1.2.1 (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof. Assume for the sake of contradiction that $\sqrt{3}$ is rational, that is there exist coprime integers p and q such that $3 = (\frac{p}{q})^2$. Then, $3q^2 = p^2$. Since p^2 is divisible by 3, p can be represented as $p = 3r$ for some integer r. So, $3q^2 = (3r)^2$ which simplifies to $q^2 = 3r^2$. Since, q^2 is divisible by 3, we have shown that both p and q are divisible by 3. However, this is a contradiction to the original claim that p and q are coprime integers. Thus, $\sqrt{3}$ is irrational. □

Yes, a similar argument can be made to show that $\sqrt{6}$ is irrational.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

It breaks down when we have that $4q^2 = p^2$ where p and q are coprime integers. It breaks down because $p = 2q$ and so p is not always divisble by 4 which means that we cannot represent p as $p = 4r$ for some $r \in \mathbb{Z}$. Thus, the proof fails.

1.2.2 Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$ ... are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

False. Let $A_n = \mathbb{N}_{\geq n}$ where $n \in \mathbb{N}$. Then, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ which is a size of 0.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$ ... are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

True. Since $\forall n \in \mathbb{N}$, $A_n \subseteq A_1$ and A_n is finite and nonempty, there exists an x such that $x \in A_n$ which means $x \in A_1$, and so x is in $\bigcap_{n=1}^{\infty} A_n$. Thus, $\bigcap_{n=1}^{\infty} A_n$ is nonempty. Also, $\forall n \in \mathbb{N}$, A_n is finite which means $\bigcap_{n=1}^{\infty} A_n$ is finite since the intersection of finite sets must be finite. Thus, $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

False, let $A = \{0\}, B = \{1\}, C = \{2\}$. Then, $A \cap (B \cup C) = \{0\} \cap (\{1\} \cup \{2\}) = \{0\} \cap \{1, 2\} = \emptyset$. But, $(A \cap B) \cup C = (\{0\} \cap \{1\}) \cup \{2\} = \emptyset \cup \{2\} = \{2\}$. $\emptyset \neq \{2\}$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

True, because set intersection is associative.

Proof. By the definition of set intersection, $A \cap (B \cap C)$ is equivalent to $(x \in A) \wedge (x \in B \wedge x \in C)$. Since conjunction is associative, this becomes $(x \in A \wedge x \in B) \wedge x \in C$ which is equivalent to $(A \cap B) \cap C$. □

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
True, because set intersection is distributive over set union.

Proof. By the definition of set union and set intersection, $A \cap (B \cup C)$ is equivalent to $(x \in A) \wedge (x \in B \vee x \in C)$. By the distributive rule for conjuction over disjunction, this becomes $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ which is equivalent to $(A \cap B) \cup (A \cap C)$. □

1.2.10 Let $y_1 = 1$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{3y_n+4}{4}$.

(a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

Proof.
Base Case ($n = 1$): $y_{n+1} = y_2 = \frac{3(1)+4}{4} = \frac{7}{4} < 4$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume that for all natural numbers $i < k$, $y_{i+1} < 4$.
* Need to prove that $y_k < 4$, that is $\frac{3y_{k-1}+4}{4} = \frac{3}{4}y_{k-1} + 1 < 4$. By the induction hypothesis, $y_{k-1} < 4$. So, $\frac{3}{4}y_{k-1} < 3$ which means $\frac{3}{4}y_{k-1} + 1 < 4$. So, $y_k < 4$.

Hence, by the principle of complete induction, $\forall n \in \mathbb{N}, y_n < 4$. □

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, ...)$ is increasing.

Proof. We'll write $p(n)$ to denote the statement "$y_n \leq y_{n+1}$". Need to prove that $\forall n \in \mathbb{N}, p(n)$.
Base Case ($n = 1$): Then, $y_n = y_1 = 1$ and $y_{n+1} = y_2 = \frac{7}{4}$. Clearly, $1 \leq \frac{7}{4}$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume that for all natural numbers $i < k$, $p(i)$ is true.
* Need to prove that $p(k)$ holds true, that is $y_k \leq y_{k+1}$. By the definition of y, $y_k = \frac{3y_{k-1}+4}{4}$ and $y_{k+1} = \frac{3y_k+4}{4}$. Need to show that $\frac{3y_{k-1}+4}{4} \leq \frac{3y_k+4}{4}$, or in more simplified terms $y_{k-1} \leq y_k$. By the induction hypothesis, $p(k-1)$ is true, that is $y_{k-1} \leq y_k$. So, $\frac{3y_{k-1}+4}{4} \leq \frac{3y_k+4}{4}$ which means $y_k \leq y_{k+1}$. Thus, $p(k)$ holds true.

Hence, by the principle of complete induction, $\forall n \in \mathbb{N}, p(n)$ is true. □