3.2.2 Let $B = \{\frac{(-1)^n n}{n+1} : n = 1, 2, 3, ...\}$.

(a) Find the limit points of B.
$\{-1, 1\}$

(b) Is B a closed set?
No, it contains neither of its limit points.

(c) Is B an open set?
No, its not possible to find an ϵ-neighborhood for every point in B such that the ϵ-neighborhood is contained in B.

(d) Does B contain any isolated points?
Every element of B is an isolated point.

(e) Find $\overline{B}$.
$B \cup \{-1, 1\}$

3.2.6 Prove Theorem 3.2.8: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof.
First we prove that if a set $F \subseteq \mathbb{R}$ is closed then every Cauchy sequence contained in F has a limit that is also an element of F. Assume $F \subseteq \mathbb{R}$ is closed, that is F contains its limit points. So, we need to show that every Cauchy sequence (a_n) contained in F has a limit in F. Assume (a_n) is an arbitrary Cauchy sequence contained in F. Since (a_n) is Cauchy, it's limit exists. So, let $a = \lim(a_n)$. Now, we need to show that a is either a limit point in F or an isolated point in F. If $a_n \neq a$ for all $n \in \mathbb{N}$, then a is a limit point and since F is closed, $a \in F$. Otherwise, $a_n = a$ for some $n \in \mathbb{N}$, and since $(a_n) \subseteq F$, $a \in F$. So, every Cauchy sequence contained in F has a limit that is also an element of F.
Next, we prove that if every Cauchy sequence contained in a set F has a limit that is also an element of F, then $F \subseteq \mathbb{R}$ is closed. Assume every Cauchy sequence contained in a set $F \subseteq \mathbb{R}$ has a limit that is also an element of F. Need to show that F is closed, that is F contains all its limit points. Let a be an arbitrary limit point of F. Then, $a = \lim a_n$ for some sequence (a_n) contained in F. Since (a_n) converges, it must be a Cauchy sequence. So, $a \in F$. Thus, F is closed. $\square$

3.2.10 (De Morgan's Laws): A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.3. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$ and $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

Proof. First, we need to show that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$, that is $\forall x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$, $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Suppose $x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Then, by definition of set complement, $\forall \lambda \in \Lambda, x \notin E_\lambda$. So, $\forall \lambda \in \Lambda, x \in E_\lambda^c$. Then, by definition of set intersection, $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. So, $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Next, we need to show that $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$, that is $\forall y \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$, $y \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Suppose $y \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then, by definition of set intersection, $\forall \lambda \in \Lambda, y \in E_\lambda^c$. So, $\forall \lambda \in \Lambda, y \notin E_\lambda$. Then, by definition of set union, $y \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. So, $y \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Thus, $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$ and $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$ means that $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$. $\square$

Proof. First, we need to show that $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$, that is $\forall x \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$, $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Suppose $x \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. So, $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$ which means that there exists at least one $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Choose $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Then, $x \in E_{\lambda'}^c$. So, $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$ which means $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Next we need to prove that $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$, that is $\forall y \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$, $y \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. Suppose $y \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Then, there exists at least one $\lambda'' \in \Lambda$ such that $y \notin E_{\lambda''}$. Choose $\lambda'' \in \Lambda$ such that $y \notin E_{\lambda''}$. Then, $y \notin \bigcap_{\lambda\in\Lambda} E_\lambda$. So, $y \in (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ which means $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$. Thus, $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$ and $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ means that $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$. $\square$

(b) Now, provide the details for the proof of Theorem 3.2.14
(i) The union of a finite collection of closed sets is closed.

Proof. Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is a collection of closed sets. Then, $\{E_\lambda : \lambda \in \Lambda\}^c$ is a collection of open sets and we know that the intersection of a finite amount of open sets is open (Theorem 3.2.3). So, taking the complement again $(\{E_\lambda : \lambda \in \Lambda\}^c)^c = \{E_\lambda : \lambda \in \Lambda\}$ gives us a closed set (since the complement of an open set is a closed set) as desired. $\square$

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof. Suppose $\{E_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of closed sets. Then, E_λ^c is open and $\forall \lambda \in \Lambda$, the union of E_λ^c is open (Theorem 3.2.3). By De Morgan's Law, we know $\bigcup_{\lambda\in\Lambda} E_\lambda^c = (\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ so $(\bigcap_{\lambda\in\Lambda} E_\lambda)^c$ is open. Then, $\bigcap_{\lambda\in\Lambda} E_\lambda$ is closed. Thus, the intersection of an arbitrary collection of closed sets is closed. $\square$

3.3.4 Show that if K is compact and F is closed, then $K \cap F$ is compact.

Proof. Suppose F is closed and K is compact, that is K is bounded and closed. Need to show that $K \cap F$ is compact, that is $K \cap F$ is bounded and closed. Since K is bounded and $K \cap F \subseteq K$, $K \cap F$ is bounded. Also, since K is closed and F is closed, $K \cap F$ is closed (since the intersection of an arbitrary collection of closed sets is closed). So, $K \cap F$ is bounded and closed which means $K \cap F$ is compact. □

3.3.8 Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K, and bisect I_0 into two closed intervals A_1 and B_1.

(a) Why must either $A_1 \cap K$ or $B_1 \cap K$ (or both) have no finite subcover consisting of sets from $\{O_\lambda : \lambda \in \Lambda\}$.

At least one of $A_1 \cap K$ or $B_1 \cap K$ must have no finite subcover since if they both did have a finite subcover then the union of them would be a finite subcover for K which would contradict the assumption that no finite subcover exists for K.

(b) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

Choose whichever of $A_1 \cap K$ or $B_1 \cap K$ does not have a finite subcover (choose any one if they both do not), then call that choice I_1. Then, bisect I_1 to give A_2 and B_2. Once again, either $A_2 \cap K$ or $B_2 \cap K$ (or both) have no finite subcover. Choose whichever of $A_2 \cap K$ or $B_2 \cap K$ does not have a finite subcover (choose either one if they both do not), then we can call that choice I_2. Repeating this over results in the sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ where $I_n \cap K$ cannot be finitely covered and as this sequence goes further, it tends towards $\lim |I_n| = 0$.

(c) Show that there exists an $x \in K$ such that $x \in I_n$ for all n.

Since K is compact, it is closed and bounded. So, $K \cap I_n \subseteq K$ is also closed and bounded for all $n \in \mathbb{N}$ which means $K \cap I_n$ is compact. Thus, by Theorem 3.3.5, the intersection of a nested sequence of nonempty compact sets is nonempty, that is $\exists x \in K$ such that $x \in K \cap I_n \subseteq I_n$ for all $n \in \mathbb{N}$. So, $x \in I_n$ for all $n \in \mathbb{N}$.

(d) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Argue that there must be an n_0 large enough to guarantee that $I_{n_0} \subseteq O_{\lambda_0}$. Explain why this furnishes us with the desired contradiction.

Since O_{λ_0} is an open set, there exists $\epsilon > 0$ such that the ϵ-neighborhood $V_\epsilon(x) \subseteq O_{\lambda_0}$. So, choose $n_0 \in \mathbb{N}$ such that $|I_{n_0}| < \epsilon$. Then, $I_{n_0} \subseteq O_{\lambda_0}$ which means I_{n_0} has a finite subcover. However, this is a contradiction to the initial claim that K has no finite subcover because $K \cap I_{n_0}$ has a finite subcover, namely O_{λ_0}.

3.3.10 Let's call a set clompact if it has the property that every closed cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbb{R}$.

All finite sets in $\mathbb{R}$ are clompact.

3.4.4 Repeat the Cantor construction from Section 3.1 starting with the open interval $[0, 1]$. This time, however, remove the open middle fourth from each component.

(a) Is the resulting set compact? Perfect?

Yes, the resulting set is both compact and perfect for similar reasons as the original Cantor Set from before (removing middle third component).

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Length: $1 - (\frac{1}{4} + 2(\frac{3}{32}) + 4(\frac{9}{256}) + ...) = 1 - (\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + ...) = 1 - (\frac{\frac{1}{4}}{1-\frac{3}{4}}) = 0.$

Dimension: Use $\frac{8}{3}$ as the magnifying amount. Then, $[0, 1]$ becomes $[0, \frac{8}{3}]$ which splits into $[0, 1] \cup [\frac{5}{3}, \frac{8}{3}]$. Since it splits into 2 and will continue to do so, we have $2 = (\frac{8}{3})^x$ which means $x = \frac{\ln 2}{\ln \frac{8}{3}} \approx 0.707$.

3.4.5 Let A and B be subsets of $\mathbb{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Proof. Let A and B be subsets of $\mathbb{R}$. Assume there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. Need to show that A and B are separated, that is $\overline{A} \cap B = \emptyset = \overline{B} \cap A$. Since U and V are disjoint, $U \cap V = \emptyset$ which means that $U \subseteq V^c$. Then, since V is open, V^c is closed which means that all of U's limit points must be in V^c, that is $\overline{U} \subseteq V^c$. So, $\overline{U} \cap V = \emptyset$. Similarly, $\overline{V} \cap U = \emptyset$. Then, since $A \subseteq U \subseteq \overline{U}$ and all of A's limit points must be limit points of U, $\overline{A} \subseteq \overline{U}$. Similarly, $\overline{B} \subseteq \overline{V}$. Thus, since $B \subseteq V$, $\overline{A} \cap B = \emptyset$. Similarly, since $A \subseteq U$, $\overline{B} \cap A = \emptyset$. So, A and B are separated. $\square$

3.4.7 (a) Find an example of a disconnected set whose closure is connected.

Let $A = (-\infty, 0)$ and let $B = (0, \infty)$. Then, $\overline{A} = (-\infty, 0]$ and $\overline{B} = [0, \infty)$. So, $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$ which means A and B are nonempty separated sets. Therefore, $E = (A \cup B) \subseteq \mathbb{R}$ is a disconnected set. Since, 0 is a limit point of E, $\overline{E} = (-\infty, \infty) = \mathbb{R}$ which is a connected set. Thus, E is a disconnected set whose closure is connected.

(b) If A is connected, is $\overline{A}$ necessarily connected? If A is perfect, is $\overline{A}$ necessarily perfect?

Yes, if A is connected then $\overline{A}$ is connected as well.

Yes, if A is perfect then $A = \overline{A}$ so $\overline{A}$ is perfect as well.

3.5.1 Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Proof.
Suppose A is a G_δ set. Then A can be written as $A = \bigcap_{n=1}^{\infty} O_n$ where each O_n for all $n \in \mathbb{N}$ is an open set. By DeMorgan's Law, $A^c = \bigcup_{n=1}^{\infty} O_n^c$ where each O_n^c for all $n \in \mathbb{N}$ is a closed set. So, A^c is an F_σ set as desired.

Suppose A is a F_σ set. Then A can be written as $A = \bigcup_{n=1}^{\infty} C_n$ where each C_n for all $n \in \mathbb{N}$ is a closed set. By DeMorgan's Law, $A^c = \bigcap_{n=1}^{\infty} C_n^c$ where each C_n^c for all $n \in \mathbb{N}$ is an open set. So, A^c is an G_δ set as desired. □

3.5.2 Replace each blank with the word finite or countable, depending on which is more appropriate.

(a) The countable union of F_σ sets is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set.

(c) The finite union of G_δ sets is an G_δ set.

(d) The countable intersection of G_δ sets is an G_δ set.

3.5.3 (This exercise has already appeared as Exercise 3.2.14.)

(a) Show that a closed interval $[a, b]$ is a G_δ set.
$[a, b]$ can be written as $[a, b] = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$ which is a countable intersection of open intervals.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.
$(a, b]$ can be written as $(a, b] = \bigcap_{n=1}^{\infty}(a, b + \frac{1}{n})$ which is a countable intersection of open intervals.
$(a, b]$ can be written as $(a, b] = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b]$ which is a countable union of closed intervals.

(c) Show that $\mathbb{Q}$ is an F_σ set, and the set of irrationals $\mathbb{I}$ forms a G_δ set.
We know that $\mathbb{Q}$ is countable so if we take the union of singleton sets each containing an element in $\mathbb{Q}$, then we get a countable union of closed sets, that is $Q = \bigcup_{n=1}^{\infty}\{q_n\}$ where $q_n \in \mathbb{Q}$. Thus, $\mathbb{Q}$ is an F_σ set. Then, since $\mathbb{I} = \mathbb{Q}^c$ and the complement of an F_σ set is a G_δ set, $\mathbb{I}$ is a G_δ set.