Tutorial 7: Green's Functions

Green's Functions

1. For the operator $L_x = \frac{d^2}{dx^2}$, find the Green's function with different boundary conditions given below $L_x G(x, x') = \delta(x - x')$:

   (a) $G(0, x') = G(1, x') = 0$,

   (b) $G(-1, x') = G(1, x') = 0$,

   (c) $G(0, x') = 0$ and $G'(1, x') = 0$.

Answers: The equation $L_x G(x, x') = \delta(x - x')$ when $x \neq x'$ reduces to $L_x G(x, x') = 0$. The solution is linear

$$G(x, x') = \begin{cases} Ax + B & x < x' \\ Cx + D & x > x'. \end{cases}$$

And G is continuous and dG/dx is discontinuous at x'. This gives us

$$\begin{aligned} Ax' + B &= Cx' + D \\ C - A &= 1. \end{aligned}$$

Solving for C and D, we get $C = A + 1$ and $D = B - x'$.

(a) Now, $G(0, x') = 0$ implies that $B = 0$ and hence $D = -x'$. $G(1, x') = 0$ implies that $C = -D = x'$ and $A = x' - 1$.
$G(x, x') = \begin{cases} x(x' - 1) & x < x' \\ (x - 1)x' & x > x' \end{cases}$

(b) Use the same procedure as part (a): $G(x, x') = \begin{cases} \frac{1}{2}(x + 1)(x' - 1) & x < x' \\ \frac{1}{2}(x - 1)(x' + 1) & x > x' \end{cases}$

(c) Now, $G(0, x') = 0$ implies that $B = 0$ and hence $D = -x'$. And $G'(1, x') = 0$ implies that $C = 0$ and hence $A = -1$.
$G(x, x') = \begin{cases} -x & x < x' \\ -x' & x > x' \end{cases}$

2. Solve the problem (1) using eigenfunction expansion method. From part (c), show that

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\left(n + \frac{1}{2}\right)\pi x\right) \sin\left(\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)^2} = \begin{cases} x, & 0 \leq x < t \\ t, & t < x \leq 1. \end{cases}$$

Answers: Only part (c)
L_x is a hermitian operator on space of functions $f:[0,1]\to\mathbb{R}$ with conditions $f(0)=0$ and $f'(1)=0$. The eigenfunctions can be found by solving $L_x u(x)=(-\lambda)\,u(x)$. The general solution is

$$u(x)=A\sin\left(\sqrt{\lambda}x\right)+B\cos\left(\sqrt{\lambda}x\right).$$

Apply BC to get $B=0$ and $\sqrt{\lambda}=\left(n+\frac{1}{2}\right)\pi$ with $n=0,1,\ldots$. Thus the eigenvalues are $\xi_n=-\left(n+\frac{1}{2}\right)^2\pi^2$ and the corresponding eigenfunctions are $u_n(x)=\sqrt{2}\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)$ (normalized by choosing $A=\sqrt{2}$. We know that for a hemitian operator, eigenfunctions form a complete basis, which means

$$\delta\left(x-x'\right)=\sum u_n(x)\,u_n\left(x'\right)=2\sum\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)\sin\left(\left(n+\frac{1}{2}\right)\pi x'\right)$$

and because G is continuous with same BC, G can be written as linear sum of eigenfunctions, that is,

$$G\left(x,x'\right)=\sum A_n u_n(x)=\sqrt{2}\sum A_n\sin\left(\left(n+\frac{1}{2}\right)\pi x\right).$$

Now,

$$\begin{aligned}
L_x G\left(x,x'\right)&=\delta\left(x-x'\right)\\
\implies\sum A_n L_x u_n(x)&=\sum u_n\left(x'\right)u_n(x)\\
\implies\sum A_n\xi_n u_n(x)&=\sum u_n\left(x'\right)u_n(x)\\
A_n&=\frac{u_n\left(x'\right)}{\xi_n}
\end{aligned}$$

Giving us,

$$\begin{aligned}
G&=\sum\frac{u_n\left(x'\right)u_n(x)}{\xi_n}\\
&=-\frac{2}{\pi^2}\sum\frac{\sin\left(\left(n+\frac{1}{2}\right)\pi x'\right)\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)^2}
\end{aligned}$$

Equating this to the solution obtained in Question 1,$G\left(x,x'\right)=\begin{cases}-x & x<x'\\ -x' & x>x'\end{cases}$, gives us the required identity.

3. Show that the Green's function for the operator $L_x=\frac{d^2}{dx^2}$ with boundary conditions $G'\left(0,x'\right)=0$ and $G'\left(1,x'\right)=0$ does not exist.

Answer:
The equation $L_x G\left(x,x'\right)=\delta\left(x-x'\right)$ when $x\neq x'$ reduces to $L_x G\left(x,x'\right)=0$. The solution is linear

$$G\left(x,x'\right)=\begin{cases}Ax+B & x<x'\\ Cx+D & x>x'.\end{cases}$$

And G is continuous and dG/dx is discontinuous at x'. This gives us

$$\begin{aligned}
Ax'+B&=Cx'+D\\
C-A&=1.
\end{aligned}$$

Solving for C and D, we get $C=A+1$ and $D=B-x'$.
Now, $G'\left(0,x'\right)=0$ implies that $A=0$. And $G'\left(1,x'\right)=0$ implies that $C=0$, and hence G' is not discontinuous at x'.

4. Find the Green's Function for the following differential operators:

(a) $Ly(x)=y''(x)+y(x)$, $x\in[0,1]$, with $y(0)=0$ and $y'(1)=0$.

(b) $Ly(x)=y''(x)-y(x)$, $x\in\mathbb{R}$, with $y(\pm\infty)<\infty$.

Answers:

(a) $LG(x,x') = 0$ implies $G(x,x') = A\sin x + B\cos x$. Then, let

$$G(x,x') = \begin{cases} A\sin x + B\cos x & x < x' \\ C\sin x + D\cos x & x > x'. \end{cases}$$

Applying BC $G(0,x') = 0$, we get $B = 0$. Applying the second BC, we get $dG/dx|_{x=1} = C\cos 1 - D\sin 1 = 0$, gives us $C = D\tan 1$.

$$G(x,x') = \begin{cases} A\sin x & x < x' \\ D(\tan 1\sin x + \cos x) & x > x'. \end{cases}$$

Thus, Also

$$\begin{aligned} \lim_{\epsilon\to 0}\left[\frac{dG}{dx}(x'+\epsilon,x') - \frac{dG}{dx}(x'-\epsilon,x')\right] &= 1 \\ D\left(\tan 1\cos x' - \sin x'\right) - A\cos x' &= 1 \\ D\left(\tan 1\sin x' + \cos x'\right) - A\sin x' &= 0 \qquad \text{continuity} \end{aligned}$$

Gives:

$$D = -\sin x', \quad A = -\left(\tan 1\sin x' + \cos x'\right)$$

Thus,

$$G(x,x') = \begin{cases} -\sin(x)\left(\cos x' + \tan 1\sin x'\right) & x < x' \\ -\sin x'\left(\cos x + \tan 1\sin x\right) & x > x' \end{cases}$$

(b) $LG(x,x') = 0$ implies $G(x,x') = Ae^{x} + Be^{-x}$. Applying BCs, we get

$$G(x,x') = \begin{cases} Ae^{x} & x < x' \\ Be^{-x} & x > x'. \end{cases}$$

Now,

$$\begin{aligned} \lim_{\epsilon\to 0}\left[\frac{dG}{dx}(x'+\epsilon,x') - \frac{dG}{dx}(x'-\epsilon,x')\right] &= 1 \\ -Be^{-x'} - Ae^{x'} &= 1 \\ Be^{-x'} - Ae^{x'} &= 0 \end{aligned}$$

Which gives $A = -\frac{1}{2}e^{-x'}$, then $B = -\frac{1}{2}e^{x'}$.

$$G(x,x') = \begin{cases} -\frac{1}{2}e^{x-x'} & x < x' \\ -\frac{1}{2}e^{x'-x} & x > x'. \end{cases}$$

5. Find the Green's functions for the differential operators

$$Ly(x) = xy''(x) + y'$$

with boundary conditions that $y(1) = 0$ and $y(0)$ should be finite. Use the Green's function, solve

$$\frac{d}{dx}\left[x\frac{dy}{dx}(x)\right] = -1.$$

Verify the solution by direct integration of the differential equation.

Answers:
$LG\left(x,t\right)=0$ implies

$$G\left(x,t\right)=\begin{cases}A\ln x+B & x<t\\ C\ln x+D & x>t.\end{cases}$$

Applying BC $\left|G\left(0,t\right)\right|<\infty$, we get $A=0$. Applying the second BC, we get $D=0$.

$$G\left(x,t\right)=\begin{cases}B & x<t\\ C\ln x & x>t.\end{cases}$$

Continuity of G gives

$$C\ln t-B=0$$

We need to be little careful with the discontinuity of G'. In this case, $LG=\frac{d}{dx}\left(x\frac{dG}{dx}\right)$ which means xG' is discontinuous by 1 unit.

$$\begin{aligned}\lim_{\epsilon\to 0}\left[\left(t+\epsilon\right)\frac{dG}{dx}\left(t+\epsilon,t\right)-\left(t+\epsilon\right)\frac{dG}{dx}\left(t+\epsilon,t\right)\right]&=1\\ t\frac{C}{t}-0&=1\end{aligned}$$

Gives:

$$C=1,\quad B=\ln t$$

Thus,

$$G\left(x,x'\right)=\begin{cases}\ln t & x<t\\ \ln x & x>t\end{cases}$$

Now,

$$\begin{aligned}y(x)&=\int_0^x(-1)\ln x\,dt+\int_x^1(-1)\ln t\,dt\\ &=-x\ln x-\left(t\ln t-t\right)\big|_x^1=1-x\end{aligned}$$

6. Find the Green's function for the differential operator

$$\begin{aligned}Ly(x)&=xy''(x)+y'(x)-\frac{n^2}{x}y(x)\\ y(0)&<\infty\\ y(1)&=0.\end{aligned}$$

Answers:
This is a Cauchy-Euler equation. The homogeneous equation can be solved by standard guess $y = r^p$. The two linearly independent solutions are $r^{\pm n}$. After applying BC,

$$G(x,t) = \begin{cases} Ar^n & x < t \\ C(r^n - r^{-n}) & x > t. \end{cases}$$

Continuity of G gives

$$At^n = C\left(t^n - t^{-n}\right)$$

In this case, $LG = \frac{d}{dx}\left(x\frac{dG}{dx}\right) - \frac{n^2}{x}G$, which means xG' is discontinuous by 1 unit.

$$nC\left(t^n + t^{-n}\right) - Ant^n = 1$$

Gives:

$$C = \frac{1}{2n}t^n, \quad A = \frac{1}{2n}\left(t^n - t^{-n}\right)$$

Thus,

$$G\left(x,x'\right) = \begin{cases} \frac{1}{2n}\left(t^n - t^{-n}\right)r^n & x < t \\ \frac{1}{2n}t^n\left(r^n - r^{-n}\right) & x > t \end{cases}$$

7. Find the Green function for associated Legendre differential operator

$$Ly(x) = \frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right] - \frac{n^2}{(1-x^2)}y \qquad x \in [-1,1]$$

with boundary condition that at ± 1, the solution must be finite.

Answers:
The two linearly independent solutions (check by substitution in $Ly = 0$) are

$$y_1(x) = A\left(\frac{1+x}{1-x}\right)^{n/2} \qquad y_1(-1) = 0$$
$$y_2(x) = B\left(\frac{1-x}{1+x}\right)^{n/2} \qquad y_2(1) = 0.$$

Clearly, y_1 is well-behaved at $x = -1$ and y_2 at $x = +1$. Now, we can apply the standard procedure to obtain the Green's function for this case

$$G(x,t) = \left[\frac{(1+x)(1-t)}{(1-x)(1+t)}\right]^{n/2} \qquad x < t$$
$$= \left[\frac{(1+t)(1-x)}{(1-t)(1+x)}\right]^{n/2} \qquad x > t.$$

8. Construct a Green's function to solve modified Helmholtz equation

$$y''(x) - k^2 y(x) = f(x)$$

where, k is some constant. The boundary condition is that the Green's function must vanish as $x \to \pm\infty$.

Answers:
$LG(x,x')=0$ implies $G(x,x')=Ae^{kx}+Be^{-kx}$. Applying BCs, we get

$$G(x,x')=\begin{cases} Ae^{kx} & x<x' \\ Be^{-kx} & x>x'. \end{cases}$$

Now,

$$\begin{aligned} \lim_{\epsilon\to 0}\left[\frac{dG}{dx}(x'+\epsilon,x')-\frac{dG}{dx}(x'-\epsilon,x')\right] &= 1 \\ -Be^{-kx'}-Ae^{kx'} &= 1 \\ Be^{-kx'}-Ae^{kx'} &= 0 \end{aligned}$$

Which gives $A=-\frac{1}{2}e^{-kx'}$, then $B=-\frac{1}{2}e^{kx'}$.

$$G(x,x')=\begin{cases} -\frac{1}{2}e^{k(x-x')} & x<x' \\ -\frac{1}{2}e^{k(x'-x)} & x>x'. \end{cases}$$

The required solution is

$$y(x)=\int_{-\infty}^{\infty} G(x,x')\,f(x')\,dx'$$

9. Prove the mean value theorem for Laplace equation: Let P be an interior point of a volume V. Let y be a solution of the Laplace equation in V. Then $y(P)$ is the average of y over the surface of any sphere in V centered about P. [Hint: Use the integral equation.] Prove that the solution of the Laplace equation cannot have a maximum or a minimum in V.

Answers:
The integral equation for Laplace equation is

$$\phi(\mathbf{r})=\frac{1}{4\pi}\oint_S\left[\frac{1}{R}\frac{\partial\phi}{\partial n'}-\phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]ds'$$

where $R=|\mathbf{r}-\mathbf{r}'|$. Let the point P be at $\mathbf{r}$. Now, take a spherical surface of radius a about P . Then, normal to the surface at $\mathbf{r}'$ is a unit vector along $(\mathbf{r}'-\mathbf{r})$. And $R=a$. The first term is

$$\frac{1}{4\pi}\oint_S\frac{1}{R}\frac{\partial\phi}{\partial n'}ds'=\frac{1}{4\pi a}\oint_S\nabla'\phi\cdot\hat{n}ds'=0$$

by Gauss law.
Now,

$$\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)=\hat{n}\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-\hat{n}\cdot\frac{(\mathbf{r}-\mathbf{r}')}{a^3}=-\frac{1}{a^2}.$$

Thus,

$$\begin{aligned} \phi(\mathbf{r}) &= -\frac{1}{4\pi}\oint_S\left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]ds' \\ &= -\frac{1}{4\pi}\oint_S\left[\phi(\mathbf{r}')\left(-\frac{1}{a^2}\right)\right]ds' \\ &= \frac{1}{4\pi a^2}\oint_S\phi(\mathbf{r}')\,ds'. \end{aligned}$$

The second part is simple consequence of the first result.

10. Consider the Laplace equation $\nabla^2\phi=0$ in a volume V with boundary S.

(a) Prove using the Green's identity, that for a function f,

$$\int_V\left(f\nabla^2 f+|\nabla f|^2\right)dv=\oint_S f\,(\nabla f\cdot\hat{\mathbf{n}})\,dS.$$

(b) Prove that the solution (assuming that it exists) to the Laplace equation in V with either Dirichlet or Neumann boundary conditions must be unique.

Answers:

(a) The divergence theorem is

$$\int_V \nabla \cdot \mathbf{A} = \oint_S \mathbf{A} \cdot \mathbf{n} ds.$$

Now, choose $\mathbf{A} = f\nabla f$. $\nabla \cdot \mathbf{A} = f\nabla^2 f + \nabla f \cdot \nabla f$. This gives the result.

(b) Let ϕ_1 and ϕ_2 be two solutions fo $\nabla^2\phi = 0$ on V with Dirichlet BC that $\phi(\mathbf{r}) = \eta(\mathbf{r})$ on S. Let $\psi = \phi_1 - \phi_2$ is solution of Laplace equation with DBC $\psi = 0$ on S. Using the result of the first part

$$\int_V \left(\psi\nabla^2\psi + |\nabla\psi|^2\right) dv = \oint_S \psi\left(\nabla\psi \cdot \hat{\mathbf{n}}\right) dS$$
$$\int_V |\nabla\psi|^2 \, dv = 0$$

since $\nabla^2\psi = 0$ on V and $\psi = 0$ on S. This means $\nabla\psi = 0$ on V and hence $\psi = \text{const} = 0$. Thus, $\phi_1 = \phi_2$.

11. Prove that the Dirichlet Green's function for Laplace equation must be symmetric under exchange of its arguments, that is, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. [Note: This result is true for all self-adjoint differential operators. Tricky proof.]

Answers:
Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two points in V bounded by S. The Dirichlet Green's function is defined by

$$\nabla^2 G\left(\mathbf{r}, \mathbf{r}'\right) = -4\pi\delta\left(\mathbf{r} - \mathbf{r}'\right) \quad \text{and} \quad G\left(\mathbf{r}, \mathbf{r}'\right) = 0$$

on S. Now,

$$G(\mathbf{r}, \mathbf{r}_1)\nabla^2 G(\mathbf{r}, \mathbf{r}_2) - G(\mathbf{r}, \mathbf{r}_2)\nabla^2 G(\mathbf{r}, \mathbf{r}_1) = -4\pi\left(G(\mathbf{r}, \mathbf{r}_1)\delta(\mathbf{r} - \mathbf{r}_2) - G(\mathbf{r}, \mathbf{r}_2)\delta(\mathbf{r} - \mathbf{r}_1)\right)$$

Integrate over the volume.

$$RHS = -4\pi\left[G(\mathbf{r}_2, \mathbf{r}_1) - G(\mathbf{r}_1, \mathbf{r}_2)\right]$$

and

$$\begin{aligned} LHS &= \int_V \left(G(\mathbf{r}, \mathbf{r}_1)\nabla^2 G(\mathbf{r}, \mathbf{r}_2) - G(\mathbf{r}, \mathbf{r}_2)\nabla^2 G(\mathbf{r}, \mathbf{r}_1)\right) dv \\ &= \int_V \nabla \cdot \left(G(\mathbf{r}, \mathbf{r}_1)\nabla G(\mathbf{r}, \mathbf{r}_2) - G(\mathbf{r}, \mathbf{r}_2)\nabla G(\mathbf{r}, \mathbf{r}_1)\right) dv \\ &= \oint_s \left(G(\mathbf{r}, \mathbf{r}_1)\nabla G(\mathbf{r}, \mathbf{r}_2) - G(\mathbf{r}, \mathbf{r}_2)\nabla G(\mathbf{r}, \mathbf{r}_1)\right) \cdot d\mathbf{S} \\ &= 0 \qquad \because G(\mathbf{r}, \mathbf{r}_1) = G(\mathbf{r}, \mathbf{r}_2) = 0 \text{ when } \mathbf{r} \in S. \end{aligned}$$