5.2.2 (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $f(x) = \frac{1}{x}$.

$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} = \lim_{x\to c} \frac{\frac{c-x}{xc}}{x-c} = \lim_{x\to c} \frac{-1}{xc} = -\frac{1}{c^2}$.

(b) Combine the result in part (a) with the chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

Proof. Let $h(x) = \frac{1}{x}$. Then, we can use the chain rule to simplify

$$\begin{aligned}
(\frac{1}{g(x)})' &= (h \circ g)'(x) \\
&= h'(g(x))g'(x) \\
&= -\frac{1}{(g(x))^2}(g'(x)) = -\frac{g'(x)}{(g(x))^2}.
\end{aligned}$$

This helps us to simplify

$$\begin{aligned}
(\frac{f}{g})'(x) &= (f(x)(h \circ g)(x))' \\
&= f'(x)h(g(x)) + f(x)(h \circ g)'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

□

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for $(\frac{f}{g})$ in a style similar to the proof of Theorem 5.2.4 (iii).

Proof.

$$\begin{aligned}
(\frac{f}{g})'(c) = \frac{(\frac{f}{g})(x) - (\frac{f}{g})(c)}{x-c} &= \frac{1}{x-c}(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}) \\
&= \frac{1}{x-c}(\frac{f(x)g(c) - g(x)f(c)}{g(x)g(c)}) \\
&= \frac{1}{g(x)g(c)}(\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - g(x)f(c)}{x-c}) \\
&= \frac{1}{g(x)g(c)}(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c})
\end{aligned}$$

Because f and g are differentiable at c, it is continuous there and thus $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Using this with the Algebraic Limit Theorem for functional limits, we can simplify the answer to $\frac{1}{(g(c))^2}(g(c)f'(c) - f(c)g'(c))$. Thus, $(\frac{f}{g})'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$ as desired. □

5.2.5 Let

$$g_a(x) = \begin{cases} x^a sin(\frac{1}{x}) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g_a^{'}$ is unbounded on $[0, 1]$.

$a = \frac{3}{2}$ satisfies both requirements.

(b) g_a is differentiable on $\mathbb{R}$ with $g_a^{'}$ continuous but not differentiable at zero.

$a = 3$ satisfies all the requirements.

(c) g_a is differentiable on $\mathbb{R}$ with $g_a^{'}$ is differentiable on $\mathbb{R}$, but such that $g_a^{''}$ is not continuous at zero.

$a = 4$ satisfies all the requirements.

5.2.8 Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If a derivative function is not constant, then the derivative must take on some irrational values.

False, take the following counterexample: $f(x) = |x|$. Then, $f'(x) = -1$ when $x < 0$ and $f'(x) = 1$ when $x > 0$ and $f'(x)$ is not defined when $x = 0$. So, we see that f' is not constant since it has two values, -1 and 1, but it is does not take on any irrational values.

(b) If f' exists on an open interval, and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

False, take the following counterexample: $f(x) = \begin{cases} x^2 sin(\frac{1}{x}) + x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$.

Then, $f'(0) = \lim_{x \to 0} \frac{x^2 sin(\frac{1}{x}) + x - 0}{x - 0} = 1 > 0$. But, the derivative at $x \neq 0$ is $f'(x) = 2xsin(\frac{1}{x}) - cos(\frac{1}{x}) + 1$ which has non-positive values at $x = \frac{1}{2n\pi}$ for $n \in \mathbb{N}$. We can see this by plugging in $\frac{1}{2n\pi}$ to get $f'(\frac{1}{2n\pi}) = \frac{1}{n\pi} sin(2n\pi) - cos(2n\pi) + 1 = 0 - 1 + 1 = 0$ and $0 > 0$ is false. We can always choose n large enough so that for any $V_\delta(0)$ with $\delta > 0$, $\frac{1}{2n\pi} \in V_\delta(0)$ and thus fails to satisfy the claim.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.
True, because if $L \neq f'(0)$ then this is a violation of Darboux's Theorem.

Proof. Assume for the sake of contradiction that $L \neq f'(0)$. We will assume that $f'(0) < L$ (the other case is shown below). Let $0 < \epsilon < \frac{L-f'(0)}{2}$. Since $\lim_{x\to 0} f'(x) = L$, we know that there exists $\delta > 0$ such that for all x with $0 < |x| < \delta$ implies $|f'(x) - L| < \epsilon$, or in more detail: $L - \epsilon < f'(x) < L + \epsilon$. Choose such a δ and choose a such that $f'(0) < a < L - \epsilon$. Then, for all x with $0 < |x| < \delta$ it is true that $f'(0) < a < f'(x)$. By Darboux's Theorem, we have $c \in (0, x)$ such that $f'(c) = a$. However, this is a contradiction since $0 < c < \delta$ but $f'(c) = a < L - \epsilon$ which means it fails to satisfy $\lim_{x\to 0} f'(x) = L$. Thus, $L = f'(0)$ must be true. □

Proof. Assume for the sake of contradiction that $L \neq f'(0)$. We will assume that $f'(0) > L$ (the other case is shown above). Let $0 < \epsilon < \frac{f'(0)-L}{2}$. Since $\lim_{x\to 0} f'(x) = L$, we know that there exists $\delta > 0$ such that for all x with $0 < |x| < \delta$ implies $|f'(x) - L| < \epsilon$, or in more detail: $L - \epsilon < f'(x) < L + \epsilon$. Choose such a δ and choose a such that $L + \epsilon < a < f'(0)$. Then, for all x with $0 < |x| < \delta$ it is true that $f'(0) > a > f'(x)$. By Darboux's Theorem, we have $c \in (0, x)$ such that $f'(c) = a$. However, this is a contradiction since $0 < c < \delta$ but $f'(c) = a > L + \epsilon$ which means it fails to satisfy $\lim_{x\to 0} f'(x) = L$. Thus, $L = f'(0)$ must be true. □

(d) Repeat conjecture (c) but drop the assumption that $f'(0)$ necessarily exists. If $f'(x)$ exists for all $x \neq 0$ and if $\lim_{x\to 0} f'(x) = L$, then $f'(0)$ exists and equals L.
False, take the counterexample: $f(x) = \frac{x^2+x}{x}$. We see that $f'(x)$ exists for all $x \neq 0$. Also, we see that for $x \neq 0, f(x) = \frac{x^2+x}{x} = x + 1$ so $\lim_{x\to 0} f'(x) = 1$. However, $f'(0)$ is undefined since f is not differentiable at $x = 0$ and so it cannot be equal to 1.

5.3.1 Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ for all $x, y \in A$. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Proof. Assume f is differentiable on a closed interval $[a, b]$ and f' is continuous on $[a, b]$. Need to show that f is Lipschitz on $[a, b]$, that is there exists an $M > 0$ such that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ for all $x, y \in A$. Suppose x, y are arbitrary such that $a <= x < y <= b$. Given the assumptions, we know that by MVT there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. Since $[a, b]$ is a compact set and f' is continuous on $[a, b]$, we know that there exists an upper bound $M > 0$ such that for all $n \in [a, b], |f'(n)| \leq M$. Thus, since $c \in [a, b]$, we have that $|f'(c)| = \left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ as desired. □

5.3.5 A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof. Assume for the sake of contradiction that f has two fixed points x and y. Then, by the provided definition of fixed points we have $f(x) = x$ and $f(y) = y$. This means that by MVT we can find a c in the provided interval such that $f'(c) = \frac{f(x)-f(y)}{x-y} = \frac{x-y}{x-y} = 1$. However, this is a contradiction to the provided assumption that $f'(x) \neq 1$. Thus, f can have at most one fixed point. $\square$

5.3.8 Assume $g : (a,b) \to \mathbb{R}$ is differentiable at some point $c \in (a,b)$. If $g'(c) \neq 0$, show that there exists a δ-neighborhood $V_\delta(c) \subseteq (a,b)$ for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$. Compare this result with Exercise 5.3.7.

Proof. Suppose $g : (a,b) \to \mathbb{R}$ is differentiable at some point $c \in (a,b)$. Assume $g'(c) > 0$ (the other case will be handled later). Since g is differentiable at c, we have $g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c} > 0$. Let $0 < \epsilon < g'(c)$. Then we know there exists a $\delta > 0$ such that $0 < |x-c| < \delta$ implies $\left|\frac{g(x)-g(c)}{x-c} - g'(c)\right| < \epsilon$. We can rewrite this to $\frac{g(x)-g(c)}{x-c} > g'(c) - \epsilon > 0$. This means that if $x > c$ then $g(x) > g(c)$ or if $x < c$, $g(x) < g(c)$ which means that the fraction will always be positive. Thus we see that this is a δ-neighborhood around c for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$ whenever $x \neq c$. Similarly, if we take the other case $[g'(c) < 0]$, then we see that we can let $f(x) = -g(x)$ and derive that there exists a neighborhood around c for which $f(x) \neq f(c)$ for all x in the neighborhood whenever $x \neq c$. Then, since $f(x) = -g(x)$, we can multiply the result by -1 to show that $g(x) \neq g(c)$ on this neighborhood for all x whenever $x \neq c$. Thus, we have proved that when $g'(c) \neq 0$ there exists a δ-neighborhood $V_\delta(c)$ for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$. $\square$

5.4.2 Fix $x \in \mathbb{R}$. Argue that the series $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ converges absolutely and thus $g(x)$ is properly defined.

We know that $h(x)$ is bounded by 1 and so $h(x) \leq 1$ for all x. Then, $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x) \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} (\frac{1}{2})^n$ which we know is a geometric series and converges since $\frac{1}{2} < 1$. By the Comparison Test, we know that $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ converges and since all of the terms are positive we know from the Absolute Convergence Test that $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ absolutely converges.

5.4.4 Show that $\frac{g(x_m)-g(0)}{x_m-0} = m+1$ where $x_m = \frac{1}{2^m}$ for $m = 0, 1, 2, ...$, and use this to prove that $g'(0)$ does not exist.

Given an m, we see that $g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m})$. Then, when $n > m$, $g(x_m) = 0$ since we are at a multiple of 2 and h has a period of 2 meaning $h(2z) = 0$ for $z \in \mathbb{Z}$. So, $g(x_m)$ simplifies to $g(x_m) = \sum_{n=0}^{m} \frac{h(2^{n-m})}{2^n} = \sum_{n=0}^{m} \frac{2^{n-m}}{2^n} = \sum_{n=0}^{m} \frac{1}{2^m}$. Then, $\frac{g(x_m)-g(0)}{x_m-0} = \frac{\sum_{n=0}^{m} \frac{1}{2^m}}{\frac{1}{2^m}} = m+1$.

To show that $g'(0)$ doesn't exist, notice that $g'(0) = \lim_{m\to\infty} \frac{g(x_m)-g(0)}{x_m-0} = \lim_{m\to\infty} m+1$ which is clearly unbounded and so the limit does not exist. Thus, $g'(0)$ doesn't exist.

5.4.5 (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(\frac{1}{2})$ does not exist.

Given an m we see that $g(1+x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(1+x_m)) = \sum_{n=0}^{\infty} \frac{h(2^n+2^{n-m})}{2^n}$. Like 5.4.4, when $n > m$, $g(1+x_m) = 0$ since we are at a multiple of 2 and h has a period of 2 meaning $h(2z) = 0$ for $z \in \mathbb{Z}$. So, $g(1+x_m) = \sum_{n=0}^{m} \frac{h(2^n+2^{n-m})}{2^n}$. Next, when $1 \le n \le m$, since h has a period of 2, we get $\frac{h(2^n+2^{n-m})}{2^n} = \frac{h(2^{n-m})}{2^n} = \frac{1}{2^m}$. So, $g(1+x_m) = \sum_{n=0}^{1} \frac{h(2^n+2^{n-m})}{2^n} + \sum_{n=1}^{m} \frac{1}{2^m}$. Then, when $n = 0$, $\frac{h(2^n+2^{n-m})}{2^n} = h(1+\frac{1}{2^m}) = h(1) - \frac{1}{2^m} = g(1) - \frac{1}{2^m}$. So, $g(1+x_m) = g(1) - \frac{1}{2^m} + \sum_{n=1}^{m} \frac{1}{2^m}$. Then, $\frac{g(1+x_m)-g(1)}{x_m} = \frac{g(1)-\frac{1}{2^m}+(\sum_{n=1}^{m}\frac{1}{2^m})-g(1)}{\frac{1}{2^m}} = -1 + \frac{\sum_{n=1}^{m}\frac{1}{2^m}}{\frac{1}{2^m}} = m-1$. Thus, like 5.4.4, as $m \to \infty$, $m-1$ is unbounded meaning that $g'(1)$ doesn't exist.

Given an m we see that $g(\frac{1}{2}+x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(\frac{1}{2}+x_m)) = \sum_{n=0}^{\infty} \frac{h(2^{n-1}+2^{n-m})}{2^n}$. Like before, when $n > m$, $g(\frac{1}{2}+x_m) = 0$ since we are at a multiple of 2 and h has a period of 2 meaning $h(2z) = 0$ for $z \in \mathbb{Z}$. So, $g(\frac{1}{2}+x_m) = \sum_{n=0}^{m} \frac{h(2^{n-1}+2^{n-m})}{2^n}$. Next, when $2 \le n \le m$, since h has a period of 2, we get $\frac{h(2^{n-1}+2^{n-m})}{2^n} = \frac{h(2^{n-m})}{2^n} = \frac{1}{2^m}$. So, $g(\frac{1}{2}+x_m) = \sum_{n=0}^{1} \frac{h(2^{n-1}+2^{n-m})}{2^n} + \sum_{n=1}^{2} \frac{h(2^{n-1}+2^{n-m})}{2^n} + \sum_{n=2}^{m} \frac{1}{2^m}$. Then, when $n = 0$, $\frac{h(2^{n-1}+2^{n-m})}{2^n} = h(\frac{1}{2}+\frac{1}{2^m}) = h(\frac{1}{2}) - \frac{1}{2^m} = g(\frac{1}{2}) - \frac{1}{2^m}$. Similarly, when $n = 1$, $\frac{h(2^{n-1}+2^{n-m})}{2^n} = \frac{h(1+2^{1-m})}{2} = \frac{h(1)-h(2*2^{-m})}{2} = \frac{1}{2} - \frac{1}{2^m}$. So, $g(\frac{1}{2}+x_m) = g(\frac{1}{2}) - \frac{1}{2^m} + \frac{1}{2} - \frac{1}{2^m} + \sum_{n=2}^{m} \frac{1}{2^m}$. Then, $\frac{g(\frac{1}{2}+x_m)-g(\frac{1}{2})}{x_m} = \frac{g(\frac{1}{2})-\frac{2}{2^m}+\frac{1}{2}+(\sum_{n=2}^{m}\frac{1}{2^m})-g(\frac{1}{2})}{\frac{1}{2^m}} = -2 + 2^{m-1} + m$. Thus, like before, as $m \to \infty$, $-2 + 2^{m-1} + m$ is unbounded meaning $g'(\frac{1}{2})$ doesn't exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = \frac{p}{2^k}$ where $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$.

Assume $m > k$ since we eventually want to see the behavior as m goes to infinity. Then, $g(x+x_m) = \sum_{n=0}^{\infty} \frac{h(2^n(\frac{p}{2^k}+\frac{1}{2^m}))}{2^n} = \sum_{n=0}^{\infty} \frac{h(p2^{n-k}+2^{n-m})}{2^n}$. Like 5.4.4 and 5.4.5a, when $n > m$, $g(x+x_m) = 0$ since we are at a multiple of 2 and h has a period of 2 meaning $h(2z) = 0$ for $z \in \mathbb{Z}$. So, $g(x+x_m) = \sum_{n=0}^{m} \frac{h(p2^{n-k}+2^{n-m})}{2^n}$. Next, when $k < n \le m$, since h has a period of 2, we get $\frac{h(p2^{n-k}+2^{n-m})}{2^n} = \frac{h(2^{n-m})}{2^n} = \frac{1}{2^m}$.

So, $g(x+x_m) = \sum_{n=0}^{k} \frac{h(p2^{n-k}+2^{n-m})}{2^n} + \sum_{n=k+1}^{m} \frac{1}{2^m}$. Then, when $0 \leq n \leq k$, $\frac{h(p2^{n-k}+2^{n-m})}{2^n} = \frac{h(p2^{n-k}) \pm 2^{n-m}}{2^n} = \frac{h(2^n x)}{2^n} \pm \frac{1}{2^m}$. So, $g(x+x_m) = (\sum_{n=0}^{k} \frac{h(2^n x)}{2^n} \pm \frac{1}{2^m}) + (\sum_{n=k+1}^{m} \frac{1}{2^m})$. Then, $\frac{g(x+x_m)-g(x)}{x_m} = \frac{(\sum_{n=0}^{k} \frac{h(2^n x)}{2^n} \pm \frac{1}{2^m}) + (\sum_{n=k+1}^{m} \frac{1}{2^m}) - g(x)}{\frac{1}{2^m}} = (\sum_{n=0}^{k} \pm 1) + (m-k-1)$. If it is supposed to be $\sum_{n=0}^{k} -1$ then we get $m-2k-1$, else if it supposed to be $\sum_{n=0}^{k} 1$ then we get $m-1$. Regardless, we see that as $m \to \infty$, both $m-2k-1$ and $m-1$ are unbounded meaning that $g'(x)$ does not exist eitherways.