CYK/2023/PH201 Mathematical Physics

QUIZ 3

Total Marks: 10 Marks, Duration: 1 Hour
Date: 30 Oct 2023, Monday

A

1. [**5 Marks**] A circular metallic (thermally conducting) disc of radius a is subjected to the boundary conditions

$$T(a,\phi) = \begin{cases} \cos\phi, & -\pi/2 < \phi < \pi/2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the steady state temperature $T(\rho,\phi)$ in the disc. Sketch isotherms.

Answer:
The solution is of the form

$$T(\rho,\phi) = \left(A' + B'\ln\rho\right)\left(C'\phi + D'\right)\left(A\rho^n + B\rho^{-n}\right)\left(C\sin n\phi + D\cos n\phi\right)$$

Since $\rho = 0$ axis is inside the disc, the requirement that Φ is bounded implies that $B' = B = 0$. The requirement that $T(\rho,\phi+2\pi) = T(\rho,\phi)$ requires that $C' = 0$ and n be an integer. Thus, we can write the general solution as

$$\Phi(\rho,\phi) = D_0 + \sum_{n=1}^{\infty} \rho^n \left(C_n \sin n\phi + D_n \cos n\phi\right)$$

And

$$\begin{aligned} C_n &= \frac{1}{\pi a^n}\int_0^{2\pi} \Phi(a,\phi)\sin n\phi\, d\phi \\ &= \frac{1}{\pi a^n}\int_{-\pi/2}^{\pi/2} \cos\phi \sin n\phi\, d\phi \\ &= 0 \end{aligned}$$

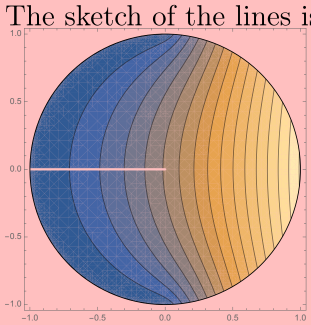
Also,

$$\begin{aligned} D_n &= \frac{1}{\pi a^n}\int_0^{2\pi} \Phi(a,\phi)\cos n\phi\, d\phi \\ &= \frac{1}{\pi a^n}\int_{-\pi/2}^{\pi/2} \cos\phi \cos n\phi\, d\phi \\ &= \frac{1}{\pi a^n}\frac{2\cos\left(\frac{\pi n}{2}\right)}{1-n^2} = \begin{cases} \frac{2(-1)^{n/2}}{\pi a^n (1-n^2)} & \text{even } n \\ 0 & n \neq 1 \\ \frac{1}{2a} & n = 1 \end{cases} \\ D_0 &= \frac{1}{2\pi}\int_0^{2\pi} \Phi(a,\phi)\, d\phi = \frac{1}{\pi} \end{aligned}$$

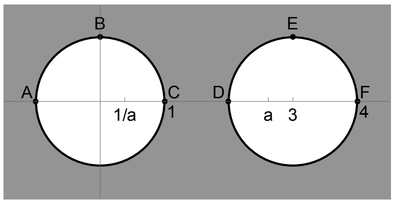
Thus,

$$T(\rho,\phi) = \frac{1}{\pi} + \frac{1}{2}\frac{\rho}{a}\cos\phi + \frac{2}{\pi}\sum_{n=1}^{\infty} (-1)^n \left(\frac{\rho}{a}\right)^{2n} \frac{\cos(2n\phi)}{1-(2n)^2}.$$

The sketch of the lines is shown below.



2. [**5 Marks**] Find the potential V in the gray region shown in the figure by completing the following steps. Both circles have a unit radius. The ABC circle is maintained at a potential of $V = 1$ and the DEF circle is at 0 potential.



(a) Find a (> 1) such that the points $(a, 0)$ and $(1/a, 0)$ are symmetric wrt the circle centered at $(3, 0)$.

(b) Consider the conformal transformation

$$w = \frac{z - a}{az - 1}.$$

Find the images of points A, B, C, D, E and F. Find the image of the gray region.

(c) Obtain the expression for the potential in w-*plane* with given boundary conditions.

(d) Obtain the expression for the potential $V(x, y)$ in z-plane (Write answer in terms of x, y and a).

Answers:

(a) Clearly, $(3 - a)(3 - 1/a) = 1$. Solving for $a = \frac{1}{2}\left(3 + \sqrt{5}\right)$.

(b) The mapping of the points will be

A	B	C	D	E	F
$\{-1, 0\}$	$\{0, 1\}$	$\{1, 0\}$	$\{2, 0\}$	$\{3, 1\}$	$\{4, 0\}$
$\{1, 0\}$	$\left\{\frac{2}{3}, \frac{\sqrt{5}}{3}\right\}$	$\{-1, 0\}$	$\left\{\frac{1}{2}\left(3\sqrt{5} - 7\right), 0\right\}$	$\left\{\frac{7}{3} - \sqrt{5}, \frac{7\sqrt{5}}{6} - \frac{5}{2}\right\}$	$\left\{\frac{1}{2}\left(7 - 3\sqrt{5}\right), 0\right\}$

The ABC circle maps to a unit circle and DEF circle maps to a circle of radius $R_0 = \frac{1}{2}\left(7 - 3\sqrt{5}\right)$ centered at the origin. The shaded region is the region between the two concentric circles.

(c) The solution in w-plane is

$$V(u, v) = 1 - \frac{\ln \rho_w}{\ln R_0}$$

(d) The solution in z-plane is

$$V(x, y) = 1 - \frac{1}{2 \ln R_0} \ln\left(\frac{(x - a)^2 + y^2}{(ax - 1)^2 + a^2 y^2}\right)$$