Roll No: | | Name: | |

CYK/2023/PH201 Mathematical Physics

End-Semester Exam

Total Marks: 50; **Duration:** 3 Hours (9AM-12NOON); **Date:** 19 Nov 2023, Sunday

Section A

[$10 \times 2 = 20$ **Marks**] Answer the following **short** questions. Write details but only briefly. Just answers will not be given marks, unless stated. Write answers on this page itself.

1. Is the function $f(z) = e^x e^{-iy}$ differentiable at any point? If yes, find the derivative of f at those points.

Answer:
Here $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$. $u_x = e^x \cos y$, $v_y = -e^x \cos y$, $u_y = -e^x \sin y$ and $v_x = -e^x \sin y$, this requires $\cos y = \sin y = 0$ which cannot happen at any value of y. Hence not differentiable at any point in $\mathbb{C}$.

2. Define $\sinh(z)$ in terms of the exponential function and show that $\sinh 2z = 2\sinh z \cosh z$.

Answer:
$\sinh z = \frac{e^z - e^{-z}}{2}$.
Then

$$2\sinh z \cosh z = 2\frac{e^z - e^{-z}}{2}\frac{e^z + e^{-z}}{2} = \frac{e^{2z} - e^{-2z}}{2} = \sinh 2z.$$

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3. Calculate $\int_1^{1+i} \pi \exp(\pi \bar{z})\, dz$ along the straight line segment between 1 and $1+i$.

Answer:
Let $C: z = 1 + iy$ with $y: 0 \to 1$.
Then
$$\int_1^{1+i} \pi e^{\pi \bar{z}} dz = \pi e^{\pi} \int_0^1 e^{-i\pi y} i dy = -e^{\pi}\left(e^{-i\pi} - 1\right) = 2e^{\pi}.$$

4. Using Cauchy residue theorem compute $\int_C \frac{\sin \pi z}{z(z-\pi/2)} dz$ where $C: |z| = \frac{3}{2}$.

Answer:
The contour contains $z = 0$ but not $z = \pi/2$. Now, at $z = 0$, the function $f(z)$ has a removable singularity. Thus the residue is 0.
$$\int_C \frac{\sin \pi z}{z\,(z - \pi/2)} dz = 2\pi i \mathrm{Res} f(0) = 0.$$
[If the residue is computed using the formula for poles, then 1 mark is deducted.]

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5. If ψ is a solution of the Laplace equation in volume V, then show that at every point $\mathbf{r}$ in V,

$$\oint_{S_\epsilon} (\nabla\psi) \cdot \hat{\mathbf{n}} ds = 0$$

where S_ϵ is a sphere of radius ϵ centered at $\mathbf{r}$. Argue that there cannot be an extremum of ψ at $\mathbf{r}$.

Answer:
Now,

$$\oint_{S_\epsilon} (\nabla\psi) \cdot \hat{\mathbf{n}} ds = \int_{V_\epsilon} \nabla^2 \psi d\tau = 0.$$

In case of a maximum(minimum), it is possible to choose an ϵ-nbd such the $\nabla\psi$ is pointing outwards (inwards) at all points of the surface and hence the surface integral will be nonzero which is contradictory to the statement above.

6. If $f(p)$ is any real valued function, then show that $f(x \pm ct)$ is a solution of the wave equation in 1D where c is the speed of the wave.

Answer:
Let $p = x \pm ct$. Let $g(x,t) = f(x \pm ct)$. Now $\partial g(x,t)/\partial x = (\partial p/\partial x) df/dp = df/dp$ etc

$$\frac{d^2 g}{dx^2} - \frac{1}{c^2}\frac{d^2 g}{dt^2} = \frac{d^2 f}{dp^2} - \frac{1}{c^2}(\pm c)^2 \frac{d^2 f}{dp^2} = 0.$$

7. Write (don't derive!) the well-behaved general solution $\psi(\rho, \theta, \phi)$ of the Laplace equation, $\nabla^2\psi = 0$ in the spherical coordinates.

Answer:
The general solution is

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta,\phi)$$

,where j_l and n_l are spherical Bessel functions and Y_{lm} are spherical harmonics.

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8. Find the region(s) in which the transformation $w = \sin z$ conformal. Mention explicitly the points at the transformation is not conformal.

Answer:
$\sin z$ is entire everywhere, but it's derivative is 0 at $z = \left(n + \frac{1}{2}\right)\pi$ where n is an integer and hence it is conformal at all points except these points.

9. Write the definition of a normal subgroup.

Answer:
A subgroup N of a group G is called a normal subgroup if $gng^{-1} \in N$ for all $n \in N$ and $g \in G$.

10. The group D_n is generated by a and b such that $a^n = b^2 = e$ and $ab = ba^{-1}$. What is the order of D_n? Is $H = \{e, ab\}$ a subgroup of D_n?

Answer:
Order is $2n$.
And $ab \cdot ab = ab \cdot ba^{-1} = e$, thus H is closed under multiplication. H is a subgroup.

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Section B

1. [10 **Marks**] Answer the following questions.

   (a) [4] Show that the group $SO(3)$ (3×3 orthogonal (or rotation) matrices with det $+1$) is a 3-parameter continuous group. What is the corresponding Lie algebra? Write a set of generators of $SO\,(3)$

Answer:
Orthogonal matrix M satisfies $M^T M = I$. These are 6 independet equation in 9 matrix elements. Thus the matrix can be specified using only 3 parameters. A rotation matrix can be written using 3 parameters, two for the direction of the axis of rotation and a parameter for angle of rotation.
If $X \in so(3)$, then $\left(e^X\right)^T = \left(e^X\right)^{-1}$, which implies that $X^T = -X$. Also $\det\left(e^X\right) = 0$ which implies that $\operatorname{tr}(X) = 0$. The Lie algebra consists of all antisymmetric matrices. The form of these matrices are

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}.$$

The basis of this algebra is

$$j_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad j_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad j_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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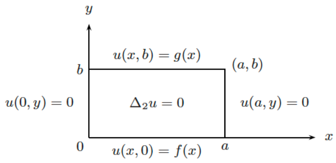
(b) [6] Use the method of separation of variables to solve the Dirichlet problem for the Laplace equation, $\nabla^2 u(x,y) = 0$ on the rectangle satisfying the boundary conditions as shown in the figure. Here, $f(x) = 0$ and $g(x) = x$. (Δ_2 is another less commonly used symbol for the Laplacian operator.) Sketch isocurves of u. [Note: You can start with the general solution without BC.]



Answer:
After separation, we get

$$u(x,y) = (A\cos kx + B\sin kx)(C\sinh ky + D\cosh ky).$$

- ▷ BC at $x = 0$ implies that $A = 0$.
- ▷ BC at $x = a$ implies that $k = n\pi/a$ for positive integer n.
- ▷ BC at $y = 0$ implies that $D = 0$

Thus,

$$u(x,y) = \sum_n C_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi}{a}x.$$

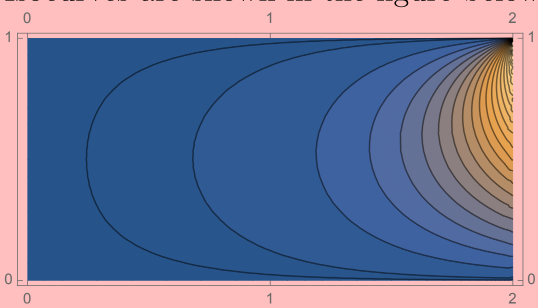
And using BC at $y = b$

$$\begin{aligned} C_n &= \frac{2}{a\sinh n\pi b/a}\int_0^a x\sin\frac{n\pi x}{a}dx \\ &= \frac{2}{a\sinh n\pi b/a}\cdot\frac{(-1)^{n+1}a^2}{\pi n} \end{aligned}$$

The full solution is then

$$u(x,y) = \frac{2a}{\pi}\sum_n \frac{(-1)^{n+1}}{n\sinh n\pi b/a}\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y.$$

Isocurves are shown in the figure below:



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2. [10 **Marks**] Answer the following questions.

   (a) [5] Find the Green's function $G(x,s)$, for the differential operator

$$Ly = \frac{d^2y}{dx^2} + k^2 y, \qquad x \in [0,1]$$

with boundary conditions that $G(0,s) = G(1,s) = 0$.

Answer:
The solution to the homogeneous equation is $A \sin kx + B \cos kx$. Let

$$G(x,s) = \begin{cases} A \sin kx + B \cos kx & x < s \\ C \sin kx + D \cos kx & x > s. \end{cases}$$

Applying BC, we get

$$G(x,s) = \begin{cases} A \sin kx & x < s \\ C' \sin k(x-1) & x > s. \end{cases}$$

Now, the conditions on G at $x = s$ are

$$C' \sin k(s-1) - A \sin ks = 0$$
$$C' \cos k(s-1) - A \cos ks = \frac{1}{k}$$

Which gives us

$$C' = \frac{1}{k \sin k} \sin ks$$
$$A = \frac{1}{k \sin k} \sin k(s-1)$$

Finally,

$$G(x,s) = \begin{cases} \frac{k}{\sin k} \sin k(s-1) \sin kx & x < s \\ \frac{k}{\sin k} \sin ks \sin k(x-1) & x > s. \end{cases}$$

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(b) [5] Find the Green's function $G(x,s)$ for the Laplace operator $L = \frac{d^2}{dx^2}$ where $0 \leq x \leq a$, using the **method of eigenfunction expansion** such that $G(0,s) = G(a,s) = 0$.

Answer:
Since $G(x,x')$ is a continuous function of x, that vanishes at the boundaries, we can expand G as

$$G(x,x') = \sum_{n=1}^{\infty} A_n(x') \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right].$$

To find the coefficients by $A_n(x')$, operate by $\mathcal{L}_x = \frac{d^2}{dx^2}$,

$$\nabla^2 G(x,x') = \sum_{n=1}^{\infty} A_n(x') \nabla^2 \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right]$$

$$\implies \delta(x-x') = -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{a^2} A_n(x') \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right]$$

Using fourier trick, we get

$$A_n(x') = -\frac{a^2}{n^2\pi^2} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x'}{a}\right)\right]$$

Thus,

$$G(x,x') = -\sum_{n=1}^{\infty} \frac{2a}{n^2\pi^2} \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

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3. [10 **Marks**] Answer the following questions.

   (a) [5] Find the solution to the wave equation on a thin square membrane of side a,

$$
\begin{aligned}
c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u\left(x,y,t\right) &= \frac{\partial^2}{\partial t^2}u\left(x,y,t\right)\\
u\left(0,y,t\right)=u\left(a,y,t\right)=u\left(x,0,t\right)=u\left(x,a,t\right) &= 0 \qquad \forall\, x,y,t\\
u\left(x,y,0\right) &= xy \qquad \forall\, x,y\\
\frac{\partial}{\partial t}u\left(x,y,0\right) &= 0 \qquad \forall\, x,y
\end{aligned}
$$

where $c=1$. [Note: You can start by merely stating the general solution without any BC. The derivation is not needed.]

Answer:

▷ The general solution (after the first two boundary conditions) is

$$u\left(x,y,t\right)=\sum\left(A_{mn}\sin\omega_{mn}t+B_{mn}\cos\omega_{mn}t\right)\sin\left(k_{x,n}x\right)\sin\left(k_{y,m}y\right)$$

where where $k_{x,n}=n\pi/a$ and $k_{y,m}=m\pi/a$ with $m,n=1,2,\ldots$,and the frequencies of these normal modes are $\omega_{nm}=ck_{nm}=c\pi\sqrt{n^2+m^2}/a$.

▷ Applying BC at $t=0$, we get

$$
\begin{aligned}
xy &= \sum_{mn}B_{mn}\sin\left(k_{x,n}x\right)\sin\left(k_{y,m}y\right)\\
0 &= \sum_{mn}A_{mn}\omega_{mn}\sin\left(k_{x,n}x\right)\sin\left(k_{y,m}y\right)
\end{aligned}
$$

▷ This implies that $A_{mn}=0$ for all m and n.

▷ And

$$
\begin{aligned}
B_{mn} &= \frac{4}{ab}\int_0^a x\sin\left(k_{x,n}x\right)dx\int_0^b y\sin\left(k_{y,m}y\right)dy\\
&= \frac{4ab(-1)^{mn}}{\pi^2 nm}
\end{aligned}
$$

▷ Thus,

$$u\left(x,y,t\right)=\frac{4ab}{\pi^2}\sum\frac{(-1)^{mn}}{nm}\cos\omega_{mn}t\sin\left(k_{x,n}x\right)\sin\left(k_{y,m}y\right)$$

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(b) [5] Find the steady-state temperature distribution in solid cylinder of height h and radius a if the top and the curved surface are held at 0° and the base at 100°. [Note: You can start by merely stating the general solution without any BC. The derivation is not needed.]

Useful Formulae:

- ▷ $\int_0^a J_m\left(\chi_{ml}\frac{\rho}{a}\right)J_m\left(\chi_{mk}\frac{\rho}{a}\right)\rho d\rho = \frac{a^2}{2}\left[J_{m+1}\left(\chi_{ml}\right)\right]^2\delta_{l,k}$
- ▷ $\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$

Answer:
The given BCs are

$$\psi(a,\phi,z)=0,\quad \psi(\rho,\phi,h)=0,\quad \psi(\rho,\phi,0)=T_0=100.$$

In addition, there are implicit conditions that $\psi(\rho,\phi,z)$ is always finite. The solution of form

$$\psi(\rho,\phi,z)=\sum_{m=0,n=1}^{\infty}\sinh\left(k_{mn}(h-z)\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right)$$

where $k_{mn}=\chi_{mn}/a$ (χ_{mn} is n^{th} zero of J_m), satisfies all conditions except the one at $z=0$. Applying this condition, we get

$$T_0=\sum_{mn}\sinh\left(k_{mn}h\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right).$$

From the orthogonality of the trigonometric functions, we can immediately set $C_{mn}=0$ for all m,n and $D_{mn}=0$ for all m,n except $m=0$. Thus,

$$\rho=\sum_{n}\sinh\left(k_{0n}h\right)D_{0n}J_0\left(k_{0n}\rho\right).$$

Using $\int_0^a\left[J_m\left(k_{mn}\rho\right)\right]^2\rho d\rho=\frac{a^2}{2}\left[J_{m+1}\left(\chi_{mn}\right)\right]^2$,

$$\begin{aligned}D_{0n}&=\frac{2T_0}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\sinh\left(k_{0n}h\right)}\int_0^a\rho J_0\left(\chi_{0n}\rho/a\right)d\rho\\&=\frac{2}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\sinh\left(k_{0n}h\right)}\left(\frac{a}{\chi_{0n}}\right)J_1\left(\chi_{0n}\right)\\&=\frac{2T_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\sinh\left(k_{0n}h\right)}\end{aligned}$$

Finally,

$$\psi(\rho,\phi,z)=\sum_{n=1}^{\infty}\frac{2T_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\sinh\left(k_{0n}h\right)}\sinh\left(k_{0n}(h-z)\right)J_0\left(k_{0n}\rho\right).$$