Tutorial 8: Group Theory

Group Theory: Properties, Equivalence Classes, Subgroups, Quotient Group.

1. Verify that each of the following sets is a group with given *group product*. Which groups are abelian?

   (a) The set of all non-zero rationals under multiplication.

   (b) The set of all complex numbers of unit magnitude under multiplication.

   (c) The set of all complex roots of the equation $z^n = 1$.

   (d) The set $\{1, 2, \ldots, p-1\}$ under multiplication modulo (p), where p is a prime number.

   (e) The set of following matrices under matrix multiplication:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

   (f) The set of functions under function composition:

$$\begin{aligned} f_1(x) &= x, & f_2(x) &= 1 - x, & f_3(x) &= x/\left(x-1\right), \\ f_4(x) &= 1/x, & f_5(x) &= 1/\left(1-x\right), & f_6(x) &= \left(x-1\right)/x, \end{aligned}$$

   (g) The set $\{T_{ab} \mid a, b \in \mathbb{R},\ a \neq 0\}$of all linear transformations on $\mathbb{R}$, such that $T_{ab}\left(x\right) = ax + b$.

   (h) Set of all isometries on $\mathbb{R}^2$ that set of all linear transformations that leave the Euclidean norm invariant.

   (i) The set of all real 2×2 matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, where $ad \neq 0$.

Answer:

(a) $(\mathbb{Q}^*, \times)$: Group, Abelian, Infinite, Discrete (countable).

(b) $G = \{z \in \mathbb{C} \mid |z| = 1\}$: Group, Abelian, Infinite, Continous, 1 parameter.

(c) $G = \{z \in \mathbb{C} \mid z^n = 1\}$: Group, Abelian, Finite, $\mathcal{O}\left(G\right) = n$, Cyclic.

(d) $G = \{1, 2, \ldots, p-1\}$, p is a prime number. The multiplication is $a \odot b = (a \times b) \mod p$:
Example: Let $p = 5$. the multiplication table is

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

To check: $3 \odot 3 = (3 \times 3) \mod 5 = 9 \mod 5 = 4$.

$(G, \odot)$: Group, Abelian, Finite, $\mathcal{O}\left(G\right) = p - 1$, Cyclic.

(e) Noting that $R_{\pi/2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the multiplication table is

$\odot$	I	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	M_x	$M_xR_{\pi/2}$	M_xR_π	$M_xR_{3\pi/2}$
I	I	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	M_x	$M_xR_{\pi/2}$	M_xR_π	$M_xR_{3\pi/2}$
$R_{\pi/2}$	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	I	$M_xR_{\pi/2}$	M_xR_π	$M_xR_{3\pi/2}$	M_x
R_π	R_π	$R_{3\pi/2}$	I	$R_{\pi/2}$	M_xR_π	$M_xR_{3\pi/2}$	M_x	$M_xR_{\pi/2}$
$R_{3\pi/2}$	$R_{3\pi/2}$	I	$R_{\pi/2}$	R_π	$M_xR_{3\pi/2}$	M_x	$M_xR_{\pi/2}$	M_xR_π
M_x	M_x	$M_xR_{3\pi/2}$	M_xR_π	$M_xR_{\pi/2}$	I	$R_{3\pi/2}$	R_π	$R_{\pi/2}$
$M_xR_{\pi/2}$	$M_xR_{\pi/2}$	M_x	$M_xR_{3\pi/2}$	M_xR_π	$R_{\pi/2}$	I	$R_{3\pi/2}$	R_π
M_xR_π	M_xR_π	$M_xR_{\pi/2}$	M_x	$M_xR_{3\pi/2}$	R_π	$R_{\pi/2}$	I	$R_{3\pi/2}$
$M_xR_{3\pi/2}$	$M_xR_{3\pi/2}$	M_xR_π	$M_xR_{\pi/2}$	M_x	$R_{3\pi/2}$	R_π	$R_{\pi/2}$	I

The group is dihedral group of order 8. Not abelian.

(f) Here, $f_k \circ f_m$ is defined as $(f_k \circ f_m)(x) = f_k(f_m(x))$. For example, $(f_2 \circ f_3) = f_2(f_3(x)) = 1 - f_3(x) = 1 - x/(x-1) = -1/(x-1) = f_5(x)$, hence $f_2 \circ f_3 = f_5$. This is a nonabelian group of order 6.

(g) Left for you.

2. Does the set of the three matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

form a group under matrix multiplication? If not, add a minimum number of matrices to complete the group. Prepare the multiplication table.

Answer:
Note that $B = A^2$. A^3 is a new element. And $A^4 = I$. So, if we add A^3, we get a cyclic group $\{E, A, B = A^2, A^3\}$.

3. Generate the matrix group (under matrix multiplication) which contains the two elements $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. What is the order of the group? Is it abelian?

Answer: Let $\psi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\sigma = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Note, that $\psi^4 = \sigma^4 = I$, $\psi^2 = \sigma^2 = -I$, $\sigma\psi\sigma^{-1} = \psi^{-1}$. Not a dihedral group. Requires a little bit of trial and error to complete the group of 8 elements. The multiplication table is

$\odot$	I	ψ	ψ^2	ψ^3	σ	$\sigma\psi$	$\sigma\psi^2$	$\sigma\psi^3$
I	I	ψ	ψ^2	ψ^3	σ	$\sigma\psi$	$\sigma\psi^2$	$\sigma\psi^3$
ψ	ψ	ψ^2	ψ^3	I	$\sigma\psi$	$\sigma\psi^2$	$\sigma\psi^3$	σ
ψ^2	ψ^2	ψ^3	I	ψ	$\sigma\psi^2$	$\sigma\psi^3$	σ	$\sigma\psi$
ψ^3	ψ^3	I	ψ	ψ^2	$\sigma\psi^3$	σ	$\sigma\psi$	$\sigma\psi^2$
σ	σ	$\sigma\psi^3$	$\sigma\psi^2$	$\sigma\psi$	ψ^2	ψ	I	ψ^3
$\sigma\psi$	$\sigma\psi$	σ	$\sigma\psi^3$	$\sigma\psi^2$	ψ^3	ψ^2	ψ	I
$\sigma\psi^2$	$\sigma\psi^2$	$\sigma\psi$	σ	$\sigma\psi^3$	I	ψ^3	ψ^2	ψ
$\sigma\psi^3$	$\sigma\psi^3$	$\sigma\psi^2$	$\sigma\psi$	σ	ψ	I	ψ^3	ψ^2

Not an abelian group.

4. Dihedral group (D_n) is a group generated by two elements A and B subject to relations $A^2 = B^n = (AB)^2 = I$. What is the order of this group? Write down the elements of D_4.

Answer:
Note that $A^{-1} = A$ and $\left(B^k\right)^{-1} = B^{n-k}$Since $BA = AB^{n-1}$. This means that all elements can be written as $A^m B^k$ where $m = 0, 1$ and $k = 0, 1, \ldots, n-1$. The order of the group is $2n$.
Obvious subgroups are

- ▷ $\left\{I, B, B^2, \ldots, B^{n-1}\right\}$, if some $p \mid n$, then $\left\{I, B^p, B^{2p}, \ldots, B^{n-p}\right\}$.
- ▷ $\left\{I, AB^k\right\}$ where $k = 0, 1, \ldots, n-1$.

Equivalence classes are

- ▷ $\{I\}$,
- ▷ $\left\{B^k, B^{n-k}\right\}$, if n is even then $B^{n/2} = B^{n-n/2}$. Number of these classes are $n/2$ or $(n-1)/2$ depending on even n or odd n. $[\left(AB^k\right) B^m \left(AB^k\right)^{-1} = AB^{k+m}AB^k = AAB^{n-k-m}B^k = B^{n-m}]$
- ▷ If n is even then $\left\{A, AB^2, \ldots, AB^{n-2}\right\}$, $\left\{AB, AB^3, \ldots, AB^{n-1}\right\}$, $[B^k \left(AB^m\right) \left(B^k\right)^{-1} = B^k AB^m B^{n-k} = AB^{n-k} B^m B^{n-k} = AB^{m-2k}]$. What will happen when n is odd?

5. Find the subgroup of the symmetric (permutation) group S_4, which leaves the polynomial $x_1x_2 + x_3 + x_4$ invariant.

Answer:
The operation of group element $(abcd)$ on $x_1x_2+x_3+x_4$ gives us $x_ax_b+x_c+x_d$on Only operations that leave this polynomial invariant are $(1234), (2134), (1243), (2143)$.

6. An element a of a group G is said to be conjugate to $b \in G$ if $a = g^{-1}bg$ for some $g \in G$ and is denoted by $a \sim b$.

   (a) Show that the conjugacy relation is an equivalence relation.

   (b) Show that all elements of a class have same order. (For $a \in G$, minimal n such that $a^n = e$ is called the order of a)

   (c) Show that in an abelian group, every element is a class by itself.

   (d) Show that a normal subgroup contains complete classes.

Answer:

Clearly, for any $a, b \in G$

- $a = eae^{-1}$, where e is identity, then $a \sim a$
- If $a \sim b$, then $\exists\, g$ s.t. $a = g^{-1}bg \Rightarrow$ $b = gag^{-1} = (g^{-1})^{-1}ag^{-1} \Rightarrow b \sim a$.
- If $a \sim b$, $b \sim c$ then $\exists\, g$ and h s.t. $a = g^{-1}bg$ and $b = h^{-1}ch \Rightarrow a = (hg)^{-1}c(hg) \Rightarrow a \sim c$

Thus conjugacy is an equivalence relation.

^ Definition

A given binary relation ~ on a set X is said to be an equivalence relation if and only if it is reflexive, symmetric and transitive.

That is, for all a, b and c in X:

- $a \sim a$. (Reflexivity)
- $a \sim b$ if and only if $b \sim a$. (Symmetry)
- if $a \sim b$ and $b \sim c$ then $a \sim c$. (Transitivity)

if order of a is n, that is $a^n = e$. If $b \sim a$ then $\exists\, g$ s.t. $b = g^{-1}ag \Rightarrow b^n = (g^{-1}ag)^n = \underbrace{g^{-1}ag \cdot g^{-1}ag \cdots g^{-1}ag}_{n \text{ times}} = g^{-1}a^ng = g^{-1}g = e$.
thus order of b is also n.

7. Find all conjugacy classes and subgroups of the following groups.

   (a) The set of following matrices under matrix multiplication:

$$\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\right\}$$

   (b) The group generated by two elements $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & i\\i & 0\end{bmatrix}$.

   (c) Group generated by σ and ψ with constraints $\sigma^2 = \psi^3 = e$ and $\sigma\psi = \psi^2\sigma$.

Answer:
There is no quick procedure to find equivalence classes, it is a brute force method. When we are performing hand-calculations, a little trial and error would help.

(a) Look at the multiplication table in $1e$. The subgroups must be of order 2 and 4 (factors of 8). Each mirror with identity forms a subgroup. R_π with identity is also a subgroup. And all rotations is a subgroup of order 4.

The five conjugacy classes are listed here.

$$\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}, \left\{\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right\}\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\right\},$$
$$\left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\right\}$$

(b) Look at the multiplication table in $Q3$. Note the onlyelement with order 2, that is an element which is it's own inverse, is ψ^2. So, $\{I, \psi^2\}$ is the subgroup of order 2. Subgroups of order 4 are $\{I, \psi, \psi^2, \psi^3\}$, $\{I, \sigma, \psi^2, \sigma\psi^2\}$.

Here is a table of conjugacy transformation (all possible conjugacy products)

a⊙b⊙a^{-1}	I	ψ	ψ^2	ψ^3	σ	$\sigma\psi$	$\sigma\psi^2$	$\sigma\psi^3$
I	I	I	I	I	I	I	I	I
ψ	ψ	ψ	ψ	ψ	ψ^3	ψ^3	ψ^3	ψ^3
ψ^2	ψ^2	ψ^2	ψ^2	ψ^2	ψ^2	ψ^2	ψ^2	ψ^2
ψ^3	ψ^3	ψ^3	ψ^3	ψ^3	ψ	ψ	ψ	ψ
σ	σ	$\sigma\psi^2$	σ	$\sigma\psi^2$	σ	$\sigma\psi^2$	σ	$\sigma\psi^2$
$\sigma\psi$	$\sigma\psi$	$\sigma\psi^3$	$\sigma\psi$	$\sigma\psi^3$	$\sigma\psi^3$	$\sigma\psi$	$\sigma\psi^3$	$\sigma\psi$
$\sigma\psi^2$	$\sigma\psi^2$	σ	$\sigma\psi^2$	σ	$\sigma\psi^2$	σ	$\sigma\psi^2$	σ
$\sigma\psi^3$	$\sigma\psi^3$	$\sigma\psi$	$\sigma\psi^3$	$\sigma\psi$	$\sigma\psi$	$\sigma\psi^3$	$\sigma\psi$	$\sigma\psi^3$

Each row is a conjugacy class of the corresponding element. Thus the distinct classes are $\{I\}, \{\psi, \psi^3\}, \{\psi^2\}, \{\sigma, \sigma\psi^2\}, \{\sigma\psi, \sigma\psi^3\}$.

(c) Left for you.

8. Show that every subgroup of index 2 is a normal subgroup.

Answer:
Let $H \subset G$ and $\mathcal{O}(H) = \mathcal{O}(G)/2$. Let $u \notin H$, thenHu is a right coset which is not H and uH a left coset which is not H. Clearly $Hu = uH$. Since *all* left cosets and right cosets are equal, H must be normal.

9. Show that the dihedral group D_4 is homomorphic to the group $\mathbb{Z}_2 = \{1, -1\}$ under multiplication.

Answer:
The group D_4 is generated by $a^4 = b^2 = e$ and $bab = a^{-1} = a^3$. All elements can be written as $b^m a^n$ where $m = 0, 1$ and $n = 0, 1, 2, 3$. Define a mapping from $\phi : D_4 \to \mathbb{Z}_2$ such that

$$\phi(b^m a^n) = (-1)^m .$$

Check for homomorphism:

$$LHS = \phi\left(b^m a^n b^l a^k\right) = \phi\left(b^{m+l} a^p\right) = (-1)^{m+l}$$
$$RHS = \phi(b^m a^n)\,\phi\left(b^l a^k\right) = (-1)^m (-1)^l = (-1)^{m+l} .$$

Here $p = n + k$ if $l = 0$ and $p = 4 - n + k$ if $l = 1$.

10. Show that the group of all positive real numbers under multiplication is isomorphic to the set of real numbers under addition. (Hint: Mapping is logarithm function)

Answer:
$G = (\mathbb{R}^+, \times)$ and $\bar{G} = (\mathbb{R}, +)$. Let $\phi : G \to \bar{G}$ such that $\phi : x \mapsto \log x$. This is a homomorphism since

$$\phi(x \times y) = \log(x \times y) = \log x + \log y = \phi(x) + \phi(y) .$$

11. Construct a homomorphism of the group S_3 onto $\mathbb{Z}_2$. What is the kernel of the homomorphism? Find the factor group of S_3 that is isomorphic to $\mathbb{Z}_2$.
Note: The permutation group, S_n, is a group of all permutations of n symbols (which we will take as integers from 1 to n). Each element is denoted by $a = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix}$ or just by $a = (a_1, \ldots, a_n)$ where a_1 to a_n are permutations of 1 to n. The product is given by $a \circ b = (a_{b_1}, a_{b_2}, \ldots, a_{b_n})$. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

And $\mathcal{O}(S_n) = n!$.

Answer:
The group $S_3 = \left\{e, \psi, \psi^2, \sigma, \sigma\psi, \sigma\psi^2\right\}$, where $\psi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. It is easy to check that the mapping $\phi(\psi) = \phi(\psi) = \phi\left(\psi^2\right) = 1$ and remaining elements to -1.

12. Let N be a normal subgroup of G. Show that G is homomorphic to G/N.

Answer:
Let, $G/N = \{N, aN, bN, \ldots\}$. Let $\phi : G \to G/N$ such that $\phi(g) = gN$. Check that

$$LHS = \phi(g_1 g_2) = g_1 g_2 N$$
$$RHS = \phi(g_1)\phi(g_2) = (g_1 N)(g_2 N) = g_1 g_2 N$$