Tutorial 7: Green's Functions

Green's Functions

1. For the operator $L_x = \frac{d^2}{dx^2}$, find the Green's function with different boundary conditions given below $L_x G(x, x') = \delta(x - x')$:
   (a) $G(0, x') = G(1, x') = 0$,
   (b) $G(-1, x') = G(1, x') = 0$,
   (c) $G(0, x') = 0$ and $G'(1, x') = 0$.

2. Solve the problem (1) using eigenfunction expansion method. From part (c), show that

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi x\right) \sin\left(\left(n+\frac{1}{2}\right)\pi t\right)}{\left(n+\frac{1}{2}\right)^2} = \begin{cases} x, & 0 \leq x < t \\ t, & t < x \leq 1. \end{cases}$$

3. Show that the Green's function for the operator $L_x = \frac{d^2}{dx^2}$ with boundary conditions $G'(0, x') = 0$ and $G'(1, x') = 0$ does not exist.

4. Find the Green's Function for the following differential operators:
   (a) $Ly(x) = y''(x) + y(x)$, $x \in [0, 1]$, with $y(0) = 0$ and $y'(1) = 0$.
   (b) $Ly(x) = y''(x) - y(x)$, $x \in \mathbb{R}$, with $y(\pm\infty) < \infty$.

5. Find the Green's functions for the differential operators

$$Ly(x) = xy''(x) + y'$$

with boundary conditions that $y(1) = 0$ and $y(0)$ should be finite. Use the Green's function, solve

$$\frac{d}{dx}\left[x \frac{dy}{dx}(x)\right] = -1.$$

Verify the solution by direct integration of the differential equation.

6. Find the Green's function for the differential operator

$$\begin{aligned} Ly(x) &= xy''(x) + y'(x) - \frac{n^2}{x} y(x) \\ y(0) &< \infty \\ y(1) &= 0. \end{aligned}$$

7. Find the Green function for associated Legendre differential operator

$$Ly(x) = \frac{d}{dx}\left[\left(1 - x^2\right) \frac{dy}{dx}\right] - \frac{n^2}{(1 - x^2)} y \qquad x \in [-1, 1]$$

with boundary condition that at ± 1, the solution must be finite.

8. Construct a Green's function to solve modified Helmholtz equation

$$y''(x) - k^2 y(x) = f(x)$$

where, k is some constant. The boundary condition is that the Green's function must vanish as $x \to \pm\infty$.

9. Prove the mean value theorem for Laplace equation: Let P be an interior point of a volume V. Let y be a solution of the Laplace equation in V. Then $y(P)$ is the average of y over the surface of any sphere in V centered about P. [Hint: Use the integral equation.] Prove that the solution of the Laplace equation cannot have a maximum or a minimum in V.

10. Consider the Laplace equation $\nabla^2\phi = 0$ in a volume V with boundary S.

    (a) Prove using the Green's identity, that for a function f,

    $$\int_V \left(f\nabla^2 f + |\nabla f|^2\right) dv = \oint_S f\left(\nabla f \cdot \hat{\mathbf{n}}\right) dS.$$

    (b) Prove that the solution (assuming that it exists) to the Laplace equation in V with either Dirichlet or Neumann boundary conditions must be unique.

11. Prove that the Dirichlet Green's function for Laplace equation must be symmetric under exchange of its arguments, that is, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. [Note: This result is true for all self-adjoint differential operators. Tricky proof.]