CYK/2022/PH201 Mathematical Physics

Tutorial 3: Series and Application of Residues

1. Verify each of the following Taylor expansions by finding a general formula for $f^{(j)}(z_0)$.

   (a) $\sinh z = \sum\limits_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$ with $z_0 = 0$.

   (b) $\cosh z = \sum\limits_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$ with $z_0 = 0$.

   (c) $\frac{1}{1-z} = \sum\limits_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$ with $z_0 = i$.

   (d) $z^3 = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$ with $z_0 = 1$.

Answer:
We are verifying the Taylor's expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}.$$

(a) $\frac{d^n}{dz^n} \sinh z = \sinh z$ if n is even and $\cosh z$ if n is odd. Thus

$$a_n = \frac{1}{n!}\frac{d^n}{dz^n} \sinh z = \begin{cases} 0 & \text{even } n \\ \frac{1}{n!} & \text{odd } n \end{cases}$$

$$\therefore\ a_{2j} = 0, \qquad a_{2j+1} = \frac{1}{(2j+1)!}$$

The radius of convergence is infinte.

(b) Similar to the first one.

(c) $\frac{d^n}{dz^n}\frac{1}{1-z} = \left(\frac{n!}{(1-z)^{n+1}}\right)$. Thus, $a_n = \frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{1-z}\Big|_{z=i} = \frac{1}{(1-i)^{n+1}}$. And, $\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$.
The radius of convergence is $\sqrt{2}$.

(d) $\frac{d}{dz} z^3\big|_{z=1} = 3$, $\frac{d^2}{dz^2} z^3\Big|_{z=1} = 6$, $\frac{d^3}{dz^3} z^3\Big|_{z=1} = 6$ and $\frac{d^n}{dz^n} z^3\big|_{z=1} = 0$ for all $n \geq 4$. Thus, $a_0 = 1$, $a_1 = 3$, $a_2 = 6/2! = 3$, $a_3 = 6/3! = 1$ and $a_n = 0$ for $n \geq 4$.

2. Find Tayor series for following functions about $z = 0$ and state the radius of convergence.

 (a) $\sin z$
 (b) $\cos z$
 (c) $\ln(1+z)$
 (d) $\tan^{-1} z$
 (e) $(1+z)^p$

Solution: (a) $f(z) = \sin z$, $z_0 = 0$.

$$f^{(n)}(z) = (\sin z)(-1)^{n/2} \quad \text{even } n$$
$$= (\cos z)(-1)^{(n-1)/2} \quad \text{odd } n$$

Then $a_n = \frac{1}{n!} f^{(n)}(z_0) = 0$ even n

$$= (-1)^{(n-1)/2} \quad \text{odd } n$$

$$\Rightarrow \sin z = \sum_n a_n (z - z_0)^n = \sum_{\text{odd } n} (-1)^{(n-1)/2} z^n / n!$$

put $n = 2j+1$ where $j = 0, 1, \cdots$ then $\sin z = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$

$\sin z$ is analytic everywhere $\Rightarrow$ Radius of convergence $= \infty$

(b) $\cos z = \sum_{j=0}^{\infty} (-1)^j z^{2j} / (2j)!$

(c) $f(z) = \ln(1+z)$ then $f^{(n)}(z) = (-1)^{n-1}(n-1)! / (1+z)^n$ for $n \geq 1$

$f^{(1)}(z) = \frac{1}{1+z}$

$f^{(2)}(z) = \frac{(-1)}{(1+z)^2}$

$f^{(3)}(z) = \frac{(-1)^2 \, 2}{(1+z)^3}$

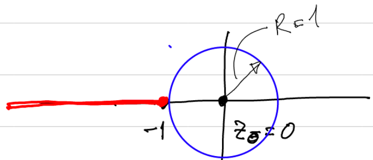
$\vdots$

Then $a_0 = f(0) = \ln(1) = 0$

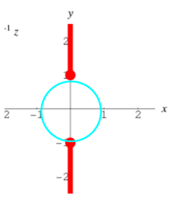
$$a_n = f^{(n)}(0)/n! = (1/n!) \cdot (-1)^{n-1}(n-1)! = (-1)^{n-1}/n$$

$$\Rightarrow \ln(1+z) = z - z^2/2 + z^3/3 + \cdots$$

Branch cut of $\ln(1+z)$ is shown with red line with $z = -1$ as the branch point. Thus the series expansion is valid inside the blue circle. $\Rightarrow R = 1$.



(d) $\tan^{-1}(z) = \sum_n \frac{(-1)^n}{2n+1} z^{2n+1}$ $\quad |z| < 1$



To see this, define $\tan^{-1} z = \frac{1}{2} i [\ln(1 - iz) - \ln(1 + iz)]$. Verify that $\tan(\tan^{-1} z) = z$ [by using $\tan z = (e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$].

Now, we can see that the branch cut of $\tan^{-1} z$ as shown in the figure Series is valid inside the blue circle.

3. Find the Laurent Series for the function $1/\left(z+z^2\right)$ for each of the following domains:

   (a) $0<|z|<1$; about $z=0$.
   (b) $1<|z|$; about $z=0$.
   (c) $0<|z+1|<1$; about $z=-1$;
   (d) $1<|z+1|$; about $z=-1$;

(a) For $z_0=0$ and $0<|z|<1$: $f(z) = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z}\sum_n (-1)^n z^n$
$= \sum_n (-1)^n z^{n-1}$

(b) For $z_0=0$ and $1<|z|$: $f(z) = \frac{1}{z^2}\frac{1}{(1+1/z)} = \sum_n (-1)^n \left(1/z^{n+2}\right)$

(c) For $z_0=-1$ and $0<|z+1|<1$: $f(z) = \left[1/(z+1)\right]\frac{1}{(z+1)-1}$
$= \frac{-1}{(z+1)}\sum_{n=0}^{\infty}(z+1)^n = -\sum_n (z+1)^{n-1}$

(d) For $z_0=-1$ and $1<|z+1|$: $f(z) = -1/(z+1)^2\cdot\frac{1}{1-1/(z+1)} = \cdots$ (For you).

4. Find Laurent series for

   (a) $\sin(2z)/z^3$ in $|z|>0$;
   (b) $z^2\cos(1/3z)$ in $|z|>0$.

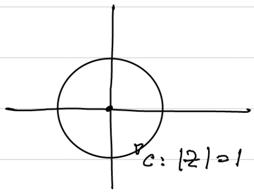
Solns: (a) $\sin(2z)/z^3 = \frac{1}{z^3}\sum_n (-1)^n\frac{(2z)^{2n+1}}{(2n+1)!}$

(b) $z^2\cos(1/3z) = z^2\sum_n (-1)^n\frac{1}{(2n)!}\cdot\left(\frac{1}{3z}\right)^{2n}$

5. Prove that the Laurent series expansion of the function $f(z)=\exp\left[\frac{\lambda}{2}\left(z-\frac{1}{z}\right)\right]$in $|z|>0$ is given by $\sum_{k=-\infty}^{\infty} J_k(\lambda)\, z^k$,where $J_k(\lambda)=\frac{1}{2\pi}\int_0^{2\pi}\cos(k\theta-\lambda\sin\theta)\,d\theta$. The functions $J_k(\lambda)$ are known as *Bessel functions* of the first kind.

by definition of Lorentz series,

$$C_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$



Here $z_0=0$ and $z=e^{i\theta}$ $\quad dz = ie^{i\theta}d\theta$

and $f(z) = \exp\left[\lambda/2\left(e^{i\theta}-e^{-i\theta}\right)\right]$

Then
$$C_n = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\exp[i\lambda\sin\theta]}{e^{i(n+1)\theta}}\cdot ie^{i\theta}\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\left(-n\theta+\lambda\sin\theta\right)\right]d\theta$$

Note that integrand $g(\theta) = \bar{g}(2\pi-\theta) \Rightarrow C_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(-\lambda\sin\theta+n\theta)\,d\theta$

6. Determine all the isolated singularities of each of the following functions and compute the residue at each singularity

   (a) $e^{3z}/(z-2)$
   (b) $(z+1)/(z^2-3z+2)$
   (c) $(\cos z)/z^2$

(d) $\left(\frac{z-1}{z+1}\right)^3$

(e) $\sin(1/3z)$

(f) $(z-1)/\sin z$

Soln. (a) Simple pole at $z=2$ with residue e^6

(b) $f = (z+1)/(z-2)(z-1)$. Simple poles at $z=2$ and $z=1$.

$\text{Res } f(2) = 3$ and $\text{Res } f(1) = -2$

(c) pole of order 2 at $z=0$. Residue $=0$

(d) pole of order 3 at $z=-1$. Residue $= \frac{1}{2} \frac{d^2}{dz^2}\left[(z-1)^3\right]\Big|_{z=-1}$

$= \frac{1}{2} \cdot 6 \cdot (z-1)\big|_{z=-1} = -6$

(e) Essential sing at $z=0$. Residue $= 1/3$.

(f) Simple poles at $m\pi$. Residue $= (m\pi - 1)/(-1)^m$

7. Evaluate each of the following integrals by means of the Cauchy residue theorem.

(a) $\int_C \frac{\sin z}{z^2-4} dz$ where $C : |z| = 5$.

(b) $\int_C \frac{e^z}{z(z-2)^3} dz$ where $C : |z| = 3$.

(c) $\int_C \tan z dz$ where $C : |z| = 2\pi$.

(d) $\int_C \frac{1}{z^2 \sin z} dz$ where $C : |z| = 1$.

Soln. (a) Poles at $z = \pm 2$. C contains both. $f(z) = \sin z/(z+2)(z-2)$

$\text{Res } f(2) = \sin(2)/4$, $\text{Res } f(-2) = -\sin(2)/(-4) = \sin(2)/4$

$\Rightarrow \int_C f(z)\,dz = 2\pi i\left[\text{Res} f(2) + \text{Res } f(-2)\right] = \pi i \sin(2)$

(b) Let $f = e^z/z(z-2)^3$. Simple pole at $z=0$, pole of order 3 at $z=2$.

$\text{Res } f(0) = -1/8$.

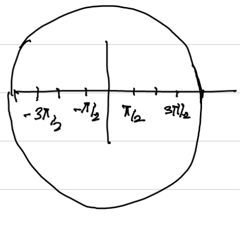
$\text{Res } f(2) = \left(\frac{1}{2!}\right) \frac{d^2}{dz^2}\left[e^z/z\right] = e^2/8$

$\Rightarrow \int_C f(z)\,dz = \pi i\,(e^2-1)/4$ C contains both poles.

(c) Now, $\tan z$ has simple poles at $z = \pi/2 + m\pi$

$\text{Res}(\tan z)\big|_{z=\pi/2+m\pi} = -1$.

$\Rightarrow \int_C \tan z\,dz = 2\pi i\,(4 \times (-1)) = -8\pi i$



(d) $f(z) = 1/z^2 \sin z$.

pole of order 3 at $z=0$ and simple poles at $z = m\pi$, $m \neq 0$

$\text{Res } f(0) = \frac{1}{2} \frac{d^2}{dz^2}\left(z/\sin z\right) = \frac{1}{2}\left[\frac{d}{dz}\right]\left[1/\sin z - z\cos z/\sin^2 z\right]$

$= \lim_{z\to 0} \frac{1}{2}\left[-\frac{2\cos z}{\sin^2 z} + \frac{z}{\sin z} + \frac{2\cos z \cdot z}{\sin^3 z}\right]$

$= \lim_{z\to 0} \frac{1}{2}\left[\frac{-2\cos z\left[\sin z - z\right]}{\sin^3 z} + \frac{z}{\sin z}\right]$

$= \frac{1}{2}(-2+1) = -1/2$.

8. Let f have an isolated singularity at z_0 (f analytic in punctured nbd of z_0). Show that the residue of the derivative f' is equal to zero.

Isolated singularity at $z_0 \Rightarrow$

$$f(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

Then $$f'(z) = \cdots - \frac{2b_2}{(z-z_0)^3} - \frac{b_1}{(z-z_0)^2} + a_1 + 2a_2(z-z_0) + \cdots$$

Thus co-eff of $\left(\frac{1}{z-z_0}\right)$ in the series for $f'(z) = 0$.

9. Using method of residues, verify each of the following.

(a) $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$.

(b) $\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$.

(c) $\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{2\pi}{ab}$

(d) $\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$

Technique used for the kind of problems: $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

Consider a circular contour centered around origin with radius 1. On the contour $z = \cos\theta + i\sin\theta$. with $\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$ and $\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

Let $F(z) = f\left((z-\bar{z})/2i, (z+\bar{z})/2\right)$. Then

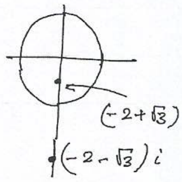
$$\int_C \frac{F(z)}{iz}\, dz = \int_0^{2\pi} \frac{f(\cos\theta, \sin\theta)\, iz\, d\theta}{iz} = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

(a) $$I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \int_C \frac{dz/iz}{2 + (z-\bar{z})/2i} = \int_C \frac{2\,dz}{4iz + z^2 - 1}$$

The integrand has two simple poles at $(-2\pm\sqrt{3})i = -3.732i$ or $-0.268i$.

The residue at $(-2+\sqrt{3})i = \frac{1}{\sqrt{3}i}$

Thus $I = 2\pi i/\sqrt{3}i = 2\pi/\sqrt{3}$.



(b) $$I = \int_0^{2\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \int \frac{dz}{iz(3+(z+\bar{z}))^2} = \frac{1}{i}\int \frac{dz \cdot z}{(z^2+3z+1)^2}$$

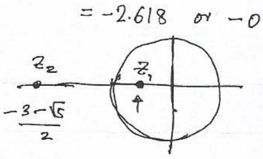
← Note this limit.

If $f(z) = z/(z^2+3z+1)$ then the poles are at $z = (-3\pm\sqrt{5})/2$

$= -2.618$ or -0.382

The poles are of order 2.

Then $$\operatorname*{Res}_{z \to z_1} f(z) = \frac{d}{dz}\left(\frac{z}{(z-z_2)^2}\right)_{z=z_1}$$



$$= \frac{1}{(z_1-z_2)^2} - \frac{2z_1}{(z_1-z_2)^3}$$

$$= \frac{\sqrt{5}+3-\sqrt{5}}{(\sqrt{5})^3} = \frac{3}{5\sqrt{5}} \Rightarrow I = \frac{6\pi\sqrt{5}}{25}$$

The required integral is $= \frac{I}{2} = \frac{3\pi\sqrt{5}}{25}$

(c)

$$I = \int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = 4\int_c \frac{dz/iz}{-a^2(z-\bar{z})^2 + b^2(z+\bar{z})^2}$$

Now,

$$a^2\sin^2\theta + b^2\cos^2\theta = -\frac{a^2}{4}(z-1/z)^2 + b^2/4(z+1/z)^2$$

$$= \frac{b^2-a^2}{4z^2}\left[z^4 + 2\gamma z^2 + 1\right] \quad \text{where} \quad \gamma = \frac{b^2+a^2}{b^2-a^2} \quad |\gamma| > 1$$

then,

$$I = \frac{4}{i(b^2-a^2)} \int_c \frac{z}{z^4 + 2\gamma z^2 + 1}\, dz \qquad \gamma^2 - 1 = \frac{(b^2+a^2)^2 - (b^2-a^2)^2}{(b^2-a^2)^2}$$

Let $f(z) = z/(z^4 + 2\gamma z^2 + 1)$ $\qquad \sqrt{\gamma^2-1} = \frac{2ab}{(b^2-a^2)}$

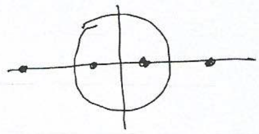
The poles are at $z = \pm\left(-\gamma \pm \sqrt{\gamma^2-1}\right)^{1/2}$.

Now, if $\gamma > 1$ then $|-\gamma - \sqrt{\gamma^2-1}| > 1$ but $|-\gamma + \sqrt{\gamma^2-1}| < 1$

and if $\gamma < -1$ then $|-\gamma + \sqrt{\gamma^2-1}| > 1$ but $|-\gamma - \sqrt{\gamma^2-1}| < 1$

For $\gamma > 1$ $\quad \text{Res } f = \frac{z}{4(z^3 + \gamma z)} = \frac{1}{4(\cancel{-\gamma} + \sqrt{\gamma^2-1} + \cancel{\gamma})} = \frac{(b^2-a^2)}{8ab}$

$$I = \frac{4}{i(b^2-a^2)} \cdot 2\pi i \cdot \left(\frac{b^2-a^2}{4ab}\right) \leftarrow \text{Two pts}$$



$$= \frac{2\pi}{ab}$$

do the same for $\gamma < -1$ case.

(d) $I = \int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \int_c \left(\frac{z^2+1}{2z}\right)^{2n} \frac{dz}{iz} = \int_c \frac{(z^2+1)^{2n}}{z^{2n}\, z^{2n+1}}\, dz$

pole of order $(2n+1)$ at $z=0$.

Then $\quad \underset{z\to 0}{\text{Res}}\left[\frac{(z^2+1)^{2n}}{z^{2n+1}}\right] = \frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}(z^2+1)^{2n}\Big|_{z=0} = \frac{1}{(2n)!}\frac{((2n)!)^2}{n!\,n!}$.

$$\Rightarrow I = \frac{2\pi\,(2n)!}{2^{2n}\,(n!)^2}$$

10. Verify the following integral formulae with the help of residues.

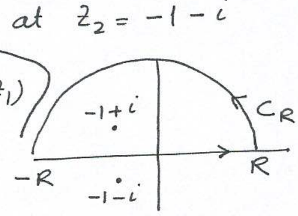
(a) $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$.

(b) $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{\pi}{54}$.

(c) $\int_0^{\infty} \frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$.

(a) Find $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$.

Let $f(z) = \frac{1}{z^2+2z+2}$. f has two poles at $z_1 = -1+i$ and at $z_2 = -1-i$



Now $\int_{-R}^{R} \frac{dx}{x^2+2x+2} + \int_{C_R} f(z)dz = 2\pi i \,\text{Res} f(z_1)$

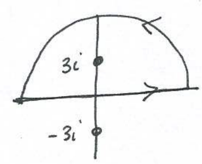
and $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

$$\Rightarrow \text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = 2\pi i \,\text{Res} f(z_1) = 2\pi i \cdot \frac{1}{2(z_1+1)} = \pi$$

(b) Find $\text{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{\pi}{54}$

Let $f(z) = \frac{1}{(z^2+9)^2}$ has two poles (2^{nd} order) $z_1 = +3i$ and $z_2 = -3i$

Again using a contour as in problem (a), we get



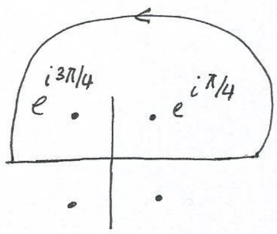
$$\text{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = 2\pi i \,\text{Res} f(z_1)$$

Now $\text{Res} f(z_1) = \frac{d}{dz}\left[\frac{1}{(z+3i)^2}\right] = -\frac{2}{(z+3i)^3}$

$$= +\frac{2}{6^3 i}$$

$$\Rightarrow \text{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{\pi}{54}$$

(c) Show $\int_0^{\infty} \frac{x^2+1}{x^4+1}dx = \pi/\sqrt{2}$



Now $f(z) = \frac{z^2+1}{z^4+1}$ has 4 simple poles

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}dx = 2\pi i\left[\frac{i+1}{4e^{3\pi/4}} + \frac{-i+1}{4e^{i\pi/4}}\right]$$

$$= 2\pi i \cdot \frac{\sqrt{2}}{4}\left[e^{-i\pi/2} - e^{i\pi/2}\right] = \pi\sqrt{2}$$

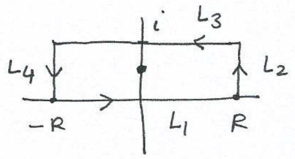
Note

11. Show that

$$pv \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx = \sec 1$$

by integrating $e^{2z}/\cosh(\pi z)$ around a rectangle with vertices at $z = \pm R, \pm R + i$ and then taking a limit $R \to \infty$.

Let $f(z) = \dfrac{e^{2z}}{\cosh(\pi z)}$



Now $\displaystyle\int_{L_1} f(z)dz = \int_{-R}^{R} \frac{e^{2x}dx}{\cosh(\pi x)}$

$$\int_{L_2} f(z)dz = \int_0^1 \frac{e^{2R}e^{2iy}\,dy}{\cosh(\pi R + \pi i y)} = \int_0^1 \frac{e^{2z}dz}{\cosh(\pi z)} = -L_4$$

$$\int_{L_3} f(z)dz = +\int_{-R}^{R} \frac{e^{2x}\cdot e^{2i}\,dx}{\cosh(\pi x)}$$

First $\displaystyle\lim_{R\to\infty}\left|\int_{L_2} f(z)dz\right| \leq \lim_{R\to\infty}\left|\frac{e^{2z}}{e^{\pi z}}\right| = \lim_{R\to\infty} e^{(2-\pi)R} \to 0.$

Same for path L_4. The only pole is at $z = i/2$.

$$\Rightarrow \left(1+e^{2i}\right)\int_{-\infty}^{\infty} \frac{e^{2x}dx}{\cosh(\pi x)} = 2\pi i \cdot \frac{e^{2i/2}}{\pi \sinh(\pi i/2)} = 2e^{i}$$

$$\Rightarrow \quad I = \frac{2e^{i}}{1+e^{2i}} = \frac{2}{e^{i}+e^{-i}} = \sec 1.$$

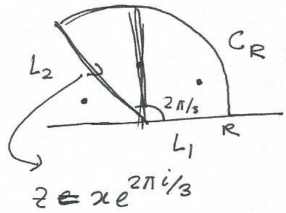
12. Show that

$$\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$

by integrating $1/\left(z^3+1\right)$ around the boundary of the circular sector $S: \left\{z = re^{i\theta} : 0 \leq \theta \leq 2\pi/3,\ 0 \leq r \leq R\right\}$ and then letting $R \to \infty$.

Show that $\displaystyle\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$

Now $\displaystyle +\int_{L_2} \frac{dx\, e^{2\pi i/3}}{x^3+1} = -e^{2\pi i/3}\int_{L_1} \frac{dx}{x^3+1}$



We already know that $\displaystyle\int_{C_R} \frac{dz}{z^3+1} \to 0$ as $R \to \infty$.

$$\Rightarrow \left(1 - e^{2\pi i/3}\right) I = 2\pi i \operatorname{Res} f\left(e^{i\pi/3}\right)$$

$$= 2\pi i\, \cancel{\operatorname{Res}}\, \frac{1}{3z^2}\bigg|_{z=e^{i\pi/3}} = \frac{2\pi i}{3}\,\frac{1}{e^{2\pi i/3}}$$

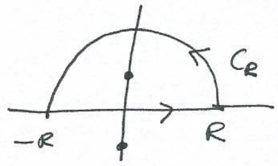
$$\Rightarrow \quad I = \frac{2\pi i}{3\left(1-e^{2\pi i/3}\right)e^{2\pi i/3}} = \frac{2\pi}{3\sqrt{3}}.$$

13. Using the method of residues, verify:

(a) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\,dx = \frac{\pi}{e^2}$.

(b) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\,(3\cos 1 + \sin 1)$.

(a) Let $f(z) = \dfrac{e^{2iz}}{z^2+1}$



Now $\displaystyle\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \,\mathrm{Res}\, f(i)$

$$\Rightarrow \int_{-R}^{R} \frac{e^{2ix}}{x^2+1}\,dx + I_R = 2\pi i \cdot \frac{e^{-2}}{2i} = \pi/e^2$$

Taking limit $R\to\infty$, we know $\lim I_R = 0$ as $R\to\infty$

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}\,dx = \pi/e^2$$

Taking real parts on both sides $\Rightarrow$ $\displaystyle \mathrm{pv}\int \frac{\cos(2x)}{x^2+1}\,dx = \frac{\pi}{e^2}$

(b) Let $f(z) = \dfrac{z e^{iz}}{z^2-2z+10}$ poles at $1\pm 3i$

Then
$$\int_{-R}^{R} f(z)\,dz + I_R = 2\pi i \,.\, \mathrm{Res}\, f(1+3i)$$

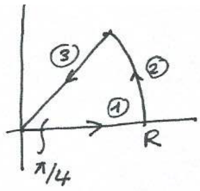
$$\Rightarrow \int_{-R}^{R} \frac{x e^{ix}}{x^2-2x+10}\,dx + I_R = 2\pi i \,\frac{(1+3i)\,e^{(i-3)}}{6i}$$
$$= \frac{\pi}{3}(1+3i)\,e^{(i-3)}$$

Then $\displaystyle \mathrm{pv}\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\left[\cos 1 - 3\sin 1 + i(3\cos 1 + \sin 1)\right]$

$$\Rightarrow \mathrm{pv}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\,(3\cos 1 + \sin 1)$$

14. Given that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$, integrate e^{iz^2} around the boundary of the circular sector $S : \{z = re^{i\theta} : 0 \le \theta \le \pi/4,\ 0 \le r \le R\}$ and letting $R \to \infty$, prove that

$$\int_0^\infty e^{ix^2}dx = \frac{\sqrt{2\pi}}{4}(1+i).$$



Now,

$$\int_1 e^{iz^2}dz + \int_2 e^{iz^2}dz + \int_3 e^{iz^2}dz = 0$$

$$\Rightarrow \int_0^R e^{ix^2}dx + \underbrace{\int_4 \frac{e^{iw}dw}{2\sqrt{w}}}_{\to 0 \text{ as } R\to\infty} - \int e^{-x^2}dx\,\frac{(1+i)}{\sqrt{2}} = 0$$

$$\Rightarrow \int_0^\infty e^{ix^2}dx = \int_0^\infty e^{-x^2}dx\cdot\left(\frac{1+i}{\sqrt{2}}\right) = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$$

15. Using the technique of residues, verify:

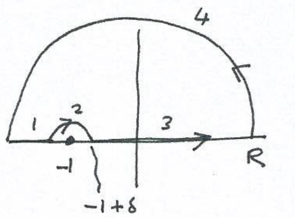
(a) $\int_{-\infty}^\infty \frac{e^{2ix}}{x+1}dx = \pi i e^{-2i}$;

(b) $\int_{-\infty}^\infty \frac{e^{ix}}{(x-1)(x-2)}dx = \pi i\left(e^{2i} - e^{i}\right)$.

(a) To show that $\int_{-\infty}^\infty \frac{e^{2ix}}{x+1}dx = \pi i e^{-2i}$

Let $f(z) = e^{2iz}/(z+1)$. f has a simple pole at $z=-1$

$I_1 = \int_{-R}^{-1-\delta} f(z)dz$, $I_2 = \int_2 f(z)dz$ etc.



Now $I_1 + I_2 + I_3 + I_4 = 0$

$\Rightarrow \quad I_1 + I_3 = -I_2 - I_4$

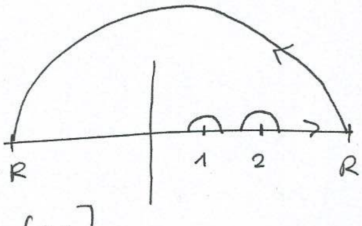
Taking two limits $\delta \to 0$ and $R \to \infty$

$$\lim_{\substack{\delta\to 0\\ R\to\infty}} I_1 + I_3 = \int_{-\infty}^\infty \frac{e^{2ix}}{x+1}dx, \quad \lim_{\delta\to 0} I_2 = -\pi i \operatorname{Res} f(z)\Big|_{z=-1} = -\pi i e^{-2i}$$

$$\lim_{R\to\infty} I_4 = 0$$

$$\Rightarrow \int_{-\infty}^\infty \frac{e^{2ix}}{x+1}dx = \pi i e^{-2i}$$

(b) Let $f(z) = \dfrac{e^{iz}}{(z-1)(z-2)}$



Then

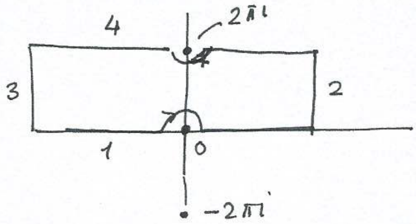
$$\int_{-\infty}^\infty \frac{e^{ix}dx}{(x-1)(x-2)} = \pi i\cdot\left[\operatorname{Res} f(1) + \operatorname{Res} f(2)\right]$$

$$= \pi i\left[-e^{i} + e^{2i}\right]$$

16. Compute pv $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x-1} dx$ for $0 < a < 1$. (Use rectangular contour).

Let $f(z) = \dfrac{e^{az}}{e^z-1}$ Simple poles at $z = 2\pi i n$ n integer.

Then $\lim\limits_{R\to\infty} \displaystyle\int_{2 \text{ or } 3} \frac{e^{az}}{e^z-1}\, dz = 0.$



$$\Rightarrow \int_1 f(z)dz = \int_{-R}^{R} \frac{e^{ax}}{e^x-1}\, dx$$

$$\int_4 f(z)dz = -\int_{-R}^{R} \frac{e^{ax}\cdot e^{2\pi a i}}{e^x-1}\, dx$$

$$\Rightarrow (1-e^{2\pi a i})\, I = +\pi i\left[\operatorname{Res} f(0) + \operatorname{Res} f(2\pi i)\right]$$

$$= +\pi i\left[1 + e^{2\pi i a}\right]$$

$$\Rightarrow \qquad I = -\pi\cot(\pi a) \checkmark$$