$$R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$

$$\Rightarrow \frac{1}{\Theta \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1)$$

$\Rightarrow$ solution Legendre's polynomial

$P_l(\cos\theta)$

$\Theta(\theta) \Rightarrow P_l(\cos\theta)$

(2) $$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

~~(2) $V(r,\theta)$~~

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l}(x^2-1)^l$$

$\boxed{x = \cos\theta}$

$P_0 = 1$

$P_1(x) = x \qquad P_2(x) = (3x^2-1)/2$

$P_3(x) = (5x^3 - 3x)/2$

$P_4(x) = (35x^4 - 30x^2 + 3)/8$

Orthonormality of Legendre polynomials

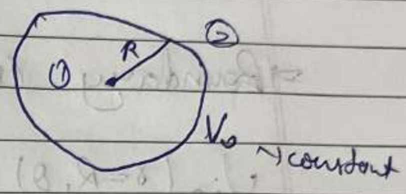
$$\int_0^{\pi} P_l(\cos\theta)\, P_n(\cos\theta) \sin\theta \, d\theta = \frac{2}{(2l+1)} \delta_{ln}$$

$$= \frac{2}{(2l+1)}, \; l = n$$

$$= 0 \quad ; \; l \neq 0$$

Question

(1) at center.



$$\frac{B_l}{r^{l+1}} \longrightarrow \infty$$ but V must be finite. $B_l \to 0$

$$\nabla^2 V = 0$$

$$V(r,\theta) = \sum \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \longrightarrow (1)$$

(1) Ist region $\to$ inside $0 \leq r \leq R$

at $r = 0$ $V \longrightarrow \infty$

$B_l = 0 \quad \forall l$

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \longrightarrow (2)$$

(2) Outside region $R \leq r \leq \infty$

$$V_{out}(r,\theta) = \sum \frac{B_l'}{r^{l+1}} P_l \cos\theta + A_0' P_0(\cos\theta)$$

$P_0 \cos\theta = 1$

$\nabla V^2 = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

For azimuthal symmetry, $V \rightarrow$ independent of ϕ

$$\cdot V(r,\theta) = R(r)\,\Theta(\theta)$$

$$\frac{\Theta\phi}{r^2 R\Theta}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) + \frac{R\phi}{r^2\sin\theta R\Theta}$$

at $r = R$

$$V_{in}(R,\theta) = V_0 = V_{out}(r=R,\theta)$$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0 .$$

$$P_m(\cos\theta)\cdot\sin\theta \sum_{l=0}^{\infty} A_l R^l P_l\cos\theta = V_0\, P_m(\cos\theta)\sin\theta$$

$$\int_0^\pi P_m(\cos\theta)\sin\theta \sum_{l=0}^{\infty} A_l R^l P_l\cos\theta\, d\theta = V_0\int_0^\pi P_m(\cos\theta)\sin\theta\, d\theta$$

$$RHS \Rightarrow V_0\int_0^\pi P_m(\cos\theta)\sin\theta\, d\theta = V_0\int_0^\pi P_m(\cos\theta)P_0(\cos\theta)\sin\theta\, d\theta =$$

$$\{P_0(\cos\theta) = 1\}$$

$$= 2V_0$$

$$\Downarrow$$

$$\underline{m=0}$$

LHS

$$\sum_{l=0}^{\infty} A_l R^l \int P_m(\cos\theta) P_l(\cos\theta) \sin\theta d\theta = 2A_0 \quad \{m=0\}$$

$\Rightarrow$ RHS = LHS

$\Downarrow$

$A_0 = V_0$

$A_l = 0$. for $l \neq 0$

$B_0 = A_0 R \qquad B_l = 0$ for $l \neq 0$

$$V_{in} = \sum A_l r^l P_l \cos\theta = A_0 \cancel{r^0} \cancel{P_0 \cos\theta}$$

$$= A_0 = V_0$$

$$\underline{V_{in} = V_0}$$

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) = \frac{B_0}{r} \cancel{P_0 \cos\theta} = \frac{V_0 R}{r}$$

$$\underline{V_{out} = \frac{V_0 R}{r}}$$

$\vec{E} = -\nabla V \qquad \vec{E}_{in} = 0$

$$\vec{E}_{out} = -\frac{dV}{dr}\hat{r}$$

$$= \frac{V_0 R}{r^2}(\hat{r})$$

$$V_{out}(r \to \infty) = A_0'$$

As A_0 is reference at ∞ so $A_0' = 0$

$$V_{in}(r,\theta) = \sum_{0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$\Rightarrow$ Boundary condition $(r = R \ , \ V = V_0)$

$$V_{in}(r=R,\theta) = V_{out}(r=R,\theta) = V_0$$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0$$

$$\Rightarrow A_l R^l = \frac{B_l}{R^{l+1}}$$

$$\Downarrow$$

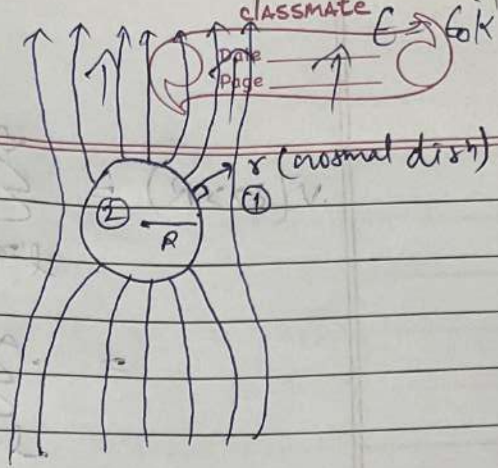
$$B_l = A_l R^{2l+1} \quad \forall l$$

$$P_l(x) = \frac{1}{2^l l!} \frac{\partial^l (x^2-1)^l}{\partial x^l}$$

$$\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1} \quad l = l'$$

$$= 0 \quad l \neq l'$$

$\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$



$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(i) $V(r \le R) = 0$

$\vec{E}(r \le R) = 0$

(ii) $\vec{E} = E_0\hat{k} = E_0(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$r >>> R \rightarrow \vec{E} \rightarrow E_0\cos\theta\,\hat{r}$ $(\hat{\theta} \longrightarrow 0)$

$V \longrightarrow -E_0 r\cos\theta$

$\Rightarrow V(r \le R) = 0 \Rightarrow V(r=R) = 0$

$\vec{E}(r>>R) = E_0\hat{k}$

$V(r>>R) = -E_0 z = -E_0 r\cos\theta$

$\frac{V_0 R}{r}$

$$V(r>>R) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \longrightarrow 1$$

$$V(R,\theta) = \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right)P_l(\cos\theta) = 0 \longrightarrow 2$$

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0$$

$\Downarrow$

$$B_l = -A_l R^{2l+1} \longrightarrow 3$$

$$V(r >> R) = \sum_{l=0}^{\infty}\left(A_l r^l + \cancel{\frac{B_l}{r^{l+1}}}\right) P_l(\cos\theta) \rightarrow ①$$

$$= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$$

$A_1 = -E_0$ $\quad\quad$ $\underline{\underline{B_1 = -A_1 R^3}}$.

$A_l = 0$ for $l \neq 1$

$A_1 = -E_0$ $\quad\quad$ $B_1 = E_0 R^3$

$A_l = 0 \quad l \neq 1 \quad B_l = 0 \quad l \neq 1$

$$V_{out} = \left(A_1 r + \frac{B_1}{r^2}\right) P_1 \cos\theta$$

ans ⇒ $V = \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right)$$

$$\vec{E}_r = -\left(\left(-E_0 - 2\frac{E_0 R^3}{r^3}\right)\cos\theta + \frac{1}{r}\left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\sin\theta\right)$$

Ans $\boxed{\vec{E} = E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\,\hat{r} - E_0\left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\hat{\theta}}$

→ ~~Surface charge density~~

$$= E_0(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) + E_0\left[\frac{2R^3}{r^3}\cos\theta\,\hat{r} + \frac{R^3}{r^3}\sin\theta\,\hat{\theta}\right.$$

$$= \epsilon_0 \hat{k} + \epsilon_0\left(\frac{2R^3}{r^3}\cos\theta\,\hat{r} + \frac{R^3}{r^3}\sin\theta\,\hat{\theta}\right)$$

→ Surface charge density

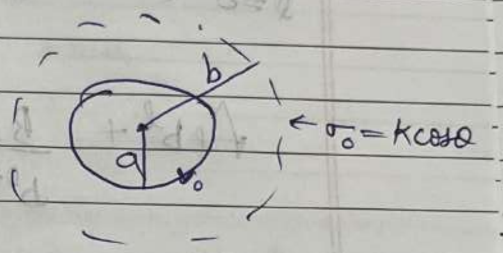
$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

$$E_1^{\perp} - \cancel{E_2^{\perp}}^{0} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = E_1^{\perp}\epsilon_0 = \underline{3\epsilon_0 E_0 \cos\theta}$$

General solution

$$V(r,\theta) = \sum\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \rightarrow ①$$



1) at $r=a$ $V \rightarrow$ continuous

2) $r=a$, $V=V_0$ $\quad$ $V \rightarrow$ finite everywhere

3) at $r=b$ $V \rightarrow$ continuous

(iv) $\epsilon_0\left[\frac{\partial V}{\partial r}\Big|_{out} - \frac{\partial V}{\partial r}\Big|_{in}\right]_{r=b} = \sigma_0 = k\cos\theta$

$$a \le r \le b \quad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l\cos\theta \longrightarrow ②$$

$$r \ge b \Rightarrow V(r \ge b, \theta) = \sum_{l=0}^{\infty}\left(\frac{C_l}{r^{l+1}}\right)P_l\cos\theta \longrightarrow ③$$

→ From 1st and 2nd boundary condition

$$\sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta) = V_0 \qquad A_0 + \frac{B_0}{a} = V_0$$

$$= E_0 \hat{R} + E_0\left(\frac{2R^3}{r^3}\cos\theta\,\hat{r} + \frac{R^3}{r^3}\sin\theta\,\hat{\theta}\right)$$

→ Surface charge density

$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

$$E_1^{\perp} - \cancel{E_2^{\perp}}^{\,0} = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = E_1^{\perp}\varepsilon_0 = \underline{3\varepsilon_0 E_0 \cos\theta}$$

classmate

Date

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$$A_l a^l + \frac{B_l}{a^{l+1}} = 0 \quad \text{for } l \neq 0$$

$$A_0 + \frac{B_0}{a} = V_0 \longrightarrow (4)$$

$$B_l = -A_l a^{2l+1} \quad \text{for } l \neq 0 \rightarrow (5)$$

$\Rightarrow$ from eqn 3 and 3rd and 4th boundary condition

$$\sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\left(\frac{C_l}{b^{l+1}}\right)P_l(\cos\theta)$$

$$A_l b^l + \frac{B_l}{b^{l+1}} = \frac{C_l}{b^{l+1}}$$

$$-\varepsilon_0\left[\sum_{l=0}^{\infty}\left(-\frac{(l+1)C_l}{b^{l+2}}\right)P_l\cos\theta - \sum_{l=0}^{\infty}\left(lA_l b^{l-1} - \frac{(l+1)B_l}{b^{l+2}}\right)P_l\cos\theta\right] = k\cos\theta$$

for $l = 1$

$$\varepsilon_0\left[\frac{2C_1}{b^3} + \left(A_1 - \frac{2B_1}{b^3}\right)\right] = k$$

$$\frac{(l+1)C_l}{b^{l+2}} + \left[lA_l b^{l-1} - \frac{(l+1)B_l}{b^{l+2}}\right] = 0$$

for $l \neq 1$

$l=0 \rightarrow A_0, B_0, C_0 \longrightarrow$ non-zero

$l=1 \rightarrow A_1, B_1, C_1 \longrightarrow$ non-zero

$l>1 \Rightarrow A_l, B_l, C_l \Longrightarrow 0$

Ans $\Rightarrow$ $r \geq b \cdot \quad V(r,\theta) = \dfrac{aV_0}{r} + \dfrac{b^3 - a^3}{3r^2\varepsilon_0}\cos\theta$

$a \leq r \leq b \quad V(r,\theta) = \dfrac{aV_0}{r} + \dfrac{k}{3\varepsilon_0}\left(r - \dfrac{a^3}{r^2}\right)\cos\theta$

Find induced charge density at $r=a$.
calculate total charge induced

Laplace's equation in spherical polar co-ordinate system

$\nabla V^2 = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

$$V(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi).$$

$$\frac{\Theta\Phi}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right) + \frac{R\Phi}{\sin\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\Rightarrow \frac{1}{R}\left(\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{R}\left(\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right) = -\left[\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}\right] = l(l+1)$$

$$\boxed{R \rightarrow A_l r^l + \frac{B_l}{r^{l+1}}}$$

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi \sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -l(l+1)$$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -l(l+1)\sin^2\theta$$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m \Rightarrow \boxed{\Phi \rightarrow e^{\pm im\phi}}$$

$\Phi \rightarrow$ single valued

$\phi \rightarrow 0$ to $2\pi \Rightarrow m = 0, \pm1, \pm2 \ldots$

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta - m^2 = 0$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right) = 0$$

$$\boxed{\Theta \rightarrow P_l^m(\cos\theta)} \; ; \; P_l^m(x)$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{|m|/2}\frac{d^{(m+l)}(x^2-1)^l}{dx^{(m+l)}} \qquad x>0$$

$$P_l^m(-x) = (-1)^{l-|m|}P_l^m(x);$$

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^{m'}(x) = \frac{2}{(2l+1)}\frac{(l+|m|)!}{(l-|m|)!}\delta_{ll'}\delta_{mm'}$$

classmate

$\Theta\Phi \longrightarrow P_l^m(\cos\theta)\, e^{\pm im\phi}$ spherical harmonic

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{\pm im\phi} \qquad m \geq 0$$

associate legendre polynomial

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$$

$$\int_{x=-1}^{x=1}\int_{\phi=0}^{2\pi} [Y_{lm}(\theta,\phi)]^* \, Y_{l'm'}(\theta,\phi)\, d(\cos\theta)\, d\phi = \delta_{ll'}\delta_{mm'}$$

$$V(r,\theta,\phi) = \sum_{l,m=-l}^{\infty,m=l}\left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right)\left(Y_{lm}(\theta,\phi)\right)$$

$\Rightarrow Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \quad Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$

$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}, \quad Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta \cdot e^{2i\phi}$

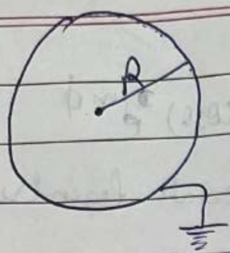
→ The situation where V is independent of θ (or $\theta \to$ constant)

$$V(r,\phi) = \sum\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)(C_m\cos\phi_m + D_m\sin\phi_m)$$

$$\Downarrow$$

$$C_m e^{im\phi} + D_m e^{-im\phi}$$

Example



$$V(r,\theta,\phi) = \frac{r^2 \sin 2\theta}{4\pi\epsilon_0} e^{i\phi}$$

$V(r \geq R, \theta, \phi) = ?$, $\vec{E} = ?$, $\sigma_2 = ?$

$$\rightarrow V(r,\theta,\phi) = \sum \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) Y_{lm}(\theta,\phi) \rightarrow (1)$$

(i) $V(r=R) = 0$

(ii) $V(r \to \infty, \theta, \phi) = \dfrac{r^2 \sin\theta}{4\pi\epsilon_0} e^{i\phi}$

(iii) $E_1^{\perp} - E_2^{\perp} = \dfrac{\sigma}{\epsilon_0}$

(iv) $E_1^{\parallel} - E_2^{\parallel} = 0$

→ From the 1st boundary condition

$$\sum \left(A_{lm} R^l + \frac{B_{lm}}{R^{l+1}}\right) Y_{lm}(\theta,\phi) = 0$$

$$\Rightarrow \left(A_{lm} R^l + \frac{B_{lm}}{R^{l+1}}\right) = 0 \quad \rightarrow \quad B_{lm} = -A_{lm} R^{2l+1} \rightarrow (2)$$

From 2nd boundary condition $(r \gg\gg\gg R)$

$$\sum (A_{lm} r^l)\, Y_{lm}(\theta,\phi) = \frac{r^2 \sin 2\theta}{4\pi\epsilon_0} e^{i\phi}$$

$\Rightarrow l = 2, \ m = 1$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}$$

$$\sin 2\theta = -2\sqrt{\frac{8\pi}{15}}\, e^{-i\phi} Y_{21}$$

$$A_{21} r^2 Y_{21} = \frac{r^2}{4\pi\epsilon_0}\left(-2\sqrt{\frac{8\pi}{15}}\right) Y_{21}$$

$$\underline{\underline{A_{21} = -\frac{2}{4\pi\epsilon_0}\sqrt{\frac{8\pi}{15}}}} \qquad B_{21} = \frac{2}{4\pi\epsilon_0}\sqrt{\frac{8\pi}{15}}\, R^5$$

$$\Rightarrow V(R,\theta,\phi) = \left(A_{21} r^2 + \frac{B_{21}}{r^3}\right) Y_{21}(\theta,\phi)$$

$$= -\frac{2}{4\pi\epsilon_0}\sqrt{\frac{8\pi}{15}}\left[r^2 - \frac{R^5}{r^3}\right] Y_{21}(\theta,\phi)$$

$$= \frac{1}{4\pi\epsilon_0}\left[r^5 - \frac{R^5}{r^3}\right] \sin 2\theta\, e^{i\phi}$$