Complex Analysis

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Contents

CHAPTER 1

Complex Numbers

1.1. Definitions

DEFINITION 1. Complex numbers are defined as ordered pairs (x, y)

Points on a complex plane. Real axis, imaginary axis, purely imaginary numbers. Real and imaginary parts of complex number. Equality of two complex numbers.

DEFINITION 2. The sum and product of two complex numbers are defined as follows:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

In the rest of the chapter use $z, z_1, z_2, \ldots$ for complex numbers and x, y for real numbers. introduce i and $z = x + iy$ notation.

1.2. Algebraic Properties

(1) Commutativity

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1$$

(2) Associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3)$$

(3) Distributive Law

$$z(z_1 + z_2) = zz_1 + zz_2$$

(4) Additive and Multiplicative Indentity

$$z + 0 = z, \qquad z \cdot 1 = z$$

(5) Additive and Multiplicative Inverse

$$\begin{aligned}-z &= (-x, -y) \\ z^{-1} &= \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right), \qquad z \neq 0\end{aligned}$$

(6) Subtraction and Division

$$z_1 - z_2 = z_1 + (-z_2), \qquad \frac{z_1}{z_2} = z_1z_2^{-1}$$

(7) Modulus or Absolute Value

$$|z| = \sqrt{x^2 + y^2}$$

(8) Conjugates and properties

$$\begin{aligned}\overline{z} &= x - iy = (x, -y) \\ \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z_1}}{\overline{z_2}}\end{aligned}$$

(9)

$$\begin{aligned}|z|^2 &= z\overline{z} \\ \mathrm{Re}z &= \frac{z + \overline{z}}{2}, \mathrm{Im}z = \frac{z - \overline{z}}{2i}\end{aligned}$$

(10) Triangle Inequality
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

1.3. Polar Coordinates and Euler Formula

(1) Polar Form: for $z \neq 0$,
$$z = r\left(\cos\theta + i\sin\theta\right)$$
where $r = |z|$ and $\tan\theta = y/x$. θ is called the argument of z. Since $\theta + 2n\pi$ is also an argument of z,the principle value of argument of z is take such that $-\pi < \theta \leq \pi$. For $z = 0$ the $\arg z$ is undefined.
(2) Euler formula: Symbollically,
$$e^{i\theta} = (\cos\theta + i\sin\theta)$$
(3)
$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ z^n &= r^n e^{n\theta} \end{aligned}$$
(4) de Moivre's Formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

1.4. Roots of Complex Numbers

Let $z = re^{i\theta}$ then
$$z^{1/n} = r^{1/n}\exp\left(i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$
There are only n distinct roots which can be given by $k = 0, 1, \ldots, n-1$. If θ is a principle value of $\arg z$ then θ/n is called the principle root.

EXAMPLE 3. The three possible roots of $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} = \left(e^{i\pi/4}\right)^{1/3}$ are $e^{i\pi/12}, e^{i\pi/12 + i2\pi/3}, e^{i\pi/12 + i4\pi/3}$.

1.5. Regions in Complex Plane

(1) $\epsilon - nbd$ of z_0 is defined as a set of all points z which satisfy
$$|z - z_0| < \epsilon$$
(2) *Deleted nbd* of z_0 is a nbd of z_0 excluding point z_0.
(3) Interior Point, Exterior Point, Boundary Point, Open set and closed set.
(4) Domain, Region, Bounded sets, Limit Points.

CHAPTER 2

Functions of Complex Variables

2.1. Functions of a Complex Variable

A *function* f defined on a set S is a rule that uniquely associates to each point z of S a a complex number w. Set S is called the *domain* of f and w is called the value of f at z and is denoted by $f(z) = w$.
$f(z) = f(x+iy) = u(x,y) + iv(x,y)$
$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) = F(r,\theta)e^{i\Theta(r,\theta)}$

EXAMPLE 4. Write $f(z) = 1/z^2$ in $u + iv$ form.
$u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ and $v(x,y) = \frac{2xy}{(x^2+y^2)^2}$
$u(r,\theta) = r^{-2}\cos 2\theta$ and $v(r,\theta) = -r^{-2}\sin 2\theta$
Domain of f is $C - \{0\}$.

A *multiple-valued function* is a rule that assigns more than one value to each point of domain.

EXAMPLE 5. $f(z) = \sqrt{z}$. This function assigns two distinct values to each $z\,(\neq 0)$. One can choose the function to be single-valued by specifying

$$\sqrt{z} = +\sqrt{r}e^{i\theta/2}$$

where θ is the principal value.

2.2. Elementary Functions

(1) **Polynomials**

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

whrere the coefficients are real. Rational Functions.

(2) **Exponential Function**

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Converges for all z. For real z the definition coincides with usual exponential function. Easy to see that $e^{i\theta} = \cos\theta + i\sin\theta$. Then

$$e^z = e^x(\cos y + i\sin y)$$

(a) $e^{z_1}e^{z_2} = e^{z_1+z_2}$.
(b) $e^{z+2\pi i} = e^z$.
(c) A line segment from $(x,0)$ to $(x,2\pi)$ maps to a circle of radius e^x centered at origin.
(d) No Zeros.

(3) **Trigonometric Functions**

Define

$$\begin{aligned}\cos z &= \frac{e^{iz}+e^{-iz}}{2}\\ \sin z &= \frac{e^{iz}-e^{-iz}}{2}\\ \tan z &= \frac{\sin z}{\cos z}\end{aligned}$$

(a) $\sin^2 z + \cos^2 z = 1$
(b) $2\sin z_1\cos z_2 = \sin(z_1+z_2) + \sin(z_1-z_2)$

(c) $2\cos z_1 \cos z_2 = \cos(z_1 + z_2) + \cos(z_1 - z_2)$
(d) $2\sin z_1 \sin z_2 = -\cos(z_1 + z_2) + \cos(z_1 - z_2)$
(e) $\sin(z + 2\pi) = \sin z$ and $\cos(z + 2\pi) = \cos z$.
(f) $\sin z = 0$ iff $z = n\pi \qquad (n = 0, \pm 1, \ldots)$
(g) $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \ldots)$
(h) These functions are not bounded.
(i) A line segment from $(0, y)$ to $(2\pi, y)$ maps to an ellipse with semimajor axis equal to $\cosh y$ under sin function.

(4) **Hyperbolic Functions**

Define

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$

(a) $\sinh(iz) = i \sin z; \qquad \cosh(iz) = \cos z$
(b) $\cosh^2 z - \sinh^2 z = 1$
(c) $\sinh(z + 2\pi i) = \sinh(z); \qquad \cosh(z + 2\pi i) = \cosh(z)$
(d) $\sinh z = 0$ iff $z = n\pi i \qquad (n = 0, \pm 1, \ldots)$
(e) $\cosh z = 0$ iff $z = \left(\frac{\pi}{2} + n\pi\right) i \qquad (n = 0, \pm 1, \ldots)$

(5) **Logarithmic Function**

Define

$$\log z = \log r + i(\theta + 2n\pi)$$

then $e^{\log z} = z$.

(a) Is multiple-valued. Hence cannot be considered as inverse of exponetial function.
(b) Priniciple value of log function is given by

$$\log z = \log r + i\Theta$$

where Θ is the principal value of argument of z.

(c) $\log(z_1 z_2) = \log z_1 + \log z_2$

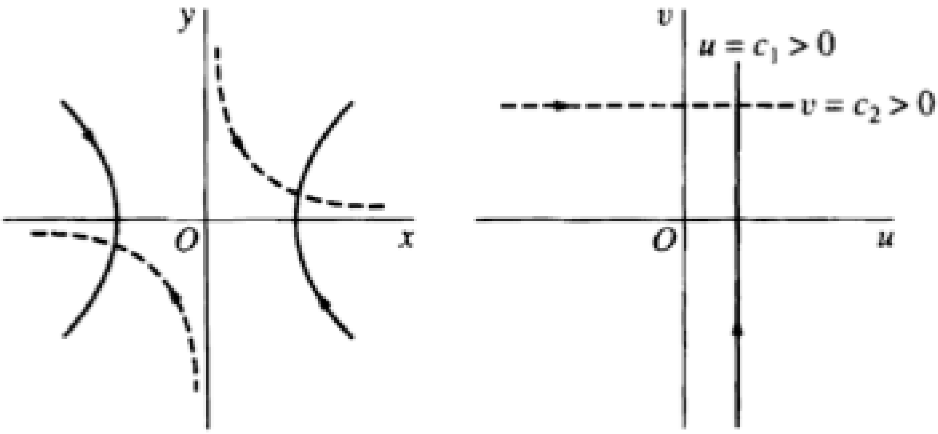
2.3. Mappings

$w = f(z)$. Graphical representation of images of sets under f is called *mapping*. Typically shown in following manner:

(1) Draw regular sets (lines, circles, geometric regions etc) in a complex plane, which we call z plane. Use $z = x + iy = re^{i\theta}$.
(2) Show its images on another complex plane, which we call w plane. Use $w = f(z) = u + iv = \rho e^{i\phi}$.

Example 6. $w = z^2$. Here $u = (x^2 - y^2)$ and $v = 2xy$.

(1) A straight line $x = t$ maps to a parabola $v^2 = -4t^2\left(u - t^2\right)$
(2) A straight line $y = t$ maps to a parabola $v^2 = 4t^2\left(u - t^2\right)$
(3) A half circle given by $z = r_0 e^{i\theta}$ where $0 \le \theta \le \pi$ maps to a full circle given by $w = r_0^2 e^{i2\theta}$. This also means that the upper half plane maps on to the entire complex plane.
(4) A hyperbola $x^2 - y^2 = c$ maps to a straight line $u = c$.

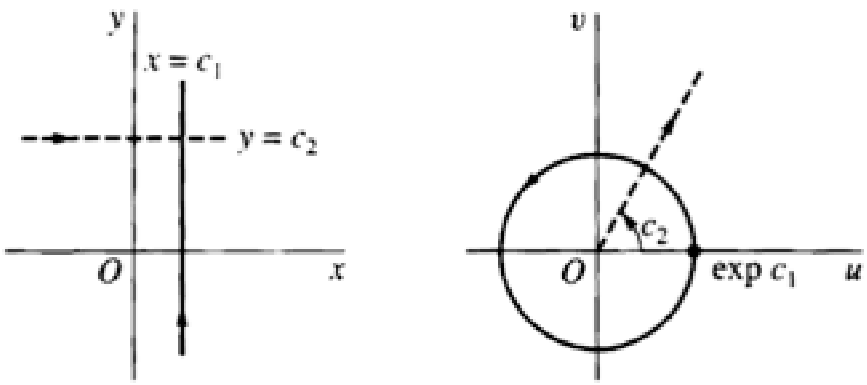


EXAMPLE 7. $w = 1/z$. Here $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$

(1) First $x = t$ maps to a circle $\left(u - \frac{1}{2t}\right)^2 + v^2 = \frac{1}{4t^2}$.
(2) $y = mx$ line maps to $v = -mu$.

2.4. Mappings by Elementary Functions.

(1) **Translation** by z_0 is given by $w = z + z_0$.
(2) **Rotation** through an angle θ_0 is given by $w = e^{i\theta_0} z$.
(3) **Relflection** through x axis is given by $w = \bar{z}$.
(4) **Exponential Function**
A vertical line maps to a circle.
A horizontal line maps to a radial line.
A horizontal strip enclosed between $y = 0$ and $y = 2\pi$ maps to the entire complex plane.



(5) **Sine Function** $\sin z = \sin x \cosh y + i \cos x \sinh y$
A vertical line maps to a branch of a hyperbola.
A horizontal line maps to an ellipse and has a period of 2π.

CHAPTER 3

Analytic Functions

3.1. Limits

A function f is defined in a deleted nbd of z_0.

Definition 8. The limit of the function $f(z)$ as $z \to z_0$ is a number w_0 if, for any given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Example 9. $f(z) = 5z$. Show that $\lim_{z\to z_0} f(z) = 5z_0$.

Example 10. $f(z) = z^2$. Show that $\lim_{z\to z_0} f(z) = z_0^2$.

Example 11. $f(z) = z/\bar{z}$. Show that the limit of f does not exist as $z \to 0$.

Theorem 12. *Let $f(z) = u(x,y) + iv(x,y)$ and $w_0 = u_0 + iv_0$. $\lim\limits_{z\to z_0} f(z) = w_0$ if and only if $\lim\limits_{(x,y)\to(x_0,y_0)} u = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v = v_0$.*

Example 13. $f(z) = \sin z$. Show that the $\lim_{z\to z_0} f(z) = \sin z_0$

Example 14. $f(z) = 2x + iy^2$. Show that the $\lim_{z\to 2i} f(z) = 4i$.

Theorem 15. *If $\lim\limits_{z\to z_0} f(z) = w_0$ and $\lim\limits_{z\to z_0} F(z) = W_0$,*

$$\begin{aligned}
\lim_{z\to z_0} [f(z) + F(z)] &= w_0 + W_0; \\
\lim_{z\to z_0} f(z) F(z) &= w_0 W_0; \\
\lim_{z\to z_0} f(z)/F(z) &= \frac{w_0}{W_0} \qquad W_0 \neq 0.
\end{aligned}$$

This theorem immediately makes available the entire machinery and tools used for real analysis to be applied to complex analysis. The rules for finding limits then can be listed as follows:

(1) $\lim\limits_{z\to z_0} c = c$.
(2) $\lim\limits_{z\to z_0} z^n = z_0^n$.
(3) $\lim\limits_{z\to z_0} P(z) = P(z_0)$ if P is a polynomial in z.
(4) $\lim\limits_{z\to z_0} \exp(z) = \exp(z_0)$.
(5) $\lim\limits_{z\to z_0} \sin(z) = \sin z_0$.

3.2. Continuity

Definition 16. A function f, defined in some nbd of z_0 is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

This definition clearly assumes that the function is defined at z_0 and the limit on the LHS exists. The function f is continuous in a region if it is continuous at all points in that region.

If funtions f and g are continuous at z_0 then $f + g$, fg and f/g $(g(z_0) \neq 0)$ are also continuous at z_0.

If a function $f(z) = u(x,y) + iv(x,y)$ is continuous at z_0 then the component functions u and v are also continuous at (x_0, y_0).

3.3. Derivative

DEFINITION 17. A function f, defined in some nbd of z_0 is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the derivative of f at z_0 and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

EXAMPLE 18. $f(z) = z^2$. Show that $f'(z) = 2z$.

EXAMPLE 19. $f(z) = |z|^2$. Show that this function is differentiable only at $z = \dot{0}$. In real analysis $|x|$ is not differentiable but $|x|^2$ is.

If a function is differentiable at z, then it is continuous at z.

The converse in not true. See Example 19.

Even if component functions of a complex function have all the partial derivatives, does not imply that the complex function will be differentiable. See Example 19.

Some rules for obtaining the derivatives of functions are listed here. Let f and g be differentiable at z.

(1) $\frac{d}{dz}(f \pm g)(z) = f'(z) \pm g(z)$.
(2) $\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z)$.
(3) $\frac{d}{dz}(f/g)(z) = (f'(z)g(z) - f(z)g'(z)) / [g(z)]^2$ if $g(z) \neq 0$.
(4) $\frac{d}{dz}(f \circ g)(z) = f'(g(z))g'(z)$.
(5) $\frac{d}{dz}c = 0$.
(6) $\frac{d}{dz}z^n = nz^{n-1}$.

3.4. Cauchy-Riemann Equations

THEOREM 20. *If $f'(z_0)$ exists, then all the first order partial derivatives of component function $u(x, y)$ and $v(x, y)$ exist and satisfy Cauchy-Riemann Conditions:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

EXAMPLE 21. $f(z) = z^2 = x^2 - y^2 + i2xy$. Show that Cauchy-Riemann Condtions are satisfied.

EXAMPLE 22. $f(z) = |z|^2 = x^2 + y^2$. Show that the Cauchy-Riemann Condtions are satisfied only at $z = 0$.

THEOREM 23. *Let $f(z) = u(x, y) + iv(x, y)$ be defined in some nbd of the point z_0. If the first partial derivatives of u and v exist and are continuous at z_0 and satisfy Cauchy-Riemann equations at z_0, then f is differentiable at z_0 and*

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

EXAMPLE 24. $f(z) = \exp(z)$. Show that $f'(z) = \exp(z)$.

EXAMPLE 25. $f(z) = \sin(z)$. Show that $f'(z) = \cos(z)$.

EXAMPLE 26. $f(z) = \frac{\bar{z}^2}{z}$. Show that the CR conditions are satisfied at $z = 0$ but the function f is not differentiable at 0.

If we write $z = re^{i\theta}$ then we can write Cauchy-Riemann Conditions in polar coordinates:

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r. \end{aligned}$$

3.5. Analytic Functions

DEFINITION 27. A function is analytic in an open set if it has a derivative at each point in that set.

DEFINITION 28. A function is analytic at a point z_0 if it is analytic in some nbd of z_0.

DEFINITION 29. A function is an entire function if it is analytic at all points of C.

EXAMPLE 30. $f(z) = 1/z$ is analytic at all nonzero points.

EXAMPLE 31. $f(z) = |z|^2$ is not analytic anywhere.

A function is not analytic at a point z_0, but is analytic at some point in each nbd of z_0 then z_0 is called the singular point of the function f.

3.6. Harmonic Functions

DEFINITION 32. A real valued function $H(x, y)$ is said to be *harmonic* in a domain of xy plane if it has continuous partial derivatives of the first and second order and satisfies *Laplace equation*:

$$H_{xx}(x, y) + H_y(x, y) = 0.$$

THEOREM 33. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions u and v are harmonic in D.*

DEFINITION 34. If two given functions $u(x, y)$ and $v(x, y)$ are harmonic in domain D and their first order partial derivatives satisfy Cauchy-Riemann Conditions

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

then v is said to be *harmonic conjugate* of u.

EXAMPLE 35. Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Show that v is hc of u and not vice versa.

EXAMPLE 36. $u(x, y) = y^3 - 3xy$. Find harmonic conjugate of u.

CHAPTER 4

Integrals

4.1. Contours

EXAMPLE 37. Represent a line segment joining points $(0,6)$ and $(3,11)$ by parametric equations.

EXAMPLE 38. Show that a half circle in upper half plane with radius R and centered at origin can be parametrized in various ways as given below:

(1) $x(t) = R\cos t, \quad y(t) = R\sin t$, where $t : 0 \to \pi$.
(2) $x(t) = t, \quad y(t) = \sqrt{R^2 - t^2}$, where $t : -R \to R$.
(3) $x(t) = R(2t-1), \quad y(t) = 2R\sqrt{t-t^2}$, where $t : 0 \to 1$.

DEFINITION 39. A set of points $z = (x, y)$ in complex plane is called an *arc* if

$$x = x(t), \qquad y = y(t), \qquad (a \le t \le b)$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t.

EXAMPLE 40. $x(t) = \cos t, \quad y(t) = \sin(2t)$, where $t : 0 \to 2\dot{\pi}$. Show that the curve cuts itself and is closed.

An arc is called *simple* if $t_1 \ne t_2 \Rightarrow z(t_1) \ne z(t_2)$.
An arc is *closed* if $z(a) = z(b)$.
An arc is *differentiable* if $z'(t) = x'(t) + iy'(t)$ exists and $x'(t)$ and $y'(t)$ are continuous. A *smooth* arc is differentiable and $z'(t)$ is nonzero for all t.

DEFINITION 41. Length of a smooth arc is defined as

$$L(C) = \int_a^b \left| \acute{z}'(t) \right| dt.$$

The length is invariant under parametrization change.

DEFINITION 42. A *contour* is a constructed by joining finite smooth curves end to end such that $z(t)$ is continuous and $z'(t)$ is piecewise continuous.

A closed simple contour has only first and last point same and does not cross itself.

4.2. Contour Integral

If C is a contour in complex plane defined by $z(t) = x(t) + iy(t)$ and a function $f(z) = u(x,y) + iv(x,y)$ is defined on it. The integral of $f(z)$ along the contour C is denoted and defined as follows:

$$\begin{aligned} \int_C f(z)\,dz &= \int_a^b f(z)\,z'(t)\,dt \\ &= \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')dt \\ &= \int (udx - vdy) + i\int (udy + vdx) \end{aligned}$$

The component integrals are usual real integrals and are well defined. In the last form appropriate limits must placed in the integrals.
Some very straightforward rules of integration are given below:

(1) $\int_C wf(z)\,dz = w\int_C f(z)\,dz$ where w is a complex constant.
(2) $\int_C (f(z) + g(z))dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(3) $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$
(4) $\left|\int_C f(z)\,dz\right| \le \int_C |f(z(t))\,z'(t)|\,dt.$
(5) If $|f(z)| \le M$ for all $z \in C$ then $\left|\int_C f(z)\,dz\right| \le ML$, where L is length of the countour C.

EXAMPLE 43. $f(z) = z^2$. Find integral of f from $(0,0)$ to $(2,1)$ along a straight line and also along st line path from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(2,1)$.

EXAMPLE 44. $f(z) = 1/z$. Find the integral from $(2,0)$ to $(-2,0)$ along a semicircular path in upper plane given by $|z| = 2$.

EXAMPLE 45. Show that $\left|\int_C f(z)\,dz\right| \le \frac{\pi}{3}$ for $f(z) = 1/\left(z^2-1\right)$ and $C : |z| = 2$ from 2 to $2i$.

4.3. Cauchy-Goursat Theorem

THEOREM 46 (Jordan Curve Theorem). *Every simple and closed contour in complex plane splits the entire plane into two domains one of which is bounded. The bounded domain is called the interior of the countour and the other one is called the exterior of the contour.*

Define a sense direction for a contour.

THEOREM 47. *Let C be a simple closed contour with positive orientation and let D be the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x and Q_y at all points on C and D, then*

$$\int_C (P(x,y)dx + Q(x,y)\,dy) = \iint [Q_x(x,y) - P_y(x,y)]\,dxdy$$

THEOREM 48 (Cauchy-Goursat Theorem). *Let f be analytic in a simply connected domain D. If C is any simple closed contour in D, then*

$$\int_C f(z)\,dz = 0.$$

EXAMPLE 49. $f(z) = z^2, \exp(z), \cos(z)$ etc are entire functions so integral about any loop is zero.

THEOREM 50. *Let C_1 and C_2 be two simple closed positively oriented contours such that C_2 lies entirely in the interior of C_1. If f is an analytic function in a domain D that contains C_1 and C_2 both and the region between them, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

EXAMPLE 51. $f(z) = 1/z$. Find $\int_C f(z)\,dz$ if C is any contour containing origin. Choose a circular contour inside C.

EXAMPLE 52. $\int_C \frac{1}{z-z_0}dz = 2\pi i$ if C contains z_0.

EXAMPLE 53. Find $\int_C \frac{2z dz}{z^2+2}$ where $C : |z| = 2$. Extend the Cauchy Goursat theorem to multiply connected domains.

4.4. Antiderivative*

If a function F is analytic in a domain D and $F'(z) = f(z)$, then F is called an antiderivative of f. (Note that it is mandatory that f be at least continuous, but later we will see that if F is analytic in D automatically implies that f is analytic too!)

THEOREM 54 (Fundamental Theorem of Integration). *Let f be defined in a simply connected domain D and is analytic in D. If z_0 and z are points in D and C is any contour in D joining z_0 and z, then the function*

$$F(z) = \int_C f(z)\,dz$$

is analytic in D and $F'(z) = f(z)$.

PROOF. Now consider

$$\begin{aligned}\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| &= \frac{1}{|\Delta z|}\left|\int_z^{z+\Delta z}(f(s)-f(z))\,ds\right| \\ &\leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z}|f(s)-f(z)|\,ds \\ &\leq \frac{1}{|\Delta z|}\epsilon\,|\Delta z| = \epsilon\end{aligned}$$

the second last step follows from the fact that f is continuous (as a result of the fact that it is analytic). □

DEFINITION 55. If f is analytic in D and z_1 and z_2 are two points in D then the definite integral is defined as

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where F is an antiderivative of f.

EXAMPLE 56. $\int_0^{2+i} z^2 dz = z^3/3\big|_0^{2+i} = \frac{2}{3} + i\frac{11}{3}$. Note that z^2 is analytic everywhere.

EXAMPLE 57. $\int_1^i \cos z = \sin z|_1^i = \sin i - \sin 1$. Note that $\cos z$ is analytic everywhere.

Now, we will state a little stronger theorem.

THEOREM 58. *If f is a continuous function on a domain D and $\int f(z)dz$ is independent of the contour joining any two points, then f has an antiderivative in D.*

PROOF. The path independence allows us to define a new function $F(z)$ as

$$F(z) = \int_{z_0}^{z} f(s)ds.$$

Now use the same argument as used in previous theorem to show that $F'(z) = f(z)$. □

EXAMPLE 59. $\int \frac{1}{z^2}dz = -\frac{1}{z}$ if $|z| > 0$. And $\oint \frac{dz}{z^2} = 0$.

EXAMPLE 60. $\int_{z_1}^{z_2} \frac{dz}{z} = \log z_2 - \log z_1$ is true only if we restrict ourselves one branch of log function. In perticular $\oint \frac{dz}{z} = 2\pi i$ if the contour is a unit circle. We can still use the idea of antiderivative to evaluate this integral. Do this in two steps:

$$\oint \frac{dz}{z} = \int_{-i}^{i}\frac{dz}{z} + \int_{i}^{-i}\frac{dz}{z}$$

for the first integral use the branch with $\theta \in (-\pi, \pi)$ and for the second integral use $\theta \in (0, \pi)$, then:

$$\begin{aligned}\oint \frac{dz}{z} &= \int_{-i}^{i}\frac{dz}{z} + \int_{i}^{-i}\frac{dz}{z} = \log i - \log(-i) + \mathrm{Log}(i) - \mathrm{Log}(-i) \\ &= i\frac{\pi}{2} - i(-\frac{\pi}{2}) + i\frac{3\pi}{2} - i\frac{\pi}{2} = 2\pi i\end{aligned}$$

4.5. Cauchy Integral Formula

THEOREM 61 (Cauchy Integral Formula). *Let f be analytic in domain D. Let C be a positively oriented simple closed contour in D. If z_0 is in the interior of C then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0}.$$

EXAMPLE 62. $f(z) = \frac{1}{z^2+4}$. Find $\int_C f(z)\,dz$ if $C : |z - i| = 2$.

EXAMPLE 63. $f(z) = \frac{z}{2z+1}$. Find $\int_C f(z)\,dz$ if C is square with vertices on $(\pm 2, \pm 2)$.

THEOREM 64. *If f is analytic at a point, then all its derivatives exist and are analytic at that point.*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

CHAPTER 5

Series*

5.1. Convergence of Sequences and Series

EXAMPLE 65. $z_n = 1/n$

EXAMPLE 66. $z_n = a^n$

EXAMPLE 67. $z_n = -2 + i\,(-1)^n\,/n^2$

DEFINITION 68. An infinite sequece $z_1, z_2, \ldots, z_n, \ldots$ of complex numbers has a *limit* z if for each positive ϵ, there exists positive integer N such that

$$|z_n - z| < \epsilon \qquad \text{whenever} \qquad n > N.$$

The sequences have only one limit. A sequence said to converge to z if z is its limit. A sequence diverges if it does not converge.

EXAMPLE 69. $z_n = z^n$ converges to 0 if $|z| < 1$ else diverges.

Let $|z| < 1$. For any given $\epsilon > 0$, choose $N > 1$ such that $|z| < \epsilon^{1/N}$. Then if $n > N$ implies that $|z^n - 0| = |z|^n < |z|^N < \epsilon$. Alternatively, if $|z| > 1$ then $|z|^n > |z|^N$ if $n > N$, thus the sequecne will not converge.

EXAMPLE 70. $z_n = 1/\sqrt{n} + i\,(n+1)\,/n$ converges to i.

Now, for $n > 2$, $|z_n - i| = \left|\frac{1}{\sqrt{n}} + i\frac{1}{n}\right| = \sqrt{\frac{1}{n} + \frac{1}{n^2}} < \sqrt{\frac{2}{n}} < \frac{2}{n}$ since, $n > 1 \implies \frac{1}{n} < 1 \implies \frac{1}{n^2} < \frac{1}{n} \implies \frac{1}{n^2} + \frac{1}{n} < \frac{2}{n}$. Thus, for a given ϵ, choose $N = \frac{2}{\epsilon}$. Then, if $n > N$, then $|z_n - i| < \frac{2}{n} < \frac{2}{N} = \epsilon$.

THEOREM 71. *Suppose that* $z_n = x_n + iy_n$ *and* $z = x + iy$. *Then,*

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y.$$

PROOF. If $\lim z_n = z$, then for every ϵ, there is N such that $|z_n - z| < \epsilon$ if $n > N$. Now, since $|x_n - x| < |z_n - z| < \epsilon$ if $n > N$. To prove converse, note that $|z_n - z| < \sqrt{(x_n - x)^2 + (y_n - y)^2} < \sqrt{\epsilon^2 + \epsilon^2} < \sqrt{2}\epsilon$. ☐

DEFINITION 72. If $\{z_n\}$ is a sequence, the infinite sum $z_1 + z_2 + \cdots + z_n + \cdots$ is called a series and is denoted by $\sum\limits_{n=1}^{\infty} z_n$.

DEFINITION 73. A series $\sum\limits_{n=1}^{\infty} z_n$ is said to converge to a sum S if a sequence of partial sums

$$S_N = z_1 + z_2 + \cdots + z_n$$

converges to S.

EXAMPLE 74. $\sum_0^\infty z^n = 1/\,(1-z)$ if $|z| < 1$

Here $S_N = \sum z^n = \frac{1-z^{N+1}}{1-z} \to \frac{1}{1-z}$ if $|z| < 1$.

THEOREM 75. *Suppose that* $z_n = x_n + iy_n$ *and* $S = X + iY$. *Then,*

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n = Y$$

PROOF. Now, the sequence of partial sums $S_n = S_n^x + iS_n^y$. Then by similar theorem for sequences, it is easy to see that $S_n \to S$ *iff* $S_n^x \to X$ and $S_n^y \to Y$. $\square$

EXERCISE 76. [Homework for you guys.] Show that if $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$, then

(1) $\sum_{n=1}^{\infty} \bar{z_n} = \bar{S}$
(2) $\sum_{n=1}^{\infty} cz_n = cS$ where c is a constant complex number
(3) $\sum_{n=1}^{\infty} (z_n + w_n) = S + T$.

5.2. Taylor Series

THEOREM 77 (Taylor Series). *If* f *is analytic in a circular disc of radius* R_0 *and centered at* z_0, *then at each point inside the disc there is a series representation for* f *given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

PROOF. A simplified proof: If $z_0 = 0$, then the series is called a Mclauren series. We will prove the theorem for Mclaurent series and with change of variable, it applies to general case of Taylor series. Now, by Cauchy formula, using a circular contour about the origin (such that z is inside the contour), we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(s)ds}{s - z} \\
&= \frac{1}{2\pi i} \oint_C \frac{f(s)ds}{s(1 - z/s)} \\
&= \frac{1}{2\pi i} \oint_C \frac{f(s)ds}{s} \sum_0^{\infty} \left(\frac{z^n}{s^n}\right) \\
&= \sum_0^{\infty} z^n \left(\frac{1}{2\pi i} \oint_C \frac{f(s)ds}{s^{n+1}}\right) \\
&= \sum_0^{\infty} \frac{1}{n!} f^{(n)}(0) z^n
\end{aligned}$$

$\square$

EXAMPLE 78. $\sin z = \sum_0^{\infty} (-1)^n z^{2n+1} / (2n+1)!$.
$\frac{d^n}{dz^n} \sin z\big|_{z=0} = (-1)^{2n+1}$ if n is odd else it is 0. Plug in to get the required expression.

EXAMPLE 79. $\frac{1}{1+z} = \sum_0^{\infty} (-1)^n z^n \qquad |z| < 1$.
Now, $\frac{d^n}{dz^n} \frac{1}{1+z} = \frac{(-1)^n n!}{(1+z)^{n+1}}$.

EXAMPLE 80. $e^z = \sum_0^\infty z^n/n!$.

EXAMPLE 81. $\frac{1}{z} = \sum_0^\infty (-1)^n (z-1)^n \qquad |z-1| < 1.$

5.3. Laurent Series

THEOREM 82 (Laurent Series). *If f is analytic at all points z in an annular region D such that $R_1 < |z - z_0| < R_2$, then at each point in D there is a series representation for f given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \end{aligned}$$

and C is any contour in D.

PROOF. Here we give a proof of the theorem with $z_0 = 0$. Look at the figure shown below. Location of z is in the annular region D. Let C_1 and C_2 be two circular contours centered about O with radii r_1 and r_2. $(R_1 < r_1 < |z| < r_2 < R_2)$. Let γ be another circular contour centered about z such that it completely lies inside the annular region formed by C_1 and C_2. Now, by Cauchy-Goursat theorem

$$\int_{C_1} \frac{f(s)ds}{s-z} + \int_{\gamma} \frac{f(s)ds}{s-z} - \int_{C_2} \frac{f(s)ds}{s-z} = 0$$

since in the region between the contours the integrand is analytic. This implies

$$f(z) = \int_{C_2} \frac{f(s)ds}{s-z} + \int_{C_1} \frac{f(s)ds}{z-s}.$$

Note that on C_2, $|s| > |z|$ and hence

$$\int_{C_2} \frac{f(s)ds}{s-z} = \sum_{n=0}^{\infty} z^n \int_{C_2} \frac{f(s)ds}{s^{n+1}}$$

resulting in

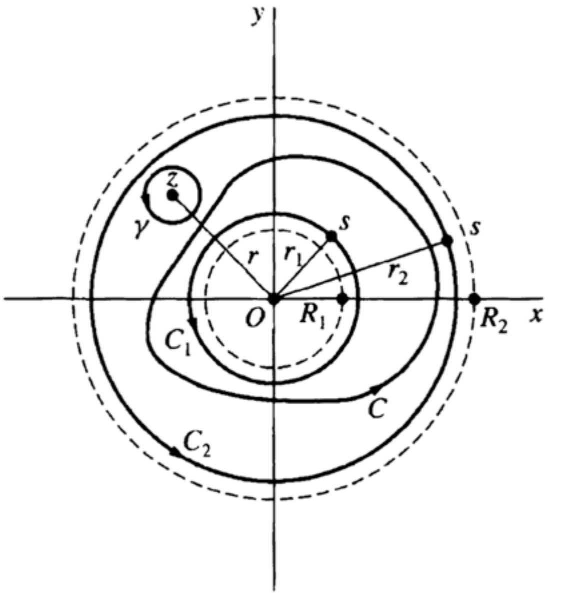
$$a_n = \int_{C_2} \frac{f(s)ds}{s^{n+1}} = \int_C \frac{f(s)ds}{s^{n+1}}.$$

The last is result of CG theorem again because $f(s)/s^{n+1}$ is analytic between C_2 and C. Similarly, on C_1 $|z| > |s|$, then

$$\int_{C_1} \frac{f(s)ds}{z-s} = \sum_{n=1}^{\infty} \int_{C_1} f(s)ds \frac{s^{n-1}}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n} \int_{C_1} \frac{f(s)ds}{s^{-n+1}}.$$

That is

$$b_n = \int_{C_1} \frac{f(s)ds}{s^{-n+1}} = \int_C \frac{f(s)ds}{s^{-n+1}}.$$



∎

Example 83. If f is analytic inside a disc of radius R about z_0, then the Laurent series for f is identical to the Taylor series for f. That is all $b_n = 0$.

Example 84. $\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n}$ where $|z| > 1$.
Clearly when $1/|z| < 1$, we have

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+(1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n}$$

Example 85. $f(z) = \frac{-1}{(z-1)(z-2)}$. Find Laurent series for all $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$.
Now,

$$\frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Example 86. Note that $b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$.

CHAPTER 6

Theory of Residues And Its Applications

6.1. Singularities

Definition 87. If a function f fails to be analytic at z_0 but is analytic at some point in each neighbourhood of z_0, then z_0 is a *singular point* of f.

Definition 88. If a function f fails to be analyitc at z_0 but is analytic at each z in $0 < |z - z_0| < \delta$ for some δ, then f is said to be an *isolated singular point* of f.

Example 89. $f(z) = 1/z$ has an isolated singularity at 0.

Example 90. $f(z) = 1/\sin(\pi z)$ has isolated singularities at $z = 0, \pm 1, \ldots$.

Example 91. $f(z) = 1/\sin(\pi/z)$ has isolated singularities at $z = 1/n$ for integral n, also has a singularity at $z = 0$.

Example 92. $f(z) = \log z$ all points of negative x-axis are singular.

6.2. Types of singularities

If a function f has an isolated singularity at z_0 then $\exists$ a δ such that f is analytic at all points in $0 < |z - z_0| < \delta$. Then f must have a Laurent series expansion about z_0. The part $\sum_{n=1}^{\infty} b_1 (z - z_0)^{-n}$ is called the principal part of f.

(1) If there are infinite nonzero b_i in the prinipal part then z_0 is called an essential singularity of f.
(2) If for some integer m , $b_m \neq 0$ but $b_i = 0$ for all $i > m$ then z_0 is called a pole of order m of f. If $m = 1$ then it is called a simple pole.
(3) If all bs are zero then z_0 is called a removable singularity.

Example 93. $f(z) = (\sin z)/z$ is undefined at $z = 0$. f has a removable isolated singularity at $z = 0$.

6.3. Residues

Suppose a function f has an isolated singularity at z_0, then there exists a $\delta > 0$ such that f is analytic for all z in deleted nbd $0 < |z - z_0| < \delta$. Then f has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

The coefficient

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

where C is any contour in the deleted nbd, is called the residue of f at z_0.

Example 94. $f(z) = \frac{1}{z - z_0}$. Then $\int_C f(z)\, dz = 2\pi i b_1 = 2\pi i$ if C contains z_0, otherwise 0.
Here the function is in LS(Laurent Series) form.

Example 95. $f(z) = \frac{1}{z(z-2)^4}$. Show $\int_C f(z)\, dz = -\frac{\pi i}{8}$ if $C : |z - 2| = 1$.

Now, function has two isolated singularities, one at 0 (simple pole) and the other at 2 (pole of order 4). The contour C contains the singularity at 2. First we find the LS of f about 2:

$$\begin{aligned}
f(z) &= \frac{1}{(z-2)^4}\frac{1/2}{1+(z-2)/2} \\
&= \frac{1}{2(z-2)^4}\sum_{n=0}\frac{(-1)^n(z-2)^n}{2^n} \\
&= \frac{1}{2(z-2)^4}-\frac{1}{4(z-2)^3}+\frac{1}{8(z-2)^2}+\frac{-1}{16(z-2)}+\frac{1}{32}+\cdots
\end{aligned}$$

The co-efficient of $\frac{1}{z-2}$ is $b_1 = -1/16$. Then

$$\int_C f(z)\,dz = 2\pi i b_1 = 2\pi i\left(\frac{-1}{16}\right)$$

Example 96. $f(z) = z\exp\left(\frac{1}{z}\right)$. Show $\int_C f(z)\,dz = \pi i$ if $C : |z| = 1$.

Clearly there is an essential isolated singularity at $z = 0$. The LS series about $z = 0$ is

$$ze^{1/z} = z\left(1+\frac{1}{z}+\frac{1}{2z^2}+\cdots\right)$$

Thus, $b_1 = 1/2$.

Example 97. $f(z) = \exp\left(\frac{1}{z^2}\right)$. Show $\int_C f(z)\,dz = 0$ if $C : |z| = 1$ even though it has a singularity at $z = 0$.

The LS is

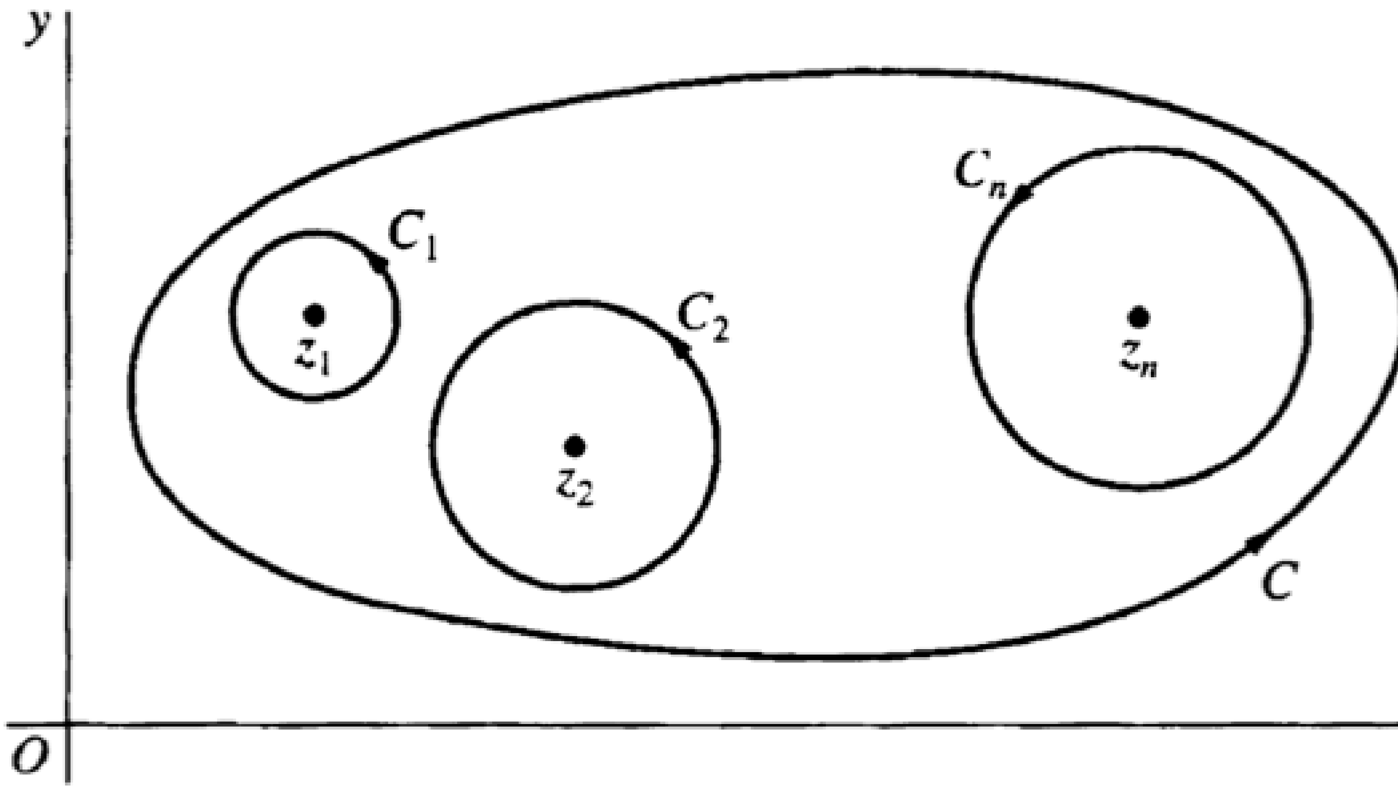
$$e^{1/z^2} = 1+\frac{1}{z^2}+\cdots$$

Then $b_n = 0$ for all odd n. Thus, $b_1 = 0$.

Theorem 98. *If a function f is analytic on and inside a positively oriented countour C, except for a finite number of points $z_1, z_2, \ldots, z_k$ inside C, then*

$$\int_C f(z)\,dz = 2\pi i\sum_{i=1}^{k} Resf(z_i).$$

Proof. Suppose the contour contains z_i $(i = 1, 2, \ldots, k)$ and the function is not analytic at these points then those must be isolated singularities (see figure).



Then we can use Cauchy-Goursat theorem to write

$$\begin{aligned}\int_C f(z)dz &= \sum_{i=1}^{k}\int_{C_i} f(z)dz \\ &= \sum_{i=1}^{k} 2\pi i \operatorname{Res}[f(z_i)]\end{aligned}$$

□

EXAMPLE 99. Show that $\int \frac{5z-2}{z(z-1)}dz = 10\pi i$ with $C: |z| = 2$.
The integrand can be written as

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{5}{z-1}$$

6.4. Residues of Poles

THEOREM 100. *If a function f has a pole of order m at z_0 then*

$$Resf(z_0) = \lim_{z\to z_0}\left[\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right].$$

PROOF. Let LS of f be

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}.$$

Let $\phi(z) = f(z)(z-z_0)^m$. Note that ϕ is analytic in a nbd of z_0 and

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + \cdots + b_m.$$

This is a Taylor series for ϕ and

$$b_1 = \frac{1}{(m-1)!}\phi^{(m-1)}(z_0)$$

□

EXAMPLE 101. $f(z) = \frac{1}{z-z_0}$. Simple pole Res $f(z_0) = 1$.

EXAMPLE 102. $f(z) = \frac{1}{z(z-z_0)^4}$. Simple pole at $z = 0$. Res $f(0) = 1/16$. Pole of order 4 at $z = 2$. Res $f(2) = -1/16$.

EXAMPLE 103. $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$. Res $f(0) = -3/\pi^4$. Res $f(\pi) = -\left(6 - \pi^2\right)/2\pi^4$.

6.5. Quotients of Analytic Functions

THEOREM 104. *If a function $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are analytic at z_0, then*

(1) *f is singular at z_0 iff $Q(z_0) = 0$.*
(2) *f has a simple pole at z_0 if $Q'(z_0) \neq 0$. Then residue of f at z_0 is $P(z_0)/Q'(z_0)$.*

Appendix References

This appendix contains the references.

Appendix Index

This appendix contains the index.