Public Key Primitives

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March 9, 2022

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1. Overview

Public Key Exchange Motivation

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- This requires that if the eavesdropper takes the transcript of their phone call, they are not able to generate the secret k

NOTE: no requirements for integrity (no protection from man in the middle) and the protocol is fully anonymous (no way to verify that Alice and Bob are talking to one another)

Anonymous Key Exchange Attack Game

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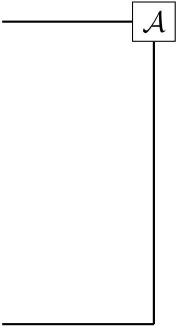


Anonymous Key Exchange Attack Game

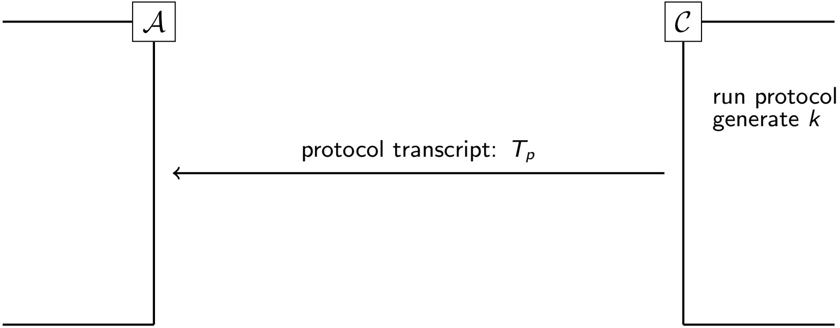
Anonymous Key Exchange Attack Game

$\mathcal{A}$

$\mathcal{C}$

run protocol
generate k

Anonymous Key Exchange Attack Game



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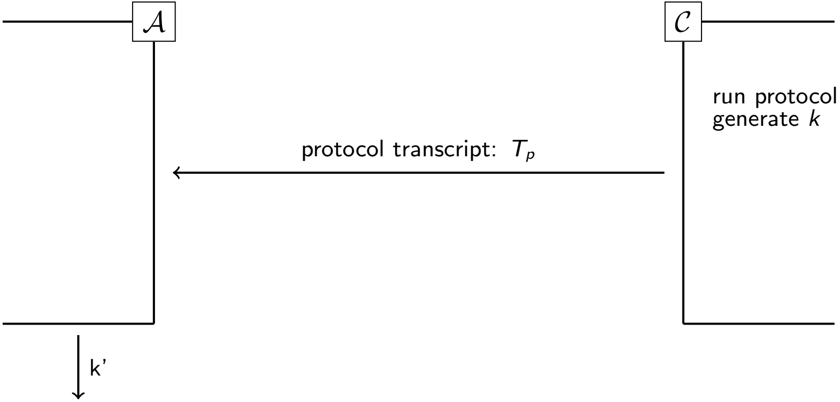

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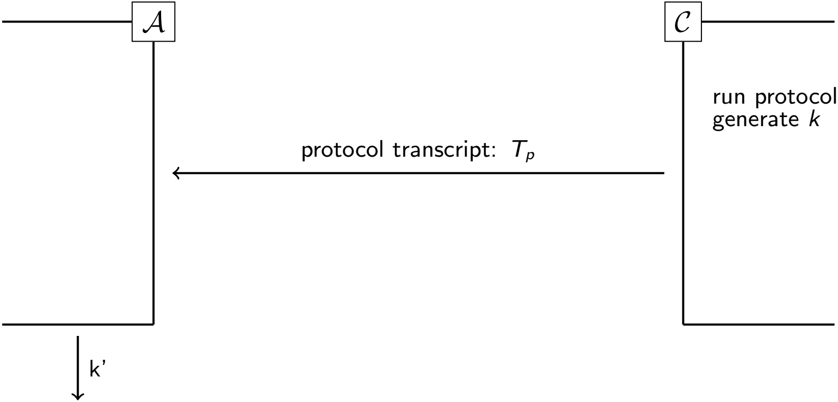

if $k' = k$, then the adversary wins

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2. Assumes that adversary cannot simply guess parts of k (i.e. no uniform randomness distinguishability requirement)
3. No identity verification

2 Trapdoor Functions

Trapdoor Functions

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- correctness: $\forall (pk, sk) : I(sk, F(pk, x)) = x$

Trapdoor Key Exchange

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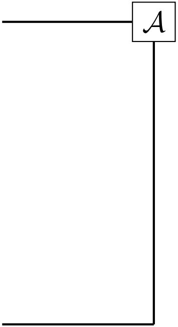

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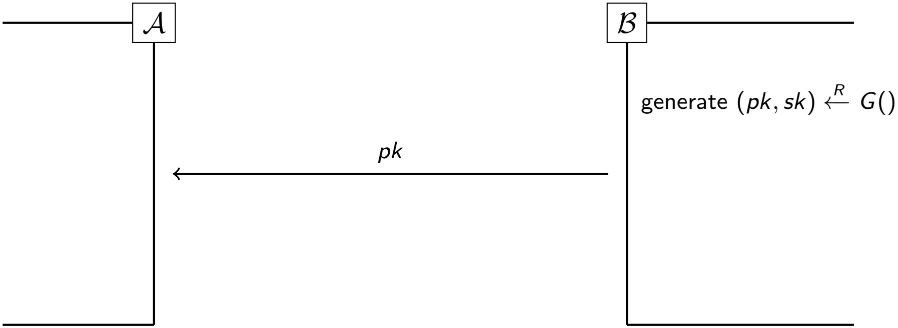

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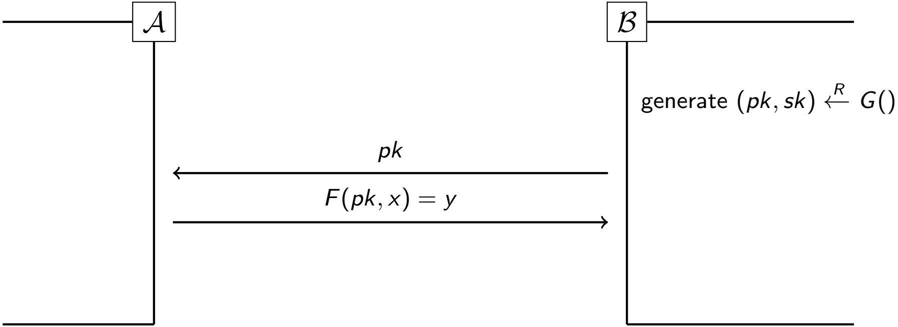

Trapdoor Key Exchange

$\mathcal{A}$ | $\mathcal{B}$

$\mathcal{B}$: generate $(pk, sk) \xleftarrow{R} G()$

$\mathcal{A} \xleftarrow{pk} \mathcal{B}$

$\mathcal{A} \xrightarrow{F(pk, x) = y} \mathcal{B}$

$\mathcal{B}$: $s \leftarrow F(sk, y) = x$

Trapdoor Key Exchange Attack Game

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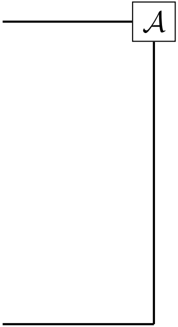

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$\mathcal{A}$

$\mathcal{C}$

generate $(pk, sk) \xleftarrow{R} G()$
generate $y = F(pk, x)$

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generate $(pk, sk) \xleftarrow{R} G()$

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protocol transcript: (pk, y) $\longleftarrow$

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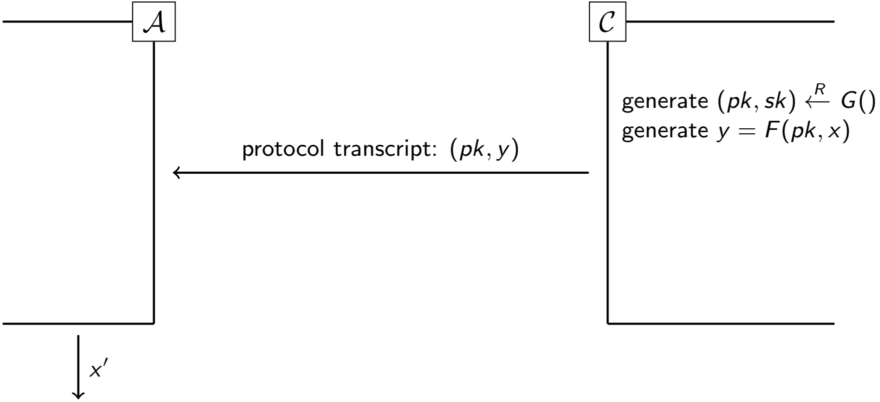

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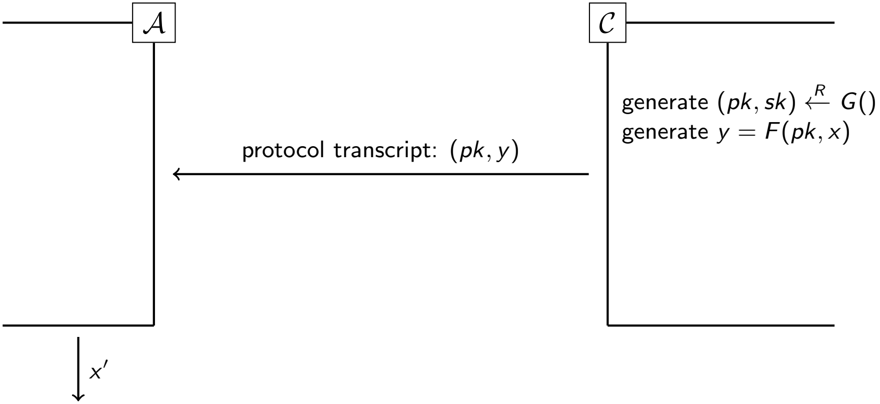

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3. RSA

RSA Background

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- Legend has it they got drunk on wine during passover at a student's house and came up with the system staying up all night
- allegedly, the british intelligence agencies came up with a similar system a few years earlier but didn't think it was feasible with the current computers

RSA Key Generation

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- $I(sk, y) := y^d \in \mathbb{Z}_n$

RSA Security

RSA Security

- given n the RSA Modulus, e the encryption exponent, d the decryption exponent, and $y = x^e$, it is mathematically hard to calculate x

4. Diffie-Hellman Key Exchange

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- NSA even sent letters to journal editors warning that authors of cryptography papers could be sentenced to prison time for violating laws around military weapon export

Diffie-Hellman Key Exchange

- start by sample two large primes: p, q s.t. q divides $p - 1$
- all math is done mod p (working in $\mathbb{Z}_{|}$)
- since q divides p, there exists a g s.t. $g^q = 1$, this will serve as the generator for a Group ($\mathbb{G} := g^a : a = 0, ..., q - 1$)

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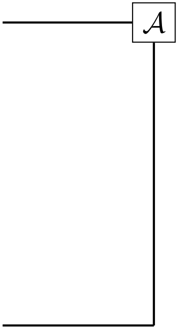

Diffie-Hellman Key Exchange

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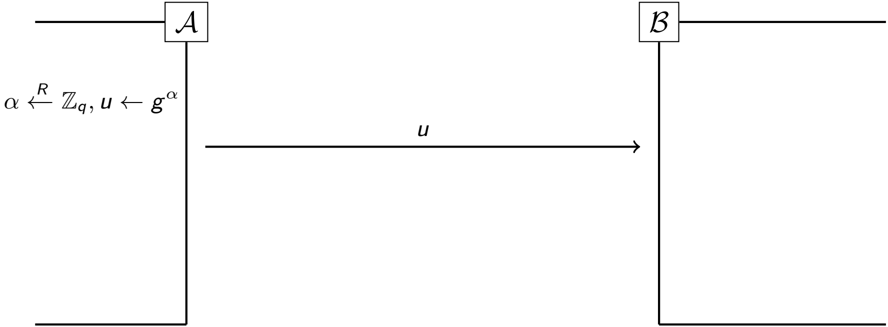
$\mathcal{A}$ $\mathcal{B}$

$\alpha \xleftarrow{R} \mathbb{Z}_q, u \leftarrow g^{\alpha}$

Diffie-Hellman Key Exchange



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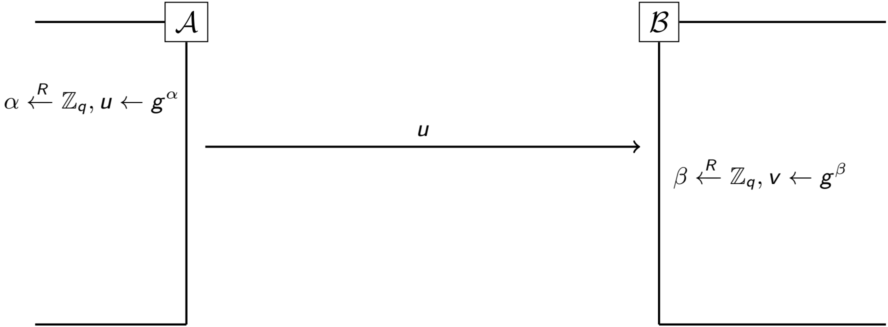

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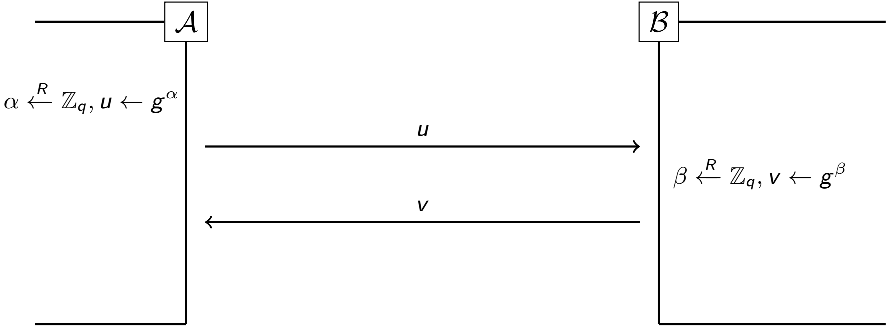

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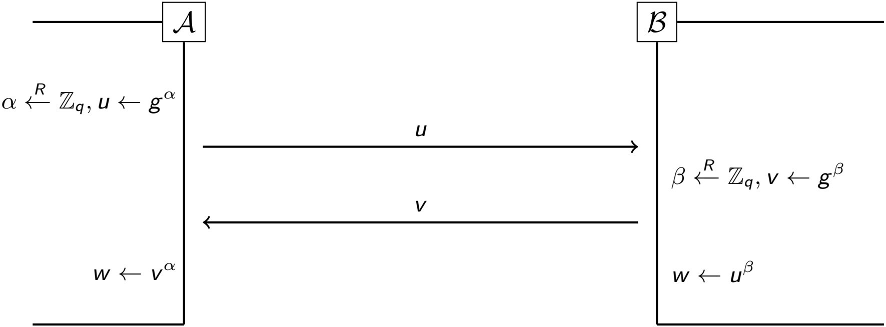

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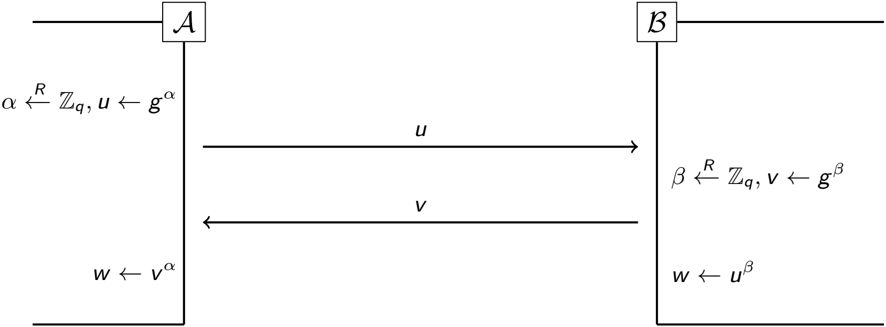

$$w = v^\alpha = u^\beta = g^{\alpha\beta}$$

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- over a cyclic group $\mathbb{G}$ it is mathematically hard to compute α given g^α, where g is a generator of $\mathcal{G}$
- this is further extended to: given (g^α, g^β) where g is a generator, $\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$, it is hard to compute $g^{\alpha\beta}$