Tutorial 1: Analytic Functions

1. Find for each function given below, the domain of definition:
   (a) $f(z) = \frac{1}{z^2+1}$;
   (b) $f(z) = Arg(\frac{1}{z})$;
   (c) $f(z) = \frac{z}{\bar{z}+z}$;
   (d) $f(z) = \frac{1}{1-|z|^2}$.

2. Write each of the following functions in the form $f(z) = u(x, y) + iv(x, y)$:
   (a) $f(z) = z^3 - 1$;
   (b) $f(z) = \sin z$;
   (c) $\mathrm{f}(z) = \log z$.

3. A line segment is given by $z_1(t) = (1, t)$ where $0 \leq t \leq 4\pi$.
   (a) Let $f(t) = \exp(z_1(t)) = u(t) + iv(t)$. Plot $u(t)$ and $v(t)$ as a function of t.
   (b) Do the same for $z_2(t) = (2, t)$ and $z_3 = (t, \pi/6)$.

4. Show that
   (a) $\sin^2 z + \cos^2 z = 1$;
   (b) $\sin^2(1 + i) = 1.2828 + 1.6489i$ and $\cos^2(1 + i) = -0.2828 - 1.6489i$;
   (c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$;
   (d) $\cosh^2 z - \sinh^2 z = 1$;
   (e) $\cosh^2(1 + i) = -.2828 + 1.6489i$ and $\sinh^2(1 + i) = -1.2828 + 1.6489i$.
   (f) $\log(z_1 z_2) = \log z_1 + \log z_2$.

5. Let $w = 1/z$ and $z = x + iy$.
   (a) Find u and v if $w = u + iv$.
   (b) Show that a curve in z-plane given by
   $$A(x^2 + y^2) + Bx + Cy + D = 0$$
   $(B^2 + C^2 > 4AD)$ transforms into a curve in w-plane given by
   $$D(u^2 + v^2) + Bu - Cv + A = 0$$
   (c) Show that a line, not passing through origin in z-plane, maps to a circle passing through origin in w-plane.
   (d) Find and sketch a level curve in z-plane for $u(x, y) = 5$.

6. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the function $w = \exp z$, where $w = \rho e^{i\phi}$.

7. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by transformation
   (a) $f(z) = z^2$;
   (b) $f(z) = z^3$;

(c) $f(z) = z^4$.

8. A particle constrained to move in a two dimensional plane, where its coordinates can be given by a complex number z. It is acted upon by a central force $F(z) = f(|z|)z$. Derive the equations of motion

$$\begin{aligned} 2r'\theta' + r\theta'' &= 0 \\ r'' - r\left(\theta'\right)^2 &= \frac{r}{m} f\left(|z|\right) \end{aligned}$$

9. Find each of the following limits.

   (a) $\lim\limits_{z \to 2+3i} (z - 5i)^2$

   (b) $\lim\limits_{z \to 2} \frac{z^2+3}{iz}$

   (c) $\lim\limits_{z \to 3i} \frac{z^2+9}{z-3i}$

   (d) $\lim\limits_{z \to 1-i} \left[x + i\left(2x + y\right)\right]$

   (e) $\lim\limits_{z \to \pi i/2} \left(z + 1\right) e^z$

10. Show that the limit of the function $f\left(z\right) = \left(z/\bar{z}\right)^2$ as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin.

11. Find $f'(z)$ when

    (a) $f\left(z\right) = 3z^2 - 2z + 4$;

    (b) $f(z) = (1 - 4z^2)^3$;

    (c) $f\left(z\right) = \frac{z-1}{2z-1}$;

12. Prove that $\frac{d}{dz} z^n = nz^{n-1}$ where n is an integer.

13. Find the derivative of the given functions using the rules of differentiation:

    (a) $e^z = 1 + \sum_1^\infty z^n/n!$.

    (b) $\sin z = \left(e^{iz} - e^{-iz}\right)/2i$.

    (c) $\cos z = \left(e^{iz} + e^{-iz}\right)/2$.

    (d) $\tan z = \sin z / \cos z$.

14. Show that the derivative of $f\left(z\right)$ does not exist for any z for each of the following:

    (a) $f\left(z\right) = \bar{z}$.

    (b) $f\left(z\right) = \operatorname{Re} z$.

    (c) $f\left(z\right) = \operatorname{Im} z$.

15. Write the following functions in the form $f\left(z\right) = u\left(x, y\right) + iv\left(x, y\right)$ and find the derivative for each:

    (a) $\cosh z$.

    (b) $\sinh z$.

    (c) $\log z$.

16. Prove *L'Hospital rule*: If $f(z)$ and $g\left(z\right)$ are analytic at z_0 and $f\left(z_0\right) = g\left(z_0\right) = 0$, but $g'\left(z_0\right) \neq 0$, then

$$\lim_{z \to z_0} \frac{f\left(z\right)}{g\left(z\right)} = \frac{f'\left(z_0\right)}{g'\left(z_0\right)}.$$

Find $\lim_{z \to i} \left(1 + z^6\right)/\left(1 + z^{10}\right)$ using L'Hospital rule.

17. Let $f(z) = z^3 + 1$, and let $z_1 = \left(-1 + i\sqrt{3}\right)/2$, $z_2 = \left(-1 - i\sqrt{3}\right)/2$. Show that there is no point w on the line segment from z_1 to z_2 such that

$$f(z_2) - f(z_1) = f'(w)(z_2 - z_1).$$

This shows that the mean-value theorem does not extend to complex functions.

18. If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic at z, then prove Cauchy-Riemann conditions

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r \end{aligned}$$

and that $f'(z) = e^{-i\theta}(u_r + iv_r)$.

19. Show that following functions are harmonic and find their harmonic conjugates. Find functions $f(z)$ of which the following are real parts.

    (a) y

    (b) xy

    (c) $\log(x^2 + y^2)$

20. If $f(z) = u(x,y) + iv(x,y)$, the equations $u(x,y) = c_1, v(x,y) = c_2$ where c_1 and c_2 are constants generate a family of curves in xy plane, namely, *level curves.*

    (a) Find the normal vector to these level curves.

    (b) Show that the two sets of level curves, one for u function and other for v function are orthogonal to each other if f is analytic.

21. $f(z) = z + 1/z$. Show that the level curve for $\operatorname{Im} f(z) = 0$ consists of a real axis (excluding $z = 0$) and the circle $|z| = 1$.

22. Consider a wedge bounded by the nonnegative real axis and a line $y = x$ $(x \geq 0)$. Find a harmonic function $\phi(x,y)$ which is zero on the sides of the wedge but is not identically zero.