(1)

Sol. Test II

(1) final answer

$$V(x,y) = \frac{2}{\sin\frac{\pi a}{b}} \sin\left(\frac{\pi}{b}x\right) \sinh\left(\frac{\pi y}{b}\right) + \frac{\sin 5\frac{\pi x}{b} \sinh\frac{5\pi y}{b}}{10 \sin\frac{5\pi a}{b}}.$$

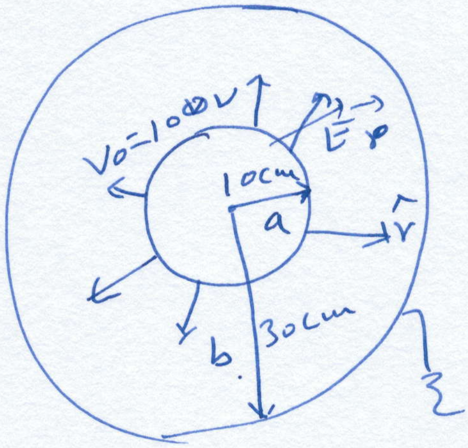
$$\vec{E} = -\vec{\nabla} V$$

(2)

(2) Only radial dependency.

∴ Laplace's eq.

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$



$$\Rightarrow \quad V = -\frac{A}{r} + B.$$

Boundary condition

at $r = 10\,cm$, $V(r=10cm) = 100V = V_0$

at $r = 30\,cm$, $V(r=30cm) = 0 \Rightarrow B = \frac{A}{b}$

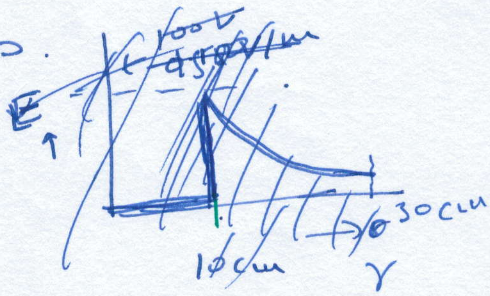
$$\Rightarrow \quad A = \frac{V_0}{\frac{1}{b} - \frac{1}{a}}$$

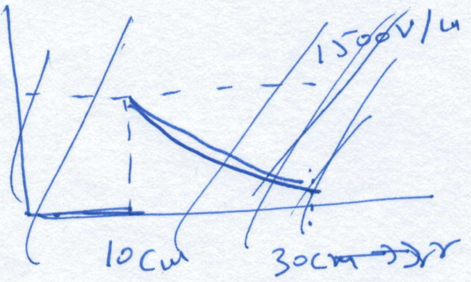
$$\therefore \quad V = V_0 \cdot \frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\frac{1}{a} - \frac{1}{b}}$$

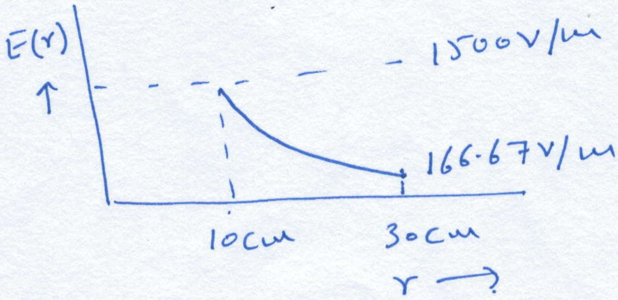
$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} = \frac{V_0}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}\hat{r}$$

on substituting V_0, a & b.

$$\vec{E} = \frac{15}{r^2}\hat{r} \quad V/m.$$





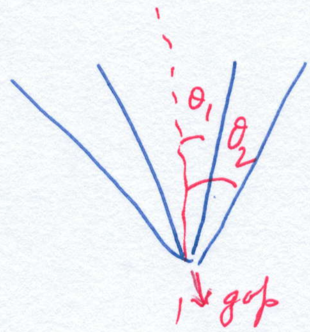


③

③ Only 'θ' dependence.

$\therefore$ Laplace's equat

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$



$$\Rightarrow V = A \ln(\tan\theta/2) + B \quad \text{—} ①$$

Boundry Conditions

$$V(\theta_1 = \frac{\pi}{10}) = 0 \quad \text{—} ②$$

$$V(\theta_2 = \frac{\pi}{6}) = 50V \quad \text{—} ③$$

from ①, ② & ②

$$B = -A \ln(\tan\theta_1/2)$$

for ③

$$A = \frac{V_0}{\ln\left(\frac{\tan\theta_2/2}{\tan\theta_1/2}\right)}, \quad V_0 = 50V$$

$$\therefore V = \frac{V_0 \ln\left(\frac{\tan\theta/2}{\tan\theta_1/2}\right)}{\ln\left(\frac{\tan\theta_2/2}{\tan\theta_1/2}\right)} = 95.1 \ln\left(\frac{\tan\theta/2}{0.1584}\right) V.$$

$$\vec{E} = -\vec{\nabla} V = \frac{1}{r}\frac{dV}{d\theta}\hat{\theta}$$

$$= -\frac{V_0}{r\sin\theta \ln\left(\frac{\tan\theta_2/2}{\tan\theta_1/2}\right)}\hat{\theta}$$

$$= \frac{95.1}{r\sin\theta}\hat{\theta} \quad V/m$$

(4)

(4) $V_{out} = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$V_{in} = \sum A_l r^l P_l(\cos\theta)$

$\left.\frac{\partial V}{\partial r}\right|_{out} = \sum \frac{-(l+1) B_l}{r^{l+2}} P_l(\cos\theta)$

$\left.\frac{\partial V}{\partial r}\right|_{in} = \sum l A_l r^{l-1} P_l \cos\theta.$

$\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta$

$= \frac{2}{2m+1}, \quad l = m$

$= 0 \quad l \neq m$

$\sigma(\theta) = -\epsilon_0 \left[\left.\frac{\partial V}{\partial r}\right|_{out} - \left.\frac{\partial V}{\partial r}\right|_{in} \right]_{r=R}$

$= -\epsilon_0 \left[-\sum \frac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta) - \sum (l A_l R^{l-1}) P_l(\cos\theta) \right.$

$\int \sigma(\theta) P_m(\cos\theta) \sin\theta \, d\theta = + \epsilon_0 \left[+ \frac{(m+1)}{R^{m+2}} \left(\frac{2}{2m+1}\right) B_m + \frac{m A_m R^{m-1} 2}{(m+1)} \right]$ — (1)

at the surface $V_{in}(r=R) = V_{out}(r=R)$

$\Rightarrow B_l = A_l R^{2l+1}$ — (2).

Substitute (2) in (1) and knowing the form of $\sigma(\theta)$, integration of L.H.S can be solved and hence A_l or B_l can be obtained and from (2) the other coefficient can be worked out.

Finally $A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma(\theta) P_l(\cos\theta) \sin\theta \, d\theta.$

$= \frac{1}{2\epsilon_0 R^{l-1}} I_l.$

and $B_l = \frac{1}{2\epsilon_0 R^{l-1}} I_l R^{2l+1} = \frac{I_l R^{l+2}}{2\epsilon_0}$

Substituting in V

$$V_{out}(r \geq R, \theta) = \frac{1}{2\epsilon_0} \sum \frac{R^{l+2}}{r^{l+1}} I_l P_l(\cos\theta)$$

and

$$V_{in}(r \leq R, \theta) = \frac{1}{2\epsilon_0} \sum \frac{r^l}{R^{(l-1)}} I_l P_l(\cos\theta)$$

~~knowing the form of $\sigma(\theta)$~~

By knowing the functional dependence of θ on $\sigma(\theta)$ integration I_l can be evaluated.

⑤ Azimuthal symmetry.

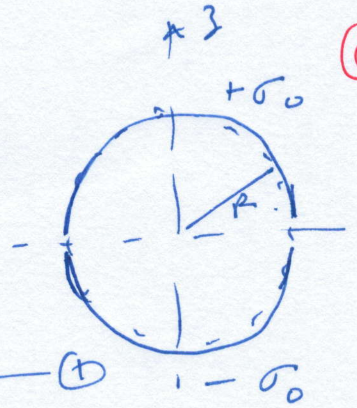
$\therefore$ General sol.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

potential to be finite everywhere

$\therefore$ for $r \le R$, $V_{in}(r,\theta) = \sum A_\ell r^\ell P_\ell(\cos\theta)$ —— ①

and for $r \ge R$ $V_{out}(r,\theta) = \sum \frac{B_\ell}{r^{\ell+1}} \cdot P_\ell(\cos\theta)$ —— ②



Potential to be continuous at the boundary.

$$\therefore V_{in}(r=R,\theta) = V_{out}(r=R,\theta)$$

from eq. ① & ②

$$\Rightarrow B_\ell = A_\ell R^{2\ell+1} \quad \text{——} ③$$

as the surface charge density is given (defined)

$$\therefore \sigma = -\epsilon_0\left[\frac{\partial V_{out}}{\partial r}\Big|_{r=R} - \frac{\partial V_{in}}{\partial r}\Big|_{r=R}\right]$$

$$= -\epsilon_0\left\{\sum_{\ell=0}^{\infty}\left[-\frac{B_\ell(\ell+1)}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta)\right\}.$$

and from ③.

$$\sigma = \epsilon_0 \sum (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) \quad \text{——} ④.$$

multiplying both sides of eq. ④ by $P_m(\cos\theta)\sin\theta$ and integrating from 0 to π.

$$\therefore A_\ell = \frac{1}{2\epsilon_0 R^{\ell-1}}\int_0^\pi \sigma P_\ell(\cos\theta)\sin\theta\, d\theta \quad \text{——} ⑤$$

Solving integration

$$I_\ell = \int_0^\pi \sigma P_\ell(\cos\theta)\sin\theta\, d\theta = \sigma_0\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta - \sigma_0\int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta\, d\theta \quad \text{——} ⑥$$

$$\cancel{I = \sigma_0 \int_1^0 P_\ell(x)\, dx.}$$ taking $\cos\theta = x$.

$$I_\ell = \sigma_0 \int_1^0 P_\ell(x)(-dx) - \sigma_0 \int_0^{-1} P_\ell(x)(-dx)$$

$$= \sigma_0 \int_0^1 P_\ell(x)\, dx - \sigma_0 \int_{-1}^0 P_\ell(x)\, dx. \quad \text{——— (7)}$$

2nd term integration

$$\int_{-1}^0 P_\ell(x)(dx) = \int_1^0 P_\ell(-y)(-dy) \quad (\text{substituting } x = -y$$

$$= \int_0^1 P_\ell(-y)\, dy.$$

$$= (-1)^\ell \int_0^1 P_\ell(x)\, dx \quad \left(\because P_\ell(-x) = (-1)^\ell P_\ell(x).\right. \quad \text{——— (8)}$$

substituting (8) in (7)

$$I_\ell = \sigma_0 \int_0^1 P_\ell(x)\, dx - \sigma_0 (-1)^\ell \int_0^1 P_\ell(x)\, dx$$

$$= \sigma_0 \left(1 - (-1)^\ell\right) \int_0^1 P_\ell(x)\, dx.$$

$\Rightarrow$ $I_\ell = 0$ for even ℓ

$\neq 0$ for odd ℓ., $\boxed{\text{for odd } \ell,\ I_\ell = 2\sigma_0 \int_0^1 P_\ell(x)\, dx.}$

from (5)

$$A_\ell = \frac{1}{\epsilon_0 R^{\ell-1}} \sigma_0 \int_0^1 P_\ell(x)\, dx, \quad \text{for } \ell = 1, 3, 5 \ldots \quad \text{——— (9)}$$

From (3)

$$B_l = \frac{R^{l+2}}{\epsilon_0}\sigma_0 \int_0^1 P_l(x)\,dx, \qquad l = 1, 3, 5 \ldots \quad (10)$$

$\therefore$ for $r \leq R$ for ~~(+)~~ eq (1) & (9).

$$\cancel{V(r \leq R, \theta) = \frac{\sigma_0}{\epsilon} \sum_{l=1,3,5} \left[\int P_l(x)\,dx \right]}$$

$$\boxed{V(r \leq R, \theta) = \frac{\sigma_0}{\epsilon} \sum_{l=1,3,5\ldots} \frac{I_l\, P_l(\cos\theta)}{R^{l-1}}}$$

For eq. (2) & (10).

$$\boxed{V(r \geq R, \theta) = \frac{\sigma_0}{\epsilon_0} \sum_{l=1,3,5} R^{l+2}\, I_l\, P_l(\cos\theta)}$$

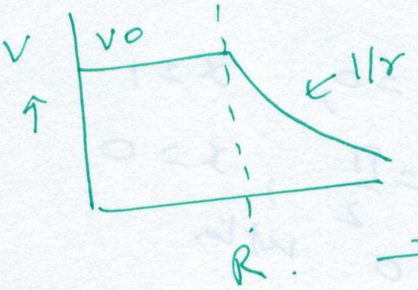
Note: Integration can be performed at least for first few orders and verify that series is convergent.

(6) $V_{in}(r \le R) = V_0$

$$V_{out}(r \ge R) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

at $r = R$

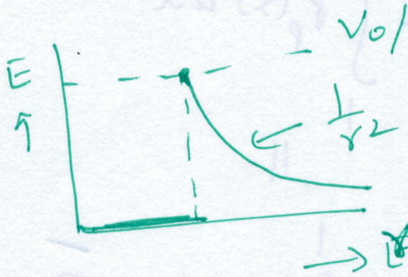
$$V_{out}(r = R) = V_{in}(r = R)$$



$$\therefore \sum \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0$$

$$\Rightarrow \quad B_0 = V_0 R$$

$$B_l = 0 \quad \text{if } l \neq 0$$



$$\therefore V_{out} = \frac{B_0}{r} = \frac{V_0 R}{r}$$

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} = \frac{V_0 R}{r^2}\hat{r}$$