Tutorial 4: Laplace Equation

1. Let function $\phi(x, y, z)$ be a solution to the Laplace equation $\nabla^2\phi = 0$ in a volume V. Define a vector field $\mathbf{E} = -\nabla\phi$. Let S be any simple closed surface inside V.

   (a) Prove Gauss theorem: $\oint_s \mathbf{E} \cdot d\mathbf{S} = 0$

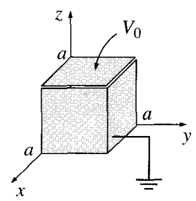
   (b) Deduce that the function ϕ cannot have a maximum or a minimum value at any point inside S.

   (c) If ϕ is constant over the points of S, it must be constant throughout the interior of S.

2. A simple closed surface S encloses a volume V. Assuming that a solution to the Dirichlet problem of the Lapalce equation in V exists, show that it must be unique.

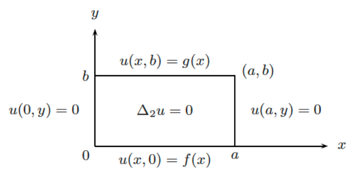
3. Consider a volume $V = \{(x, y, z) \,|x, y \geq 0\}$ bounded by *grounded conducting plates* at surfaces $x = 0$ and $y = 0$. An infinite wire of uniform linear charge density λ is kept parallel to z-axis and passes through point $(1, 2, 0)$. Solve this problem using the method of images and sketch the equipotential lines in XY plane. If you can, use some software to make the sketch (for example, use Mathematica, Maple, Matlab, Gnuplot etc. Or write your own code using Java, Javascript etc.)

4. **Griffiths Problem 3.15** A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (See Fig). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.



5. Consider a semi-infinite plate (bounded by $x = 0$, $x = \pi$ and $y = 0$). The bottom edge is held at temperature $T(x) = \cos x$ and the other sides are at $0°$. Find the steady state temperature in the plate and sketch isotherms.

6. Use the method of separation of variables to solve the Dirichlet problem for the Laplace equation, $\nabla^2 u(x, y) = 0$ on the rectangle $R = \{(x, y) \,|0 < x < a, 0 < y < b\}$, satisfying the boundary conditions as shown in the figure. Here, $f(x) = 0$ and $g(x) = x$. (Δ_2 is another less commonly used symbol for the Laplacian operator.) Sketch isocurves of u.



7. **Jackson 2.14** A variant of the preceding two-dimensional problem (we discussed problem the problem 2.13 in the class) is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$. Show that the potential inside the cylinder is

$$\Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin\left[(4n+2)\,\phi\right]}{2n+1}$$

8. A circular metallic (thermally conducting) disc of radius a is subjected to the following boundary conditions. Find the steady state temperature $T(\rho, \phi)$ in the disc. Sketch isotherms.

   (a) $T(a, \phi) = \frac{1}{2}\left(1 + \cos^3\phi\right), \qquad -\pi < \phi < \pi$

   (b) $T(a, \phi) = |\phi|, \qquad -\pi < \phi < \pi$

9. Solve the Laplace equation $\nabla^2 u(\rho, \phi) = 0$ in the wedge with three sides $\phi = 0, \phi = \beta$, and $\rho = a$ (see Figure) and the boundary conditions $u(\rho, 0)) = 0 = u(\rho, \beta)$, and $u(a, \phi) = f(\phi)$ for $0 < \phi < \beta$.