Torque free $\Rightarrow$ No external torque $\vec{\tau} = 0$,

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{const.}$$

Euler's Eq^n $\Rightarrow$ $I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 = 0$

$$= \omega_1 \left(I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 \right) = 0$$

$$= \frac{1}{2}\frac{d}{dt}(I_1 \omega_1^2) - (I_2 - I_3)\,\omega_2 \omega_3 \omega_1 = 0 \quad \text{— (i)}$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}(I_2 \omega_2^2) - (I_3 - I_1)\,\omega_3 \omega_1 \omega_2 = 0 \quad \text{— (ii)}$$

$$\frac{1}{2}\frac{d}{dt}(I_3 \omega_3^2) - (I_1 - I_2)\,\omega_1 \omega_2 \omega_3 = 0 \quad \text{— (iii)}$$

(i) + (ii) + (iii) $\Rightarrow$ $\frac{1}{2}\frac{d}{dt}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = 0$

$$\frac{d}{dt}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = 0$$

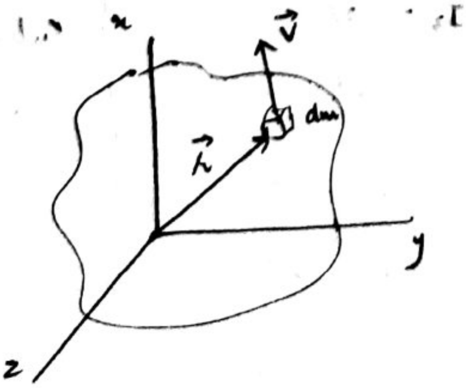
$$\frac{dT}{dt} = 0$$

$$T = \text{const.}$$

KE is constant.

Kinetic energy of rigid body

$$T = \frac{1}{2}\int dm\, v^2 = \frac{1}{2}\int dm\,(\vec{v}\cdot\vec{v})$$



$$= \frac{1}{2}\int dm\,\left[\vec{v}\cdot(\vec{\omega}\times\vec{r})\right]$$

$$= \frac{1}{2}\int dm\,\left[\vec{\omega}\cdot(\vec{r}\times\vec{v})\right]$$

$$= \frac{1}{2}\int dm\,\vec{\omega}\cdot\left[\vec{r}\times(\vec{\omega}\times\vec{r})\right]$$

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$$= \frac{1}{2}\int dm \cdot \left[\vec{\omega} \cdot \left[(\vec{r}\cdot\vec{r})\vec{\omega} - \vec{r}(\vec{r}\cdot\vec{\omega}) \right] \right]$$

$$= \frac{1}{2}\int dm \left[(\vec{\omega}\cdot\vec{\omega})(\vec{r}\cdot\vec{r}) - (\vec{\omega}\cdot\vec{r})(\vec{r}\cdot\vec{\omega}) \right]$$

$$= \frac{1}{2}\int dm \left[\omega^2 r^2 - (\vec{\omega}\cdot\vec{r})^2 \right]$$

$\vec{\omega} = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z$

$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$= \frac{1}{2}\int dm \left[(\omega_x^2 + \omega_y^2 + \omega_z^2)(x^2 + y^2 + z^2) - (\omega_x x + \omega_y y + \omega_z z)^2 \right]$$

$$= \frac{1}{2}\int dm \left[\cancel{\omega_x^2 x^2} + \omega_x^2 y^2 + \omega_x^2 z^2 + \omega_y^2 x^2 + \cancel{\omega_y^2 y^2} + \omega_y^2 z^2 + \omega_z^2 x^2 + \omega_z^2 y^2 + \cancel{\omega_z^2 z^2} - \left(\cancel{\omega_x^2 x^2} + \cancel{\omega_y^2 y^2} + \cancel{\omega_z^2 z^2} + 2\omega_x\omega_y xy + 2\omega_y\omega_z yz + 2\omega_z\omega_x xz \right) \right]$$

$$= \frac{1}{2}\int dm \left[\omega_x^2(y^2+z^2) + \omega_y^2(x^2+z^2) + \omega_z^2(x^2+y^2) - 2\omega_x\omega_y yx - 2\omega_y\omega_z yz - 2\omega_x\omega_z xz \right]$$

$$T = \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{xz}\omega_x\omega_z \right]$$

$(x, y, z) \to$ body fixed

$I_{xy} = I_{yz} = I_{xz} = 0$

Choose principle axes (x_1, x_2, x_3).

$I_{xx} = I_1, \quad I_{yy} = I_2, \quad I_{zz} = I_3$

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \right)$$

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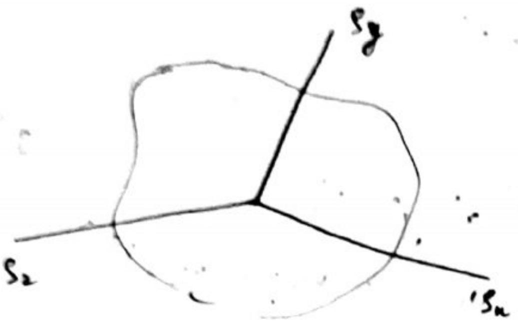
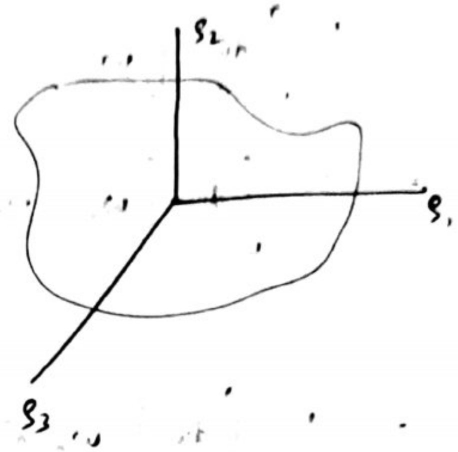
$$I_{op} = I_{xx} n_x^2 + I_{yy} n_y^2 + I_{zz} n_z^2 + 2I_{xy} n_x n_y + 2I_{yz} n_y n_z + 2I_{zx} n_z n_x$$

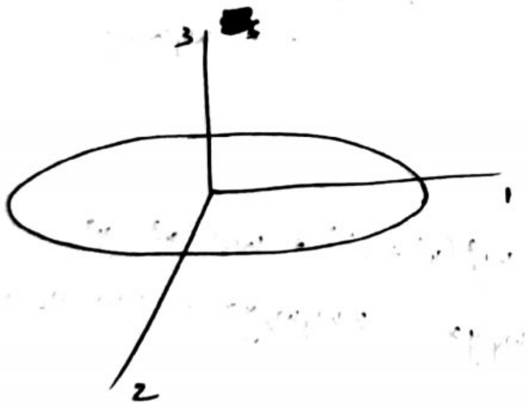
$$I_{xx} S_x^2 + I_{yy} S_y^2 + I_{zz} S_z^2 + 2I_{xy} S_x S_y + 2I_{yz} S_y S_z + 2I_{zx} S_z S_x$$



$\Rightarrow$



$$\Rightarrow I_1 S_1^2 + I_2 S_2^2 + I_3 S_3^2 = 1$$



$$I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \quad \text{(i)}$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0 \quad \text{(ii)}$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \quad \text{(iii)}$$

$$I_1 = I_2 < I_3$$

(iii) $\Rightarrow$

$$I_3 \dot{\omega}_3 = 0$$

$$\omega_3 = \text{const.}$$

(i) & (ii) $\rightarrow$

$$I_1 \dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 = 0$$

$$I_1 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0$$

$$\dot{\omega}_1 + \left(\frac{I_3 - I_1}{I_1}\right)\omega_2\omega_3 = 0$$

$$\dot{\omega}_2 - \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1 = 0$$

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$$\cancel{\frac{I_3 - I_1}{I_1}} \Rightarrow \quad \frac{I_3 - I_1}{I_1} \omega_3 = \Omega \quad \text{(let)}$$

$$\dot{\omega}_1 + \Omega \omega_2 = 0 \quad \Rightarrow \quad \ddot{\omega}_1 + \Omega \dot{\omega}_2 = 0 \quad \Rightarrow \quad \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

$$\dot{\omega}_2 - \Omega \omega_1 = 0$$

$$\omega_1(t) = A \cos(\Omega t + \alpha)$$

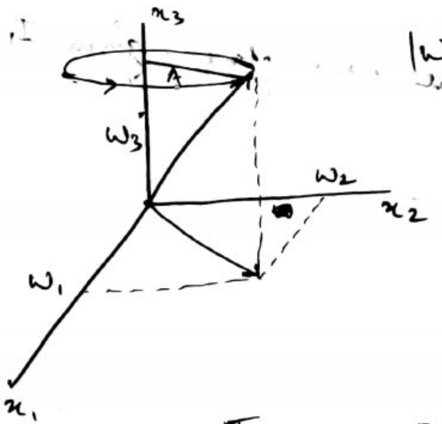
$$\omega_2 = \frac{-1}{\Omega} \dot{\omega}_1 = \frac{1}{\Omega} A \Omega \sin(\Omega t + \alpha)$$

$$\omega_2(t) = A \sin(\Omega t + \alpha)$$

$$\omega_1^2 + \omega_2^2 = A^2 = \text{const.}$$

$$\omega_3 = \text{const.}$$

Body



$$|\omega|^2 = \text{const.}$$

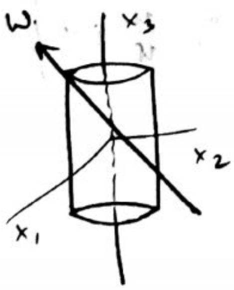
$$\omega_3 = 7.3 \times 10^{-5} \text{ rad/s}$$

$$\frac{I_3 - I_1}{I_1} = 0.0033$$

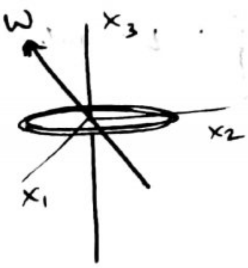
$$T_{precision} = \frac{2\pi}{\Omega} = 300 \text{ days}$$

$$T_{actual} = 440 \text{ days}$$

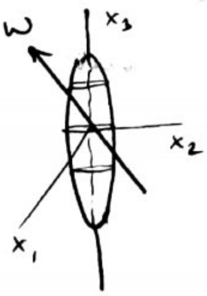
$I_1 = I_2 \neq I_3$ $\quad x_3 \Rightarrow$ symmetry axis



$I_1 > I_3$

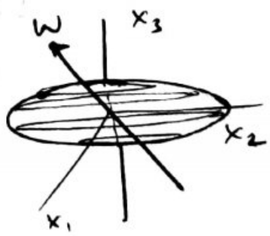


$I_3 > I_1$



Prolate spheroid

$I_1 > I_3$



Oblate spheroid

$I_3 > I_1$

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x_3, $\vec{\omega}$, $\vec{L}$

$(\hat{k})$

$$\vec{A} = \hat{k} \times \vec{\omega} = \hat{k} \times (\hat{i}\omega_1 + \hat{j}\omega_2 + \hat{k}\omega_3)$$

$$= \hat{j}\omega_1 - \hat{i}\omega_2$$

$$\vec{A}\cdot\vec{L} = (-\hat{i}\omega_2 + \hat{j}\omega_1)\cdot(\hat{i}L_1 + \hat{j}L_2 + \hat{k}L_3)$$

$$= -\omega_2 L_1 + \omega_1 L_2$$

$$= -\omega_2(I_1\omega_1) + \omega_1(I_2\omega_2)$$

$$= \omega_1\omega_2(-I_1 + I_2)$$

If $I_1 = I_2 \Rightarrow \vec{A}\cdot\vec{L} = 0$

$$\vec{L} \perp \vec{A}$$

$\vec{L}$, $\hat{k}$, $\vec{\omega}$ lie in the same plane (given $I_1 = I_2$)

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

$I_1 = I_2 \Rightarrow I_3\dot{\omega}_3 = 0$

$$\omega_3 = \text{const.}$$

$$L_3 = I_3\omega_3$$

$$L_3 = \text{const.}$$

$$\vec{L}\cdot\vec{k} = \text{const.}$$

$$L\cos\theta = \text{const.}$$

$\theta = \text{const.}$ $\left\{ \because L = \text{const. by } \frac{dL}{dt} = 0 \right\}$

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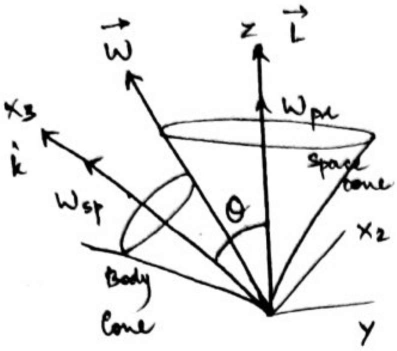

$W_{sp} \Rightarrow W_{spin}$

$W_{pe} \Rightarrow$ ~~Wprecision~~ $W_{precession}$

$$\vec{W} = \vec{W}_{pe} + \vec{W}_{sp}$$

$$\vec{W} = W_{pe}\hat{z} + W_{sp}\hat{k}$$

$$= W_{pe}(\hat{k}\cos\theta + \hat{j}\sin\theta) + W_{sp}\hat{k}$$

$$\vec{W} = \hat{j}\,W_{pe}\sin\theta + \hat{k}(W_{sp} + W_{pe}\cos\theta)$$

$$\vec{L} = \hat{i}L_1 + \hat{j}L_2 + \hat{k}L_3$$

$$= \hat{j}L\sin\theta + \hat{k}L\cos\theta$$

$$L_2 = I_2W_2$$

$$L\sin\theta = I_2W_{pe}\sin\theta$$

$$W_{pe} = \frac{L}{I_2} = \frac{L}{I_1} = \text{const.}$$

$$L_3 = I_3W_3$$

$$L\cos\theta = I_3(W_{sp} + W_{pe}\cos\theta)$$

$$L\cos\theta = I_3\left(W_{sp} + \frac{L}{I_1}\cos\theta\right)$$

$$L\cos\theta\left(1 - \frac{I_3}{I_1}\right) = I_3W_{sp}$$

$$W_{sp} = \frac{L\cos\theta\,(1 - I_3/I_1)}{I_3}$$

$$W_{sp} = \frac{(W_{pe}I_1)\cos\theta(1 - I_3/I_1)}{I_3}$$

$$W_{sp} = W_{pe}\left(\frac{I_1}{I_3} - 1\right)\cos\theta$$

$$W_{sp} = \text{const.}$$

If $\theta < 90^\circ$, $\cos\theta > 0$

$I_1 > I_3$, W_{sp} & W_{pe} have same sign.

$I_3 > I_1$, W_{sp} & W_{pe} have opp. sign, Body cone will be inside space cone

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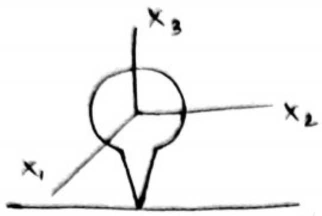
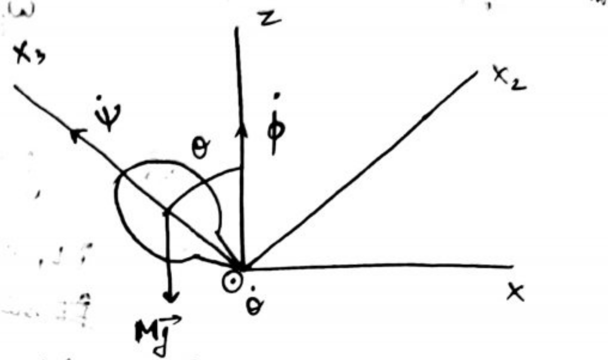
Spinning Top



$I_1 = I_2 \neq I_3$

Inclined due to nutation



$$\vec{\omega} = \dot\theta \hat{x}_1 + \dot\phi \sin\theta \hat{x}_2 + \dot\phi \cos\theta \hat{x}_3 + \dot\psi \hat{x}_3$$

$$\omega_1 = \dot\theta$$

$$\omega_2 = \dot\phi \sin\theta$$

$$\omega_3 = \dot\psi + \dot\phi \cos\theta$$

$$I_1 \dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1 = Mgh \sin\theta$$

$$I_2 \dot\omega_2 - (I_3 - I_1)\omega_1\omega_3 = \tau_2 = 0$$

$$I_3 \dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3 = 0$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$= h Mg \sin(\pi - \theta) \hat{x}_1$$
$$= Mg \sin\theta \, x_1$$

$$I_3 \dot\omega_3 = 0$$

$$I_3 \omega_3 = \text{const.}$$

$$\omega_3 = \text{const.}$$

$$L_3 = \text{const.}$$

$$I_1 \omega_1 \dot{\omega}_1 - (I_1 - I_3)\,\omega_1 \omega_2 \omega_3 = Mgh \sin\theta\, \omega_1 \quad \text{—(I)}$$

$$I_1 \omega_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_1 \omega_2 \omega_3 = 0 \quad \text{—(II)}$$

Adding (I) & (II)

$$I_1 \omega_1 \dot{\omega}_1 + I_1 \omega_2 \dot{\omega}_2 = Mgh \sin\theta\, \omega_1$$

$$I_1 (\omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2) = Mgh \sin\theta\, \dot{\theta}$$

$$\frac{1}{2} I_1 \frac{d}{dt}(\omega_1^2 + \omega_2^2) = \frac{-d}{dt}(Mgh\cos\theta)$$

$$\frac{d}{dt}\left[\frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + Mgh\cos\theta\right] = 0$$

$$\frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + Mgh\cos\theta = W = \text{const}$$

$$E_{Total} = W + \frac{1}{2} I_3 \omega_3^2$$

$$\frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + Mgh\cos\theta = W$$

$$L_z = L_2 \sin\theta + L_3 \cos\theta$$

$$= I_2 \omega_2 \sin\theta + I_3 \omega_3 \cos\theta$$

$$L_z = I_1 \dot{\phi} \sin^2\theta + L_3 \cos\theta$$

$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{I_1 \sin^2\theta}$$

$$\frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + Mgh\cos\theta = W$$

$$\frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \frac{(L_z - L_3\cos\theta)^2}{I_1^2 \sin^2\theta} + Mgh\cos\theta = W$$

$$\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{I_1^2} + \frac{2Mgh\cos\theta}{I_1 \sin^2\theta} = \frac{2W}{I_1}$$

$$\dot{\theta}^2 = \frac{2}{I_1}(W - Mgh\cos\theta) - \left(\frac{L_z - L_3\cos\theta}{I_1 \sin\theta}\right)^2$$

$$\dot{\theta}^2 = (A - B\cos\theta) - \left(\frac{C - D\cos\theta}{\sin\theta}\right)^2$$

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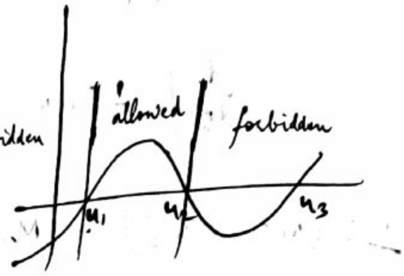
$$A = \frac{2W}{I}, \quad B = \frac{2Mgh}{I_1}, \quad I_1$$

$$\cos\theta = u$$
$$-\sin\theta \, d\theta = du$$
$$-\sqrt{1-u^2}\,\dot{\theta} = \dot{u}$$

$$\frac{\dot{u}^2}{1-u^2} = A - Bu - \frac{(C-Du)^2}{1-u^2}$$

$$\dot{u}^2 = (A-Bu)(1-u^2) - (C-Du)^2 = f(u)$$

$$f(u) \sim Bu^3$$
$$f(\infty) \to \infty$$
$$f(-\infty) \to -\infty$$

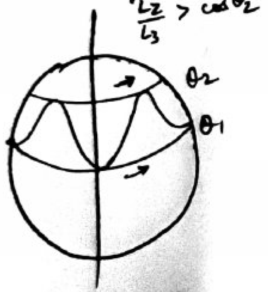


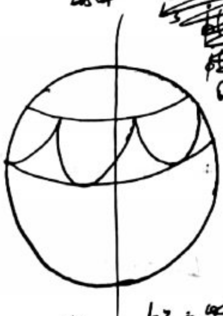
$$\dot{\phi} = \frac{L_2 - L_3\cos\theta}{I_1 \sin^2\theta}$$

$$f(u) = \dot{u}^2 > 0$$

$\dot{\phi} > 0$

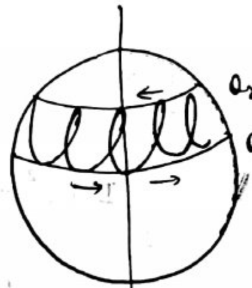
$\frac{L_2}{L_3} > \cos\theta_2$





$\frac{L_2}{L_3} = \cos\theta_2$

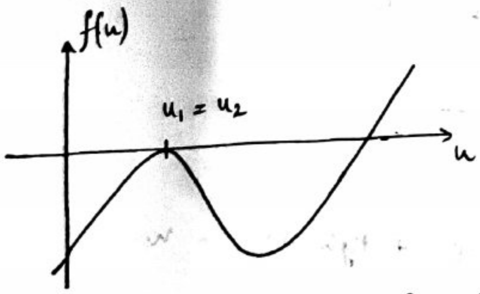
$\dot{\phi}(u_2) = 0$



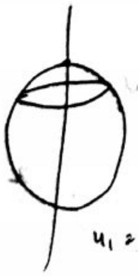
$\cos\theta_1 < \frac{L_2}{L_3} < \cos\theta_2$

$\dot{\phi}(u_2) < 0$

$\dot{\phi}(u_1) > 0$



pure precession



$u_1 = u_2$

uniform precession

$$f = (u-u_1)^2 F(u)$$
$$f' = 2(u-u_1)F(u) + (u-u_1)^2 F'(u)$$
$$= (u-u_1)\left[2F(u) + (u-u_1)F'(u)\right]$$

$$f(u_1) = f'(u_1) = 0$$

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$$f(u) = (A - Bu)(1-u^2) - (C - Du)^2$$

$$0 = (A - Bu_1)(1-u_1^2) - (C - Du_1)^2$$

$$f'(u) = -B(1-u^2) - 2u(A-Bu) + 2D(C-Du)$$

$$0 = -B(1-u_1^2) - 2u_1(A-Bu_1) + 2D(C-Du_1)$$

$$A - Bu_1 = \alpha, \quad C - Du_1 = \beta \quad \Rightarrow \quad \alpha(1-u_1^2) - \beta^2 = 0$$

$$\alpha = \frac{\beta^2}{1-u_1^2}$$

$$-B(1-u_1^2) - 2u_1\alpha + 2\beta D = 0$$

$$-B(1-u_1^2) - \frac{2u_1\beta^2}{1-u_1^2} + 2\beta D = 0$$

$$\frac{2u_1}{1-u_1^2}\beta^2 - 2D\beta + B(1-u_1^2) = 0$$

$$\beta = \frac{2D \pm \sqrt{4D^2 - 4B(1-u_1^2)\cdot\frac{2u_1}{1-u_1^2}}}{2\cdot\frac{2u_1}{1-u_1^2}}$$

$$\beta = \frac{2D \pm \sqrt{4D^2 - 8Bu_1}}{\frac{4u_1}{1-u_1^2}}$$

$$\beta = \frac{1-u_1^2}{4u_1}\left[2D \pm \sqrt{4D^2 - 8u_1B}\right]$$

$$C - Du_1 = \beta = \frac{1-u_1^2}{2u_1}\cdot D\left[1 \pm \sqrt{1 - \frac{2u_1B}{D^2}}\right]$$

$$\dot{\phi} = \frac{C - D\cos\theta_1}{\sin^2\theta_1} = \frac{D}{2u_1}\left[1 \pm \sqrt{1 - \frac{2u_1B}{D^2}}\right]$$

$$\dot{\phi} = \frac{I_3\omega_3}{2I_1\cos\theta_1}\left[1 \pm \sqrt{1 - \frac{4I_1 mgh\cos\theta_1}{I_3^2\omega_3^2}}\right]$$

$$\omega_3 = \underbrace{\dot{\psi}}_{\text{very large}} + \dot{\phi}\cos\theta \qquad \omega_3 \Rightarrow \text{very large}$$

Binomially expanding, we get

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$$\dot{\phi}_{\pm} \approx \frac{I_3 \omega_3}{2 I_1 \cos\theta_1} \left[1 \pm \left(1 - \frac{2 I_1 mgh \cos\theta_1}{I_3^2 \omega_3^2} \right) \right]$$

$$\dot{\phi}_{+} \approx \frac{I_3 \omega_3}{I_1 \cos\theta_1} \qquad (\omega_3 \Rightarrow \text{very large})$$

$$\dot{\phi}_{-} \approx \frac{I_3 \omega_3}{2 I_1 \cos\theta_1} \cdot \frac{2 I_1 mgh \cos\theta_1}{I_3^2 \omega_3^2}$$

$$\dot{\phi}_{-} \approx \frac{mgh}{I_3 \omega_3}$$

$\dot{\phi}_{+} \Rightarrow$ fast [redacted] precession

$\dot{\phi}_{-} \Rightarrow$ slow precession

Only slow precession is observed

Hamilton's Principle

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

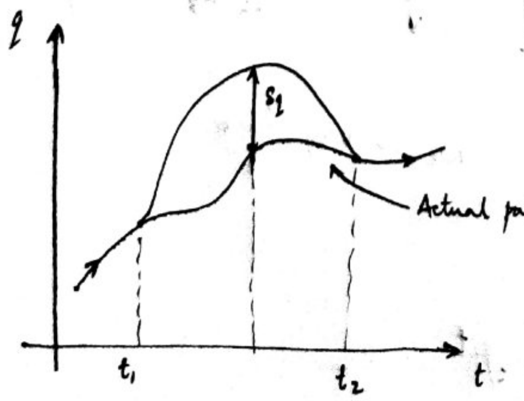
Action

Action is $\underset{\text{min / max.}}{\underline{\text{extremum}}}$ for actual path

$$\delta S = 0$$

$$q \longrightarrow q + \delta q$$

$$\delta q(t_1) = 0$$
$$\delta q(t_2) = 0$$



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$$\delta S = \delta \int L(q, \dot q, t)\, dt$$

$$= \int \delta L(q, \dot q, t)\, dt$$

$$= \int \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \delta \dot q + \frac{\partial L}{\partial t} \delta t \right] dt$$

$f = f(x, y, z)$

$\delta f = f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z)$

$= \cancel{f(x,y,z)} + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z} + \delta x^2 \frac{\partial^2 f}{\partial x^2} + \cdots - \cancel{f(x,y,z)}$

Neglecting higher order terms

$\delta t = 0$ because varying q on vertical axis

$\delta q = \varepsilon \cdot f(t)$

$\delta \dot q = \frac{d}{dt}(\delta q)$

$= \varepsilon \dot f(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \partial q \right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \cdot \partial q + \frac{\partial L}{\partial \dot q} \partial \dot q$$

$$\Rightarrow \frac{\partial L}{\partial \dot q} \partial \dot q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \partial q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \cdot \partial q$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \cdot \delta q + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \partial q \right) \right] dt$$

$$= \int_{t_1}^{t_2} \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \right] dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \delta q \right) dt$$

$$\delta S = \int_{t_1}^{t_2} \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \right] dt + \int_{t_1}^{t_2} d\left(\frac{\partial L}{\partial \dot q} \delta q \right)$$

$$+ \frac{\partial L}{\partial \dot q} \delta q(t) \Big|_{t_1}^{t_2}$$

$$+ 0 \quad \left(\because \begin{matrix} \delta q(t_1) = 0 \\ \delta q(t_2) = 0 \end{matrix} \right)$$

$$\delta S = 0 \Rightarrow \int_{t_1}^{t_2} \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \right) \right] dt = 0$$

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$$\therefore \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$\therefore$ Euler - Langrange Eq^n

Legendre Transformation

$$f = f(x, y, z)$$

$$df = u\,dx + v\,dy + w\,dz$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$\therefore u = \frac{\partial f}{\partial x}, \quad v = \frac{\partial f}{\partial y}, \quad w = \frac{\partial f}{\partial z}.$$

$$d(ux) = u\,dx + x\,du.$$

$$d(f - ux) = -x\,du + v\,dy + w\,dz$$

$$dg = -x\,du + v\,dy + w\,dz \qquad (g = f - ux)$$

$$g = g(u, y, z)$$

Legendre Transformation for Lagrangian $\Rightarrow$

$$L = L(q, \dot{q}, t)$$

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$= \frac{\partial L}{\partial q} dq + p\,d\dot{q} + \frac{\partial L}{\partial t} dt$$

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$$d(p\dot{q}) = p\,d\dot{q} + \dot{q}\,dp$$

$$d(L - p\dot{q}) = \frac{\partial L}{\partial q}dq - \dot{q}\,dp + \frac{\partial L}{\partial t}dt$$

$$-dH = \frac{\partial L}{\partial q}dq - \dot{q}\,dp + \frac{\partial L}{\partial t}dt \quad \text{— (i)}$$

$$H = H(q, p, t)$$

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt \quad \text{— (ii)}$$

Comparing (i) & (ii)

$$\frac{\partial H}{\partial q} = \frac{-\partial L}{\partial q}, \quad \frac{\partial H}{\partial p} = +\dot{q}, \quad \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$$

$$\Downarrow$$

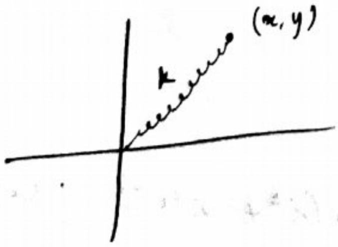
Hamiltonian eqⁿ

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}(p) = 0$$

$$\frac{\partial L}{\partial q} = \dot{p} \Rightarrow \frac{\partial H}{\partial q} = -\dot{p}$$

Particle in 2-D



$$l_0 = 0$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

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$$H = p_x \dot{x} + p_y \dot{y} - L$$

$$= m\dot{x}^2 + m\dot{y}^2 - \left[\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)\right]$$

$$H = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k(x^2 + y^2)$$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2} k(x^2 + y^2)$$

Plane - polar co-ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} k r^2$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= m\dot{r}^2 + m r^2\dot{\theta}^2 - \left[\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} k r^2\right]$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} k r^2$$

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$$H = \frac{1}{2} m \left[\left(\frac{p_r}{m}\right)^2 + r^2 \left(\frac{p_\theta}{m r^2}\right)^2 \right] + \frac{1}{2} k r^2$$

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + \frac{1}{2} k r^2$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2}$$

$$\dot{p}_r = \frac{-\partial H}{\partial r} = + \frac{p_\theta^2}{m r^3} - k r$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = 0$$

$\boxed{\text{H.W.}}$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} k (x^2 + y^2 + z^2)$$

Convert this into

cylindrical coords (s, ϕ, z)

spherical co-ords (r, θ, ϕ)

Find H for all cases

$$F = F(q, p, t)$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q} \frac{dq}{dt} + \frac{\partial F}{\partial p} \frac{dp}{dt} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial p} \dot{p} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q} \cdot \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}$$

$$[F, G] = \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q}$$

Poisson bracket

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

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Properties $\Rightarrow$ $[F, F] = 0$

$$[uF, \lambda G] = u\lambda [F, G]$$

$$[F, G_1 + G_2] = [F, G_1] + [F, G_2]$$

$$[F_1 + F_2, G] = [F_1, G] + [F_2, G]$$

$$[F_1 F_2, G] = F_1 [F_2, G] + F_2 [F_1, G]$$

$$[F, G_1 G_2] = G_1 [F, G_2] + G_2 [F, G_1]$$

$$[F, G] = -[G, F]$$

Jacobi's Identity

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$$

Particle in 3-D

$x, y, z \Downarrow x_1, x_2, x_3$ $\quad$ $p_x, p_y, p_z \Downarrow p_1, p_2, p_3$

x_i $\quad$ p_i $\quad$ $(i = 1, 2, 3)$

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} \right)$$

$$[F, G] = \sum_k \left(\frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \cdot \frac{\partial G}{\partial x_k} \right)$$

$$[x_i, x_j] = \sum_k \left(\frac{\partial x_i}{\partial x_k} \cancelto{0}{\frac{\partial x_j}{\partial p_k}} - \cancelto{0}{\frac{\partial x_i}{\partial p_k}} \cdot \frac{\partial x_j}{\partial x_k} \right)$$

$$= 0$$

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$$[x_i , p_j] = \sum_k^3 \left(\frac{\partial x_i}{\partial x_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial x_i}{\partial p_k} \frac{\partial p_j}{\partial x_k} \right)$$

$$\frac{\partial x_i}{\partial x_k} = 1 \quad \text{for } i = k \quad \Rightarrow \quad \frac{\partial x_i}{\partial x_k} = \delta_{ik} = 1 \quad i = k$$

$$= 0 \quad \text{for } i \neq k \qquad\qquad = 0 \quad i \neq k$$

$$[x_i , p_j] = \sum_k^3 \delta_{ik} \delta_{jk}$$

$$= \cancel{\sum} \delta_{ij}$$

$$[x_i , p_j] = \delta_{ij}$$

One particle in 3-D

$$x, y, z \longrightarrow x_1, x_2, x_3 = \{x_i\}$$

$$p_x, p_y, p_z \longrightarrow p_1, p_2, p_3 = \{p_i\}$$

Kronecker

$$[x_i , p_j] = \delta_{ij} = 1 \quad i = j$$

$$= 0 \quad i \neq j$$

$$\vec{L} = L_x, L_y, L_z \Rightarrow L_1, L_2, L_3 = \{L_i\}$$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ x_1 & x_2 & x_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

$$\vec{L} = \hat{e}_1 (x_2 p_3 - x_3 p_2) + \hat{e}_2 (x_3 p_1 - x_1 p_3) + \hat{e}_3 (x_1 p_2 - x_2 p_1)$$

$$\vec{L} = \hat{e}_1 L_1 + \hat{e}_2 L_2 + \hat{e}_3 L_3$$

$$L_1 = x_2 p_3 - x_3 p_2$$

$$L_2 = x_3 p_1 - x_1 p_3$$

$$L_3 = x_1 p_2 - x_2 p_1$$

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Levi-Civita Symbol

ε_{ijk}

$\varepsilon_{123} = +1$

$\varepsilon_{213} = -1$

$\varepsilon_{113} = 0$

$\varepsilon_{121} = 0$

$\varepsilon_{331} = 0$

$$L_i = \sum_{jk} \varepsilon_{ijk} x_j p_k$$

$\delta_{ij} = \delta_{ji}$ (symmetric)

$\varepsilon_{ijk} = -\varepsilon_{jik}$ (Anti-symmetric)

$$[x_i, L_j] = \left[x_i, \sum_{kl} \varepsilon_{jkl} x_k p_l\right]$$

$$= \sum_{kl} \varepsilon_{jkl} [x_i, x_k p_l]$$

$$= \sum_{kl} \varepsilon_{jkl} \left(\overset{0}{\cancel{[x_i, x_k] p_l}} + x_k [x_i, p_l] \right)$$

$$= \sum_{kl} \varepsilon_{jkl} \left(0 + x_k \delta_{il} \right)$$

$$= \sum_{k} \varepsilon_{jki} x_k$$

$$[x_i, L_j] = \sum_{k} \varepsilon_{ijk} x_k$$

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$$[p_i , L_j] = \left[p_i , \sum_{kl} \varepsilon_{jkl}\, x_k p_l\right]$$

$$= \sum_{kl} \varepsilon_{jkl} [p_i , x_k p_l]$$

$$= \sum_{kl}' \varepsilon_{jkl} \left[[p_i , x_k] p_l + \cancel{x_k [p_i , p_l]} \right]$$

$$= \sum_{kl} \varepsilon_{jkl} \left(p_l \cdot (-\delta_{ik}) \right)$$

$$= \sum_{l} \varepsilon_{jil} (-p_l)$$

$$= \sum_{l} \varepsilon_{ijl} \cdot p_l$$

$$[p_i , L_j] = \sum_{k} \varepsilon_{ijk} \cdot p_k$$

$$[L_i , L_m] = \left[\sum_{jk} \varepsilon_{ijk}\, x_j p_k , L_m \right]$$

$$= \sum_{jk} \varepsilon_{ijk} [x_j p_k , L_m]$$

$$= \sum_{jk} \varepsilon_{ijk} \left(x_j [p_k , L_m] + [x_j , L_m] p_k \right)$$

$$= \sum_{jk} \varepsilon_{ijk} \left(x_j \sum_n \varepsilon_{kmn} p_n + p_k \sum_n \varepsilon_{jmn} x_n \right)$$

$$= \sum_{jkn} \varepsilon_{ijk} \varepsilon_{kmn}\, x_j p_n + \sum_{jkn} \varepsilon_{ijk} \varepsilon_{jmn}\, x_n p_k$$

$$\left(\sum_k \varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \right)$$

$$= \sum_{jn} \left(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \right) x_j p_n - \sum_{kn} \left(\delta_{im} \delta_{kn} - \delta_{in} \delta_{mk} \right) x_n p_k$$

$$= \cancel{\sum_j \delta_{im} x_j p_j} - x_m p_i \cancel{- \sum_k \delta_{im} x_k p_k} + x_i p_m$$

$$= x_i p_m - x_m p_i$$

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$$[L_i, L_m] = \sum_k \varepsilon_{imk} L_k$$

$$[L_i, L_j] = \sum_k \varepsilon_{ijk} L_k \Rightarrow [L_1, L_2] = L_3$$

$$[L_2, L_3] = L_1$$

$$[L_3, L_1] = L_2$$

Example $[L^2, L_i] = 0 \quad i = 1, 2, 3$

$i = 1 \Rightarrow [L^2, L_1]$

$$= [L_1^2 + L_2^2 + L_3^2, L_1]$$

$$= [L_1^2, L_1] + [L_2^2, L_1] + [L_3^2, L_1]$$

$$= 0 + [L_2 L_2, L_1] + [L_3 L_3, L_1]$$

$$= L_2[L_2, L_1] + [L_2, L_1]L_2 + L_3[L_3, L_1] + [L_3, L_1]L_3$$

$$= \cancel{-L_2 L_3} \cancel{-L_3 L_2} + \cancel{L_3 L_2} + \cancel{L_2 L_3}$$

$$= 0$$

$$[L^2, L_i] = 0$$

$\boxed{\text{H.W.}}$

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2}k(x^2 + y^2 + z^2)$$

Find $[p_i, L_j]$, $[x_i, L_j]$, $[H, L_j]$, $[H, x_i]$, $[H, p_i]$

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Canonical Transformation

$$\{q, p, t\} \longrightarrow \{Q, P, t\}$$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{Q} = \frac{\partial K}{\partial P}$$

$$\dot{p} = \frac{-\partial H}{\partial q} \qquad \dot{P} = \frac{-\partial K}{\partial Q}$$

$$S = \int_{t_1}^{t_2} L \, dt$$

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt$$

$$= \delta \int_{t_1}^{t_2} \left[L(q, \dot{q}, t) + \frac{d}{dt} F(q, t) \right] dt$$

$$= \delta \int_{t_1}^{t_2} L \, dt + \delta \int_{t_1}^{t_2} \frac{dF}{dt} \, dt$$

$$= \delta \int_{t_1}^{t_2} L \, dt$$

$$\delta \int_{t_1}^{t_2} dF(q, t) = \delta \left[F(q_2, t_2) - F(q_1, t_1) \right] = 0 - 0 = 0$$

$$S = \int (p\dot{q} - H) \, dt$$

$$\cancel{S} = \int (P\dot{Q} - K) \, dt$$

$$S = \int \left(P\dot{Q} - K + \frac{dF}{dt} \right) dt$$

$$p\dot{q} - H = P\dot{Q} - K + \frac{dF}{dt}$$

$$\frac{dF}{dt} = p\dot{q} - H - P\dot{Q} + K$$

$$dF = p \, dq - P \, dQ + (K - H) \, dt$$

$$F = F(q, Q, t)$$

Generating function of canonical transformation

$$F = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial Q} dQ + \frac{\partial F}{\partial t} dt$$

$$p = \frac{\partial F}{\partial q}, \quad -P = \frac{\partial F}{\partial Q}, \quad K - H = \frac{\partial F}{\partial t}$$

$$p = \frac{\partial F}{\partial q}(q, Q, t) \qquad -P = \frac{\partial F}{\partial Q}(q, Q, t)$$

$$p = f(q, Q, t) \qquad -P = g(q, Q, t)$$

$$Q = f(p, q, t) \qquad -P = g(q, f(p, q, t), t)$$

$$d(PQ) = P\,dQ + Q\,dP$$

$$dF + d(PQ) = \cancel{P\,dQ} + Q\,dP + (p\,dq - \cancel{P\,dQ} + (K-H)\,dt)$$

$$d(F + PQ) = p\,dq + Q\,dP + (K-H)\,dt$$

$$G = F + PQ$$

$$dG = p\,dq + Q\,dP + (K-H)\,dt$$

$$G = G(q, P, t)$$

$$dG = \frac{\partial G}{\partial q} dq + \frac{\partial G}{\partial P} dP + \frac{\partial G}{\partial t} dt$$

$$p = \frac{\partial G}{\partial q}, \quad Q = \frac{\partial G}{\partial P}, \quad K - H = \frac{\partial G}{\partial t}$$

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$$d(pq) = p\,dq + q\,dp$$

$$d\phi = d(F - pq) = p\,dq - P\,dQ + (K-H)\,dt - (p\,dq + q\,dp)$$

$$d\phi = -P\,dQ - q\,dp + (K-H)\,dt$$

$$\phi = \phi(p, Q, t)$$

$$d\phi = \frac{\partial \phi}{\partial Q}\,dQ + \frac{\partial \phi}{\partial p}\,dp + \frac{\partial \phi}{\partial t}\,dt$$

$$-P = \frac{\partial \phi}{\partial Q}, \quad -q = \frac{\partial \phi}{\partial p}, \quad K - H = \frac{\partial \phi}{\partial t}$$

$$dG = p\,dq + Q\,dP + (K-H)\,dt$$

$$d(pq) = p\,dq + q\,dp$$

$$d\psi = d(G - pq) = p\,dq + Q\,dP + (K-H)\,dt - (p\,dq + q\,dp)$$

$$= Q\,dP - q\,dp + (K-H)\,dt$$

$$d\psi = \frac{\partial \psi}{\partial P}\,dP + \frac{\partial \psi}{\partial p}\,dp + \frac{\partial \psi}{\partial t}\,dt$$

$$\psi = \psi(P, p, t)$$

$$Q = \frac{\partial \psi}{\partial P}, \quad -q = \frac{\partial \psi}{\partial p}, \quad K - H = \frac{\partial \psi}{\partial t}$$

$$S = \int_{t_1}^{t_2} L\,dt \Rightarrow \frac{dS}{dt} = L = p\dot{q} - H$$

$$dS = p\,dq - H\,dt$$

$$S = S(q, t)$$

$$dS = \frac{\partial S}{\partial q}\,dq + \frac{\partial S}{\partial t}\,dt$$

$$p = \frac{\partial S}{\partial q}, \quad -H = \frac{\partial S}{\partial t}$$

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$$\frac{\partial S}{\partial t} + H(q, p, t) = 0$$

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 = K$$

PDE

Hamilton - Jacobi Eq^n

$\boxed{\text{H.W.}}$

$$H = p_x \dot{x} - L$$

Verify $\quad H(x, p_x, t) = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$

$$S(x, t)$$

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}, t\right) = 0$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}kx^2 = 0$$

Separation of variable $\Rightarrow$

$$S(x, t) = W(x) - Et$$

$$\frac{\partial S}{\partial t} = 0 - E$$

$$\frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} - 0$$

$$-E + \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = 0$$

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 = E - \frac{1}{2}kx^2$$

$$\frac{dW}{dx} = \pm\sqrt{2m}\sqrt{E - \frac{1}{2}kx^2}$$

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$$W(x) = \pm\sqrt{2m}\int\sqrt{E-\frac{1}{2}kx^2}\,dx + c$$

$$K = H + \frac{\partial G}{\partial t} = 0$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \rightarrow Q = \text{const}$$

$$S = W(x) - Et$$

$$S = \pm\sqrt{2m}\int\sqrt{E-\frac{1}{2}kx^2}\,dx - Et$$

$$Q = \frac{\partial G}{\partial P} \Rightarrow G = \frac{\partial S}{\partial E}$$

$$\frac{\partial S}{\partial E} = \text{const}$$

$$\frac{\partial S}{\partial E} = \pm\sqrt{2m}\int\frac{dx}{2\sqrt{E-\frac{1}{2}kx^2}} - t = C_1$$

$$C_1 = \pm\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{a}\right) - t \qquad a = \frac{2E}{k}$$

Projectile Motion

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$H = p_x\dot{x} + p_y\dot{y} - L$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

$$\frac{\partial S}{\partial t} + H\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, t\right) = 0$$

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$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy = 0$$

$$S(x,y,t) = W(y) + p_x x - Et$$

$$\frac{\partial S}{\partial x} = p_x, \quad \frac{\partial S}{\partial y} = \frac{\partial W}{\partial y}, \quad \frac{\partial S}{\partial t} = -E$$

$$\Rightarrow \quad -E + \frac{p_x^2}{2m} + \frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + mgy = 0$$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 = E - \frac{p_x^2}{2m} - mgy$$

$$\frac{dW}{dy} = \pm\sqrt{2m}\sqrt{E - \frac{p_x^2}{2m} - mgy}$$

$$W = \pm\sqrt{2m}\int\sqrt{E - \frac{p_x^2}{2m} - mgy}\, dy + c$$

$$S = \pm\sqrt{2m}\int\sqrt{E - \frac{p_x^2}{2m} - mgy}\, dy + c + p_x x - Et$$

$$C_1 = \frac{\partial S}{\partial p_x} = \pm\sqrt{2m}\int\frac{-2p_x/2m \, dy}{2\sqrt{E - \frac{p_x^2}{2m} - mgy}} + 0 + x + 0$$

$$C_2 = \frac{\partial S}{\partial E} = \pm\sqrt{2m}\int\frac{dy}{2\sqrt{E - \frac{p_x^2}{2m} - mgy}} + 0 + 0 - t$$

$$C_1 - x = \dots$$

$$C_2 + t = \dots$$

$$\frac{C_1 - x}{C_2 + t} = \frac{P_x}{m} \Rightarrow x - x_0 = V_0 (t - t_0)$$

Kepler Problem

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

$$L = L(r, \cancel{\theta}, \dot{r}, \dot{\theta}, \cancel{t}) \qquad P_\theta = \frac{\partial L}{\partial \theta} = \text{const.} = P_\theta$$

$$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$$

$$\frac{\partial S}{\partial t} + H(r, \theta, P_r, P_\theta, t) = 0$$

$$\frac{\partial S}{\partial t} + H\left(r, \theta, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, t\right) = 0 \quad , \quad S(r, \theta, t)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + V(r) = 0$$

$$S(r, \theta, t) = W(r) + P_\theta \theta - Et$$

$$\frac{\partial S}{\partial t} = -E \quad , \quad \frac{\partial S}{\partial r} = \frac{dW}{dr} \quad , \quad \frac{\partial S}{\partial \theta} = P_\theta$$

$$-E + \frac{1}{2m}\left(\frac{dW}{dr}\right)^2 + \frac{1}{2mr^2} P_\theta^2 + V(r) = 0$$

$$\frac{1}{2m}\left(\frac{dW}{dr}\right)^2 = E - \frac{P_\theta^2}{2mr^2} - V(r)$$

$$\frac{dW}{dr} = \pm \sqrt{2m} \sqrt{E - \frac{P_\theta^2}{2mr^2} - V(r)}$$

$$W = \pm \sqrt{2m} \int \sqrt{E - \frac{P_\theta^2}{2mr^2} - V(r)}\, dr + C_0$$

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$G(q, P, t)$

$$p = \frac{\partial G}{\partial q} \qquad Q = \frac{\partial G}{\partial P}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = 0 \qquad H + \frac{\partial G}{\partial t} = K(Q, P, t)$$

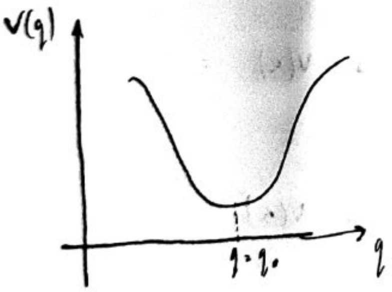
$$\dot{P} = \frac{-\partial K}{\partial Q} = 0$$

$$G = \frac{\partial S}{\partial E} = \sqrt{2m}\int \frac{2\, dr}{\sqrt{E - \frac{P_\theta^2}{2mr^2} - V(r)}} - t$$

$$C_2 = \frac{\partial S}{\partial P_\theta} = \sqrt{2m}\int \frac{2\, dr \cdot \left(\frac{-P_\theta}{mr^2}\right)}{\sqrt{E - \frac{P_\theta^2}{2mr^2} - V(r)}} + \theta$$

$$\frac{G + t}{C_2 - \theta} = \frac{-mr^2}{P_\theta}$$

Small Oscillations



$$V(q) = V(q_0) + \left(\frac{\partial V}{\partial q}\right)_{q=q_0}(q - q_0) + \frac{1}{2!}\left(\frac{\partial^2 V}{\partial q^2}\right)_{q=q_0}(q-q_0)^2$$

$$V(q) = V(q_0) + \frac{1}{2}k(q - q_0)^2 + \dots$$

$$\left(\frac{\partial V}{\partial q}\right)_{q=q_0} = 0$$

$$V(q) = V(q_0) + \frac{1}{2}k\eta^2$$

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$$T = \frac{1}{2} m \dot{q}^2 \qquad \eta = q - q_0$$

$$T = \frac{1}{2} m \dot{\eta}^2 \qquad \dot{\eta} = \dot{q}$$

$$L = \frac{1}{2} m \dot{\eta}^2 - V(q_0) - \frac{1}{2} k \eta^2$$

$$\cancel{\frac{\partial L}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right)}$$

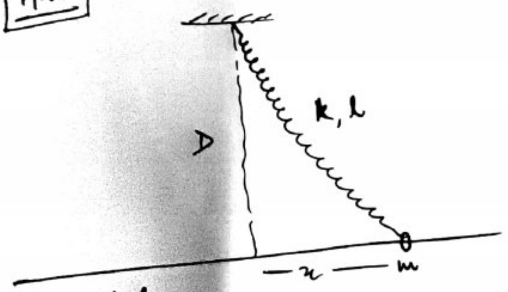
$$\frac{\partial L}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = 0 \implies -k\eta - \frac{d}{dt}\left(m\dot{\eta}\right) = 0$$

$$\implies -k\eta - m\ddot{\eta} = 0$$

$$\ddot{\eta} + \frac{k}{m}\eta = 0$$

$$\eta(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$\boxed{\text{H.W.}}$



Horizontal surface

$l \Rightarrow$ unstretched length

Find (i) $D \gtrsim l$

Find $V = ?$ $T = ?$ for small x

E – L eqⁿ

(ii) $D = l$

(iii) $D \gtrsim l$ by newtonian mechanics

(iv) $D = l$ by newtonian mechanics

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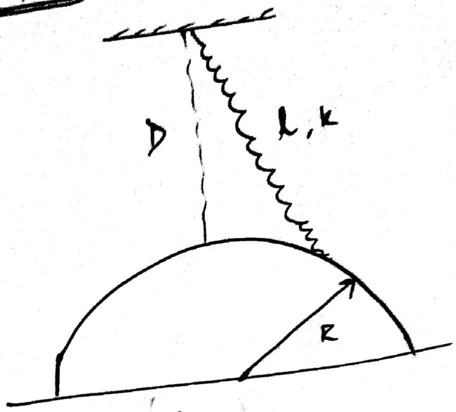
$\boxed{\text{H.W.}}$



E :

(i) $D \gtrsim l$

Find $V =$

$T =$

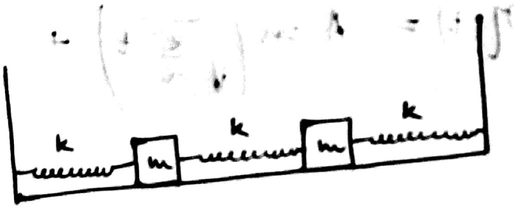
E-L eq^n

(ii) $D = l$

(iii) $D \gtrsim l$
Newtonian mechanics

(iv) $D = l$
Newtonian mechanics

$\boxed{\text{H.W.}}$

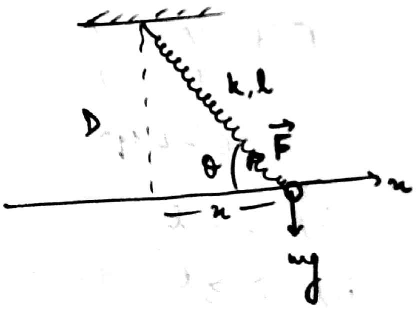


$T =$

$V =$

E-L eq^n

H.W. Sol^n



$$|\vec{F}| = k\Delta l$$

$$\Delta l = \sqrt{D^2+x^2} - l$$

$$= D\left(1+\frac{x^2}{D^2}\right)^{1/2} - l$$

$$= D\left(1+\frac{x^2}{2D^2}+\cdots\right) - l$$

$$= D - l + \frac{x^2}{2D^2} + \cdots$$

$$F\sin\theta = N + mg$$

$$F\cos\theta = -m\ddot{x}$$

$$\cos\theta = \frac{x}{\sqrt{D^2+x^2}} = \frac{x}{D}\left(1+\frac{x^2}{D^2}\right)^{-1/2}$$

$$= \frac{x}{D}\left(1 - \frac{x^2}{2D^2} + \cdots\right)$$

$$\cos\theta = \frac{x}{D} - \frac{x^3}{2D^3} \cdots$$

$$-m\ddot{x} = F\cos\theta$$

$$-m\ddot{x} = k\left[(D-l) + \frac{x^2}{2D^2} + \cdots\right]\left(\frac{x}{D} - \frac{x^3}{2D^3} \cdots\right)$$

$$= k\left[(D-l)\frac{x}{D} + \cancel{\frac{x^3}{2D^2}} - (D-l)\frac{x^3}{2D^3} + O(x^5)\right]$$

$$m\ddot{x} + k\left[\left(1-\frac{l}{D}\right)x + \frac{l\,x^3}{2D^3} + O(x^5)\right] = 0$$

$x << D$

$\frac{x}{D} << 1$

$$m\ddot{x} + k\left(1-\frac{l}{D}\right)x = 0$$

$$\omega = \sqrt{\frac{k}{m}\left(1-\frac{l}{D}\right)}$$

$(D \gtrsim l)$

$(D = l)$

$$m\ddot{x} + \frac{l}{2D^3}x^3 = 0$$

$$m\ddot{x} + k_1 x^3 = 0$$

$$m\dot{x}\ddot{x} + k_1 \dot{x} x^3 = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{4}k_1 x^4\right) = 0$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{4}k_1 x^4 = E$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{4}k_1 x^2}$$

$$t = \int dt = \pm\sqrt{\frac{m}{2}}\int \frac{dx}{\sqrt{E - \frac{1}{4}kx^2}}$$

$$\int_0^1 \frac{dz}{\sqrt{1-z^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

$\Gamma(1/4) = 0.626$

$\Gamma(3/4) = 0.225$

Aliter

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$V = \frac{1}{2}k(\Delta l)^2$$

$$\Delta l = \sqrt{D^2+x^2} - l$$

$$(\Delta l)^2 = (D^2+x^2) + l^2 - 2l\sqrt{D^2+x^2}$$

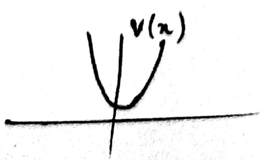
$$= D^2 + x^2 + l^2 - 2lD\left(1 + \frac{x^2}{2D^2} + \cdots\right)$$

$$= (D-l)^2 + x^2 - 2lD\left(\frac{x^2}{2D^2} - \frac{1}{8}\frac{x^4}{D^4} + \cdots\right)$$

$$(\Delta l)^2 = (D-l)^2 + \left(1 - \frac{l}{D}\right)x^2 + \frac{1}{4}\frac{l}{D^3}x^4$$

$$V = \frac{1}{2}k(\Delta l)^2 = \frac{1}{2}k\left[(D-l)^2 + \left(1-\frac{l}{D}\right)x^2 + \frac{1}{4}\frac{lx^4}{D^3}\right]$$

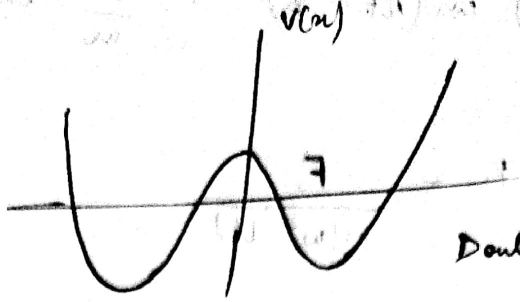
$$V(x) = A + Bx^2 + Cx^4 \qquad (D \gtrsim l)$$



$B > 0$

$C > 0$

$D \leq l \Rightarrow B < 0$



Double-well potential

Forced Oscillation

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega^2 = \frac{k}{m}$$

Let $F(t) = F_0 \cos(\Omega t + \beta)$

$$\ddot{X}_{hom} + \omega^2 X_{hom} = 0$$

$$X_{hom}(t) = A\cos(\omega t + \alpha)$$

$$X(t) = X_{hom}(t) + X_{PI}(t)$$

$$X_{PI}(t) = B\cos(\Omega t + \beta)$$

$$X(t) = A\cos(\omega t + \alpha) + B\cos(\Omega t + \beta)$$

$$\dot{X}(t) = -A\omega \sin(\omega t + \alpha) - B\Omega \sin(\Omega t + \beta)$$

$$\ddot{X}(t) = -A\omega^2 \cos(\omega t + \alpha) - B\Omega^2 \cos(\Omega t + \beta)$$

$$\cancel{m\ddot{x} = -kx + F(t)}$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

$$-A\omega^2 \cos(\omega t + \alpha) - B\Omega^2 \cos(\Omega t + \beta) + \omega^2\big(A\cos(\omega t + \alpha) + B\cos(\Omega t + \beta)\big)$$

$$= \frac{F_0 \cos(\Omega t + \beta)}{m}$$

$$B(\omega^2-\Omega^2)\cos(\Omega t+\beta) = \frac{F_0}{m}\cos(\Omega t+\beta)$$

$$B = \frac{F_0}{m(\omega^2-\Omega^2)}$$

$$x(t) = A\cos(\omega t+\alpha) + \frac{F_0}{m(\omega^2-\Omega^2)}\cos(\Omega t+\beta)$$

Let $\Omega = \omega + \varepsilon$

$$x(t) = A'\cos(\omega t+\alpha) + \frac{F_0}{m(\omega^2-\Omega^2)}\left[\cos(\Omega t+\beta) - \cos(\omega t+\beta)\right]$$

$$\begin{aligned}\cos(\Omega t+\beta) &= \cos(\omega t+\varepsilon t+\beta)\\ &= \cos(\omega t+\beta)\cancel{\cos\varepsilon t}^{\,1} - \sin(\omega t+\beta)\cancel{\sin\varepsilon t}^{\,\varepsilon t}\\ &= \cos(\omega t+\beta) - \varepsilon t\sin(\omega t+\beta)\end{aligned}$$

$$\omega^2-\Omega^2 = (\omega+\Omega)(\omega-\Omega) = (2\omega)(-\varepsilon)$$

$$x(t) = A'\cos(\omega t+\alpha) + \frac{F_0}{m}\,\frac{\cancel{-\varepsilon}t\sin(\omega t+\beta)}{(2\omega)\cancel{(-\varepsilon)}}$$

$$x(t) \approx A'\cos(\omega t+\alpha) + \frac{F_0 t}{2m\omega}\sin(\omega t+\beta)$$

$$\begin{aligned}A\cos(\omega t+\alpha) &= Re\left[Ae^{i(\omega t+\alpha)}\right]\\ &= Re\left[(Ae^{i\alpha})e^{i\omega t}\right]\\ &= Re\left[Ce^{i\omega t}\right]\end{aligned}$$

$$B\cos(\Omega t+\beta) = Re\left[(Be^{i\beta})e^{i\Omega t}\right] = Re\left[De^{i\Omega t}\right]$$

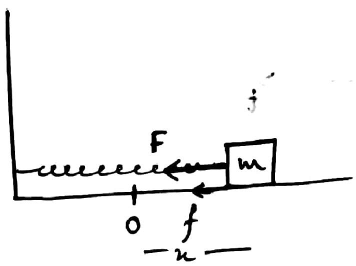
$$x(t) = A\cos(\omega t+\alpha) + B\cos(\Omega t+\beta)$$
$$= \mathrm{Re}\left[A e^{i\alpha} e^{i\omega t} + B e^{i\beta} e^{i\Omega t}\right]$$

$$\Omega = \omega + \varepsilon$$

$$x(t) = \mathrm{Re}\left[A e^{i\alpha} e^{i\omega t} + B e^{i\beta} e^{i\varepsilon t} e^{i\omega t}\right]$$
$$= \mathrm{Re}\left[\left(A e^{i\alpha} + B e^{i(\beta+\varepsilon t)}\right) e^{i\omega t}\right]$$
$$= \mathrm{Re}\left[K e^{i\omega t}\right]$$

$$K = A e^{i\alpha} + B e^{i(\beta+\varepsilon t)}$$
$$|K| = \left|A + B e^{i(\beta-\alpha+\varepsilon t)}\right|$$
$$|K| = \sqrt{A^2 + B^2 + 2AB\cos(\beta-\alpha+\varepsilon t)}$$

Damped Oscillation



$$m\ddot{x} = -F - f$$
$$= -kx - \gamma\ddot{x}$$

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$
$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0$$

$$\lambda = \frac{\gamma}{m}, \quad \omega^2 = \frac{k}{m}$$

$$x(t) = A e^{\alpha t}$$
$$\dot{x} = \alpha A e^{\alpha t}$$
$$\ddot{x} = \alpha^2 A e^{\alpha t}$$

$$\alpha^2 A + \lambda\alpha A + \omega^2 A = 0$$
$$\alpha^2 + \lambda\alpha + \omega^2 = 0$$
$$\Rightarrow \alpha = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\omega^2}}{2}$$

$$\sqrt{\lambda^2-4\omega^2} = i\sqrt{4\omega^2-\lambda^2} = 2i\sqrt{\omega^2-\frac{\lambda^2}{4}}$$

$$\alpha = \frac{-\lambda}{2} \pm i\omega' = \alpha_1, \alpha_2$$

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

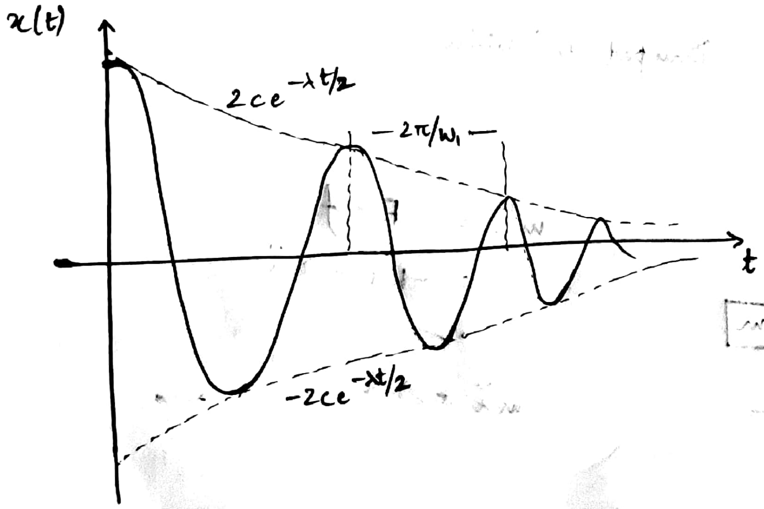
$$= e^{-\lambda t/2}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right)$$

$$B = A^*$$

$$A = ce^{i\delta}$$

$$x(t) = ce^{-\lambda t/2}\left(e^{i(\omega_1 t+\delta)} + e^{-i(\omega_1 t+\delta)}\right)$$

$$x(t) = 2ce^{-\lambda t/2}\left(\cos(\omega_1 t+\delta)\right)$$



Logarithmic Decrement

$$L.D. = \ln\frac{A_n}{A_{n+1}}$$

$$= \ln\frac{2ce^{-\lambda t_n/2}}{2ce^{-\lambda(t_n+T)/2}}$$

$$= \ln e^{\lambda T/2} = \frac{\lambda T}{2}$$

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$$L\cdot D := \frac{\lambda}{2}\cdot\frac{2\pi}{\omega_1} = \frac{\gamma}{m}\frac{\pi}{\sqrt{\omega^2-\frac{\lambda^2}{4}}} = \frac{\gamma}{m}\cdot\frac{\pi}{\sqrt{\omega^2-\frac{\gamma^2}{4m^2}}}$$

Case I Overdamped $\frac{\lambda^2}{4} > \omega^2$

$$\alpha = \frac{-\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4}-\omega^2}$$

$$= \frac{-\lambda}{2} \pm \Gamma$$

$$x(t) = e^{-\lambda t/2}\left(Ae^{\Gamma t} + Be^{-\Gamma t}\right)$$

$$= Ae^{-(\lambda/2-\Gamma)t} + B^{-(\lambda/2+\Gamma)t}$$

Case II Critically Damped $\lambda^2 = 4\omega^2$

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0$$

$$x(t) = A(t)\,e^{\alpha t}$$

$$\dot{x} = \dot{A}e^{\alpha t} + \alpha A e^{\alpha t}$$

$$\ddot{x} = \ddot{A}e^{\alpha t} + 2\alpha\dot{A}e^{\alpha t} + \alpha^2 A e^{\alpha t}$$

$$\ddot{A} + 2\alpha\dot{A} + \alpha^2 A + \lambda\dot{A} + \lambda\alpha A + \omega^2 A = 0$$

$$\ddot{A} + (2\alpha+\lambda)\dot{A} + (\alpha^2+\lambda\alpha+\omega^2)A = 0$$

$$\downarrow 0 \qquad\qquad \downarrow 0$$

$$\ddot{A} = 0$$

$$\dot{A} = c$$

$$A = ct + D$$

$$x(t) = (ct+D)\,e^{-\lambda t/2}$$

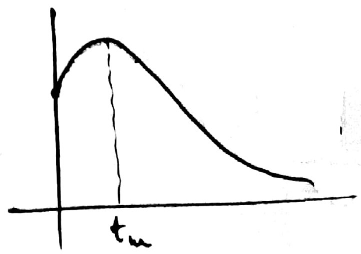
$$\dot{x} = ce^{-\lambda t/2} - (ct+D)\frac{\lambda}{2}e^{-\lambda t/2} = 0$$

$$c = (ct+D)\frac{\lambda}{2}$$

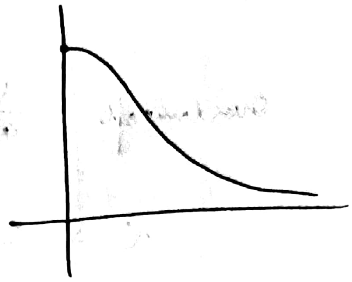
$$2c - D\lambda = c\lambda t$$

$$t_m = \frac{2c-D\lambda}{c\lambda} = \frac{2}{\lambda} - \frac{D}{c}$$

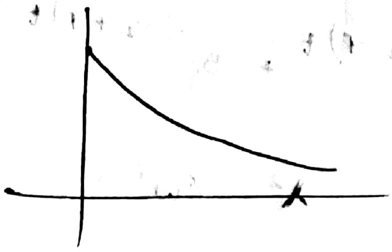
Case II a) $\frac{2}{\lambda} > \frac{D}{2}$



Case II b) $\frac{2}{\lambda} = \frac{D}{2}$



Case II c) $\frac{2}{\lambda} < \frac{D}{2}$



Forced & Damped Harmonic Oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) = F_0 \cos(\Omega t)$$

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = \frac{F_0}{m}\cos(\Omega t)$$

$$x(t) = A\cos(\Omega t) + B\sin(\Omega t)$$

$$A = R\cos\phi \quad , \quad B = R\sin\phi \quad , \quad A^2 + B^2 = R^2$$

$$\tan\phi = B/A$$

$$x(t) = R\cos(\Omega t - \phi)$$

$$\dot{x} = -\Omega A\sin\Omega t + \Omega B\cos\Omega t$$

$$\ddot{x} = -\Omega^2 A\cos\Omega t - \Omega^2 B\sin\Omega t$$

$$\left(-\Omega^2 A + \Omega\lambda B + \omega^2 A\right)\cos\Omega t + \left(-\Omega^2 B - \lambda\Omega A + \omega^2 B\right)\sin\Omega t = \frac{F_0}{m}\cos\Omega t$$

Equating coeff. of cos & sin,

$$\left[(\omega^2 - \Omega^2)A + \lambda\Omega B = \frac{F_0}{m}\right]$$

$$(\omega^2 - \Omega^2)B - \lambda\Omega A = 0$$

$$A = \frac{F_0}{m}\frac{\omega^2 - \Omega^2}{(\omega^2-\Omega^2)^2 + \lambda^2\Omega^2}$$

$$B = \frac{F_0}{m}\frac{\lambda\Omega}{(\omega^2-\Omega^2)^2 + \lambda^2\Omega^2}$$

$$R^2 = A^2 + B^2$$

$$R = \frac{F_0/m}{\sqrt{(\omega^2-\Omega^2)^2 + \lambda^2\Omega^2}}$$

$$\tan\phi = \frac{\lambda\Omega}{\omega^2 - \Omega^2}$$

~~Patru~~ Particular Integral $\Rightarrow$

$$x(t) = R\cos(\Omega t - \phi)$$

Complete Solution $\Rightarrow$

$$x(t) = x_h(t) + x_p(t)$$

$$= 2C_1 e^{-\lambda t/2}\cos(\omega_1 t + \delta) + R\cos(\Omega t - \phi)$$

$$x(t) = \underbrace{2ce^{-\lambda t/2}\cos(\omega_1 t + \delta)}_{\text{Transient term}} + R\cos(\Omega t - \phi)$$

large time $(t \to \infty) \Rightarrow$ Transient term vanishes

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$$\frac{dR}{d\Omega} = 0 \quad \Rightarrow \quad \frac{d}{d\Omega}\left(\frac{F_0/m}{\sqrt{(\omega^2-\Omega^2)^2 + \lambda^2\Omega^2}}\right)$$

$$= \frac{F_0}{m} \, \frac{-4\Omega(\omega^2-\Omega^2) + 2\lambda^2\Omega}{\left[(\omega^2-\Omega^2)^2 + \lambda^2\Omega^2\right]^{3/2}} = 0$$

$$2\lambda^2\Omega = 4\Omega(\omega^2-\Omega^2)$$

$$\frac{\lambda^2}{2} = \omega^2 - \Omega^2$$

$$\Omega = \sqrt{\omega^2 - \frac{\lambda^2}{2}}$$

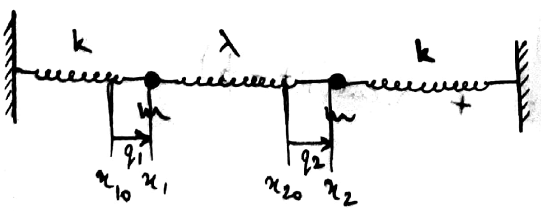
$$\Omega = \omega\left(1 - \frac{\lambda^2}{2\omega^2}\right)^{1/2}$$

$$\omega^2 >> \lambda^2$$

$$\Omega = \omega\left(1 - \frac{\lambda^2}{4\omega^2} - \cdots\right)$$

$$\Omega = \omega - \frac{\lambda^2}{4\omega}$$

Coupled Oscillation



$x_{10}, x_{20} \Rightarrow$ equilibrium position

$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k (x_1 - x_{10})^2 - \frac{1}{2} k (x_2 - x_{20})^2$$

$$- \frac{1}{2} \lambda \left[(x_2 - x_1) - (x_{20} - x_{10}) \right]^2$$

$$q_1 = x_1 - x_{10} \Rightarrow \dot{q}_1 = \dot{x}_1$$

$$q_2 = x_2 - x_{20} \Rightarrow \dot{q}_2 = \dot{x}_2$$

$$L = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2} k q_1^2 - \frac{1}{2} k q_2^2 - \frac{1}{2} \lambda (q_2 - q_1)^2$$

Equation of Motion $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

For 1, $\frac{d}{dt}(m\dot{q}_1) - \left[-kq_1 - \lambda (q_2 - q_1)(0-1) \right]$

$$m\ddot{q}_1 + kq_1 + \lambda q_1 - \lambda q_2 = 0$$

For 2, $\frac{d}{dt}(m\dot{q}_2) - \left[-kq_2 - \lambda (q_2 - q_1)(1-0) \right]$

$$m\ddot{q}_2 + kq_2 + \lambda q_2 - \lambda q_1 = 0$$

$$m\ddot{q}_1 + (k+\lambda) q_1 - \lambda q_2 = 0$$

$$m\ddot{q}_2 + (k+\lambda) q_2 - \lambda q_1 = 0$$

$$q_1 = A_1 e^{i\omega t}, \quad q_2 = A_2 e^{i\omega t}$$

$$\dot{q}_1 = i\omega A_1 e^{i\omega t} \quad \dot{q}_2 = i\omega A_2 e^{i\omega t}$$

$$\ddot{q}_1 = -\omega^2 A_1 e^{i\omega t} \quad \ddot{q}_2 = -\omega^2 A_2 e^{i\omega t}$$

$$-m\omega^2 A_1 + (k+\lambda) A_1 - \lambda A_2 = 0$$

$$-m\omega^2 A_2 + (k+\lambda) A_2 - \lambda A_1 = 0$$

$$(-m\omega^2 + k + \lambda)A_1 - \lambda A_2 = 0$$

$$(-m\omega^2 + k + \lambda)A_2 - \lambda A_1 = 0$$

$$\begin{bmatrix} -m\omega^2 + k + \lambda & -\lambda \\ -\lambda & -m\omega^2 + k + \lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

$$\det \begin{pmatrix} -m\omega^2 + k + \lambda & -\lambda \\ -\lambda & -m\omega^2 + k + \lambda \end{pmatrix} = 0$$

$$(-m\omega^2 + k + \lambda)^2 - \lambda^2 = 0$$

$$-m\omega^2 + k + \lambda = \pm\lambda$$

$$-m\omega^2 + k + \lambda = \lambda \qquad\qquad -m\omega^2 + k + \lambda = -\lambda$$

$$\omega_+^2 = \frac{k}{m} \qquad\qquad \omega_-^2 = \frac{k + 2\lambda}{m}$$

$\omega_+ , \omega_- \Rightarrow$ eigenfrequencies

$$m\omega_+^2 = k \Rightarrow \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

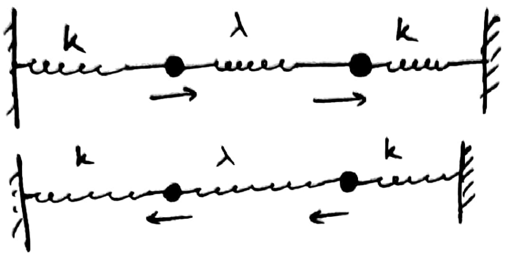
$$\left.\begin{aligned} \lambda A_1 - \lambda A_2 &= 0 \\ -\lambda A_1 + \lambda A_2 &= 0 \end{aligned}\right\} \begin{aligned} &A_1 = A_2 \\ &\text{In phase} \end{aligned}$$

$$m\omega_-^2 = k + 2\lambda \Rightarrow \begin{bmatrix} -\lambda & -\lambda \\ -\lambda & -\lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

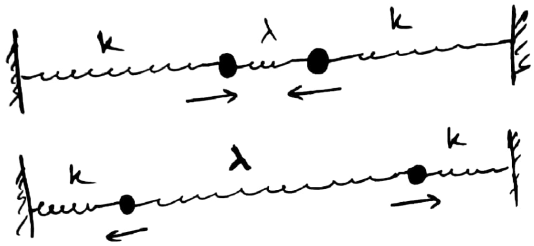
$$A_1 + A_2 = 0$$

$A_1 = -A_2$ Opposite phase

$$\left.\begin{aligned} A_1 &= A_2 \\ A_1 &= -A_2 \end{aligned}\right\} \text{eigenmodes}$$



$A_1 = A_2$ (ω_+)



$A_1 = -A_2$ (ω_-)

Normal Modes (eigenmodes)

$$Q = q_1 + q_2 \qquad q = q_1 - q_2$$

$$m\ddot{q}_1 + (k+\lambda) q_1 - \lambda q_2 = 0$$

$$m\ddot{q}_2 + (k+\lambda) q_2 - \lambda q_1 = 0$$

$$+ \Rightarrow \quad m\ddot{Q} + (k+\lambda) Q - \lambda Q = 0$$

$$m\ddot{Q} + kQ = 0$$

$$- \Rightarrow \quad m\ddot{q} + (k+\lambda) q + \lambda q = 0$$

$$m\ddot{Q} + kQ = 0$$

$$m\ddot{q} + (k+2\lambda) q = 0$$

$$\ddot{Q} + \omega_+^2 Q = 0$$

$$\ddot{q} + \omega_-^2 q = 0$$

$$Q(t) = A e^{i\omega_+ t} + B e^{-i\omega_+ t} = a \cos(\omega_+ t + \alpha)$$

$$q(t) = C e^{i\omega_- t} + D e^{-i\omega_- t} = b \cos(\omega_- t + \beta)$$