Tutorial 3: Series and Application of Residues

1. Verify each of the following Taylor expansions by finding a general formula for $f^{(j)}(z_0)$.
   (a) $\sinh z = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$ with $z_0 = 0$.
   (b) $\cosh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$ with $z_0 = 0$.
   (c) $\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$ with $z_0 = i$.
   (d) $z^3 = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$ with $z_0 = 1$.

2. Find Tayor series for following functions about $z = 0$ and state the radius of convergence.
   (a) $\sin z$
   (b) $\cos z$
   (c) $\ln(1+z)$
   (d) $\tan^{-1} z$
   (e) $(1+z)^p$

3. Find the Laurent Series for the function $1/(z+z^2)$ for each of the following domains:
   (a) $0 < |z| < 1$; about $z = 0$.
   (b) $1 < |z|$; about $z = 0$.
   (c) $0 < |z+1| < 1$; about $z = -1$;
   (d) $1 < |z+1|$; about $z = -1$;

4. Find Laurent series for
   (a) $\sin(2z)/z^3$ in $|z| > 0$;
   (b) $z^2 \cos(1/3z)$ in $|z| > 0$.

5. Prove that the Laurent series expansion of the function $f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$in $|z| > 0$ is given by $\sum_{k=-\infty}^{\infty} J_k(\lambda) z^k$,where $J_k(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\theta - \lambda \sin\theta)\, d\theta$. The functions $J_k(\lambda)$ are known as *Bessel functions* of the first kind.

6. Determine all the isolated singularities of each of the following functions and compute the residue at each singularity
   (a) $e^{3z}/(z-2)$
   (b) $(z+1)/(z^2 - 3z + 2)$
   (c) $(\cos z)/z^2$
   (d) $\left(\frac{z-1}{z+1}\right)^3$
   (e) $\sin(1/3z)$
   (f) $(z-1)/\sin z$

7. Evaluate each of the following integrals by means of the Cauchy residue theorem.
   (a) $\int_C \frac{\sin z}{z^2-4} dz$ where $C : |z| = 5$.

(b) $\int_C \frac{e^z}{z(z-2)^3} dz$ where $C: |z| = 3$.

(c) $\int_C \tan z dz$ where $C: |z| = 2\pi$.

(d) $\int_C \frac{1}{z^2 \sin z} dz$ where $C: |z| = 1$.

8. Let f have an isolated singularity at z_0 (f analytic in punctured nbd of z_0). Show that the residue of the derivative f' is equal to zero.

9. Using method of residues, verify each of the following.

(a) $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$.

(b) $\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$.

(c) $\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{2\pi}{ab}$

(d) $\int_0^{2\pi} (\cos\theta)^{2n} d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$

10. Verify the following integral formulae with the help of residues.

(a) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$.

(b) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{\pi}{54}$.

(c) $\int_0^{\infty} \frac{x^2+1}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$.

11. Show that

$$pv \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx = \sec 1$$

by integrating $e^{2z}/\cosh(\pi z)$ around a rectangle with vertices at $z = \pm R, \pm R + i$ and then taking a limit $R \to \infty$.

12. Show that

$$\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$

by integrating $1/(z^3+1)$ around the boundary of the circular sector $S: \left\{z = re^{i\theta} : 0 \le \theta \le 2\pi/3,\ 0 \le r \le R\right\}$ and then letting $R \to \infty$.

13. Using the method of residues, verify:

(a) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1} dx = \frac{\pi}{e^2}$.

(b) $\mathrm{pv}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10} dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$.

14. Given that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$, integrate e^{iz^2} around the boundary of the circular sector $S: \left\{z = re^{i\theta} : 0 \le \theta \le \pi/4,\ 0 \le r \le R\right\}$ and letting $R \to \infty$, prove that

$$\int_0^{\infty} e^{ix^2} dx = \frac{\sqrt{2\pi}}{4}(1+i).$$

15. Using the technique of residues, verify:

(a) $\int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1} dx = \pi i e^{-2i}$;

(b) $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = \pi i\left(e^{2i} - e^{i}\right)$.

16. Compute pv $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x-1} dx$ for $0 < a < 1$. (Use rectangular contour).