5.2.2 (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $f(x) = \frac{1}{x}$.

(b) Combine the result in part (a) with the chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for $(\frac{f}{g})$ in a style similar to the proof of Theorem 5.2.4 (iii).

5.2.5 Let

$$g_a(x) = \begin{cases} x^a sin(\frac{1}{x}) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g_a^{'}$ is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with $g_a^{'}$ continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ with $g_a^{'}$ is differentiable on $\mathbb{R}$, but such that g_a^{n} is not continuous at zero.

5.2.8 Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If a derivative function is not constant, then the derivative must take on some irrational values.

(b) If f' exists on an open interval, and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

(d) Repeat conjecture (c) but drop the assumption that $f'(0)$ necessarily exists. If $f'(x)$ exists for all $x \neq 0$ and if $\lim_{x\to 0} f'(x) = L$, then $f'(0)$ exists and equals L.

5.3.1 Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ for all $x, y \in A$. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

5.3.5 A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

5.3.8 Assume $g : (a, b) \to \mathbb{R}$ is differentiable at some point $c \in (a, b)$. If $g'(c) \neq 0$, show that there exists a δ-neighborhood $V_\delta(c) \subseteq (a, b)$ for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$. Compare this result with Exercise 5.3.7.

5.4.2 Fix $x \in \mathbb{R}$. Argue that the series $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ converges absolutely and thus $g(x)$ is properly defined.

5.4.4 Show that $\frac{g(x_m)-g(0)}{x_m-0} = m+1$, and use this to prove that $g'(0)$ does not exist.

5.4.5 (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(\frac{1}{2})$ does not exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = \frac{p}{2^k}$ where $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$.