2.2.1 Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1. $\lim \frac{1}{6n^2+1} = 0$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\sqrt{6\epsilon}}$. To verify let $n \geq N$ which means $\left|\frac{1}{6n^2+1}\right| < \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| = \left|\frac{1}{\frac{1}{\epsilon}+1}\right| = \frac{\epsilon}{\epsilon+1} < \epsilon$ as desired. □

2. $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{13-10\epsilon}{4\epsilon}$. To verify let $n \geq N$ and so $\left|\frac{3n+1}{2n+5}\right| < \left|\frac{3(\frac{13-10\epsilon}{4\epsilon})+1}{2(\frac{13-10\epsilon}{4\epsilon})+5}\right| = \frac{3}{2} - \epsilon$. So, $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon$ as desired. □

3. $\lim \frac{2}{\sqrt{n+3}} = 0$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{4}{\epsilon^2} - 3$. To verify let $n \geq N$ which means $\left|\frac{2}{\sqrt{n+3}}\right| < \left|\frac{2}{\sqrt{(\frac{4}{\epsilon^2}-3)+3}}\right| = \epsilon$ as desired. □

2.2.7 Informally speaking, the sequence $\sqrt{n}$ "converges to infinity."

(a) Imitate the logical structure of Definition 2.2.3 to create a rigorous definition for the mathematical statement $\lim x_n = \infty$. Use this definition to prove $\lim \sqrt{n} = \infty$.
A sequence (a_n) "converges to infinity" if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $a_n > \epsilon$.

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \epsilon^2$. To verify let $n \geq N$ which means $\sqrt{n} > \sqrt{\epsilon^2} = \epsilon$ as desired. □

(b) What does your definition in (a) say about the particular sequence $(1, 0, 2, 0, 3, 0, 4, 0, 5, 0, ...)$?
It does not converge to infinity.

2.2.8 Here are two useful definitions:

1. A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
2. A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
Frequently

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

Eventually is stronger; eventually implies frequently.

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is eventually in the set $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

x_n is not necessarily eventually in the interval $(1.9, 2.1)$, but it is frequently in $(1.9, 2.1)$. For example, $(2, 2, 2, 2, ...)$ is eventually in the interval $(1.9, 2.1)$, but $(2, -2, 2, -2, ...)$ is only frequently in the interval $(1.9, 2.1)$.

2.3.3 **(Squeeze Theorem)**. Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if lim $x_n =$ lim $z_n = l$, then lim $y_n = l$ as well.

Proof. Assume that $\forall n \in \mathbb{N}, x_n \leq y_n \leq z_n$ and $\lim x_n = \lim z_n = l$. Suppose $n \in \mathbb{N}$. Then, by the Order Limit Theorem, since $x_n \leq y_n$, $\lim y_n \geq \lim x_n = l$. Similarly, since $y_n \leq z_n$, $\lim y_n \leq \lim z_n = l$. So, $l \leq \lim y_n \leq l$. Thus, $\lim y_n = l$. □

2.3.5 Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with lim $x_n =$ lim y_n.

Proof.
First we prove that if (z_n) is convergent then (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$. Assume (z_n) is convergent to z. Let $\epsilon > 0$ be arbitrary. Because (z_n) is convergent to z we know there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|z_n - z| < \epsilon$. So, $\epsilon > |x_n - z|$ for all $n \geq N$ because $x_n = z_{2n-1}$ and $2n - 1 \geq N$. Similarly, $\epsilon > |y_n - z|$ for all $n \geq N$ because $y_n = z_{2n}$ and $2n \geq N$. Therefore, $\lim x_n = z = \lim y_n$.

Next we prove that if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$ then (z_n) is convergent. Assume (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = l$. Let $\epsilon > 0$ be arbitrary. Then, we know that there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n - l| < \epsilon$. Similarly, we know that there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|y_n - l| < \epsilon$. Let $N = \max\{2N_1, 2N_2\}$. Then, for all $n \geq N$, we have that $|z_n - l| < \epsilon$ since (z_n) consists of alternating elements from (x_n) and (y_n) which means after N both $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$. Thus, (z_n) is convergent. □

2.3.10 If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, then show that $(b_n) \to b$.

Proof. Assume $(a_n) \to 0$ and $|b_n - b| \leq a_n$. Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N, |a_n| < \epsilon$. Then, for all $n \geq N$, $|b_n - b| \leq a_n < \epsilon$ which means $|b_n - b| < \epsilon$, and so $(b_n) \to b$. □

2.4.1 Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Proof. Assume the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Then the monotone sequence of its partial sums $(t_k) = b_1 + 2b_2 + ... + 2^k b_{2^k}$ must be unbounded. We need to show that $\sum_{n=1}^{\infty} b_n$ diverges, namely that $(s_m) = b_1 + b_2 + ... + b_m$ is unbounded. Fix m and choose an arbitrary k. Then,

$$\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + ... + (b_{2^{k-1}+1} + ... + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + ... + (b_{2^k} + ... + b_{2^k}) \\
&= b_1 + b_2 + 2b_4 + 4b_8 + ... + 2^{k-1} b_{2^k} \\
&= \frac{1}{2}(2b_1 + 2b_2 + 4b_4 + 8b_8 + ... + 2^k b_{2^k}) \\
&= \frac{1}{2}(b_1 + t_k).
\end{aligned}$$

Since (t_k) is unbounded and $s_m \geq \frac{t_k}{2}$, it must mean that (s_m) is unbounded, and so $\sum_{n=1}^{\infty} b_n$ diverges. □

2.4.2 (a) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4 - x_n}$ converges.

Proof. We need to show that the sequence is monotone and bounded, then we can apply the Monotone Convergence Theorem. First, we need to use induction to show that the sequence is monotone.
Base Case ($n = 1$): $x_{1+1} = x_2 = \frac{1}{4-x_1} = \frac{1}{4-3} = 1$. $3 > 1$ so $x_1 > x_2$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume for all natural numbers $i < k$, $x_i > x_{i+1}$.
* Need to show that $x_k > x_{k+1}$. $x_k = \frac{1}{4-x_{k-1}}$ and $x_{k+1} = \frac{1}{4-x_k}$. By the induction hypothesis, we know that $x_{k-1} > x_k$. So, $4 - x_{k-1} < 4 - x_k$ which means $\frac{1}{4-x_{k-1}} > \frac{1}{4-x_k}$. Thus, $x_k > x_{k+1}$.

By the principle of complete induction, the sequence is decreasing. Now, it is clear that $x_1 = 3$ is the upper bound for the sequence. Next, we need to use induction to show that the sequence is bounded below by 0.
Base Case ($n = 1$): $x_{1+1} = x_2 = \frac{1}{4-x_1} = \frac{1}{4-3} = 1 > 0$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume for all natural numbers $i < k$, $x_i > 0$.
* Need to show that $x_k > 0$. $x_k = \frac{1}{4-x_{k-1}}$. By the induction hypothesis, we know that $x_{k-1} > 0$. Since, $3 \geq x_{k-1} > 0$, $\frac{1}{4-x_{k-1}} > 0$. Thus, $x_k > 0$.

By the principle of complete induction, the sequence is always greater than 0. So, the sequence is bounded below by 0. Thus, the sequence is monotone and bounded. So, by the Monotone Convergence Theorem, the sequence converges. □

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Since $\lim x_n$ exists, $\lim x_{n+1}$ must also exist and be the same value because these sequences only differ by their starting value and an index of 1. This means that their end behavior (where the limit will end up) will remain identical.

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim x_n$.

Let $L = \lim x_n = \lim x_{n+1}$. Then, $L = \frac{1}{4-L}$ which becomes $L^2 - 4L + 1 = 0$ which results in $L = 2 \pm \sqrt{3}$. Since the sequence is decreasing, the only answer is $L = 2 - \sqrt{3}$.

2.4.6 **(Limit Superior)**. Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. First, we need to prove that (y_n) is montonely decreasing, that is $y_{i+1} \leq y_i$ for all $i \in \mathbb{N}$. Suppose $i \in \mathbb{N}$. $y_{i+1} = \sup\{a_k : k \geq i+1\}$ and $y_i = \sup\{a_k : k \geq i\}$. Since $\{a_k : k \geq i+1\} \subseteq \{a_k : k \geq i\}$, $\sup\{a_k : k \geq i+1\} \leq \sup\{a_k : k \geq i\}$. So, $y_{i+1} \leq y_i$. Next, we need to prove that (y_n) is bounded, that is it has a lower bound and upper bound. Since (a_n) is bounded, let L be the lower bound of (a_n). Then, by the way y_n is defined, all of its elements come from (a_n) which means L is still a lower bound for (y_n). Since (y_n) is decreasing, it is also bounded from above, namely its first element is an upper bound. So, (y_n) is bounded. Thus, (y_n) converges by the Monotone Convergence Theorem. □

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by $\limsup a_n = \lim y_n$, where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

$\liminf a_n = z_n$ where $z_n = \inf\{a_k : k \geq n\}$. It always exists for any bounded sequence because of the Monotone Convergence Theorem. We can apply it here because we know (z_n) is bounded which can be shown by applying similar logic to the proof above, namely that (a_n) is bounded. We also know that (z_n) is increasing by applying similar logic to the proof above, namely that $\{a_k : k \geq$

$n+1\} \subseteq \{a_k : k \geq n\}$ where $n \in \mathbb{N}$ and since we are taking infinums of the smaller set, then it must hold true that $z_{n+1} \geq z_n$. So, since it is both bounded and monotone, we apply the Monotone Convergence Theorem to show it converges.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

Proof. Suppose $n \in \mathbb{N}$. Let $y_n = \sup\{a_k : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$. Then, $z_n \leq y_n$ because z_n is the infimums and y_n is the supremums. So, by the Order Limit Theorem, $\lim z_n \leq \lim y_n$. Thus $\liminf a_n \leq \limsup a_n$. □

Example: Let $(a_n) = (-1, 1, -1, 1, -1, 1, ...)$. Then, $\liminf a_n = -1$ and $\limsup a_n = 1$. Since $-1 < 1$, $\liminf a_n < \limsup a_n$ in this case.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof.
First, we prove that if $\liminf a_n = \limsup a_n$, then $\lim a_n$ exists. Assume $\liminf a_n = \limsup a_n$. Let $\epsilon > 0$ be arbitrary. Then, we know that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\inf a_n - L| < \epsilon$ and $|\sup a_n - L| < \epsilon$. Since $\inf a_n \leq a_n \leq \sup a_n$, it must be true that $|a_n - L| < \epsilon$ for all $n \geq N$. Thus, $\lim a_n$ exists.

Next, we prove that if $\lim a_n$ exists, then $\liminf a_n = \limsup a_n$. Assume $\lim a_n$ exists. Let $\epsilon > 0$ be arbitrary. Then, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. Since every element after a_n must be within the bounds $L - \epsilon$ and $L + \epsilon$, it follows that $\inf a_n$ and $\sup a_n$ must exist within these bounds as well. Then, $L - \epsilon \leq \liminf a_n \leq L + \epsilon$ and $L - \epsilon \leq \limsup a_n \leq L + \epsilon$. Thus, since ϵ is arbitrary, we get that $\liminf a_n = L = \limsup a_n$. □