2.2.1 Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1. $\lim \frac{1}{6n^2+1} = 0$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\sqrt{6\epsilon}}$. To verify let $n \geq N$ which means $\left|\frac{1}{6n^2+1}\right| < \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| = \left|\frac{1}{\frac{1}{\epsilon}+1}\right| = \frac{\epsilon}{\epsilon+1} < \epsilon$ as desired. □

2. $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{13-10\epsilon}{4\epsilon}$. To verify let $n \geq N$ and so $\left|\frac{3n+1}{2n+5}\right| < \left|\frac{3(\frac{13-10\epsilon}{4\epsilon})+1}{2(\frac{13-10\epsilon}{4\epsilon})+5}\right| = \frac{3}{2} - \epsilon$. So, $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon$ as desired. □

3. $\lim \frac{2}{\sqrt{n+3}} = 0$

   Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \frac{4}{\epsilon^2} - 3$. To verify let $n \geq N$ which means $\left|\frac{2}{\sqrt{n+3}}\right| < \left|\frac{2}{\sqrt{(\frac{4}{\epsilon^2}-3)+3}}\right| = \epsilon$ as desired. □

2.2.7 Informally speaking, the sequence $\sqrt{n}$ "converges to infinity."

(a) Imitate the logical structure of Definition 2.2.3 to create a rigorous definition for the mathematical statement $\lim x_n = \infty$. Use this definition to prove $\lim \sqrt{n} = \infty$.
A sequence (a_n) "converges to infinity" if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $a_n > \epsilon$.

Proof. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > \epsilon^2$. To verify let $n \geq N$ which means $\sqrt{n} > \sqrt{\epsilon^2} = \epsilon$ as desired. □

(b) What does your definition in (a) say about the particular sequence $(1, 0, 2, 0, 3, 0, 4, 0, 5, 0, ...)$?
It does not converge to infinity.

2.2.8 Here are two useful definitions:

1. A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
2. A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
Frequently

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

Eventually is stronger; eventually implies frequently.

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is eventually in the set $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

x_n is not necessarily eventually in the interval $(1.9, 2.1)$, but it is frequently in $(1.9, 2.1)$. For example, $(2, 2, 2, 2, ...)$ is eventually in the interval $(1.9, 2.1)$, but $(2, -2, 2, -2, ...)$ is only frequently in the interval $(1.9, 2.1)$.

2.3.3 **(Squeeze Theorem)**. Show that if $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof. Assume that $\forall n \in \mathbb{N}, x_n \le y_n \le z_n$ and $\lim x_n = \lim z_n = l$. Suppose $n \in \mathbb{N}$. Then, by the Order Limit Theorem, since $x_n \le y_n$, $\lim y_n \ge \lim x_n = l$. Similarly, since $y_n \le z_n$, $\lim y_n \le \lim z_n = l$. So, $l \le \lim y_n \le l$. Thus, $\lim y_n = l$. □

2.3.5 Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Proof.
First we prove that if (z_n) is convergent then (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$. Assume (z_n) is convergent to z. Let $\epsilon > 0$ be arbitrary. Because (z_n) is convergent to z we know there exists an $N \in \mathbb{N}$ such that for all $n \ge N$, $|z_n - z| < \epsilon$. So, $\epsilon > |x_n - z|$ for all $n \ge N$ because $x_n = z_{2n-1}$ and $2n - 1 \ge N$. Similarly, $\epsilon > |y_n - z|$ for all $n \ge N$ because $y_n = z_{2n}$ and $2n \ge N$. Therefore, $\lim x_n = z = \lim y_n$.

Next we prove that if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$ then (z_n) is convergent. Assume (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = l$. Let $\epsilon > 0$ be arbitrary. Then, we know that there exists an $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|x_n - l| < \epsilon$. Similarly, we know that there exists an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|y_n - l| < \epsilon$. Let $N = \max\,\{2N_1, 2N_2\}$. Then, for all $n \ge N$, we have that $|z_n - l| < \epsilon$ since (z_n) consists of alternating elements from (x_n) and (y_n) which means after N both $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$. Thus, (z_n) is convergent. □

2.3.10 If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, then show that $(b_n) \to b$.

Proof. Assume $(a_n) \to 0$ and $|b_n - b| \leq a_n$. Let $\epsilon > 0$ be arbitrary. Since $(a_n) \to 0$, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N, |a_n| < \epsilon$. Then, for all $n \geq N$, $|b_n - b| \leq a_n < \epsilon$ which means $|b_n - b| < \epsilon$, and so $(b_n) \to b$. $\square$