, $n \geq 4$.

✓ Closures of Relations:

Smallest

Reflexive: Reflexive closure of $R = R \cup \Delta$

Symmetric: Symmetric Closure of $R = R \cup R^{-1}$

Transitive: Transitive Closure of $R$, $R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Relation

On $A = \{1,2,3\}$

$R = \{(1,2), (2,3), (3,3)\}$

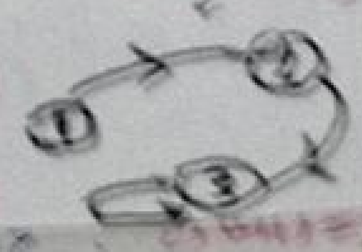
$R^2 = \{(1,3), (2,3), (3,3)\}$

$R^3 = \{(1,3), (2,3), (3,3)\}$

$R^* = R \cup R^2 \cup R^3$

$= \{(1,2), (2,3), (3,3), (1,3)\}$

| $A \times A$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | (1,1) | (1,2) | (1,3) |
| 2 | (2,1) | (2,2) | (2,3) |
| 3 | (3,1) | (3,2) | (3,3) |




$\Delta = \{(1,1), (2,2), (3,3)\}$

$R^{-1} = \{(2,1), (3,2), (3,3)\}$

$R \cup \Delta = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ $R \cup R^{-1} = \{(1,2), (2,3), (3,3), (2,1), (3,2)\}$

EX. Let N be the set of natural numbers. A relation R on $N \times N$ is defin

$(a,b) R (c,d)$ if and only if $ad = bc$

Sol: 1. For $\forall (a,b) \in N\times N$, we have $ab = ba \Rightarrow (a,b)R(a,b)$, So R reflexive

2. For $(a,b), (c,d) \in N\times N$

Let $(a,b)R(c,d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,d)R(a,b)$

$\therefore$ R is symmetric.

3. For $(a,b), (c,d), (e,f) \in N\times N$

Let

$(a,b) R (c,d) \Rightarrow ad = bc \Rightarrow \frac{a}{b} = \frac{c}{d}$

& $(c,d) R (e,f) \Rightarrow cf = de \Rightarrow \frac{c}{d} = \frac{e}{f}$

$\Bigg\} \Rightarrow \frac{a}{b} = \frac{e}{f} \Rightarrow af = be \Rightarrow (a,b)R(e,f)$

Thus $(a,b)R(c,d) \wedge (c,d)R(e,f) \Rightarrow (a,b)R(e,f)$

$\therefore$ R is a transitive relation.

So R is an equivalence Relation.

$[(a,b)]_R = \{(c,d) \mid (a,b)R(c,d)\} = \{(c,d) \mid ad = bc\}$

$= \{(c,d) \mid \frac{c}{d} = \frac{a}{b}\}$

$\therefore [(3,5)]_R = \{(c,d) \mid \frac{c}{d} = \frac{3}{5}\} = \{(c,d) \in N\times N \mid \frac{c}{d} = \frac{3}{5}\}$ | $a,b,c,d \in N$.

$= \{(c,d) \in N\times N \mid d = \frac{5}{3}c\}$

$= \{(3,5), (6,10), (9,15), (12,20), \cdots\}$

Partitions of $S = \{1,2,3,4,5,6\}$

$A_1 = \{1,2,3\}, A_2 = \{4,5\}$

$A_3 = \{6\}$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

Q1. Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$. Define R on A by

$(x_1, y_1) R (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$

(a) Verify that R is an equivalence relation on A.

(b) Determine the equivalence classes: $[(1,3)], [(2,4)]$, and $[(1,1)]$

(c) Determine the partition of A induced by R

Q2. Let R be a relation on the set of integers defined by

$aRb$ if and only if $3a + 4b = 7n$, for some integer n.

Examine R for an equivalence relation.

Theorem: Let R be an equivalence relation on a set A. These ~~elemen~~ statements for elements a and b are equivalent:

(i) $aRb$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \phi$

Scanned by TapScanner

Consider the statements. (True or False; how to prove?)

- In 11-digit number (in decimal system), at least two digits are the same
- In a 27-letter word (in English alphabet) at least two letters are the same.
- At least two of any eight students in this class have birthdays that occur on the same day of the week.
- At least twelve of 78 MCA students have birthdays that occur on the same day of the week.

- $D$ = set of digits in 11-digit number, $|D| = 11$, so $|D| > |R|$
  $R$ = set of digits in decimal number system, $|R| = 10$
  Define $f : D \to R$ by $f(d) = r$, a digit in decimal number system

  We compute, $i = \lceil \frac{|D|}{|R|} \rceil = \lceil \frac{11}{10} \rceil = 2$

  By the Pigeonhole principle, there are at two distinct digits, $d_i \neq d_j$ in $D$ such that $f(d_i) = f(d_j)$

  Therefore, one decimal digit occurs at least two times in 11-digit number.

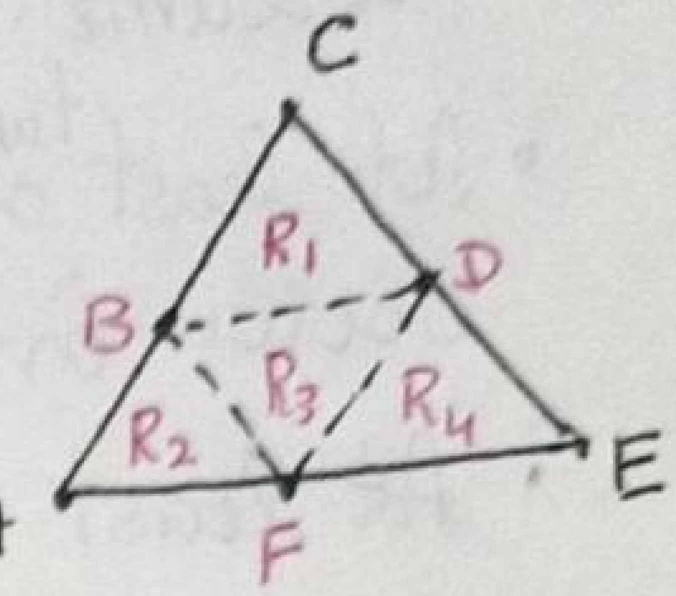
- $D$ = set of letters in 27-letter word
  $R$ = set of letters in English alphabet
  $|D| = 27 > |R| = 26$. Define $f : D \to R$ by $f(d) = r$, a letter in English Alphabet.
  $i = \lceil \frac{27}{26} \rceil = 2$, $\therefore$ there are at least distinct letters $d_i \neq d_j$ in $D$ such that $f(d_i) = f(d_j)$ $(f(d_i), f(d_j) \in R)$

  $\therefore$ Therefore, one English letter appears at least two times in 27-letter word.

- $D$ = Set of students; $|D| > |R|$
  $R$ = set of days in a week
  $f : D \to R$ by $f(d) = d$, a day of the week on which birthday of student d occurs
  $i = \lceil \frac{8}{7} \rceil = 2$
  For $d_i \neq d_j$ in $D$ $f(d_i) = f(d_j)$

- $D$ = set of MCA students, $|D| = 78 > |R| = 7$
  $R$ = set of days in a week
  $f : D \to R$ by $f(d) = \varepsilon$, a day of the week on which the birthday of student d occurs.
  $i = \lceil \frac{78}{7} \rceil = 12$
  At least 12 distinct students $d_i \in D$ s.t. $f(d_1) = f(d_2) = \dots = f(d_{12})$

EX. Triangle ACE is equilateral with $AC = 1$. If five points are selected from the interior of the triangle, there are at least whose distance apart is less than $\frac{1}{2}$.




SOL. $D = \{p_1, p_2, p_3, p_4, p_5\}$ = set of 5 points

$R = \{R_1, R_2, R_3, R_4\}$ = set of 4 regions

$|D| = 5 > |R| = 4$.

$f : D \rightarrow R$ by $f(p_m) = R_k$, a region in which point $p_m$ lies.

$i = \lceil \frac{|D|}{|R|} \rceil = \lceil \frac{5}{4} \rceil = 2$

By the pigeonhole principle there are at least two distinct points $p_m \neq p_n$ such that

$$f(p_m) = f(p_n)$$

$\therefore$ Two distinct points $p_m, p_n$ lies in the same region, so the result.

EX: 14 students of a class appear at End-semester examinations. There are at least two students whose seat numbers differ by a multiple of 13.

SOL: D : set of students ; $R = \{0, 1, 2, \cdots, 12\}$ = set of remainders upon division by 13.

$|D| = 14 > |R| = 13$

$f : D \rightarrow R$ by $f(d) = r$, a remainder when seat number of student d is divided by 13.

$k = \lceil \frac{|D|}{|R|} \rceil = 2$, $\therefore$ By the pigeonhole principle, there are at least two students $d_i \neq d_j$ such that $f(d_i) = f(d_j)$

Let m & n be the seat numbers of students $d_i$ & $d_j$ respectively.

By the Division algorithm / Lemma

$$m = 13k_1 + r_1 \Rightarrow m = 13k_1 + f(d_i)$$
$$n = 13k_2 + r_2 \Rightarrow n = 13k_2 + f(d_j)$$
$$\overline{(m-n) = 13k, \text{ where } k = k_1 - k_2}$$

Hence the result.

EX There are 38 different time periods during which classes at a university can be scheduled. If there are 667 different classes, how many different rooms will be needed?

SOL. Ans. 18

D = set of different classes

R = set of time periods,

$|D| = 667 > |R| = 38$

$f : D \rightarrow R$ by $f(d) = r$, a room in which class d is assigned

$k = \lceil \frac{|D|}{|R|} \rceil = \lceil \frac{667}{38} \rceil = 18$ ✓

By the pigeonhole, there are at least 18 distinct classes $d_i$ s.t.

$f(d_1) = f(d_2) = \cdots = f(d_{18})$

Thus, classes $d_1, d_2, \cdots, d_{18}$ are assigned in

Scanned by TapScanner

**EX.** Use Warshall's algorithm to compute transitive closure of a relation $R$ on the set $A = \{1,2,3,4\}$ given by

$$R = \{(1,1),(1,4),(2,2),(2,3),(3,2),(3,3),(4,1),(4,4)\}$$

**SOL.**

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

K=1

1. Transfer all 1s from $W_0$ to $W_1$
2. Non-zero entries in column 1 : 1, 4; Row 1 : 1, 4

$\therefore W_1$ has 1 in position: (1,1) (1,4) (4,1) (4,4)

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

K=2

1. Transfer all 1s from $W_1$ to $W_2$
Non-zero Entries in column 2 : 2, 3; Row 2 : 2, 3

$\therefore W_2$ has 1 in positions (2,2) (2,3) (3,2) (3,3)

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

K=3

Transfer all 1s from $W_2$ to $W_3$

Non-zero entries in column 3: 2, 3; Row 3 : 2, 3

$\therefore W_3$ has 1 in locations (2,2) (2,3) (3,2) (3,3)

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

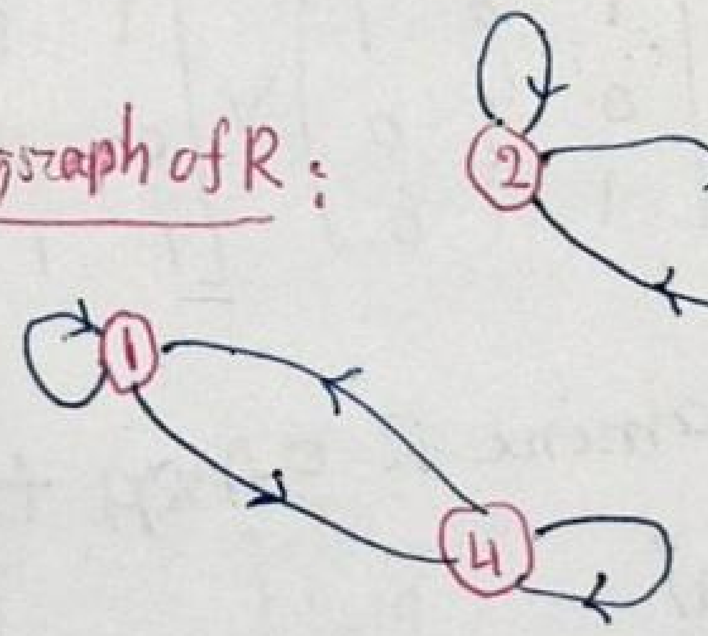
Transfer all 1s from $W_3$ to $W_4$

Non-zero entries in column 4: 1, 4; Row 4 : 1, 4

$\therefore W_4$ has 1 in positions (1,1) (1,4) (4,1) (4,4). ($\because$ already 1, no new entry needed)

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \therefore R^\infty = \{(1,1),(1,4),(2,2),(2,3),(3,2),(3,3),(4,1),(4,4)\}$$

Digraph of R:



**EX.** Find the connectivity relation / transitive closure $R^\infty$ of a relation:
$R = \{(1,2),(2,1),(2,3),(3,4)\}$ on $A = \{1,2,3,4\}$

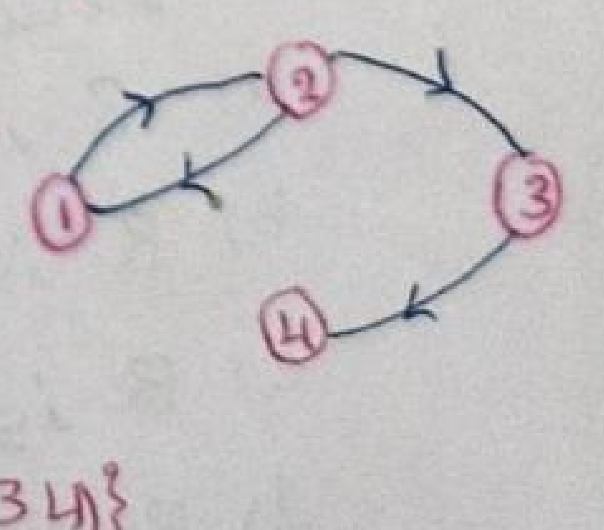
**SOL.**

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \text{ Compute } M_{R^2} = (M_R)^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{R^3} = (M_R)^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^4} = (M_R)^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{R^5} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ Thus } (M_R)^n = \begin{cases} (M_R)^2 & \text{if } n \text{ even} \\ (M_R)^3 & \text{if } n \text{ odd} \end{cases}$$

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

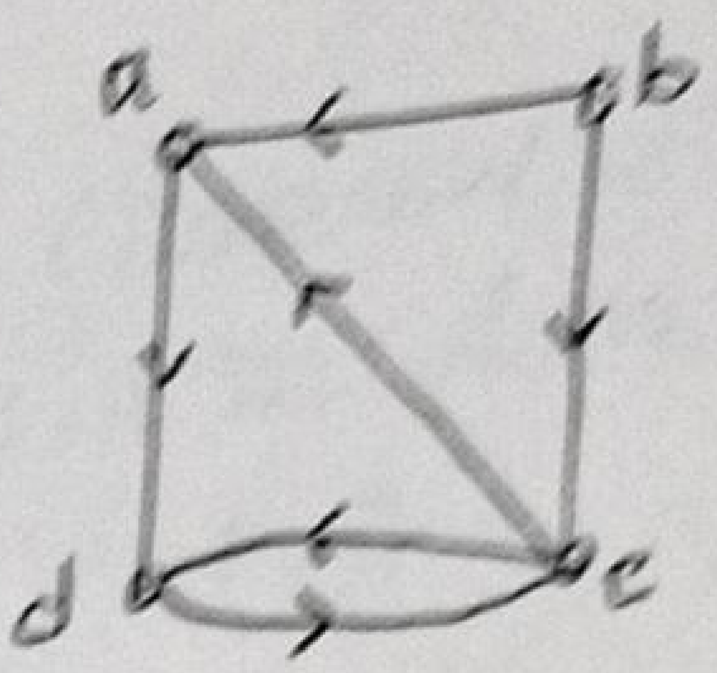
Digraph of R:



$\therefore R^\infty = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$

**Theorem**: Let $M_R$ be the zero-one matrix of the relation $R$ on a set with $n$ elements. Then the zero-one matrix of the transitive closure $R^*$ is
$$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots \vee M_{R^n}.$$

**EX.** A relation R is represented by the directed graph:




$$W_0 = M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{} W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_1$: 2,3
$R_1$: 4

$$\xrightarrow{} W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{} W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{} W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$C_2$: 0
$R_2$: 1,2,4

$C_3$: 2,4
$R_3$: 1,4

$C_4$: 1,2,3,4
$R_4$: 1,3,4

the matrix of the transitive closure

$R^* = R^\infty = \{(a,a),(a,c),(a,d),(b,a),(b,c),(b,d),$
$(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}$

**EX.** Find the zero-one matrix of the transitive closure of the relation R, where

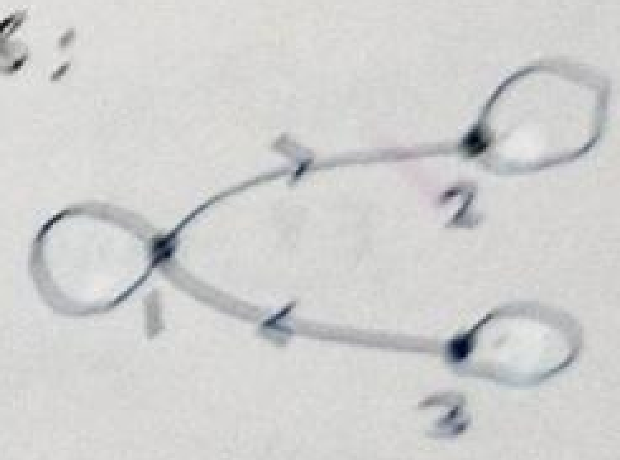
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

**SOL.** $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

**EX.** On a set $A = \{1,2,3\}$, examine $R \subseteq A \times A$ for its properties:

(i) $R = \{(1,1),(1,2),(2,2),(3,3),(3,1)\}$ $\quad M_R = [\quad]$

Reflexive, antisymmetric, neither symmetric nor transitive

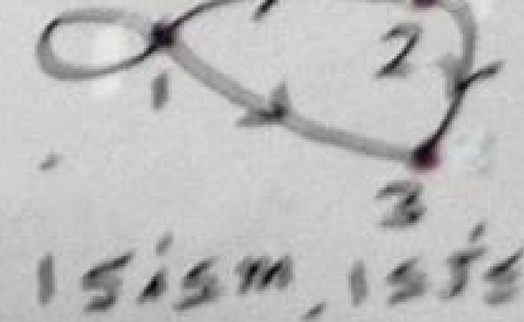



(ii) $R = \{(1,1),(1,2),(2,1),(2,3),(3,2)\}$ $\quad M_R = [\quad]$

Symmetric, not reflexive, antisymmetric, or transitive


(iii) $R = \{(1,1),(1,2),(2,3),(1,3)\}$ $\quad M_R = [\quad]$

Transitive, antisymmetric, not reflexive or symmetric




**DEF:** $E = (e_{ij})_{m \times n}$, $F = (f_{ij})_{m \times n}$ are $m \times n$ (0,1)-matrices.
E precedes, or is less than F, $E \leq F$, if $e_{ij} \leq f_{ij}$, $\forall\ 1 \leq i \leq m$, $1 \leq j \leq n$

**THEOREM:** Given a set A with $|A| = n$, and a relation R on A, let M denote the relation for R. Then,

a) R is reflexive if and only if $I_n \leq M$.

b) R is symmetric if and only if $M = M^{tr}$.

c) R is transitive if and only if $M \cdot M = M^2 \leq M$

d) R is antisymmetric if and only if $M \wedge M^{tr} \leq I_n$

$I_n = (\delta_{ij})_{n \times n}$, not
$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

# n-ary Relations

Consider Relation

3-ary $R \subseteq N \times N \times N$ defined by $R = \{(a,b,c) \mid a < b < c\}$ : degree of relation $R = 3$
Its domains = set of natural numbers.

3-ary $R \subseteq Z \times Z \times Z$ defined by $R = \{(a,b,c) \mid b = a+k \wedge c = a+2k, k \text{ is some integer}\}$
degree of $R = 3$
domains : set of integers

3-ary $R \subseteq Z \times Z \times Z^+$ defined by $R = \{(a,b,m) \mid a \equiv b \pmod m \wedge m \geq 1\}$ : degree of $R = 3$
First two domain $= Z$
III domain $= Z^+$

3-ary $R \subseteq Z \times Z \times Z$ defined by $R = \{(a,b,c) \mid 0 < a < b < c < 5\}$

4-ary $R \subseteq Z^+ \times Z^+ \times Z^+ \times Z^+$ defined by $R = \{(a,b,c,d) \mid abcd = 6\}$

Table: Flights.

| Airline | Flight-number | Gate | Destination | Departure-time |
|---|---|---|---|---|
| Nadir | 122 | 34 | Detroit | 08:10 |
| Acme | 221 | 22 | Denver | 08:17 |
| Acme | 122 | 33 | Anchorage | 08:22 |
| Acme | 323 | 34 | Honolulu | 08:30 |
| Nadir | 199 | 13 | Detroit | 08:47 |
| Acme | 222 | 22 | Denver | 09:10 |
| Nadir | 322 | 34 | Detroit | 09:44 |

Let
A: set of airlines
N: set of flight numbers
G: set of gates
D: set of destinations
T: set of departure times
Then Table represents a relation
$R \subseteq A \times N \times G \times D \times T$

List the 5-tuples in the relation in this table. $\therefore R = \{(a,n,g,d,t) \mid t \geq 0\}$

$R = \{$(Nadir, 122, 34, Detroit, 08:10), (Acme, 221, 22, Denver, 08:17),

**DEF**. Let $A_1, A_2, \ldots, A_n$ be sets. An $n$-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and $n$ is called its degree.

These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases.

EX. Let S: set of student names ; N: set of student ID numbers
F: set of fields of study, G: set of GPA marks.

A relation (4-ary relation), $R \subseteq S \times N \times M \times G$ is represented as set of 4-tuples of the form (STUDENT NAME, ID NUMBER, MAJOR, GPA)

A sample database of six such records is represented by a 4-ary relation: 6(4-tuples)

$R_1$ = {(Aman, 231445, Computer Science, 3.88), (Atal, 888323, Physics, 3.45), (Rahul, 102147, Computer Science, 3.49), (Gagan, 453876, mathematics, 3.45), (Rao, 678543, mathematics, 3.90), (Salil, 786576, Psychology, 2.99)}

TABLE-1. Students.

| Student Name | ID Number | Major | GPA |
|---|---|---|---|
| Aman | 231445 | Computer Science | 3.88 |
| Atal | 888323 | Physics | 3.45 |
| Rahul | 102147 | Computer Science | 3.49 |
| Gagan | 453876 | mathematics | 3.45 |
| Rao | 678543 | mathematics | 3.90 |
| Salil | 786576 | Psychology | 2.99 |

Each column of the table corresponds to an ATTRIBUTES of the database.
Attributes of this database are: Student Name, ID Number, Major, and GPA.

DEF A domain of an n-ary relation is called a primary key when the value of the n-tuple from this domain determines the n-tuple. That is, a domain is a primary key when no two n-tuples in the relation have the same value from this domain.

Ex. In this 4-ary relation, the domain of student names (unique) is a primary key.
the domain of ID numbers (unique) is a primary key.
The domain of major fields of study is not a primary key.
The domain of grade point averages is not a primary key.

DEF When the values of a set of domains (combination of domains) determine an n-tuple in a relation (uniquely), the Cartesian product of these domains is called a composite key.

Ex. In the above 4-ary relation
no two 4-tuples have both the same major and the same GPA.
the cartesian product, $M \times G$ is a COMPOSITE KEY.