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Sol. Answered Quiz I   dtd. Feb 07, 2024

Q1 Azimuthal Symmetry.

i General Solution to the Laplace's eq.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \quad \text{--- (1)} \qquad \text{--- 1}$$

Boundary condition.

ii (a) Potential to be finite everywhere.

(b) Potential to be continuous at $r = R$ — 2

c at $r = R$, $V(R,\theta) = 2\cos^2\theta$

iii From 1st boundary condition, for eq (1).

$$V_{in}(r \le R, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{--- (1)}$$

$$V_{out}(r \ge R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad \text{--- (2)}$$

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From 2nd boundary condition.

$$V_{in}(r=R,\theta) = V_{out}(r=R,\theta)$$

$$\therefore \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

equating the coefficients of $P_l(\cos\theta)$ — 1

$$A_l R^l = \frac{B_l}{R^{l+1}}$$

$$\Rightarrow \quad B_l = A_l R^{2l+1} \quad \text{--- (3)}$$

From 3rd condition, equation (1) (or one can proceed with equation 2)

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = 2\cos^2\theta \quad \text{--- (4)} \qquad \text{--- 1}$$

Recall $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$$\Rightarrow \cos^2\theta = \frac{2}{3}P_2\cos\theta + \frac{1}{3}$$

$$= \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta) \quad \text{--- (5)}$$

Substituting eq. 5 in eq. 4

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = 2\left[\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right]$$

$\Rightarrow$ for $l \neq 0$ & 2, $A_l = 0$. —— 1

for $l=0$, $A_0 = \frac{2}{3}$ —— 5

for $l=2$ $A_2 = \frac{4}{3R^2}$ —— 6

—— 1

From ~~3~~, eqs 3, 5 & 6.

$$B_0 = \frac{2}{3}R. \quad \text{—— 7}$$

$$B_2 = \left(\frac{4}{3R^2}\right)R^5 = \frac{4R^3}{3} \quad \text{—— 8}$$

—— 1

Our soln substituting the coefficients in eqs. 1 & 2

$$\boxed{\begin{aligned} V_{in}(r \le R, \theta) &= \frac{2}{3} + \frac{4r^2}{3R^2}P_2(\cos\theta) \\ V_{out}(r > R, \theta) &= \frac{2R}{3r} + \frac{4R^3}{3r^3}P_2(\cos\theta) \end{aligned}}$$

—— 1

Q2 $\frac{d^2\Phi}{d\phi^2} = 0$ for the azimuthal symmetry, i.e. Potential is independent of ϕ —— 2

and

$\frac{d^2\Phi}{d\phi^2} \neq 0$ when potential is not independent of ϕ. —— 1

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Q 3

$$\vec{E} = y\hat{i} + x\hat{j}$$

$$V = -\int \vec{E}\cdot d\vec{l} + c$$ 1

$$= -\int (y\hat{i} + x\hat{j})\cdot(dx\hat{i} + dy\hat{j}) + c$$

$$= -\int (y\,dx + x\,dy) + c$$

$$= -\int d(xy) + c$$

$$= -xy + c$$

$$= -(r\cos\varphi)(r\sin\varphi) + c$$

$$= -r^2\cos\varphi\sin\varphi + c$$ 1

Note: Full credit for ~~the~~ not considering the constant of integration (i.e. $c = 0$)