3.2.2 Let $B = \{\frac{(-1)^n n}{n+1} : n = 1, 2, 3, ...\}$.

(a) Find the limit points of B.
$\{-1, 1\}$

(b) Is B a closed set?
No, it contains neither of its limit points.

(c) Is B an open set?
No, its not possible to find an ϵ-neighborhood for every point in B such that the ϵ-neighborhood is contained in B.

(d) Does B contain any isolated points?
Every element of B is an isolated point.

(e) Find $\overline{B}$.
$B \cup \{-1, 1\}$

3.2.6 Prove Theorem 3.2.8: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof.
First we prove that if a set $F \subseteq \mathbb{R}$ is closed then every Cauchy sequence contained in F has a limit that is also an element of F. Assume $F \subseteq \mathbb{R}$ is closed, that is F contains its limit points. So, we need to show that every Cauchy sequence (a_n) contained in F has a limit in F. Assume (a_n) is an arbitrary Cauchy sequence contained in F. Since (a_n) is Cauchy, it's limit exists. So, let $a = \lim(a_n)$. Now, we need to show that a is either a limit point in F or an isolated point in F. If $a_n \neq a$ for all $n \in \mathbb{N}$, then a is a limit point and since F is closed, $a \in F$. Otherwise, $a_n = a$ for some $n \in \mathbb{N}$, and since $(a_n) \subseteq F$, $a \in F$. So, every Cauchy sequence contained in F has a limit that is also an element of F.
Next, we prove that if every Cauchy sequence contained in a set F has a limit that is also an element of F, then $F \subseteq \mathbb{R}$ is closed. Assume every Cauchy sequence contained in a set $F \subseteq \mathbb{R}$ has a limit that is also an element of F. Need to show that F is closed, that is F contains all its limit points. Let a be an arbitrary limit point of F. Then, $a = \lim a_n$ for some sequence (a_n) contained in F. Since (a_n) converges, it must be a Cauchy sequence. So, $a \in F$. Thus, F is closed. □

3.2.10 (De Morgan's Laws): A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.3. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$ and $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

Proof. First, we need to show that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$, that is $\forall x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$, $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Suppose $x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Then, by definition of set complement, $\forall \lambda \in \Lambda, x \notin E_\lambda$. So, $\forall \lambda \in \Lambda, x \in E_\lambda^c$. Then, by definition of set intersection, $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. So, $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Next, we need to show that $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$, that is $\forall y \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$, $y \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Suppose $y \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then, by definition of set intersection, $\forall \lambda \in \Lambda, y \in E_\lambda^c$. So, $\forall \lambda \in \Lambda, y \notin E_\lambda$. Then, by definition of set union, $y \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. So, $y \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Thus, $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$ and $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$ means that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$. □

Proof. First, we need to show that $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$, that is $\forall x \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$, $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Suppose $x \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. So, $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$ which means that there exists at least one $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Choose $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Then, $x \in E_{\lambda'}^c$. So, $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$ which means $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Next we need to prove that $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$, that is $\forall y \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$, $y \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. Suppose $y \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Then, there exists at least one $\lambda'' \in \Lambda$ such that $y \notin E_{\lambda''}$. Choose $\lambda'' \in \Lambda$ such that $y \notin E_{\lambda''}$. Then, $y \notin \bigcap_{\lambda\in\Lambda} E_\lambda$. So, $y \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ which means $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. Thus, $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$ and $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ means that $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$. □

(b) Now, provide the details for the proof of Theorem 3.2.14

(i) The union of a finite collection of closed sets is closed.

Proof. Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is a collection of closed sets. Then, $\{E_\lambda : \lambda \in \Lambda\}^c$ is a collection of open sets and we know that the intersection of a finite amount of open sets is open (Theorem 3.2.3). So, taking the complement again $(\{E_\lambda : \lambda \in \Lambda\}^c)^c = \{E_\lambda : \lambda \in \Lambda\}$ gives us a closed set (since the complement of an open set is a closed set) as desired. □

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof. Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of closed sets. Then, E_λ^c is open and $\forall \lambda \in \Lambda$, the union of E_λ^c is open (Theorem 3.2.3). By De Morgan's Law, we know $\bigcup_{\lambda\in\Lambda} E_\lambda^c = (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ so $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ is open. Then, $\bigcap_{\lambda\in\Lambda} E_\lambda$ is closed. Thus, the intersection of an arbitrary collection of closed sets is closed. □

3.3.4 Show that if K is compact and F is closed, then $K \cap F$ is compact.

Proof. Suppose F is closed and K is compact, that is K is bounded and closed. Need to show that $K \cap F$ is compact, that is $K \cap F$ is bounded and closed. Since K is bounded and $K \cap F \subseteq K$, $K \cap F$ is bounded. Also, since K is closed and F is closed, $K \cap F$ is closed (since the intersection of an arbitrary collection of closed sets is closed). So, $K \cap F$ is bounded and closed which means $K \cap F$ is compact. □

3.3.8 Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K, and bisect I_0 into two closed intervals A_1 and B_1.

(a) Why must either $A_1 \cap K$ or $B_1 \cap K$ (or both) have no finite subcover consisting of sets from $\{O_\lambda : \lambda \in \Lambda\}$.

At least one of $A_1 \cap K$ or $B_1 \cap K$ must have no finite subcover since if they both did have a finite subcover then the union of them would be a finite subcover for K which would contradict the assumption that no finite subcover exists for K.

(b) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

Choose whichever of $A_1 \cap K$ or $B_1 \cap K$ does not have a finite subcover (choose any one if they both do not), then call that choice I_1. Then, bisect I_1 to give A_2 and B_2. Once again, either $A_2 \cap K$ or $B_2 \cap K$ (or both) have no finite subcover. Choose whichever of $A_2 \cap K$ or $B_2 \cap K$ does not have a finite subcover (choose either one if they both do not), then we can call that choice I_2. Repeating this over results in the sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$ where $I_n \cap K$ cannot be finitely covered and as this sequence goes further, it tends towards $\lim |I_n| = 0$.

(c) Show that there exists an $x \in K$ such that $x \in I_n$ for all n.

Since K is compact, it is closed and bounded. So, $K \cap I_n \subseteq K$ is also closed and bounded for all $n \in \mathbb{N}$ which means $K \cap I_n$ is compact. Thus, by Theorem 3.3.5, the intersection of a nested sequence of nonempty compact sets is nonempty, that is $\exists x \in K$ such that $x \in K \cap I_n \subseteq I_n$ for all $n \in \mathbb{N}$. So, $x \in I_n$ for all $n \in \mathbb{N}$.

(d) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Argue that there must be an n_0 large enough to guarantee that $I_{n_0} \subseteq O_{\lambda_0}$. Explain why this furnishes us with the desired contradiction.

Since O_{λ_0} is an open set, there exists $\epsilon > 0$ such that the ϵ-neighborhood $V_\epsilon(x) \subseteq O_{\lambda_0}$. So, choose $n_0 \in \mathbb{N}$ such that $|I_{n_0}| < \epsilon$. Then, $I_{n_0} \subseteq O_{\lambda_0}$ which means I_{n_0} has a finite subcover. However, this is a contradiction to the initial claim that K has no finite subcover because $K \cap I_{n_0}$ has a finite subcover, namely O_{λ_0}.

3.3.10 Let's call a set clompact if it has the property that every closed cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbb{R}$.

All finite sets in $\mathbb{R}$ are clompact.

3.4.4

3.4.5

3.4.7

3.5.1

3.5.2

3.5.3