CYK/2023/PH201 Mathematical Physics

# QUIZ 1

Total Marks: 10 Marks, Duration: 50 Mins
Date: 22 Aug 2023, Tuesday

**B**

1. [$4 \times 1$ **Marks**] Answer the following short questions (You can write the answers directly):

   (a) Find $\lim\limits_{z \to i/2} \dfrac{(2z-3)(iz+1)}{(iz-1)^2}$.

   (b) Stub: Using the rules of the differentiation, find the derivative of $\cos\left(5z^3+3z\right)$.

   (c) Sketch the map of the unit circle $|z|=1$ under the transformation $w = e^{-i\pi/3}\left(1-\sqrt{2}e^{i\pi/4}z\right)$.

   (d) Find the numerical value of $\cos^2(2i+3)$.

Answers:

(a) $\lim\limits_{z \to i/2} \dfrac{(2z-3)(iz+1)}{(iz-1)^2} = \dfrac{2}{9}(i-3)$.

(b) $\dfrac{d}{dz}\cos\left(5z^3+3z\right) = -\left(15z^2+3\right)\sin\left(5z^3+3z\right)$

(c) The transformation can be written as $w - e^{-i\pi/3} = -\sqrt{2}e^{-i\pi/12}z$. Thus, $\left|w - e^{-i\pi/3}\right| = \sqrt{2}$. The image of the set is again a circle of radius $\sqrt{2}$ with center at $e^{-i\pi/3}$.

(d) $\cos^2(2i+3) = 13.6103 + 3.81261i$

2. [**3 Marks**] Determine if the function $u(x,y) = -y(6x+1)$ is harmonic. If it is harmonic, find the conjugate harmonic function $v(x,y)$ and express $u+iv$ as an analytic function of $z$.

Answer:
Since $\left(\partial_x^2 + \partial_y^2\right)u = 0$, the function $u$ is harmonic. Since $v_y = u_x$,

$$v_y = -6y \implies v(x,y) = -3y^2 + g(x).$$

And from $v_x = -u_y$, we get

$$g'(x) = (6x+1) \implies g(x) = \left(3x^2+x\right).$$

And hence $v(x,y) = 3x^2 - 3y^2 + x$. The complex function $u+iv$, will be

$$\begin{aligned} u+iv &= -y(6x+1) + i\left(3x^2-3y^2+x\right) \\ &= iz(1+3z). \end{aligned}$$

3. [**3 Marks**]A function $f(z)$ is defined as

$$f(z) = \begin{cases} 3\bar{z}^2 & \mathrm{Im}(z) > 0 \\ 4z & \mathrm{Im}(z) < 0 \end{cases}$$

and $C$ is the anticlockwise circular arc of unit radius with center at $-1$. Write a parametrization for $C$ and using primary definition, find $\displaystyle\int_C f(z)dz$ .

Answer:
The whole contour can be written as two contours $C_1 : z = -1 + e^{i\theta}$ with $\theta : -\pi \to 0$ and $C_2 : z = -1 + e^{i\theta}$ with $\theta : 0 \to \pi$. Now, $dz = izd\theta$ for both cases, and

$$\int_{C_1} f(z)dz = \int_{C_1} 4zdz = \int_{-\pi}^{0} 4i\left(-1 + e^{i\theta}\right)e^{i\theta}d\theta = -8$$

And

$$\int_{C_2} 3\bar{z}^2 dz = \int_0^{\pi} 3i\left(-1 + e^{-i\theta}\right)^2 \cdot e^{i\theta}d\theta = -6\pi i$$

Thus,

$$\int_C f(z)dz = -8 - 6\pi i.$$