4.2.2 Assume a particular $\delta > 0$ has been constructed as a suitable response to a particular $\epsilon$ challenge. Then, any larger/smaller (pick one) $\delta$ will also suffice.

Smaller.

4.2.4 Review the definition of Thomae's function $t(x)$ from Section 4.1.

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} - \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Construct three different sequences $(x_n)$, $(y_n)$, and $(z_n)$, each of which converges to 1 without using the number 1 as a term in the sequence.
$(x_n) = 1 - \frac{1}{n}$ for all $n \in N$ such that $n > 1$.
$(y_n) = 1 + \frac{1}{n}$ for all $n \in N$ such that $n > 1$.
$(z_n) = 1 - \frac{1}{n^2}$ for all $n \in N$ such that $n > 1$.

(b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.
$\lim t(x_n) = \lim t(y_n) = \lim t(z_n) = 0$

(c) Make an educated conjecture for $\lim_{x \to 1} t(x)$, and use Definition 4.2.1B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbb{R} : t(x) \geq \epsilon\}$. Argue that all the points in this set are isolated.)
My conjecture is that $\lim_{x \to 1} t(x) = 0$.

*Proof.* Suppose $\epsilon > 0$. Define $(x_n)$ to be the sequence $\frac{p+n-1}{q+n-1}$ where $p, q \in \mathbb{Z}$ and $\frac{p+n-1}{q+n-1} < 1$. □

4.2.5 (a) Supply the details for how Corollary 4.2.4 part (ii) follows from the sequential criterion for functional limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

*Proof.* Suppose $f$ and $g$ are functions defined on the domain $A \subseteq \mathbb{R}$ and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Need to show that $\lim_{x \to c}[f(x) + g(x)] = L + M$, that is as $x_n$ approaches $c$, $f(x_n) + g(x_n)$ approaches $L + M$. Since $f(x_n) \to L$ and $g(x_n) \to M$, we can use the algebraic limit theorem to get $f(x_n) + g(x_n) \to L + M$. □

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

*Proof.* Suppose $f$ and $g$ are functions defined on the domain $A \subseteq \mathbb{R}$ and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Let $\epsilon > 0$. Need to show that $\lim_{x \to c}[f(x) + g(x)] =$

$L + M$, that is there exists $\delta$ such that whenever $0 < |x - c| < \delta$, it follows that $|(f(x) + g(x)) - (L + M)| < \epsilon$. Since $\lim_{x\to c} f(x) = L$, we know that there exists a $\delta_1$ such that whenever $0 < |x - c| < \delta_1$, it follows that $|f(x) - L| < \frac{\epsilon}{2}$. Similarly, since $\lim_{x\to c} g(x) = M$, we know that there exists a $\delta_2$ such that whenever $0 < |x - c| < \delta_2$, it follows that $|g(x) - M| < \frac{\epsilon}{2}$. So, choose $\delta$ such that $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x - c| < \delta$:

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |(f(x) - L)| + |(g(x) - M)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, $0 < |x - c| < \delta$ implies $|(f(x) + g(x)) - (L + M)| < \epsilon$ as desired. □

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).