6.2.11 Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

Proof. Suppose $\epsilon > 0$ is arbitrarily chosen. Since (f_n) is a uniformly convergent sequence of functions, we can choose $N_1 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ whenever $n, m \geq N_1$ and $x \in A$. Similarly, since (g_n) is a uniformly convergent sequence of functions, we can choose $N_2 \in \mathbb{N}$ such that $|g_n(x) - g_m(x)| < \frac{\epsilon}{2}$ whenever $n, m \geq N_2$ and $x \in A$. Let $N = \max(N_1, N_2)$. Then, $|f_n(x) + g_n(x) - f_m(x) - g_m(x)| \leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n, m \geq N$ and $x \in A$. Thus, by the Cauchy Criterion for Uniform Convergence, $(f_n + g_n)$ is a uniformly convergent sequence of functions. □

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.
Let $f_n = \frac{1}{n(1+x)}$ which the textbook says uniformly converges to 0 on $\mathbb{R}$.
Let $g_n = \frac{1}{n} + x$ which we can see converges uniformly to x on $\mathbb{R}$.
Then $f_n * g_n = \frac{1}{n^2(1+x)} + \frac{x}{n(1+x)}$. The limit of this as $n \to \infty$ is 0. To see that it is not uniformly convergent notice that $|f_n(x) - f(x)| = \left|\frac{xn+1}{n^2(1+x)}\right|$ which means that $N > \frac{xn+1}{n^2(1+x)}$. So, N depends on x and thus there is no way to pick an N that will satisfy all x. Thus, it fails to be uniformly convergent.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

Proof. Suppose $\epsilon > 0$ is arbitrarily chosen. Assume there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$. Need to show that $(f_n g_n)$ does converge uniformly, that is there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in A$, it holds true that $|f_n(x)g_n(x) - f_m(x)g_m(x)| < \epsilon$ which when simplified becomes

$$\begin{aligned}
|f_n(x)g_n(x) - f_m(x)g_m(x)| &= |f_n(x)g_n(x) - f_n(x)g_m(x) + f_n(x)g_m(x) - f_m(x)g_m(x)| \\
&= |f_n(x)(g_n(x) - g_m(x)) + g_m(x)(f_n(x) - f_m(x))| \\
&\leq |f_n(x)(g_n(x) - g_m(x))| + |g_m(x)(f_n(x) - f_m(x))| \\
&\leq |f_n(x)||(g_n(x) - g_m(x))| + |g_m(x)||f_n(x) - f_m(x)| < \epsilon
\end{aligned}$$

Since (f_n) is a uniformly convergent sequence of functions, we can choose $N_1 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{2M}$ whenever $n, m \geq N_1$ and $x \in A$. Since (g_n) is a uniformly convergent sequence of functions, we can choose $N_2 \in \mathbb{N}$ such that $|g_n(x) - g_m(x)| < \frac{\epsilon}{2M}$ whenever $n, m \geq N_2$ and $x \in A$. Then, let $N = \max(N_1, N_2)$. So, we have that $|f_n(x)||(g_n(x) - g_m(x))| + |g_m(x)||f_n(x) - f_m(x)| < |f_n(x)|\frac{\epsilon}{2M} + |g_m(x)|\frac{\epsilon}{2M}$ whenever $n, m \geq N$ and $x \in A$. Since $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, this simplifies to $M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$. So, $|f_n(x)g_n(x) - f_m(x)g_m(x)| < \epsilon$ for all $m, n \geq N$ and $x \in A$. Thus, $(f_n g_n)$ converges uniformly. □

6.3.3 Consider the sequence of functions $f_n(x) = \frac{x}{1+nx^2}$. Exercise 6.2.4 contains some advice for how to show that (f_n) converges uniformly on $\mathbb{R}$. Review or complete that exercise. Now, let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

$f_n'(x) = \frac{1-nx^2}{(nx^2+1)^2}$.

$f'(x) = \lim f_n'(x)$ for all $x \neq 0$. We can see this because $f = \lim f_n$ and $f_n \to 0$ for all x which means that $\lim f_n'(x) = f'(x) = 0$ when $x \neq 0$, but when $x = 0$, $f'(x) = 1$ and $\lim f_n'(x) = 0$ – thus they are not equal at $x = 0$.

6.4.7 Let $h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$.

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.

Proof. Let $M_n = \frac{1}{n^2}$. Notice that $\sum_{n=1}^{\infty} M_n$ converges by the p-series test. Then, since $\left|\frac{1}{x^2+n^2}\right| \leq M_n$ and $\sum_{n=1}^{\infty} M_n$ converges, we know by the Weierstrass M-Test that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly. Thus, since it converges uniformly and $\frac{1}{x^2+n^2}$ is continuous for all $x \in \mathbb{R}$ and $n \geq 1$, h must be a continuous function defined on all of $\mathbb{R}$. □

(b) Is h differentiable? If so, is the derivative function h' continuous?

Yes it is differentiable and the derivative function h' is continuous. First, we look at the derivative of $\frac{1}{x^2+n^2}$ which is $\frac{-2x}{(x^2+n^2)^2}$. Notice that $\sum_{n=1}^{\infty} \frac{-2x}{(x^2+n^2)^2} = \sum_{n=1}^{\infty} \frac{|-2x|}{x^2+n^2}\frac{1}{x^2+n^2} \leq \sum_{n=1}^{\infty} \frac{2|x|}{n^2x^2+n^4} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$ which we know converges by the p-series test. Let $M_n = \frac{1}{n^3}$. So, by the Weierstrass M-Test, $\sum_{n=1}^{\infty} \frac{-2x}{(x^2+n^2)^2}$ converges uniformly for all x meaning that h is differentiable. Similar to part a, we see that since h' converges uniformly and $\frac{-2x}{(x^2+n^2)^2}$ is continuous for all $x \in \mathbb{R}$ with $n \geq 1$, we know that h' must be continuous.

6.5.9 Use Theorem 6.5.7 to argue that power series are unique. If we have $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2....$ (Start by showing that $a_0 = b_0$.)

Proof. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Assume $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, in other words $f(x) = g(x)$, for all x in an interval $(-R, R)$. Then, for $x = 0$, we have $a_0 = f(0)$ and $b_0 = g(0)$ which means $a_0 = b_0$. Next, take the derivative of f and g to get $\sum_{n=1}^{\infty} na_n x^{n-1}$ and $\sum_{n=1}^{\infty} nb_n x^{n-1}$ respectively. Once again, for $x = 0$, we have $a_1 = f'(0)$ and $b_1 = g'(0)$ which means that $a_1 = b_1$. We can inductively continue this process to find that the $k'th$ derivative of f and g are $\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$ and $\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} b_n x^{n-k}$ respectively. Then, once again, for $x = 0$, we have $a_k = f^k(0)$ and $b_k = g^k(0)$ which means that $a_k = b_k$. Thus, $a_n = b_n$ for all $n = 0, 1, 2...$ as desired. □