GRAPH THEORY

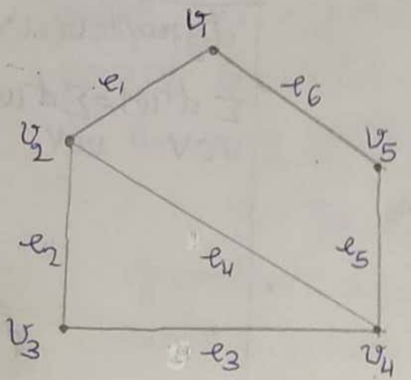


Fig. 1

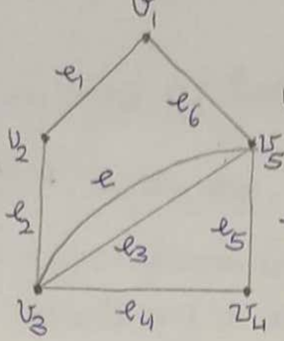


Fig. 2

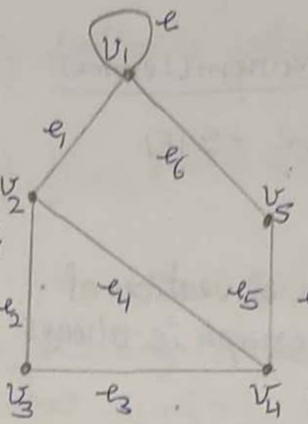


Fig. 3

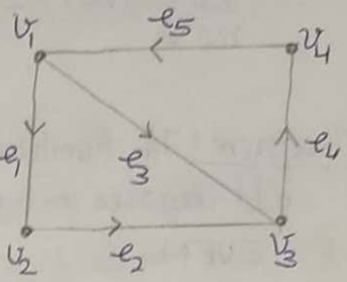


Fig. 4

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$S = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_5\}\}$

: a set/collection of 2-element sets of elements in V.

$\psi : E \longrightarrow S$ by

$\psi(e_1) = \{v_1, v_2\} = \{v_2, v_1\}$

$\psi(e_2) = \{v_2, v_3\} = \{v_3, v_2\}$

etc.

A triplet (V, E, ψ):

Undirected Graph.

An undirected graph (V, E) without multiple edges and self-loops

: SIMPLE GRAPH

An undirected graph (V, E) with parallel/multiple edges

: MULTIGRAPH

An undirected graph (V, E) with self-loop(s) (with or without parallel edges)

: PSEUDO GRAPH.

Let $G = (V, E)$

If $|V|$: finite & $|E|$: finite, then G FINITE Graph.

If $E = \phi$, then G is null graph

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

$S = \{\langle v_1, v_2\rangle, \langle v_1, v_3\rangle, \langle v_2, v_3\rangle, \langle v_3, v_4\rangle, \langle v_1, v_4\rangle\}$

$\subseteq V \times V$

$\psi : E \longrightarrow S$ by

$\psi(e_1) = \langle v_1, v_2\rangle$

$\psi(e_2) = \langle v_2, v_3\rangle$

$\psi(e_3) = \langle v_1, v_3\rangle$

$\psi(e_4) = \langle v_3, v_4\rangle$

$\psi(e_5) = \langle v_4, v_1\rangle$

A triplet (V, E, ψ):

Directed Graph.

If $\psi(e) = \{v_i, v_j\}$, then

v_i & v_j : end vertices of e
: adjacent vertices

e is incident with vertices v_i and v_j

If $\psi(e) = \{v_i, v_i\}$, then e is called a loop

A vertex v that has no incident edges is called an isolated vertex.

If $\psi(e) = \{v_i, v_j\} = \psi(e')$, e and e' are called parallel/multiple edges.

If $\psi(e) = \{v_i, v_j\}$ & $\psi(e') = \{v_j, v_k\}$ then e and e' are adjacent edges.

Degree(v) = deg(v),
= Number of edges incident with v

If deg(v) = 1, then v : pendant vertex

If deg(v) = 0, v : isolated vertex.

If $\psi(e) = \langle v_i, v_j\rangle$

v_i & v_j : adjacent vertices

v_i : initial vertex

v_j : terminating vertex

Edge e is incident with vertices v_i, v_j

v_i is adjacent to v_j whereas v_j is adjacent to v_i.

Out-degree

$d^+(v)$ = Number of edges incident out of v

In-degree

$d^-(v)$ = Number of edges incident into v.

The Handshaking Theorem (Lemma):

$$\sum_{v \in V} \deg(v) = 2e = 2|E|$$

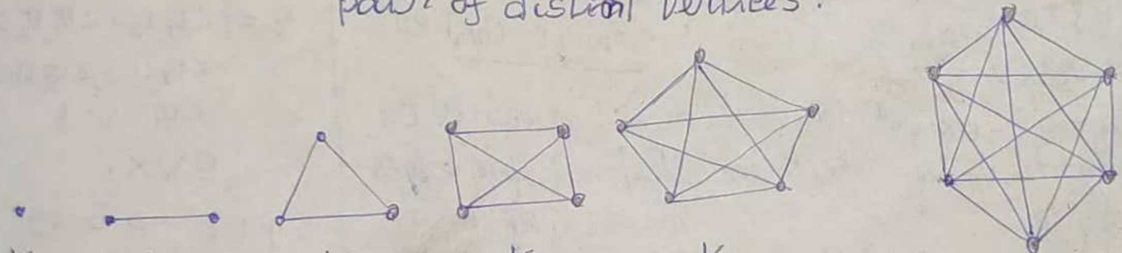
Theorem: The number of vertices of odd degree in a graph is always EVEN.

Theorem: In a digraph $G=(V,E)$,

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$$

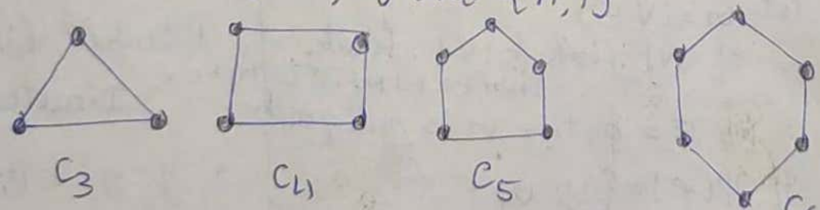
Regular Graph: If $\deg(v) = r$, $\forall v \in V$, then $G=(V,E)$ is called r-regular graph.

Complete Graph, K_n: A simple graph that contains exactly one edge between each pair of distinct vertices.



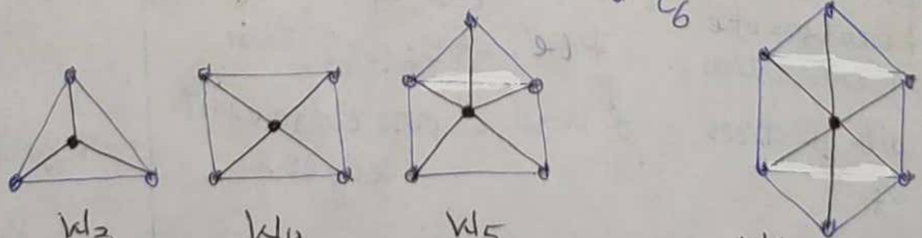
K_1 K_2 K_3 K_4 K_5 K_6

Cycle. The cycle C_n, $n \geq 3$, consists of n vertices $1, 2, \ldots, n$ and edges $\{1,2\}$, $\{2,3\}$, ..., $\{n-1, n\}$ and $\{n, 1\}$



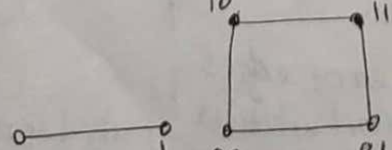
C_3 C_4 C_5 C_6

Wheel, W_n



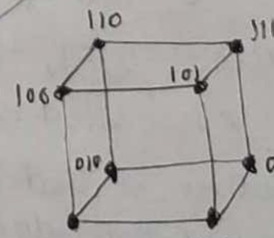
W_3 W_4 W_5 W_6

n-Cube: Q_n

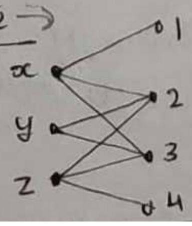


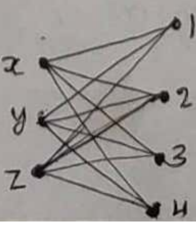
Q_1 Q_2

Q_3:



Bipartite →

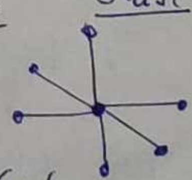




: $K_{3,4}$

← Complete Bipartite $K_{m,n}$

star



$K_{1,6}$

Walk: Let x, y be (not necessarily distinct) vertices in an undirected graph $G=(V,E)$.

An x-y walk in G is a (loop-free) finite alternating sequence:

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y and involving n edges $e_i = \{x_{i-1}, x_i\}$, where $1 \le i \le n$.

Length of the walk = the number of edges in the walk $= n$.

Any x-y walk where $x=y$ (and $n>1$) is called a closed walk. Otherwise, the walk is called open.

Note that a walk may repeat both vertices and edges.

$e_i \neq e_j, i \neq j$ If no edge in the x-y walk is repeated, then the walk is called an x-y trail. A closed x-x trail is called a circuit.

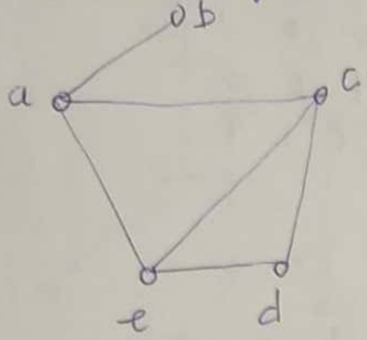
$x_i \neq x_j, i \neq j, i,j = 0,n$ If no vertex of the x-y walk occurs more than once, then the walk is called an x-y path. A closed x-x path is called the cycle.

Theorem: Let $G=(V,E)$ be an undirected graph with $a,b \in V$, $a \neq b$. If there exists a trail (in G) from a to b, then there is a path (in G) from a to b.

DEF. Let $G=(V,E)$ be an undirected graph.
If there is a path between any two distinct vertices of G, then G is called connected.
A graph that is not connected is called disconnected.

Theorem: An undirected graph $G=(V,E)$ is disconnected if and only if V can be partitioned into at least two subsets V_1, V_2 such that there is no edge in E of the form $\{x,y\}$ where $x \in V_1$ any $y \in V_2$.

Representing Graphs:



A simple Graph

:

Adjacency List:

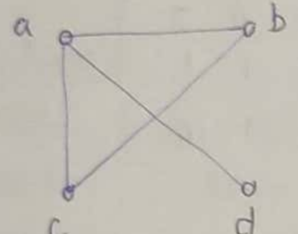
Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

&

Adjacency Matrix:

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$a_{ij} = a_{ji}$



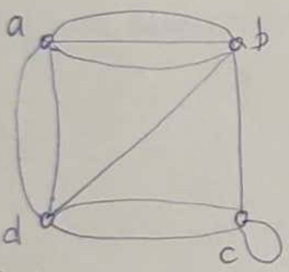
Simple Graph

:

Vertex	adjacent vertices
a	b, c, d
b	a, c
c	a, b
d	a

&

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



A Pseudograph

:

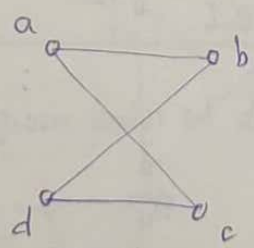
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

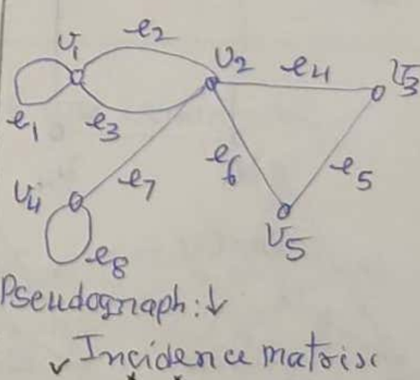
Adjacency matrix

Ordering of vertices: a, b, c, d

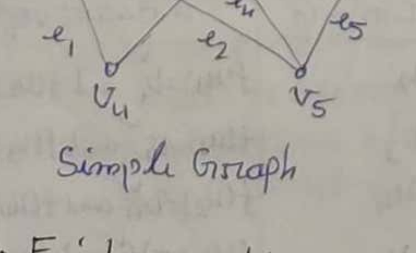
:



Simple Graph



Pseudograph: ↓



Simple Graph

:

Incidence Matrix:

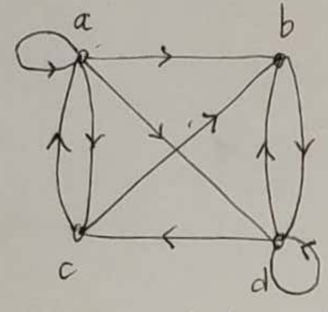
$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Q. Find an adjacency matrix for each of the graphs:
a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

: Parallel edges:
Self-loops:



Directed graph

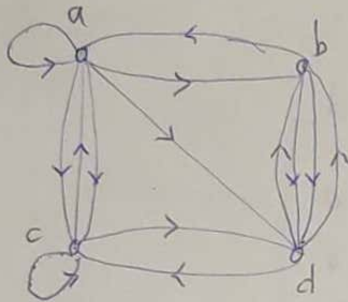
Adjacency List:

vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

&

Adjacency Matrix:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0 & 0 & 0 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 1 & 1 & 1 \end{array}$$



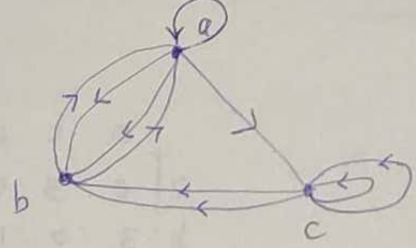
vertex	Terminal vertices

&

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 2 & 1 \\ b & 1 & 0 & 0 & 2 \\ c & 1 & 0 & 1 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$$

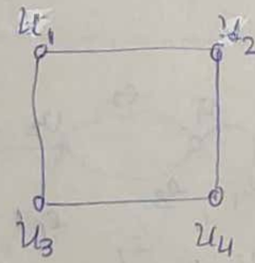
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

Adjacency matrix

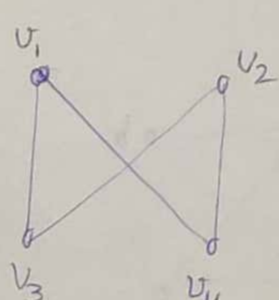


Digraph.

EX:



$G=(V,E)$



$H=(W,F)$

Define a function $f: V \longrightarrow W$ by

$f(u_1) = v_1$

$f(u_2) = v_4$

$f(u_3) = v_3$

$f(u_4) = v_2$

(i) one-to-one correspondence between V and W

(ii) ? correspondence preserves adjacency.

$$A(G) = \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 1 & 0 \\ u_2 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_4 & 0 & 1 & 1 & 0 \end{array}$$

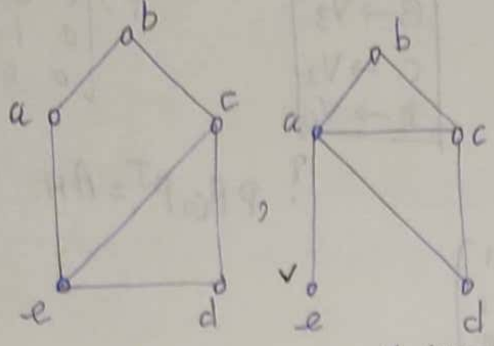
$$A(H) = \begin{array}{c|cccc} & v_1 & v_4 & v_3 & v_2 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 1 & 0 \end{array}$$

Identical

$G=(V,E)$ and $H=(W,F)$ are ISOMORPHIC.

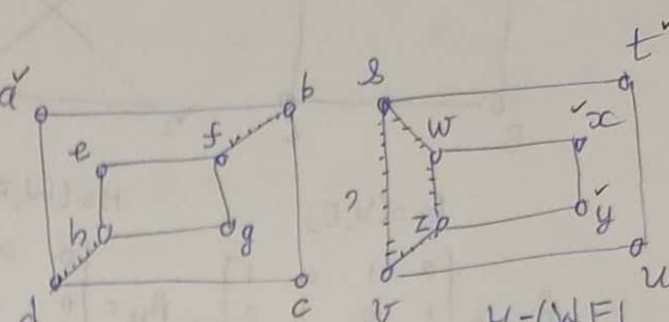
adjacent vertices in G	adjacent vertices in H
u_1 and u_2	$f(u_1)=v_1$ and $f(u_2)=v_4$
u_1 and u_3	$f(u_1)=v_1$ and $f(u_3)=v_3$
u_2 and u_4	$f(u_2)=v_4$ and $f(u_4)=v_2$
u_3 and u_4	$f(u_3)=v_3$ and $f(u_4)=v_2$

EX: Determine whether the graphs shown in Fig. 1 and Fig. 2 are isomorphic.



$G=(V,E)$ $H=(W,F)$

Fig. 1



$G=(V,E)$ $H=(W,F)$

Fig. 2

Solution:

(i) $|V|=$ $|W|=$

(ii) $|E|=$ $|F|=$

(iii) Degree sequence of G:

Degree sequence of H:

Solution:

(i) $|V|=$ $|W|=$

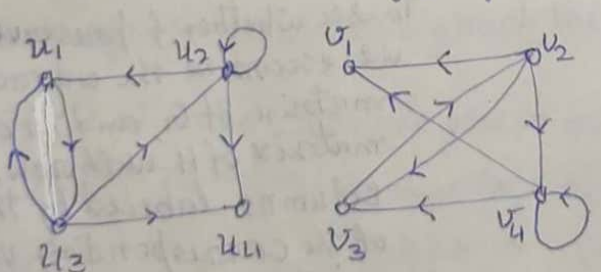
(ii) $|E|=$ $|F|=$

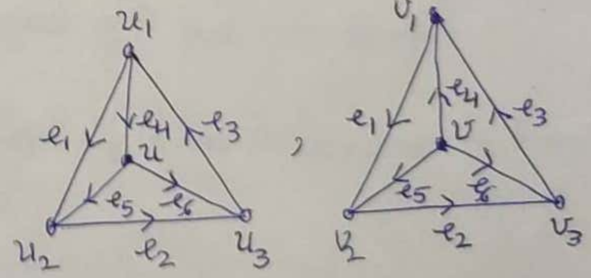
(iii) Degree sequence of G:

Degree sequence of H:

subgraphs formed by/made up of vertices of degree 3 and the edges connecting are not ISOMORPHIC.

EX: Determine whether the given pair of directed graphs are isomorphic.





Sol: To be isomorphic

1. Corresponding undirected graphs must be isomorphic
2. The directions of the corresponding edges must also agree.

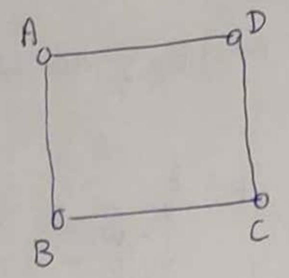
$d^+(u)=2$; $d^+(v)=3$

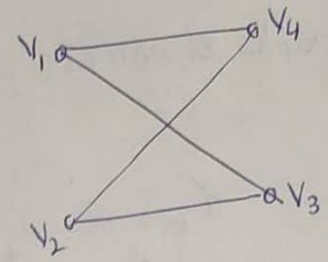
$d^-(u)=1$; $d^-(v)=0$

ISOMORPHIC

NOT ISOMORPHIC

EX:





MAPPING Assume

$A \to V_1$
$B \to V_3$
$C \to V_2$
$D \to V_4$

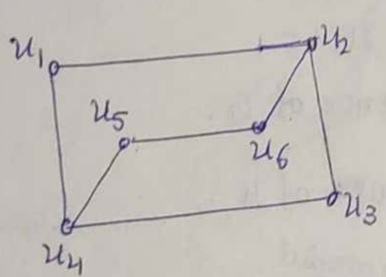
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

? $P A_G P^T = A_H$

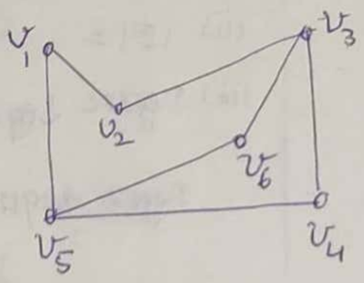
$G = (V, E)$ $\qquad$ $H = (W, F)$

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad A_H = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

EX: Determine whether the following pair of graphs are isomorphic.



$G = (V, E)$



$H = (W, F)$

(1) $|V| =$ $\qquad$ $|W| =$

(2) $|E| =$ $\qquad$ $|F| =$

(3) Degree sequence of G:

Degree sequence of H:

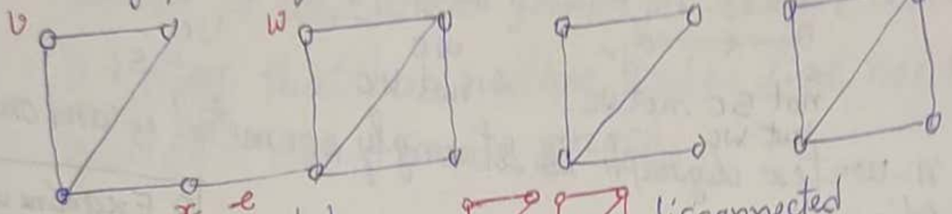
(4) $f : V \to W$ by

$f(u_1) = v_6$
$f(u_2) = v_3$
$f(u_3) = v_4$
$f(u_4) = v_5$
$f(u_5) = v_1$
$f(u_6) = v_2$

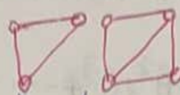
To see whether f preserves edges, we examine the adjacency matrix of G, and the adjacency matrix of H with rows and columns labeled by the images of the corresponding vertices in G.

Connectedness in Undirected Graphs:

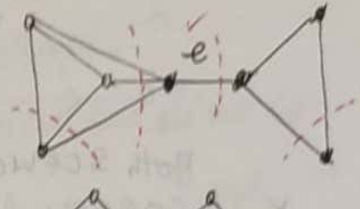
DEF. An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph. Otherwise disconnected.

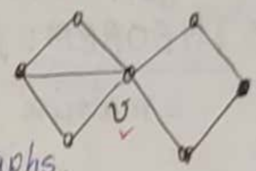


connected



disconnected





A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

Theorem: A graph is disconnected if and only its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2

Theorem: If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Theorem: If G is a bipartite graph, then each cycle of G has even length.

Theorem: Let G be a simple graph on n vertices. If G has k components, then the number m of edges of graph G satisfies

$$(n-k) \leq m \leq (n-k)(n-k+1)/2$$

Corollary: Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.

Q. Determine $k(G)$ and $\lambda(G)$ for each of : (i) C_6 (ii) W_6 (iii) $K_{4,7}$ (iv) Q_4.

A disconnecting set in a connected graph is a set of edges whose removal disconnects G.

A disconnecting set, no proper subset of which is a disconnecting set is called a ✓cutset. A cutset with only one edge is called a bridge.

Edge connectivity, $\lambda(G)$ = size of the smallest cutset in G. G is k-edge connected if $\lambda(G) \geq k$.

A separating set in a connected graph G is a set of vertices whose deletion disconnects G. A separating set with only one vertex is called a CUT-VERTEX.

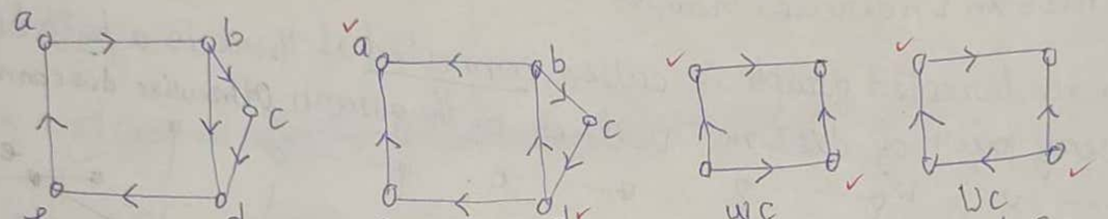
(Vertex) connectivity $k(G)$ = size of small separating set in G. G is k-connected if $k(G) \geq k$

Connectedness in Directed Graphs:

DEF. A directed graph is <u>strongly connected</u> if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

DEF. A directed graph is <u>weakly connected</u> if there is a path between every two vertices in the underlying undirected graph.

DEF. A digraph is unilaterally connected if for every pair of vertices one is reachable from the other



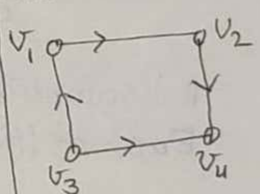
Both SC & WC | not SC, not WC. but WC | WC not UC | UC not SC

✓ **THEOREM:** An n-vertex digraph is strongly connected if and only if the matrix M defined by

$$M = A + A^2 + A^3 + \cdots + A^n$$

has no zero entry, A is the adjacency matrix.

Q. Examine whether....

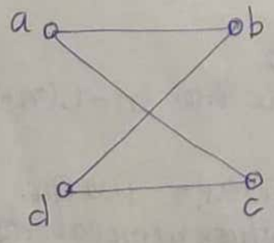


is strongly connected.

Counting Paths Between Vertices:

✓ **Theorem:** Let G be a graph with adjacency matrix A with respect to the ordering $1, 2, 3, \ldots, n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from i to j, where r is a positive integer, equals the (i,j)th entry of A^r.

EX. Determine the number of paths of length four from a to d in the simple graph:



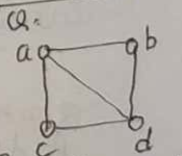
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix};$$

The number of paths of length four from a to d is the $(1,4)$th entry of A^4:

$$A^2 = \begin{bmatrix} \quad \end{bmatrix}; A^3 = \begin{bmatrix} \quad \end{bmatrix}; A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

? $X = A + A^2 + A^3 = [\quad]$

Q.



? $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

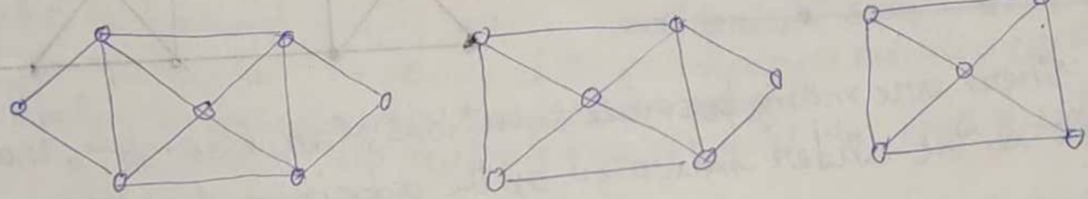
✓ **THEOREM:** If A is the adjacency matrix of an undirected graph G with n vertices, and

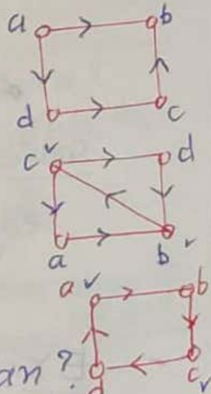
$$X = A + A^2 + A^3 + \cdots + A^{n-1}$$

Then, G is disconnected iff and only if there exists at least one entry in matrix that is zero.

EULER GRAPHS

- A closed walk in a graph that contains every edge of the graph exactly once is called an Euler line/Euler circuit. and a graph that consists of an Euler line is called an Euler graph.
- A open walk in a graph that includes (or traces or covers) all edges of the graph without retracing any edge is called a unicursal line or an open Euler or an Euler path.
  A (connected) graph that has a unicursal line will be termed/called a unicursal graph or semi-Euler graph.
- A graph that has neither Euler line nor unicursal line is called non-Euler graph.





Q. Discuss Königsberg bridges problem.

Q. Which of the following graphs are Eulerian? semi-Eulerian?
(i) K_5 (ii) $K_{2,3}$ (iii) the graph of the cube (iv) the graph of the octahedron (v) the Petersen graph.

Q. Examine each of the following for an Euler graph.
(i) K_n (ii) $K_{m,n}$ (iii) W_n (iv) Q_k (v) Platonic graphs.

<u>Theorem</u>: A connected graph G an Euler graph if and only if all vertices of G are of even degree.

<u>Theorem</u>. A connected graph G is an Euler graph if and only if it can be decomposed into circuits/its set of edges can be split up into disjoint cycles.

<u>Corollary</u>: A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

<u>Theorem</u>: In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Fleury's Algorithm :

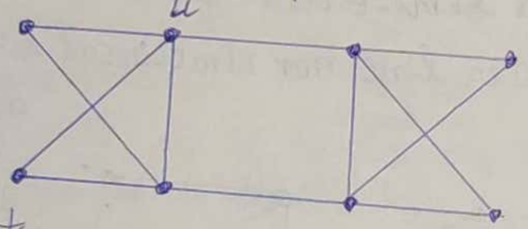
Theorem: Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian line of G.

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules.:

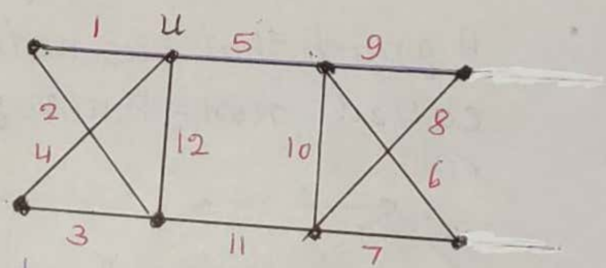
(i) erase the edges as they are traversed, and if any isolated vertices result, erase them too;

(ii) at each stage, use a bridge only if there is no alternative.

EX: Use Fleury's algorithm to produce an Eulerian trail / line / circuit for the graph.

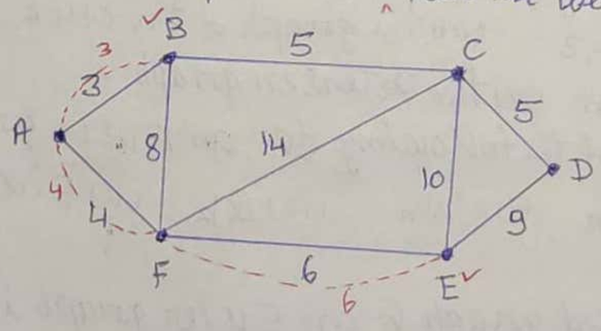


Solution →

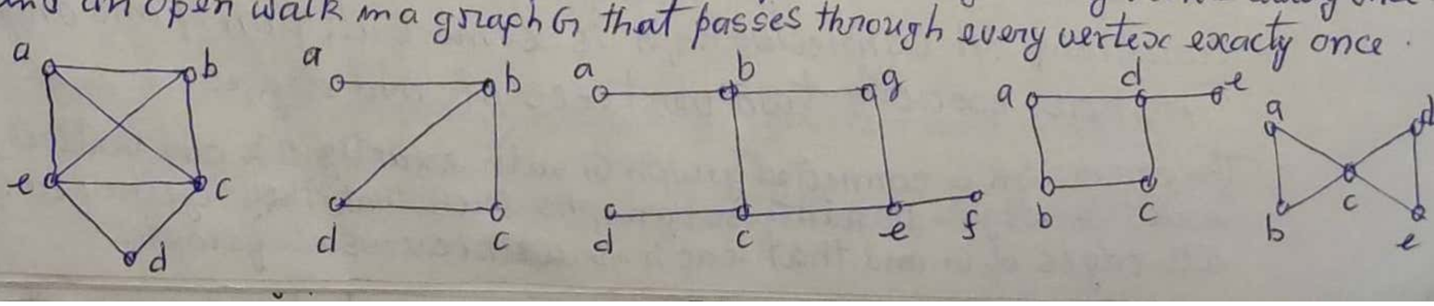


Solution: There are many possible solutions; for example, traverse the edges in the order indicated by the adjoining diagram.

EX: Solve the Chinese postman problem for the weighted graph:



EX. Find a closed walk in a graph G that passes through every vertex exactly once.
Find an open walk in a graph G that passes through every vertex exactly once.

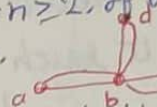


HAMILTONIAN CIRCUITS AND PATHS:

- A Hamiltonian circuit in a connected graph G is defined as a closed walk that traverses / visits every vertex of G exactly once, except of course, the starting vertex, at which the walk also terminates. A graph with a Hamiltonian circuit is called Hamiltonian graph.
- An open walk in a connected graph G (without self-loop & parallel edges) that traverses every vertex of G exactly once is called a Hamiltonian path. A graph which contains a Hamiltonian path is called a semi-Hamiltonian graph.
- A connected graph which is neither Hamiltonian nor semi-Hamiltonian is called non-Hamiltonian.

Theorem: Let $G=(V,E)$ be a loop-free graph with $|V|=n\geq 2$. If $\deg(x)+\deg(y)\geq n-1$ for all $x,y\in V$, $x\neq y$ (non-adjacent), then G has a Hamilton path.

Theorem: Let $G=(V,E)$ be a loop-free graph with $|V|=n\geq 2$. If $\deg(v)\geq (n-1)/2$ for all $v\in V$, then G has a Hamilton path.



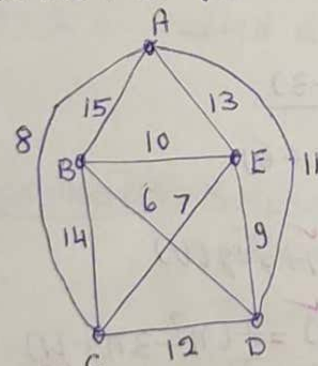
$\deg(x)+\deg(y)\geq 4>3=4-1$

→ Q. Which of the following graphs are Hamiltonian? semi-Hamiltonian?
(i) K_5 (ii) $K_{2,3}$ (iii) the graph of the octahedron, (iv) K_6 (v) the 4-cube, Q_4.

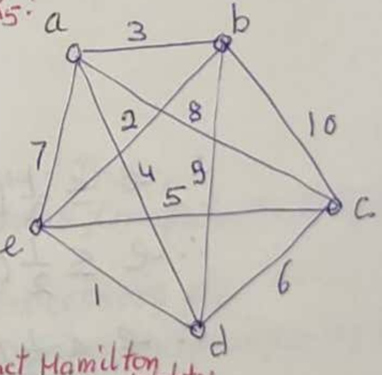
→ Q. Examine the following for Hamiltonian graph?
(i) K_n, (ii) $K_{m,n}$ (iii) Platonic graphs, (iv) W_n, (v) the k-cube, Q_k.

EX. Solve the TSP for the weighted graph. | No. of different Hamilton circuits in K_5 or permutations $=\frac{(5-1)!}{2}=\underline{\underline{12}}$

Weighted K_5



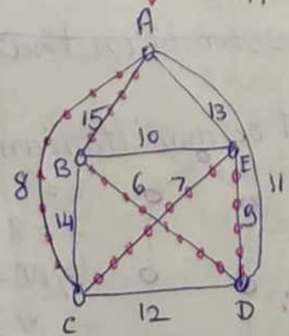
Weighted K_5:

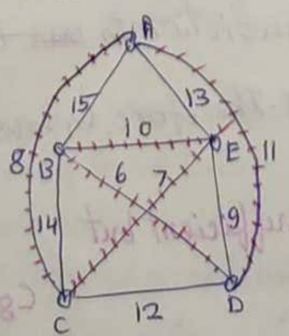


Method of Nearest-Neighbourhood:
(Approx. Solution)

List all distinct Hamilton circuits and their weights:

Permutation	Weight	Permutation	Weight
A, B, C, D, E, A	63		





Total distance = 115 Approx.

Total distance = 42 Correct Sol.

ORE'S THEOREM. Let G be a simple connected graph with $n \geq 3$ verti[ces]
then G is Hamiltonian if

$$\boxed{\deg(u) + \deg(v) \geq n}$$

for every pair of non-adjacent vertices u and v.

DIRAC'S THEOREM. A simple connected graph with $n \geq 3$ vertices is Hamiltonian if

$$\boxed{\deg(v) \geq \frac{n}{2}}, \; \cancel{\forall v} \text{ for every } v \text{ in } G.$$

Corollarly: The connected graph G with $n \geq 3$ vertices has a Hamiltonian circuit provided the number of edges in G

$$\boxed{e \geq \frac{1}{2}(n^2 - 3n + 6)}$$

Proof. If possible, let the graph G be non-Hamiltonian. Then by Dirac's theorem, there will exist a pair of non-adjacent vertices u and v such that

$$\deg(u) + \deg(v) \leq n - 1$$

Let H be the subgraph of G obtained by deleting the u and v from G. the graph H will have $(n-2)$ vertices and $e - \deg(u) - \deg(v)$ edges. the maximum number of edges in H can be ${}^{n-2}C_2$.

$$\therefore [e - \deg(u) - \deg(v)] \leq {}^{n-2}C_2$$
$$= \frac{(n-2)(n-3)}{2}$$
$$= \frac{1}{2}(n^2 - 5n + 6)$$

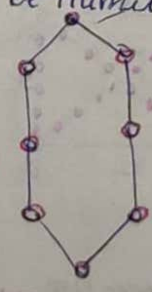
$$\therefore e \leq \frac{1}{2}(n^2 - 5n + 6) + \deg(u) + \deg(v)$$
$$\therefore e \leq \frac{1}{2}(n^2 - 5n + 6) + (n-1) = \frac{1}{2}(n^2 - 3n + 4)$$
$$\therefore e < \frac{1}{2}(n^2 - 3n + 6)$$

which is a contradiction to ~~our assumption that~~

$\therefore$ Our assumption is wrong. Therefore G must be Hamiltonian.

Illustration (Given conditions are sufficient but not necessary)

C_8:



$n = 8$
$e = 8$
$\deg(v) = 2$
$\forall v$.

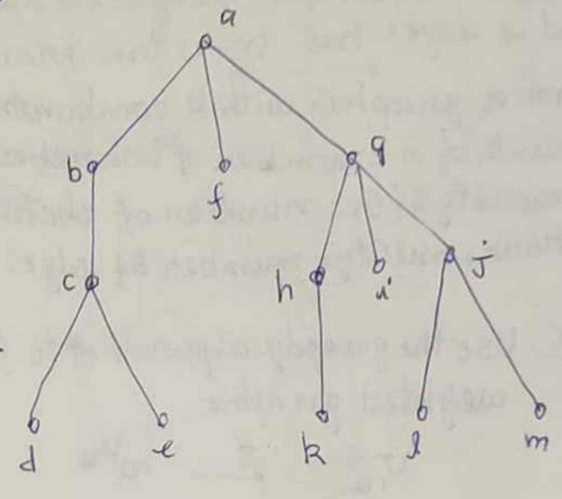
THEOREM: Let D be a strongly connected digraph with n vertices. If $\text{outdeg}(v) \geq n/2$ and $\text{indeg}(v) \geq n/2$ for each vertex v, then G is Hamiltonian.

TREE: A tree is a connected undirected graph with no circuits

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

DEF. A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

EX: Given a Rooted tree T: with root a



The parent of c is b. The children of g are h, i, and j. The siblings of h are i and j. The ancestors of e are c, b, and a. The descendants of b are c, d, and e. The internal vertices are a, b, c, g, h, and j. The leaves are d, e, f, i, k, l, and m.

DEF: A rooted tree is called an m-ary tree if every internal vertex has no more than m children. The tree is a full m-ary tree if every internal vertex has exactly m children. An m-ary tree with $m=2$ is called a binary tree.

Trees as Models: Saturated Hydrocarbons and Trees, Representing Organizations; Computer File Systems; Tree-Connected Parallel Processors etc

Applications of Trees: Binary search trees, Decision Trees, Huffman coding, Game Trees etc.

Properties of Trees:

Theorem: There is one and only one path between every pair of vertices in a tree, T.

Theorem: If in a graph G there is one and only one path between every pair of vertices, G is tree.

Theorem: A tree with n vertices has n-1 edges.

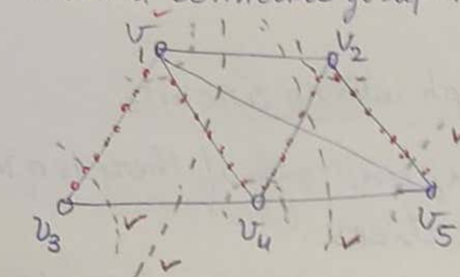
Theorem: Any connected graph with n vertices and n-1 edges is a tree.

Theorem: A graph is a tree if and only if it is minimally connected.

Theorem: A graph G with n vertices, n-1 edges, and no circuits is connected.

Spanning Trees:

Given a connected graph:



Branches: $\{v_1, v_3\}$ $\{v_1, v_4\}$ $\{v_2 v_4\}$ $\{v_2 v_5\}$

Fundamental cutset: $\{\{v_1,v_3\},\{v_3,v_4\}\}$ $\{\{v_2,v_5\},\{v_4,v_5\}\}$

Chords: $\{v_1, v_2\}$ $\{v_3, v_4\}$ $\{v_4, v_5\}$

Fundamental circuit: v_1, v_2, v_4, v_1 v_4, v_5, v_2, v_4

Theorem: Every connected graph has at least one spanning tree.

Theorem: With respect to any of its spanning tree, a connected graph of n vertices and e edges has $(n-1)$ tree branches and $(e-n+1)$ chords.

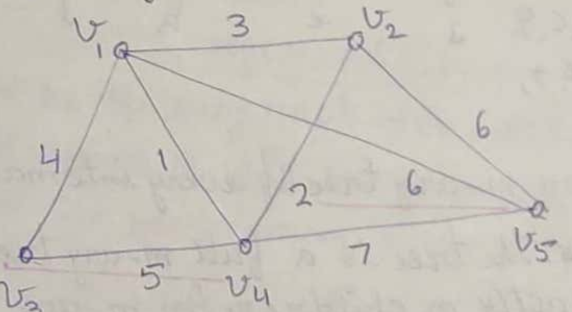
For a graph G with k components, cutset rank $= n-k$; cycle rank nullity $\mu = e-n+k$

rank of G = number of branches in any spanning tree (or forest) of G = Number of fundamental cutsets

nullity of G = number of chords in G = Number of fundamental circuits.

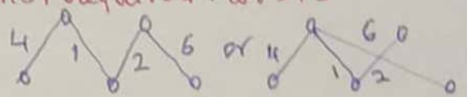
rank + nullity = number of edges in G.

EX. Use the greedy algorithm to find a minimum weight spanning tree for the weighted graph:

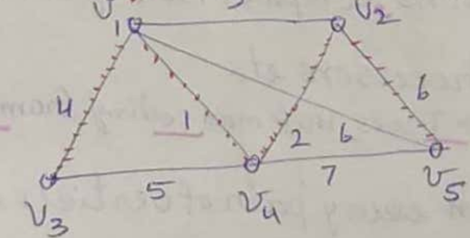


method-1: $w\{v_1,v_2\}=3$ ~~III~~ × (creates circuit)
$w\{v_1, v_3\}=4$ IV ✓
$w\{v_1, v_4\}=1$ I ✓
$w\{v_1, v_5\}=6$ ⟩ or VI ✓
$w\{v_2, v_5\}=6$
$w\{v_3, v_4\}=5$ ~~V~~ × (creates ckt)
$w\{v_4, v_5\}=7$
$w\{v_2, v_4\}=2$ II ✓

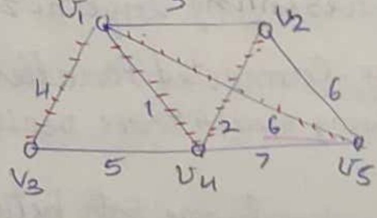
MST required: wt = 13



method-II:



OR

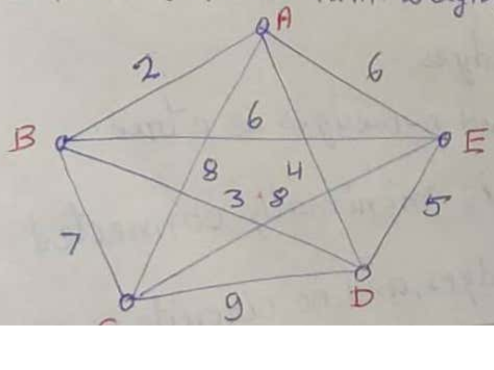


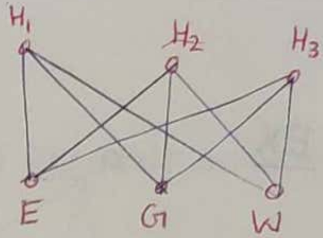
method-II (matrix version)

$$\begin{array}{c|ccccc} & \mathrm{I}\ v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & - & 3 & 4 & 1 & 6 \\ v_2 & 3 & - & \infty & 2 & 6 \\ v_3 & 4 & \infty & - & 5 & \infty \\ v_4 & 1 & 2 & 5 & - & 7 \\ v_5 & 6 & 6 & \infty & 7 & - \end{array}$$

or

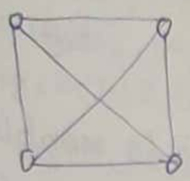
Q. Find a minimum-weight spanning tree for the weight graph: (MST: minimal spanning tree)



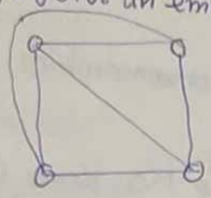


PLANAR GRAPHS:

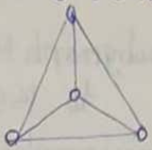
DEF: A graph is called planar if it can be drawn in the plane or on the surface of a sphere without any edges crossing. Such a drawing is called a planar representation or plane drawing of G or an embedding of G in the plane.



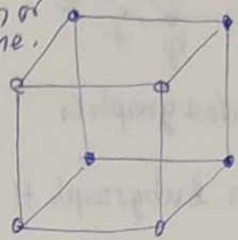
K_4



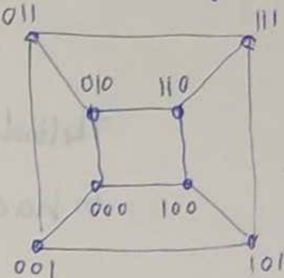
K_4 (plane graph) (redrawn)



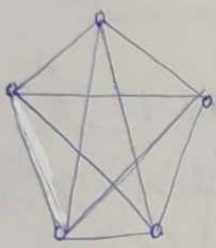
K_4 (plane graph) (Redrawn) ;



Q_3

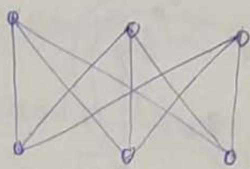


Q_3: Planar representation of Q_3 (plane graph)



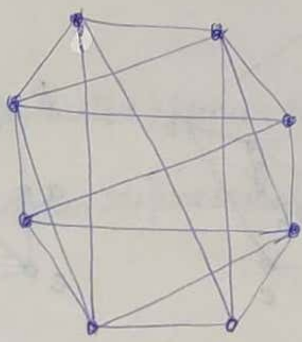
K_5

$\nu(K_5)=1$
$\theta(K_5)=2$



$K_{3,3}$

$\nu(K_{3,3})=1$
$\theta(K_{3,3})=2$



Theorem: $K_{3,3}$ and K_5 are non-planar.

DEF. Let $G=(V,E)$ be a loop-free undirected graph, where $E\neq\phi$. An elementary subdivision of G results when an edge $e=\{u,w\}$ is removed from G and then the edges $\{u,v\}, \{v,w\}$ are added to $G-e$, where $v\notin V$.

DEF. The loop-free undirected graphs $G_1=(V_1,E_1)$ and $G_2=(V_2 E_2)$ are called HOMEOMORPHIC if they are isomorphic or if they can both be obtained from the same loop-free undirected graph H by a sequence of elementary subdivisions (or by inserting new vertices of degree 2 into its edges). For example, any two cycle graphs are homeomorphic.

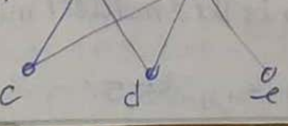
THEOREM (Kuratowski, 1930): A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

OR

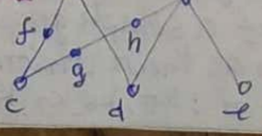
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Homeomorphic Graphs G_1, G_2, G_3
empty seq. of el. subdivisions,

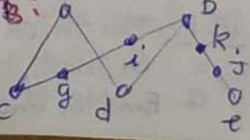
G_1:



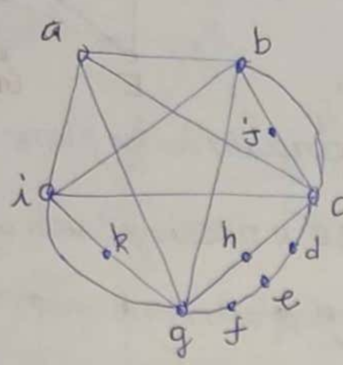
G_2:



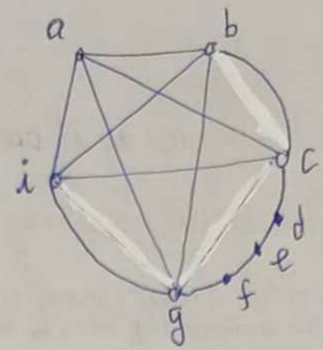
G_3:



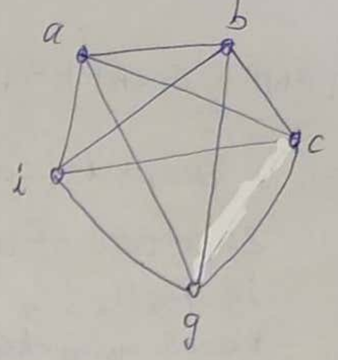
EX.



The Undirected graph G



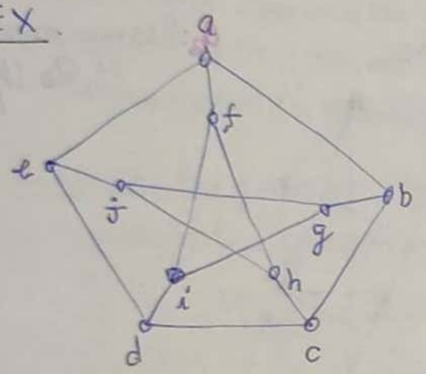
A subgraph H homeomorphic to K_5



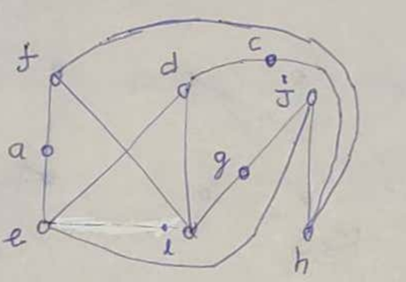
K_5

G has a subgraph H homeomorphic to K_5. Hence G is nonplanar.

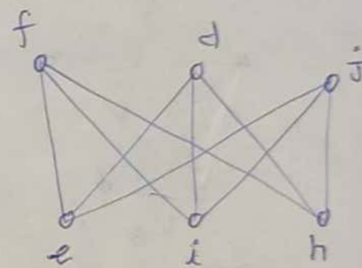
EX.



(a) Petersen graph: G



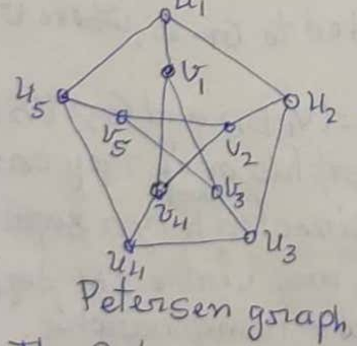
(b) H: Subgraph of G, homeomorphic to $K_{3,3}$



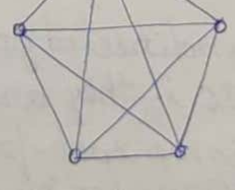
(c) $K_{3,3}$

THEOREM: A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

EX.



Petersen graph



K_5

The Petersen graph is contractible to K_5 (contract the five 'spokes' joining the inner and outer 5-cycles)

∴ The Petersen graph is nonplanar.

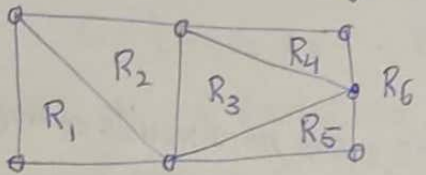
DEF: The crossing number $cr(G)$ of a simple graph G is the minimum number of crossings that can occur when G is drawn in the plane. $cr(K_5)=1=cr(K_{3,3})$

✓ Q. Determine the crossing number of: K_5, K_6, K_7, $K_{3,4}$, $K_{4,4}$, $K_{5,5}$.

DEF. The thickness of a simple graph G is the smallest number of planar subgraphs of G that have G as their union. $\theta(K_5)=2=\theta(K_{3,3})$

✓ Q. Find the thickness of: K_5, K_6, K_7, $K_{3,4}$, $K_{4,4}$, $K_{5,5}$.

- A planar representation of a graph splits the plane into REGIONS, including an unbounded region.



$Deg(R_1)=3$, $Deg(R_2)=3$
$Deg(R_3)=3$, $Deg(R_4)=3$
$Deg(R_5)=3$, $Deg(R_6)=7$

$\sum_{i=1}^{6} Deg(R_i) = 22 = 2\times 11 = 2e$

$\therefore \sum_{i=1}^{n} Deg(R_i) = 2e$

THEOREM (EULER'S FORMULA): Let G be a connected planar simple graph with e edges, v vertices and r regions in a planar representation of G. Then

$$\boxed{v - e + r = 2}$$

Proof: Try yourself.

Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $\boxed{e \leq 3v-6}$ ✓

Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof: If G has one or two vertices, the result is true.
If G has at least three vertices, then $e \leq 3v-6$, so $2e \leq 6v-12$
If the degree of every vertex were at least six i.e. $deg(v_i) \geq 6$.
$2e = \sum_{v\in V} deg(v)$, then $2e \geq 6v$
↳ contradicts the inequality: $2e \leq 6v-12$

It follows that there must be a vertex with degree no greater than five.

Ex: show that K_5 is nonplanar.

Sol. The graph K_5 has five vertices and 10 edges.
$e = 10$ and $3v-6 = 3\times 5 - 6 = 9$ → $10 \leq 9$ is contradiction to
If K_5 were planar, then
$e \leq 3v-6$
But $10 \nleq 9$ i.e. $10 \nleq 3\times 5-6$ i.e. $e \leq 3v-6$ is not satisfied. Therefore K_5 non-planar

G planar → $e \leq 3v-6$
$\equiv$ $e > 3v-6$ → G nonplanar (method of contraposition)

For K_5
$10 > 3\times 5 - 6$ i.e. $10 > 9$
$\therefore$ K_5 nonplanar.

Corollary 3. If a connected planar simple graph has e edges and v vertices with $v \geq 3$, and no circuits of length three, then

$$\boxed{e \leq 2v-4}$$

EX: show that $K_{3,3}$ is nonplanar.

SOL. $K_{3,3}$ has six vertices and nine edges. Also, $K_{3,3}$ has no circuits of length three.

$e = 9$; $2v-4 = 2 \times 6 - 4 = 8$

$\boxed{9 < 8 \text{ is a contradiction}}$

G is planar $\longrightarrow e \leq 2v-4$

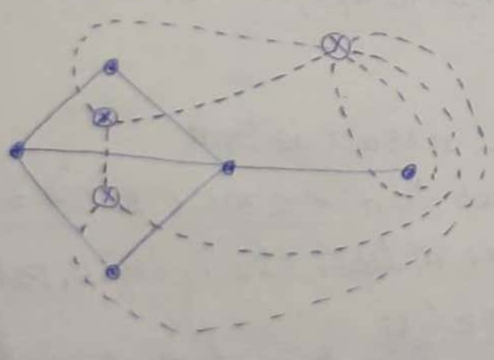
$\equiv e > 2v-4 \longrightarrow$ G is nonplanar

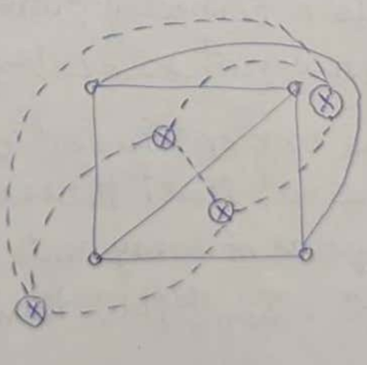
For $K_{3,3}$

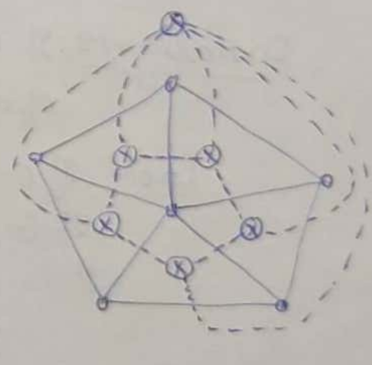
$e = 9 > 8 = 2v-4$ is satisfied.

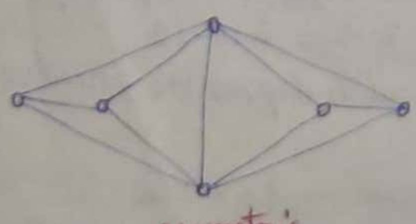
$\therefore K_{3,3}$ is nonplanar.

GEOMETRIC DUAL:









Q. Find the geometric dual of (i) a wheel, W_n, (ii) the cube graph, (iii) the dodecahedron graph.

23/01/2021

GRAPH COLOURING

DEF. Let $G=(V,E)$ be an undirected graph with no multiple edges & loops and $C=\{c_1, c_2, c_3, \ldots, c_n\}$ a set of colors.

A function $f: V \to C$ is called vertex colouring (colouring(v)) of graph G if $f(v_i) \neq f(v_j)$, for adjacent vertices $v_i, v_j \in V$.

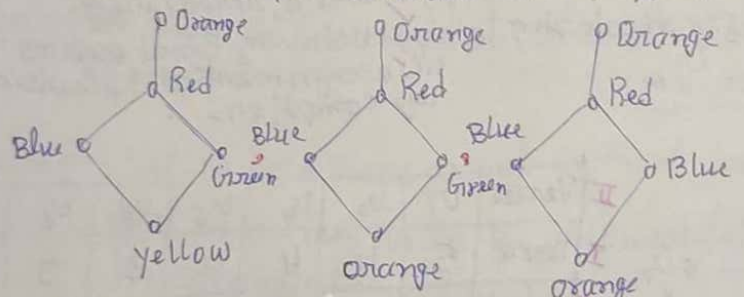
A colouring(v) is called proper coloring if any two adjacent vertices have different colors. The minimum number of colours needed for a proper colouring of a graph G is called the chromatic number of G. $\chi(G)=k$.

G is k-colourable if one of k colors is assigned to each vertex so that adjacent vertices have different colours.

If G is k-colourable but not (k−1) colourable, then G is k-chromatic, $\chi(G)=k$.

Thus, G is k-colourable if $\chi(G) \leq k$

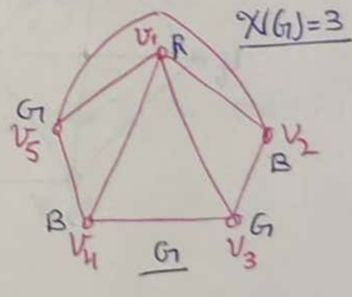
G is k-chromatic if $\chi(G)=k$.



5 colors required, 4 colours required, 3 colours required

$\therefore \chi(G)=3$

G is 3-chromatic.



Standard Results:

(a) $\chi(G)=1$ iff G is a null graph, (b) $\chi(K_n)=n$

(c) $\chi(G)=2$ iff G is a non-null bipartite graph i.e. $\chi(K_{m,n})=2$.

(d) For $n \geq 3$ $\chi(C_n)=\begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$ (e) For $n \geq 3$ $\chi(W_n)=\begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$

Theorem: If G is a simple graph with largest vertex-degree Δ, then G is $(\Delta+1)$-colourable.

Theorem (Brooks, 1941): If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is $\Delta (\geq 3)$, then G is Δ-colourable.

Q. Find the chromatic number of:
(i) each of the Platonic graphs, (ii) $K_{r,s,t}$ (iii) Q_k

Application of Vertex Colourings:

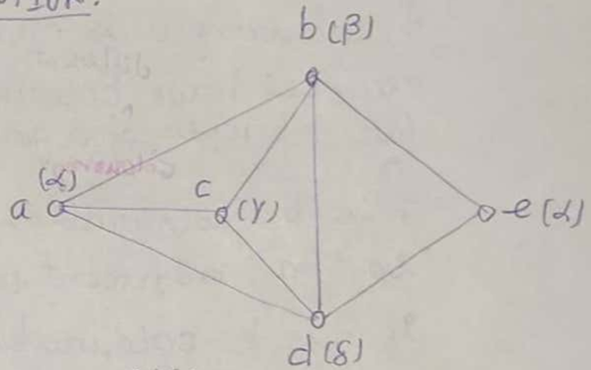
EX: suppose that a chemist wishes to store five chemicals a, b, c, d, and e in various areas of a warehouse. Some of these chemicals react violently when in contact, and so must be kept in separate areas.

In the following table, an asterisk indicates those pairs of chemicals that must be separated.

	a	b	c	d	e
a	–	*	*	*	–
b	*	–	*	*	*
c	*	*	–	*	–
d	*	*	*	–	*
e	–	*	–	*	–

How many areas are needed?

SOLUTION:

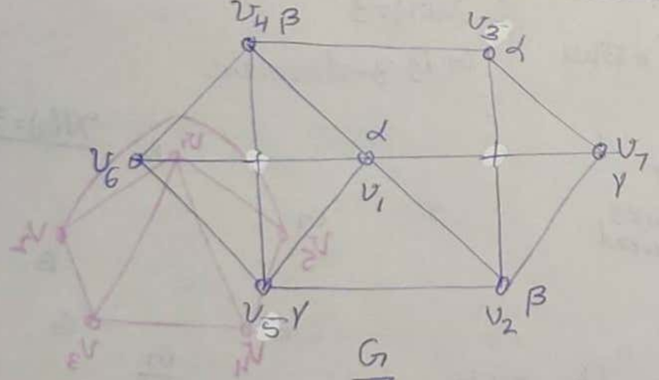


$\chi(G) = 4. \Rightarrow$ 4 areas are needed.

SOL: For example, chemicals a and e can be stored in area α, and chemicals b, c and d can be stored in areas β, γ and δ, respectively.

Q: Illustrate applications of graph coloring to (i) Scheduling Final exams
(ii) assignment of television channels
(iii) compilers.

Welch and Powell Algorithm:



II Vertex	v_1	v_2	v_4	v_5	v_3	v_6	v_7
I Degree	5	4	4	4	3	3	3
III Colour	α	β	β	γ	α	δ	γ

$\chi(G) = 4$

Theorem: Every simple planar graph is 6-colourable.

Theorem: Every simple planar graph is 5-colourable.

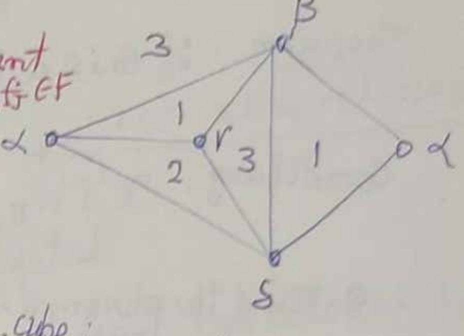
Theorem (Four Color Problem)
Every simple planar graph is 4-colourable.

Colouring Maps:

We define a MAP to be 3-connected plane graph (so a map contains no cut-sets with 1 or 2 edges and in particular no vertices of degree 1 or 2).

A map is defined to be k-colourable(f) if its faces can be coloured with k colours so that no two faces with a boundary edge in common have the same colour. $\phi : F \longrightarrow C$, $\phi(f_i) \neq \phi(f_j)$ for adjacent faces $f_i, f_j \in F$

EX: The map is 3-colourable(f), 4-colourable(v).



Q. Find the minimum number of colours needed to colour the faces of each of the Platonic graphs: tetrahedron, octahedron, cube; icosahedron, dodecahedron; so that neighbouring faces are coloured differently.

Q. Give an example of a plane graph that is both 2-colourable(f) and 2-colourable(v).

Theorem: A map G is 2-colourable(f) if and only if G is an Eulerian graph.

Theorem: Let G be a plane graph without loops, and let G^* be a geometric dual of G. Then G is k-colourable(v) if and only if G^* is k-colourable(f).

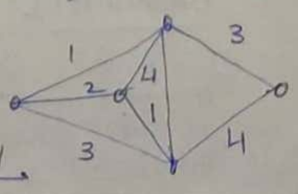
Corollary: The four-colour theorem for maps is equivalent to the four-colour theorem for planar graphs.

Colouring edges:

A graph G is k-colourable(e) (or k-edge colourable) if its edges can be coloured with k colours so that no two adjacent edges have the same colour. $\psi : E \longrightarrow C$, $\psi(e_i) \neq \psi(e_j)$, for adjacent edges $e_i, e_j \in E$.

If G is k-colourable(e) but not (k−1)-colourable(e), then the chromatic index of G is k. and we write $\chi'(G) = k$.

$\chi'(C_n) = \begin{cases} 2 & \text{for even } n; \\ 3 & \text{for odd } n \end{cases}$ $\chi'(K_n) = n-1$; $n \geq 4$ | $\chi'(G) = 4$



Theorem (Vizing's Theorem; 1964): If G is a simple graph with largest vertex degree Δ, then $\Delta \le \chi'(G) \le \Delta+1$.

Theorem: $\chi'(K_n) = n$ if n is odd $(n \ne 1)$, and $\chi'(K_n) = n-1$ if n is even.

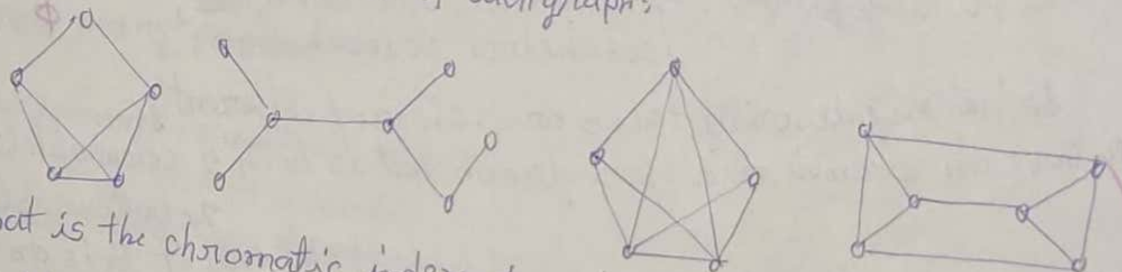
Theorem: The four-colour theorem is equivalent to the statement that

$$\chi'(G) = 3 \text{ for each cubic map.}$$

Theorem: If G is a bipartite graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$.

Corollary: $\chi'(K_{r,s}) = \max(r, s)$

Q. Find the chromatic index of each graph:

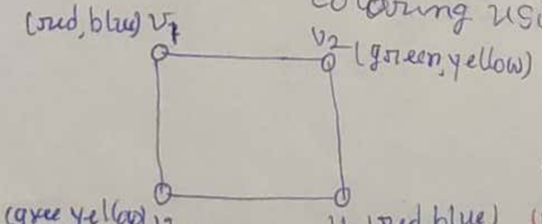


Q. What is the chromatic index of each of the Platonic graphs?

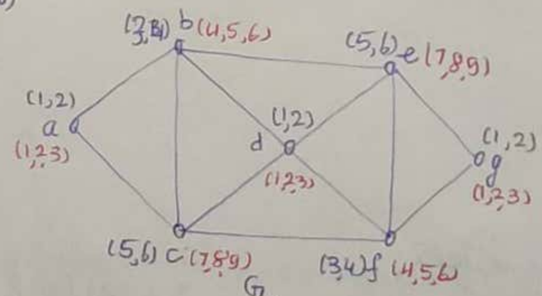
Q. Find the edge chromatic number of (a) K_n, (b) $K_{m,n}$ (c) C_n (d) W_n.

DEF: A k-tuple coloring of a graph G is an assignment of a set of k different colors to each of the vertices of G such that no two adjacent vertices are assigned a common color.

$\chi_k(G)$: the smallest positive integer n such that G has a k-tuple coloring using n colors



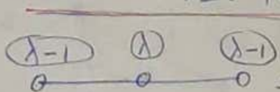
$\chi_2(C_4) = 4$.



$\chi_2(G) = 6$; $\chi_3(G) = 9$

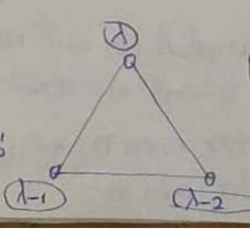
Q. Find: a) $\chi_2(K_3)$, b) $\chi_2(K_4)$ c) $\chi_2(W_4)$ d) $\chi_2(C_5)$, e) $\chi_2(K_{3,4})$, f) $\chi_3(K_5)$ g) $\chi_3(C_5)$, h) $\chi_3(K_{4,3})$

CHROMATIC POLYNOMIALS:

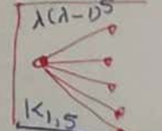


$P_{P_3}(\lambda) = P_G(\lambda) = \lambda(\lambda-1)^2$

For any tree T with n vertices; $P_T(\lambda) = \lambda(\lambda-1)^{n-1}$; $= P_{P_n}(\lambda)$



$P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)$



$\therefore P_{K_n}(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$

CHROMATIC POLYNOMIAL:

$P_n(\lambda)$: Chromatic polynomial of a graph G with n vertices

= number of ways of proper coloring using atmost λ colours.

$= \sum_{k=1}^{n}$ number of different ways of proper colouring G using exactly k different colours $\times$ total number of ways of selecting k colours out of λ colours

$$= \sum_{k=1}^{n} c_k \cdot \binom{\lambda}{k} = \sum_{k=1}^{n} \binom{\lambda}{k} \cdot c_k$$

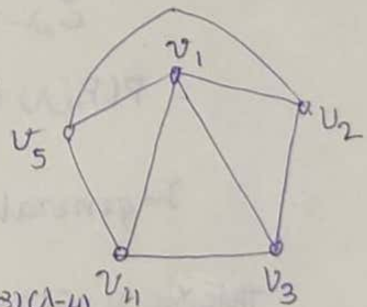
$$\therefore P_n(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \dots + c_n\frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}$$

For each graph G, c_k will be evaluated.

A graph of one edge requires at least two colours for its proper colouring and so $c_1 = 0$

A graph with n vertices and n different colours can be properly coloured in $n!$ ways.

Ex. Find the chromatic polynomial of the graph:



SOL. The graph consists of 5 vertices

$$\therefore P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

The graph contains/has a triangle, at least three different colours are required for its proper colouring. therefore

$$c_1 = 0, \; c_2 = 0 \text{ and also } c_5 = 5!$$

To evaluate c_3 :

let three colours : a, b, and c be assigned properly to vertices v_1, v_2, v_3 (of a triangle)

This can be done in $3!$ different ways.

Now vertex v_5 must have the same colour as v_3

" v_4 " " " " " as v_2

$$\therefore c_3 = 3! = 6$$

To evaluate c_4 :

With four colours, v_1, v_2, and v_3 can be properly coloured in

PTO

$^4C_1 \times {}^3C_1 \times {}^2C_1 = 4 \times 3 \times 2 = 24$ different ways.

The remaining fourth colour can be assigned to v_4 or v_5 and thus provide two choices. Therefore.

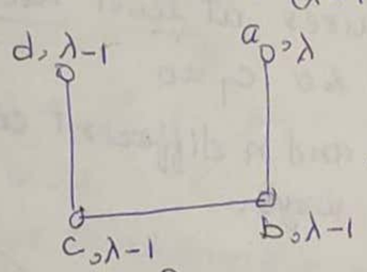
$C_4 = 2 \times 24 = 48$ ways. (The Fifth vertex provides no additional choices)

$$\therefore P_5(\lambda) = 0 + 0 + \cancel{3!}\,\frac{\lambda(\lambda-1)(\lambda-2)}{\cancel{3!}} + \overset{2}{\cancel{48}}\,\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\cancel{4!}} + \cancel{5!}\,\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{\cancel{5!}}$$

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2)[1 + 2(\lambda-3) + (\lambda-3)(\lambda-4)]$$
$$= \lambda(\lambda-1)(\lambda-2)[2\lambda - 5 + \lambda^2 - 7\lambda + 12]$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$

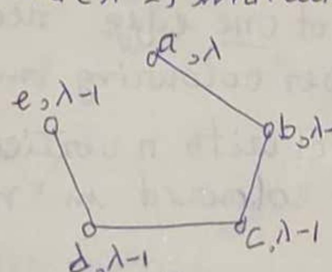
The presence of $(\lambda-1)$ and $(\lambda-2)$ indicates that is at least 3-chromatic.

EX.



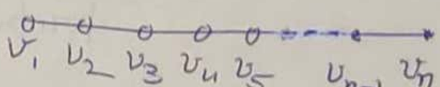
P_4

$P(P_4, \lambda) = \lambda(\lambda-1)^3$;



P_5

$P(P_5, \lambda) = \lambda(\lambda-1)^4$



P_n

$P(P_n, \lambda) = \lambda(\lambda-1)^{n-1}$

In general, if G is a path on n vertices, then $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$

Theorem: If G is a disconnected graph with k components: $G_1, G_2, \cdots G_k$, then

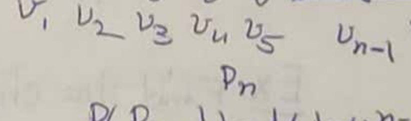
$$P(G,\lambda) = P(G_1,\lambda)\,P(G_2,\lambda) \cdots P(G_k,\lambda).$$

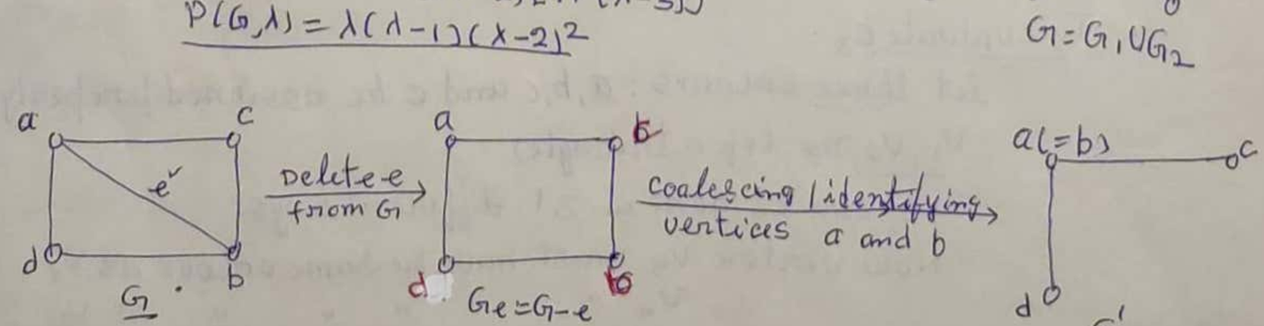
EX. $P(G,\lambda) = P(G_1,\lambda)P(G_2,\lambda) = \lambda^{(4)} \times \lambda^{(3)}$

$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) \times \lambda(\lambda-1)(\lambda-2)$

$= \lambda(\lambda-1)(\lambda-2)[1 + (\lambda-3)]$

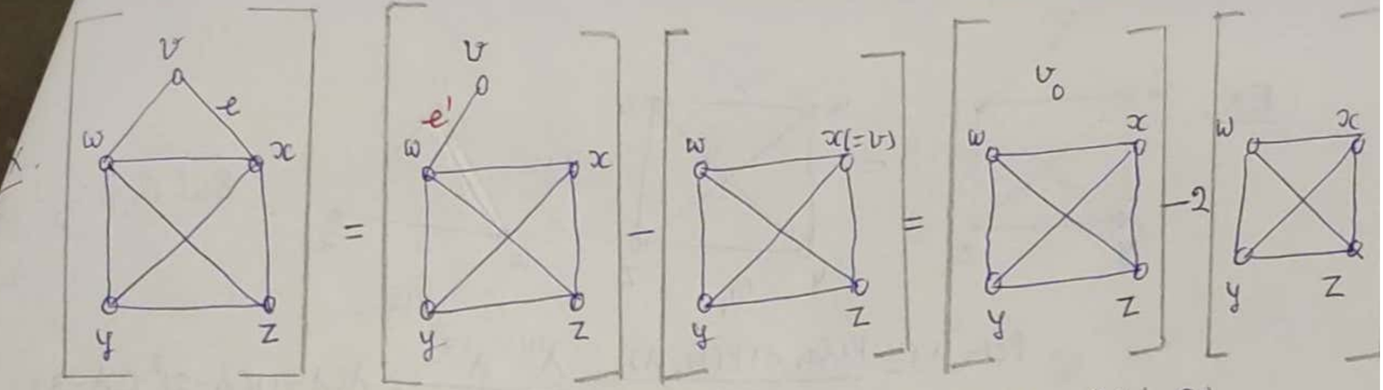
$P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)^2$





Decomposition Theorem for chromatic Polynomials: If $G=(V,E)$ is a connected graph and $e \in E$, then

$\boxed{P(G_e,\lambda) = P(G,\lambda) + P(G'_e,\lambda)}$ or $P(G,\lambda) = P(G_e,\lambda) - P(G'_e,\lambda)$



$$P(G,\lambda) = (\lambda)(\lambda^{(4)}) - 2\lambda^{(4)} = (\lambda-2)\lambda^{(4)} = \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$$

For each integer λ, $1 \le \lambda \le 3$, $P(G,\lambda)=0$, but $P(G,\lambda)>0$ for all $\lambda \ge 4$

$\therefore \chi(G)=4$

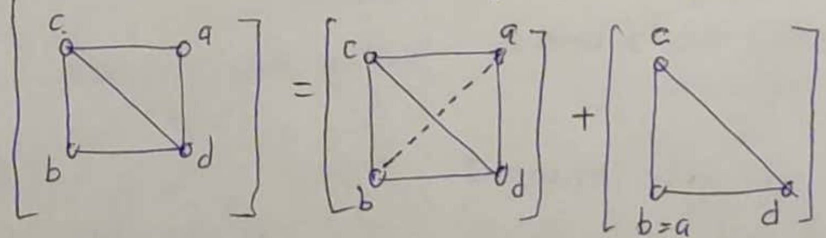
Theorem: For each graph G, the constant term in $P(G,\lambda)$ is zero.

Theorem: Let $G=(V,E)$ with $|E|>0$. The sum of coefficients in $P(G,\lambda)$ is zero.

Theorem: Let $G=(V,E)$ with $a,b \in V$ but $\{a,b\}=e \notin E$. Let
G_e^+ be the graph obtained from G by adding the edge $e=\{a,b\}$
G_e^{++} be the subgraph obtained by coalescing the vertices a and b in G. Under these circumstances,

$$P(G,\lambda) = P(G_e^+,\lambda) + P(G_e^{++},\lambda)$$

Ex.



$\therefore P(G,\lambda) = \lambda^{(4)} + \lambda^{(3)}$
$= \lambda(\lambda-1)(\lambda-2)^2$

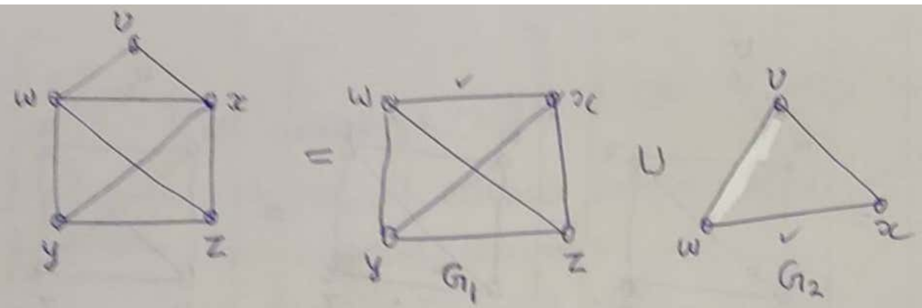
$\chi(G)=3$

If $\lambda=6$ colors are available, the vertices in G can be properly coloured in $\chi(\lambda=6) = (6)(5)(4)^2 = 480$ ways.

THEOREM: Let G be an undirected with subgraphs G_1, G_2. If $G=G_1 \cup G_2$, and $G_1 \cap G_2 = K_n$, for some $n \in Z^+$, then

$$P(G,\lambda) = P(G_1,\lambda) \cdot P(G_2,\lambda) / (\lambda^{(n)}).$$

EX.

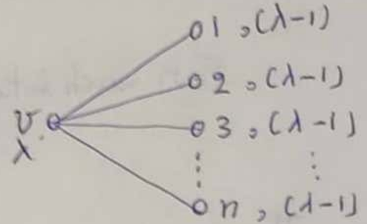


; But $G_1 \cap G_2 = K_2$ (edge w—x) $= \{w, x\}$.

$$\therefore P(G,\lambda) = \frac{P(G_1,\lambda)\,P(G_2,\lambda)}{\lambda^{(2)}} = \frac{\lambda^{(4)} \cdot \lambda^{(3)}}{\lambda^{(2)}} = \lambda(\lambda-1)(\lambda-2)^2(\lambda-3).$$

EX. For $n \in \mathbb{Z}^+$, $P(K_{1,n}, \lambda) = \lambda(\lambda-1)^n$

$\chi(K_{1,n}) = 2$



EX. $P(K_{2,3}, \lambda) = \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)^3$

↑ a & b have same colour ↑ a and b have different colours