1.2.1 (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof. Assume for the sake of contradiction that $\sqrt{3}$ is rational, that is there exist coprime integers p and q such that $3 = (\frac{p}{q})^2$. Then, $3q^2 = p^2$. Since p^2 is divisible by 3, p can be represented as $p = 3r$ for some integer r. So, $3q^2 = (3r)^2$ which simplifies to $q^2 = 3r^2$. Since, q^2 is divisible by 3, we have shown that both p and q are divisible by 3. However, this is a contradiction to the original claim that p and q are coprime integers. Thus, $\sqrt{3}$ is irrational. □

Yes, a similar argument can be made to show that $\sqrt{6}$ is irrational.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

It breaks down when we have that $4q^2 = p^2$ where p and q are coprime integers. It breaks down because $p = 2q$ and so p is not always divisble by 4 which means that we cannot represent p as $p = 4r$ for some $r \in \mathbb{Z}$. Thus, the proof fails.

1.2.2 Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$ ... are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

False. Let $A_n = \mathbb{N}_{\geq n}$ where $n \in \mathbb{N}$. Then, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ which is a size of 0.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$ ... are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

True. Since $\forall n \in \mathbb{N}$, $A_n \subseteq A_1$ and A_n is finite and nonempty, there exists an x such that $x \in A_n$ which means $x \in A_1$, and so x is in $\bigcap_{n=1}^{\infty} A_n$. Thus, $\bigcap_{n=1}^{\infty} A_n$ is nonempty. Also, $\forall n \in \mathbb{N}$, A_n is finite which means $\bigcap_{n=1}^{\infty} A_n$ is finite since the intersection of finite sets must be finite. Thus, $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

False, let $A = \{0\}, B = \{1\}, C = \{2\}$. Then, $A \cap (B \cup C) = \{0\} \cap (\{1\} \cup \{2\}) = \{0\} \cap \{1, 2\} = \emptyset$. But, $(A \cap B) \cup C = (\{0\} \cap \{1\}) \cup \{2\} = \emptyset \cup \{2\} = \{2\}$. $\emptyset \neq \{2\}$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

True, because set intersection is associative.

Proof. By the definition of set intersection, $A \cap (B \cap C)$ is equivalent to $(x \in A) \wedge (x \in B \wedge x \in C)$. Since conjunction is associative, this becomes $(x \in A \wedge x \in B) \wedge x \in C$ which is equivalent to $(A \cap B) \cap C$. □

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
True, because set intersection is distributive over set union.

Proof. By the definition of set union and set intersection, $A \cap (B \cup C)$ is equivalent to $(x \in A) \wedge (x \in B \vee x \in C)$. By the distributive rule for conjuction over disjunction, this becomes $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ which is equivalent to $(A \cap B) \cup (A \cap C)$. □

1.2.10 Let $y_1 = 1$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{3y_n+4}{4}$.

(a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

Proof.
Base Case ($n = 1$): $y_{n+1} = y_2 = \frac{3(1)+4}{4} = \frac{7}{4} < 4$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume that for all natural numbers $i < k$, $y_{i+1} < 4$.
* Need to prove that $y_k < 4$, that is $\frac{3y_{k-1}+4}{4} = \frac{3}{4}y_{k-1} + 1 < 4$. By the induction hypothesis, $y_{k-1} < 4$. So, $\frac{3}{4}y_{k-1} < 3$ which means $\frac{3}{4}y_{k-1} + 1 < 4$. So, $y_k < 4$.

Hence, by the principle of complete induction, $\forall n \in \mathbb{N}, y_n < 4$. □

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, ...)$ is increasing.

Proof. We'll write $p(n)$ to denote the statement "$y_n \leq y_{n+1}$". Need to prove that $\forall n \in \mathbb{N}, p(n)$.
Base Case ($n = 1$): Then, $y_n = y_1 = 1$ and $y_{n+1} = y_2 = \frac{7}{4}$. Clearly, $1 \leq \frac{7}{4}$.
Induction Step:

* Suppose $k \in \mathbb{N}$ such that $k \geq 2$.
* Assume that for all natural numbers $i < k$, $p(i)$ is true.
* Need to prove that $p(k)$ holds true, that is $y_k \leq y_{k+1}$. By the definition of y, $y_k = \frac{3y_{k-1}+4}{4}$ and $y_{k+1} = \frac{3y_k+4}{4}$. Need to show that $\frac{3y_{k-1}+4}{4} \leq \frac{3y_k+4}{4}$, or in more simplified terms $y_{k-1} \leq y_k$. By the induction hypothesis, $p(k-1)$ is true, that is $y_{k-1} \leq y_k$. So, $\frac{3y_{k-1}+4}{4} \leq \frac{3y_k+4}{4}$ which means $y_k \leq y_{k+1}$. Thus, $p(k)$ holds true.

Hence, by the principle of complete induction, $\forall n \in \mathbb{N}, p(n)$ is true. □

1.3.2 (a) Write a formal definition in the style of Definition 1.3.2 for the infimum or greatest lower bound of a set.

A real number s is an infimum for a set $A \subseteq \mathbb{R}$ if it meets these two criteria:
(i) s is a lower bound for A
(ii) if b is any lower bound for A, then $s \geq b$

(b) Now, state and prove a version of Lemma 1.3.7 for greatest lower bounds.

Assume $s \in \mathbb{R}$ is a lower bound for the set $A \subseteq \mathbb{R}$. Then, $s = \inf A$ iff for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s + \epsilon > a$.

Proof.
($\Rightarrow$) Assume $s = \inf A$. Need to show that $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$. Assume $\epsilon > 0$. Since s is an infimum for A and $s + \epsilon > s$, $s + \epsilon$ cannot be a lower bound for A which means that $\exists a \in A$ such that $a < s + \epsilon$ (because otherwise $s + \epsilon$ would be a lower bound). Thus, the claim is satisfied.

($\Leftarrow$) Assume s is a lower bound for A such that $\forall \epsilon > 0$, $s + \epsilon$ is not a lower bound for A. So, for all lower bounds l, $l \leq s$. This satisfies both parts of the definition of $s = \inf A$, that is s is a lower bound and for all lower bounds l, $l \leq s$. □

1.3.8 If sup $A <$ sup B, then show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Assume $\sup A < \sup B$. Need to show that $\exists b \in B$ such that b is an upper bound for A. Let $\epsilon = \sup B - \sup A > 0$. Then, we know that $\exists b \in B$ such that $b > \sup B - \epsilon = \sup A$. Thus, since $b > \sup A$, b is an upper bound for A. □

1.3.9 Without worrying about formal proofs for the moment, decide if the following statements about suprema and infima are true or false. For any that are false, supply an example where the claim in question does not appear to hold.

(a) A finite, nonempty set always contains its supremum.

True, the last element of the set is the supremum.

(b) If $a < L$ for every element a in the set A, then $\sup A < L$.

False, let $A = (0, 1)$ which means $\sup A = 1$. Let $L = 1$. Then, $\forall a \in A, a < L$, but $\sup A \not< L$.

(c) If A and B are sets with the property that $a < b$ for every $a \in A$ and every $b \in B$, then it follows that $\sup A < \inf B$.

False, let $A = (0, 1)$ and $B = (1, 2)$. Then, $\forall a \in A$ and $\forall b \in B$, $a < b$, but $\sup A = 1 = \inf B$.

(d) If $\sup A = s$ and $\sup B = t$, then $\sup(A + B) = s + t$. The set $A + B$ is defined as $A + B = \{a + b : a \in A \text{ and } b \in B\}$.

True

(e) If $\sup A \leq \sup B$ then there exists a $b \in B$ that is an upper bound for A.

False, let $A = [0, 1]$ and $B = (0, 1)$. Then, $\sup A = \sup B = 1$. But, there is no $b \in B$ such that b is an upper bound for A.