5.2.2 (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $f(x) = \frac{1}{x}$.
$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} = \lim_{x\to c} \frac{\frac{c-x}{xc}}{x-c} = \lim_{x\to c} \frac{-1}{xc} = -\frac{1}{c^2}$.

(b) Combine the result in part (a) with the chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

*Proof.* Let $h(x) = \frac{1}{x}$. Then, we can use the chain rule to simplify

$$\begin{aligned}(\frac{1}{g(x)})' &= (h \circ g)'(x) \\ &= h'(g(x))g'(x) \\ &= -\frac{1}{(g(x))^2}(g'(x)) = -\frac{g'(x)}{(g(x))^2}.\end{aligned}$$

This helps us to simplify

$$\begin{aligned}(\frac{f}{g})'(x) &= (f(x)(h \circ g)(x))' \\ &= f'(x)h(g(x)) + f(x)(h \circ g)'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

□

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for $(\frac{f}{g})$ in a style similar to the proof of Theorem 5.2.4 (iii).

*Proof.*

$$\begin{aligned}(\frac{f}{g})'(c) = \frac{(\frac{f}{g})(x) - (\frac{f}{g})(c)}{x-c} &= \frac{1}{x-c}(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}) \\ &= \frac{1}{x-c}(\frac{f(x)g(c) - g(x)f(c)}{g(x)g(c)}) \\ &= \frac{1}{g(x)g(c)}(\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - g(x)f(c)}{x-c}) \\ &= \frac{1}{g(x)g(c)}(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c})\end{aligned}$$

Because $f$ and $g$ are differentiable at $c$, it is continuous there and thus $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Using this with the Algebraic Limit Theorem for functional limits, we can simplify the answer to $\frac{1}{(g(c))^2}(g(c)f'(c) - f(c)g'(c))$.
Thus, $(\frac{f}{g})'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$ as desired. □

5.2.5 Let

$$g_a(x) = \begin{cases} x^a sin(\frac{1}{x}) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for $a$ so that

(a) $g_a$ is differentiable on $\mathbb{R}$ but such that $g_a^{'}$ is unbounded on $[0, 1]$.

$a = \frac{3}{2}$ satisfies both requirements.

(b) $g_a$ is differentiable on $\mathbb{R}$ with $g_a^{'}$ continuous but not differentiable at zero.

$a = 3$ satisfies all the requirements.

(c) $g_a$ is differentiable on $\mathbb{R}$ with $g_a^{'}$ is differentiable on $\mathbb{R}$, but such that $g_a^{''}$ is not continuous at zero.

$a = 4$ satisfies all the requirements.

5.2.8 Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If a derivative function is not constant, then the derivative must take on some irrational values.

False, take the following counterexample: $f(x) = |x|$. Then, $f'(x) = -1$ when $x < 0$ and $f'(x) = 1$ when $x > 0$ and $f'(x)$ is not defined when $x = 0$. So, we see that $f'$ is not constant since it has two values, -1 and 1, but it is does not take on any irrational values.

(b) If $f'$ exists on an open interval, and there is some point $c$ where $f'(c) > 0$, then there exists a $\delta$-neighborhood $V_\delta(c)$ around $c$ in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

False, take the following counterexample: $f(x) = \begin{cases} x^2 sin(\frac{1}{x}) + x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$.

Then, $f'(0) = \lim_{x \to 0} \frac{x^2 sin(\frac{1}{x}) + x - 0}{x - 0} = 1 > 0$. But, the derivative at $x \neq 0$ is $f'(x) = 2x sin(\frac{1}{x}) - cos(\frac{1}{x}) + 1$ which has non-positive values at $x = \frac{1}{2n\pi}$ for $n \in \mathbb{N}$. We can see this by plugging in $\frac{1}{2n\pi}$ to get $f'(\frac{1}{2n\pi}) = \frac{1}{n\pi} sin(2n\pi) - cos(2n\pi) + 1 = 0 - 1 + 1 = 0$ and $0 > 0$ is false. We can always choose $n$ large enough so that for any $V_\delta(0)$ with $\delta > 0$, $\frac{1}{2n\pi} \in V_\delta(0)$ and thus fails to satisfy the claim.

(c) If $f$ is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.
True, because if $L \neq f'(0)$ then this is a violation of Darboux's Theorem.

*Proof.* Assume for the sake of contradiction that $L \neq f'(0)$. We will assume that $f'(0) < L$ (A similar proof holds true for the other case as shown below this proof). Let $0 < \epsilon < \frac{L-f'(0)}{2}$. Since $\lim_{x\to 0} f'(x) = L$, we know that there exists $\delta > 0$ such that for all $x$ with $0 < |x| < \delta$ implies $|f'(x) - L| < \epsilon$, or in more detail: $L - \epsilon < f'(x) < L + \epsilon$. Choose such a $\delta$ and choose $a$ such that $f'(0) < a < L-\epsilon$. Then, for all $x$ with $0 < |x| < \delta$ it is true that $f'(0) < a < f'(x)$. By Darboux's Theorem, we have $c \in (0, x)$ such that $f'(c) = a$. However, this is a contradiction since $0 < c < \delta$ but $f'(c) = a < L - \epsilon$ which means it fails to satisfy $\lim_{x\to 0} f'(x) = L$. Thus, $L = f'(0)$ must be true. □

*Proof.* Assume for the sake of contradiction that $L \neq f'(0)$. We will assume that $f'(0) > L$ (A similar proof holds true for the other case as shown below this proof). Let $0 < \epsilon < \frac{f'(0)-L}{2}$. Since $\lim_{x\to 0} f'(x) = L$, we know that there exists $\delta > 0$ such that for all $x$ with $0 < |x| < \delta$ implies $|f'(x) - L| < \epsilon$, or in more detail: $L - \epsilon < f'(x) < L + \epsilon$. Choose such a $\delta$ and choose $a$ such that $L+\epsilon < a < f'(0)$. Then, for all $x$ with $0 < |x| < \delta$ it is true that $f'(0) > a > f'(x)$. By Darboux's Theorem, we have $c \in (0, x)$ such that $f'(c) = a$. However, this is a contradiction since $0 < c < \delta$ but $f'(c) = a > L + \epsilon$ which means it fails to satisfy $\lim_{x\to 0} f'(x) = L$. Thus, $L = f'(0)$ must be true. □

(d) Repeat conjecture (c) but drop the assumption that $f'(0)$ necessarily exists. If $f'(x)$ exists for all $x \neq 0$ and if $\lim_{x\to 0} f'(x) = L$, then $f'(0)$ exists and equals $L$.
False, take the counterexample: $f(x) = \frac{x^2+x}{x}$. We see that $f'(x)$ exists for all $x \neq 0$. Also, we see that for $x \neq 0, f(x) = \frac{x^2+x}{x} = x + 1$ so $\lim_{x\to 0} f'(x) = 1$. However, $f'(0)$ is undefined since $f$ is not differentiable at $x = 0$ and so it cannot be equal to 1.

5.3.1 Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on $A$ if there exists an $M > 0$ such that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ for all $x, y \in A$. Show that if $f$ is differentiable on a closed interval $[a, b]$ and if $f'$ is continuous on $[a, b]$, then $f$ is Lipschitz on $[a, b]$.

5.3.5 A fixed point of a function $f$ is a value $x$ where $f(x) = x$. Show that if $f$ is differentiable on an interval with $f'(x) \neq 1$, then $f$ can have at most one fixed point.

5.3.8 Assume $g : (a, b) \to \mathbb{R}$ is differentiable at some point $c \in (a, b)$. If $g'(c) \neq 0$, show that there exists a $\delta$-neighborhood $V_\delta(c) \subseteq (a, b)$ for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$. Compare this result with Exercise 5.3.7.

5.4.2 Fix $x \in \mathbb{R}$. Argue that the series $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ converges absolutely and thus $g(x)$ is properly defined.

5.4.4 Show that $\frac{g(x_m)-g(0)}{x_m-0} = m+1$, and use this to prove that $g'(0)$ does not exist.

5.4.5 (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(\frac{1}{2})$ does not exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = \frac{p}{2^k}$ where $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$.