Partial Differential Eqn

Laplace Eqn

$$\nabla^2 \phi = 0$$

Wave Eqn

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

← speed of wave

Heat / Diffusion Eqn

$$k\nabla^2 \phi = \frac{\partial \phi}{\partial t} \qquad k \Rightarrow \text{diffusion constant}$$

Laplace Eqn

1) Electrostatics

$$\nabla \times E = 0, \quad \nabla \cdot E = 0$$

$$\nabla^2 \phi = 0 \qquad (\rho = 0)$$

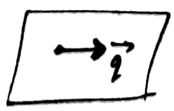
2) Gravitation

3) Steady State Heat Flow

$$\nabla^2 T = 0$$

4) Fluid Flow

$$\rho(x, y, z, t), \quad \vec{q}(x, y, z, t)$$



$$\left.\begin{aligned} \nabla \cdot \vec{q} &= 0 \\ \nabla \times \vec{q} &= 0 \end{aligned}\right\}$$

$$\vec{q} = \nabla \psi$$

$$\nabla^2 \psi = 0$$

5) Magnetostatics

Linear magnetic material

$$\nabla \times H = J_{free}$$

$$B = \mu H \qquad (J_{free} = 0)$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot (\mu H) = 0$$

$$\Rightarrow \nabla \cdot H = 0$$

$$H = -\nabla \phi_M$$

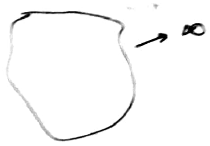
$$\nabla^2 \phi_M = 0$$

Q

$$\phi = \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r'$$

$$\phi_{Mono} = \frac{Q_{Tot.}}{\vec{r}_0} \qquad \phi_{dipole} = \frac{\vec{p}\cdot\vec{r}}{|r|^3}$$

Q

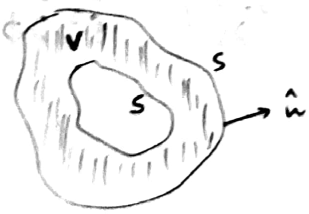


$$\phi = \phi_0$$

$$\phi = ax + b$$

$$\phi = ay + b$$

Q



Dirichlet Boundary Conditions

$$\nabla^2\phi = 0$$

$$\phi(\vec{r}) = \text{given}, \quad \vec{r} \in S$$

~~Interior~~ Exterior

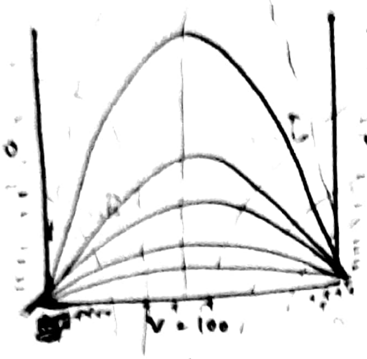
unique solⁿ exists if $u \to 1/r$ as $r \to \infty$ & $|\nabla\phi| \to 1/r^2$

Neumann Boundary Conditions

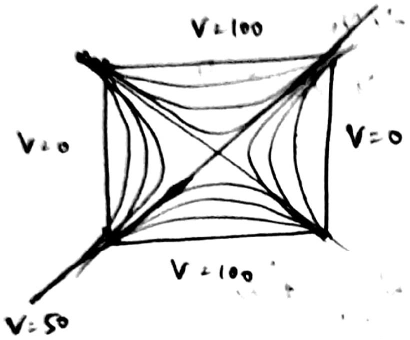
$$\nabla^2\phi = 0 \qquad \frac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{n} = \text{given}$$

Interior $\Rightarrow$ solⁿ exists if $\frac{\partial\phi}{\partial n}$ satisfies Gauss Law

Q



Q



Cartesian Co-ordinate System

$$\nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Co-ordinate System

$$\nabla^2 f(\rho, \phi, z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Co-ordinate System

$$\nabla^2 f(r, \theta, \phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Laplace Eq^n : Variable separation

2-D Region D

$$\nabla^2\phi = 0 \Rightarrow \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V = 0$$

$$V(x,y) = X(x)\cdot Y(y) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} \quad \forall x, y \in D$$

$$\frac{1}{\phi}\nabla^2\phi = \frac{1}{XY}\left[\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2}\right] = 0$$

$$\frac{d^2}{dx^2}f + w^2 f = 0$$

$$f(x) = A\sin Wx + B\cos Wx$$
$$= Ae^{iWx} + Be^{-iWx}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$
$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda$$

$$X'' = +\lambda X$$
$$Y'' = -\lambda Y$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}V$$

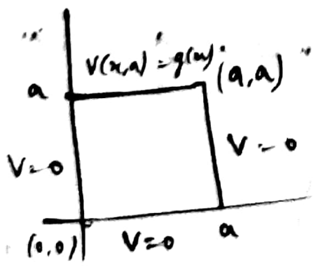
$$V(\rho,\phi) = R(\rho)\cdot P(\phi)$$

$$\frac{1}{V}\nabla^2 V = \frac{1}{\rho R}\left(\frac{d}{d\rho}\rho\right)\frac{d}{d\rho}R + \frac{1}{\rho^2}\underbrace{\left(\frac{1}{P}\frac{d^2P}{d\phi^2}\right)}_{\lambda} = 0$$

$$P''(\phi) = \lambda P(\phi)$$

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} + \frac{\lambda}{\rho^2}R = 0$$

Q



$$V(x,0) = 0 \quad \forall x \in [0,a]$$
$$V(0,y) = 0 \quad \forall x \in [0,a]$$
$$V(a,y) = 0 \quad \forall x \in [0,a]$$

$$V(x,0) = X(x)\,Y(0) \quad \forall x$$
$$Y(0) = 0$$
$$X(x) = 0$$

$$X(0) = X(a) = 0$$
$$X(0)\cdot Y(y) = 0 \quad \forall y$$

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$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

$$X(0) = 0 , \quad X(a) = 0$$

$$A + B = 0 \qquad Ae^{\sqrt{\lambda}a} + Be^{-\sqrt{\lambda}a} = 0$$

$$A\left(e^{\sqrt{\lambda}a} - e^{-\sqrt{\lambda}a}\right) = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2 i^2$$

$$n \in Z^+$$

$$\cancel{X(x) = A \sin\left(\frac{n\pi a}{x}\right)}$$

$$X(x) = A' \sin\left(\frac{n\pi x}{a}\right)$$

$$y'' = -\lambda y$$

$$y'' = \left(\frac{n\pi}{a}\right)^2 y$$

$$y = Ce^{\frac{n\pi}{a}y} + De^{-\frac{n\pi}{a}y}$$

$$y = C' \sinh\left(\frac{n\pi}{a}y\right)$$

$$V_n(x,y) = A_n \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)$$

$$V(x,a) = g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) . \sinh(n\pi)$$

$$\int_0^a g(x) \sin\left(\frac{m\pi x}{a}\right) dx = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$\int_0^a g(x) \sin\left(\frac{m\pi x}{a}\right) dx = A_m \sinh(n\pi) \cdot \frac{a}{2} \quad (m = n)$$

$$A_m = \frac{2}{a \sinh(n\pi)} \int_0^a g(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

Q $a = 1$, $g(x) = V_0$

$$A_m = \frac{2V_0}{\sinh(n\pi)} \int_0^a \sin(m\pi x)\, dx$$

$$= \frac{2V_0}{\sinh(n\pi)} \int_0^1 \sin(m\pi x)\, dx$$

$$= \frac{2V_0}{\sinh(n\pi)} \left[\frac{\cos(m\pi x)}{m\pi}\right]_1^0$$

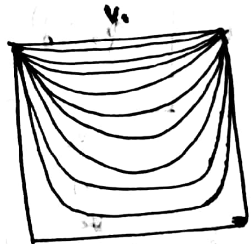
$$A_m = \frac{2V_0}{\sinh(n\pi)} \left(\frac{1}{m\pi} - \frac{\cos(m\pi)}{m\pi}\right)$$

$$A_m = \frac{2V_0}{\sinh(n\pi)\cdot m\pi} \left[1 - (-1)^n\right]$$

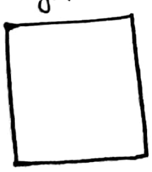
$$A_m = \frac{4V_0}{m\pi \cdot \sinh(n\pi)} \quad \text{if } m \text{ is odd}$$

$$= 0 \quad \text{if } m \text{ is even}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh(n\pi)}$$



Q



$a = 1$

$$A_m = \frac{2}{a \sinh(m\pi)} \int_0^1 x \sin(m\pi x)\, dx$$

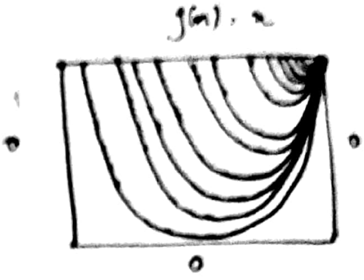
$$A_m = \frac{1}{a \sinh(m\pi)} \int_0^1 \sin(m\pi x)\, dx$$

$$= \frac{1}{a \sinh(m\pi)} \left[\frac{\cos m\pi x}{m\pi}\right]_1^0$$

$$= \frac{1}{\sinh(m\pi)} \left[\frac{1}{m\pi} - \frac{\cos m\pi}{m\pi}\right]$$

$$\frac{1}{\sinh(n\pi)\cdot(n\pi)}\left(1-(-1)^{n}\right)$$

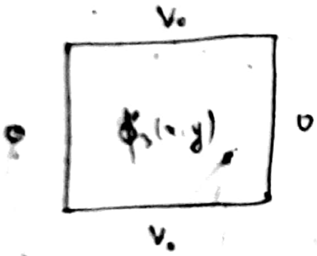
$$\frac{2}{n\pi\,\sinh(n\pi)}(-1)^{n}$$



Q

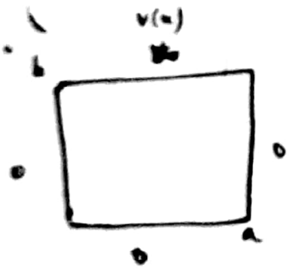
$$\phi_2(x,y) = \phi_1(x, 1-y)$$

Q



$$\phi_3(x,y) = \phi_1(x,y) + \phi_2(x,y)$$

Q



$$A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a g(x)\cdot\sin\left(\frac{n\pi x}{a}\right)dx$$

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Variable Separation in Cartesian Co-ordinates (2-D)

$$D = \{ (x,y) / 0 \le x \le a , \ 0 \le y \le a \}$$

$\nabla^2 V = 0$ in D

$V = 0$ on AB, BC, AD

$V(x,a) = g(x)$

$V(x,y) = X(x) \cdot Y(y)$

$$X'' = -\alpha^2 X , \quad X(0) = X(a) = 0$$

$$Y'' = \alpha^2 Y , \quad Y(0) = 0$$

$$X = A \sin \alpha x + B \cos \alpha x$$

$$X_m = A_m \sin\left(\frac{m\pi x}{a}\right)$$

$$\langle f, g \rangle = \int_a^b \overline{f(x)} \cdot g(x) \, dx$$

$$\langle X_m , X_n \rangle = \int_a^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = 0 \quad (m \ne n)$$
$$= \frac{a}{2} \quad (m = n)$$

$$V = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) \cdot \sinh\left(\frac{m\pi y}{a}\right)$$

$$\nabla^2 V = \sum_{m=1}^{\infty} A_m \nabla^2 V_m = 0$$

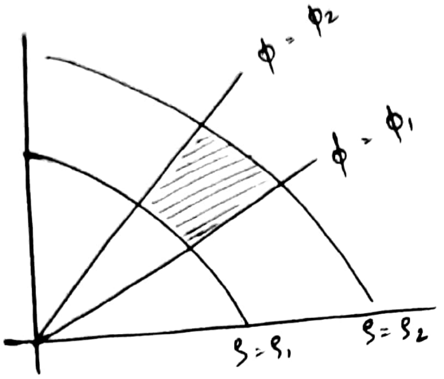
$$V(x,a) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) \cdot \sinh(m\pi)$$

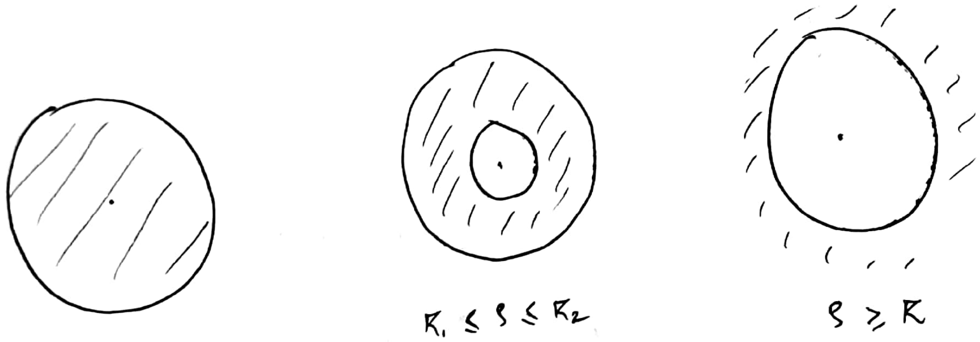
$$g(x) = \sum_m A_m \sinh(m\pi) \cdot \sin\left(\frac{m\pi x}{a}\right)$$

Multiplying $\sin\left(\frac{n\pi x}{a}\right)$ on both sides & integrating

$$\int_0^a g(x) \cdot \sin\left(\frac{n\pi x}{a}\right) = A_n \sinh(n\pi) \cdot \frac{a}{2} \quad (m = n)$$

Plane - Polar Co-ordinates (ρ, ϕ)





$\rho < R$ $\qquad$ $R_1 \leq \rho \leq R_2$ $\qquad$ $\rho \geq R$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}V\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}V = 0$$

$$V(\rho, \phi) = R(\rho) . P(\phi)$$

$$\frac{\nabla^2 V}{V} = \frac{1}{R \cdot P}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}R\right)P + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}RP\right] = 0$$

$$= \frac{1}{R}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}R\right)\right) + \frac{1}{P}\frac{1}{\rho^2}\frac{\partial^2 P}{\partial\phi^2} = 0$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - m^2 R = 0$$

$$\frac{d^2 P}{d\phi^2} = -m^2 P$$

$$P = A\sin(m\phi) + B\cos(m\phi)$$

$$R = C\rho^m + D\rho^{-m}$$

Boundary conditions $\Rightarrow$ $V(\rho, \phi + 2\pi) = V(\rho, \phi)$ $\forall \rho, \phi$

$\Rightarrow$ m must be integer $m = 0, 1, 2, \ldots$

$m = 0 \Rightarrow$ $P = A\phi + B$

$R = C + D \ln \rho$

(Boundary Condition) B.C : $|V(\rho = 0, \phi)| < \infty$

B.C : $V(\rho \to \infty, \phi) = 0$

$\cancel{V(\rho, \phi) = (C_0 + D_0 \ln \rho)}$

$$V(\rho, \phi) = (C_0 + D_0 \ln \rho) \cdot (A_0 \phi + B_0) + \sum_{m=1}^{\infty} \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right) \cdot \left(C_m \rho^m + D_m \rho^{-m}\right)$$

$$\int_0^{2\pi} \sin(m\phi) \sin(n\phi)\, d\phi = \delta_{m,n} \pi$$

$$\int_0^{2\pi} \cos(m\phi) \cdot \cos(n\phi)\, d\phi = \delta_{m,n} \pi$$

$$\int_0^{2\pi} \cos(m\phi) \cdot \sin(m\phi)\, d\phi = 0$$

B.C $\because$ $V(\rho, \phi + 2\pi) = V(\rho, \phi)$

$$\therefore \quad V(\rho, \phi) = C_0 + D_0 \ln \rho + \sum_{m=1}^{\infty} \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right) \cdot \left(C_m \rho^m + D_m \rho^{-m}\right)$$

B.C. $\because$ $|V(\rho, \phi)| < \infty$ as $\rho \to 0$, $D_0 = 0$

$D_m = 0 \quad \forall m$

B.C. $\because$ $V(\rho, \phi) \to 0$ as $\rho \to \infty$ $\therefore$ $D_0 = 0$

$C_0 = 0$

$C_m = 0 \quad \forall m$

Example $\quad \rho > R$, $V(R, \phi) = V_0$

$V(\rho, \phi)$ must be independent of ϕ.

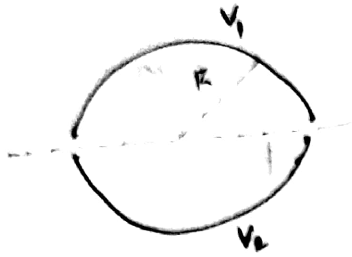
$$V(\rho, \phi) = C_0 + D_0 \ln \rho$$

$$C_0 + D_0 \ln R = V_0$$

$$D_0 = \frac{V_0 - C_0}{\ln R}$$

$$V(\rho,\phi) = C_0 + \frac{V_0 - C_0}{\ln R} \ln \rho$$

Example



$D_0 = 0$

$D_m = 0 \ \forall m$

$$V(\rho,\phi) = C_0 + \sum_{m=1}^{\infty} \rho^m \left(A_m \sin(m\phi) + B_m \cos\ldots\right.$$

$$V(R,\phi) = V_1 \quad \text{if } 0 \le \phi \le \pi$$
$$ = V_2 \quad \text{if } -\pi < \phi < 0$$

Fourier trick $\rightarrow$ $\displaystyle\int_0^{2\pi} \sin(n\phi)\, V(R,\phi)\, d\phi = \sum_{m=1}^{\infty} R^m A_m \pi \delta_{m,n}$

$$\int_0^{2\pi} \sin(n\phi)\, V(R,\phi)\, d\phi = R^n A_n \pi$$

$$A_n = \frac{1}{\pi R^n} \int_0^{2\pi} \sin(n\phi)\, V(R,\phi)\, d\phi$$

$$B_n = \frac{1}{\pi R^n} \int_0^{2\pi} \cos(n\phi)\, V(R,\phi)\, d\phi$$

$$C_0 = \frac{1}{2\pi} \int_0^{2\pi} V(R,\phi)\, d\phi$$

$$C_0 = \frac{1}{2\pi}\left[\int_0^{\pi} V_1\, d\phi + \int_{\pi}^{2\pi} V_2\, d\phi\right] = \frac{V_1 + V_2}{2}$$

$$A_n = \frac{1}{\pi R^n}\left[V_1 \int_0^{\pi} \sin(n\phi)\, d\phi + V_2 \int_{\pi}^{2\pi} \sin(n\phi)\, d\phi\right] = \cancel{\frac{1}{\pi R^n}\left[\frac{V_1 \cdot 2}{n} + \frac{V_2(-2)}{n}\right]}$$

$$= \cancel{\frac{2(V_1 - V_2)}{\pi n}}$$

$$= \frac{1}{\pi R^n}\left[\frac{V_1}{n}\left[\cos n\phi\right]_{\pi}^{0} + V_2\left[\cos n\phi\right]_{2\pi}^{\pi}\right]$$

$$= \frac{1}{\pi R^n}\left[\frac{V_1}{n}\left(1 - \cos n\pi\right) + \frac{V_2}{n}\left(\cos n\pi - 1\right)\right]$$

$$A_n = \frac{1}{\pi R^n}\left[\frac{V_1 - V_2}{n} + -\frac{V_1}{n}(-1)^n + \frac{V_2}{n}(-1)^n\right]$$

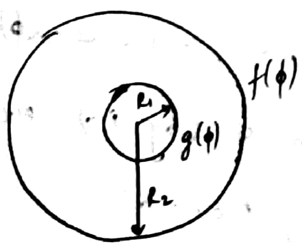
$$A_n = \frac{(V_1 - V_2)2}{n\pi R^n} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

$$B_n = 0 \quad \forall n$$

$$V(\rho, \phi) = \frac{V_1 + V_2}{2} + \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \frac{\rho^m}{R^m} \cdot \frac{2}{m}\left(\frac{V_1 - V_2}{\pi}\right) \sin(m\phi)$$

Example



$$V(R_1, \phi) = g(\phi)$$
$$V(R_2, \phi) = f(\phi)$$

$$V(\rho, \phi) = C_0 + D_0 \ln \rho + \sum_{m=1}^{\infty}\left(A_m \sin(m\phi) + B_m \cos(m\phi)\right)\left(C_m \rho^m + D_m \rho^{-m}\right)$$

$$A_m \cdot \pi\left(C_m R_1^m + D_m R_1^{-m}\right) = \int_0^{2\pi} \sin(m\phi)\, V(R_1, \phi)\, d\phi$$

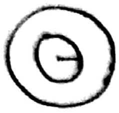
$$A_m \cdot \pi\left(C_m R_2^m + D_m R_2^{-m}\right) = \int_0^{2\pi} \sin(m\phi) \cdot V(R_2, \phi)\, d\phi$$

$$\int_0^{2\pi} V(R_1, \phi)\, d\phi = \left(C_0 + D_0 \ln R_1\right)\int_0^{2\pi} d\phi$$

$$\int_0^{2\pi} V(R_2, \phi)\, d\phi = \left(C_0 + D_0 \ln R_2\right)\int_0^{2\pi} d\phi$$

Plane Polar co-ordinates

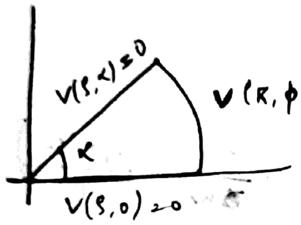
$$V(\rho,\phi) = (C_0 + D_0 \ln \rho)(A_0\phi + B_0) + \left(A_m \sin(m\phi) + B_0 \cos(m\phi)\right)\cdot\left(C_m \rho^{m} + D_m \rho^{-m}\right)$$



$m = 1, 2, 3, \ldots$

$$V(\rho,\phi) = C_0 + D_0 \ln\rho + \sum_{m=1}^{\infty} \left(C_m \rho^{m} + D_m \rho^{-m}\right)\cdot\left(A_m \sin(m\phi) + B_m \cos(m\phi)\right)$$

Example



$$V(\rho,\phi) = R(\rho)\cdot P(\phi)$$

$P(\phi) \Rightarrow$

$P(0) = P(\alpha) = 0$

$P(\phi) = A_0\phi + B_0 \qquad m = 0$

$\quad = A_m \sin(m\phi) + B_m \cos(m\phi) \qquad m \neq 0$

$A_0 = 0$, $B_0 = 0$ (by $P(0) = P(\alpha) = 0$)

$B_m = 0$ (by $P(0) = 0$)

$P(\phi) = A_m \sin(m\phi)$

$P(\alpha) = 0 = \cancel{A_m \sin(m\phi)}$ $A_m \sin(m\alpha)$

$m\alpha = n\pi$

$m = \dfrac{n\pi}{\alpha}$ $\quad n$ is integer

$R(\rho) \Rightarrow \quad R(\rho) = C_m \rho^{m} + D_m \rho^{-m}$

$$D_m = 0$$

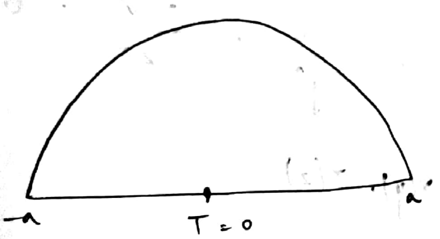
$$V(\rho,\phi) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\alpha}\phi\right) \cdot \rho^{\frac{n\pi}{\alpha}}$$

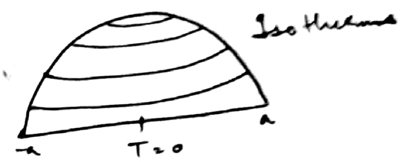
$$V(R,\phi) = f(\phi) \Rightarrow \sum_{n=1}^{\infty} A_n R^{\frac{n\pi}{\alpha}} \cdot \sin\left(\frac{n\pi}{\alpha}\phi\right) = f(\phi)$$

$$A_n = \frac{2}{\alpha \cdot R^{n\pi/\alpha}} \int_0^{\alpha} f(\phi) \sin\left(\frac{n\pi}{\alpha}\phi\right) d\phi$$

Example

$T = k\phi(\pi - \phi)$ on the surface





$$A_n = \frac{2k}{\alpha R^{n\pi/\alpha}} \int_0^{\alpha} \phi(\pi - \phi) \sin\left(\frac{n\pi}{\alpha}\phi\right) d\phi \qquad \alpha = \pi$$

$$= \cancel{\frac{2k}{\alpha R^{n\pi/\alpha}} \int_0^{\alpha}} = \frac{2k}{\pi R^n} \int_0^{\pi} \phi(\pi - \phi) \cdot \sin(n\phi)\, d\phi$$

$$= \frac{2k}{\pi R^n}\left[-\frac{\phi \cos(n\phi)}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{\cos(n\phi)}{n} d\phi - \left(\frac{-\phi^2 \cos(n\phi)}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{2\phi \cos(n\phi)}{n} \right)\right]$$

$$= \frac{2k}{\pi R^n}\left[\frac{-\phi(-1)^n}{n} + \frac{\phi}{n} + \frac{\sin(n\phi)}{n^2}\Big|_0^{\pi} - \left(\frac{-\phi^2 \cos(n\phi)}{n}\Big|_0^{\pi} + \frac{2}{n}\left(\frac{\phi \sin(n\phi)}{n^2}\Big|_0^{\pi} - \frac{2}{n}\int \frac{\sin(n\phi)}{n^2} d\phi \right)\right)\right]$$

$$= \frac{2k}{\pi R^n}\left[\frac{-\phi(-1)^n}{n} + \cancel{\frac{\phi}{n}} - \left(\frac{-\phi^2(-1)^n}{n} + \cancel{\frac{\phi^2}{n}} + \frac{2}{n}\left[\frac{-2}{n^2} \cos n\phi \Big|_0^{\pi} \right]\right)\right]$$

$$= \frac{2k}{\pi R^n}\left(\cancel{\frac{-\phi(-1)^n}{n}} + \cancel{\frac{\phi}{n}} + \cancel{\frac{\phi^2(-1)^n}{n}} - \cancel{\frac{\phi^2}{n}} + \frac{4}{n^3}\left((-1)^n - 1\right)\right)$$

$$= \cancel{\frac{2k}{\pi R^n}\left[\frac{\phi}{n} - \frac{\phi^2}{n} + \frac{4}{n^5}\left((-1)^n - 1\right)\right]}$$

$$= \frac{2k}{\pi R^n} \cdot \frac{4}{n^3}\left((-1)^n - 1\right)$$

$$= \frac{8k}{\pi R^n} \frac{1}{n^3}\left((-1)^n - 1\right)$$

$$T(s,\phi) = \frac{8k}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}\left(\frac{s}{R}\right)^{2n+1} \cdot \sin((2n+1)\phi)$$

Cartesian in 3-D

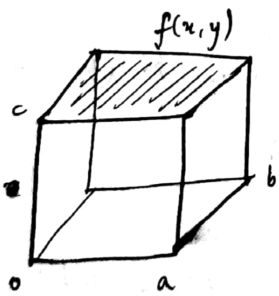
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V = X(x) \cdot Y(y) \cdot Z(z)$$

$$X''(x) = -k_x^2 \cdot X = -\sin(k_x x), \cos(k_x x)$$

$$Y''(y) = -k_y^2 \cdot Y = \quad " \quad "$$

$$Z'' = \underbrace{(k_x^2 + k_y^2)}_{k_z^2} Z = \sinh(k_z z), \cosh(k_z z)$$



$$V(x,y,c) = f(x,y)$$

$$V(0,y,z) = 0$$

$$V(a,y,z) = 0$$

$$X(0) = X(a) = 0 \Rightarrow X = A \sin\left(\frac{m\pi}{a} x\right)$$

$$Y(0) = Y(b) = 0 \Rightarrow Y = A \sin\left(\frac{n\pi}{b} y\right)$$

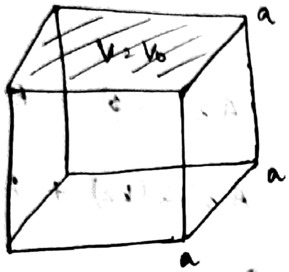
$$Z = A \sinh\left(\pi\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} z\right)$$

$$V(x,y,z) = \sum_{m=1,n=1}^{\infty\ \infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\pi\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2} z\right)$$

$$V(x,y,c) = f(x,y)$$

$$A_{mn} = \frac{4}{ab} \cdot \frac{1}{\sinh\left(\pi\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2} c\right)} \int_0^a\int_0^b f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx\, dy$$

Example



$$V(0,y,z) = V(a,y,z) = V(x,0,z) = V(x,a,z) = V(x,y,0) = 0$$

$$V(x,y,a) = V_0$$

$$V(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\frac{\pi}{a}(m^2+n^2)^{1/2} z\right)$$

$$A_{mn} = \frac{4}{a^2} \cdot \frac{1}{\sinh\left(\pi(m^2+n^2)^{1/2}\right)} \int_0^a\int_0^a dx\, dy\, V_0 \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{a}\right)$$

$$A_{mn} = \frac{4}{a^2} \cdot \frac{1}{\sinh\left(\pi(m^2+n^2)^{1/2}\right)} \cdot \frac{\cos\frac{m\pi x}{a}\Big|_0^a \cdot \cos\frac{n\pi y}{a}\Big|_0^a \, a^2}{mn\pi^2}$$

$$= \frac{4}{\cancel{a^2} \sinh\left(\pi(m^2+n^2)^{1/2}\right)} \cdot \frac{((-1)^m-1)\cdot((-1)^n-1)\,\cancel{a^2}}{mn\pi^2}$$

$$A_{mn} = \frac{16 V_0}{\pi^2 mn \sinh\left(\pi(m^2+n^2)^{1/2}\right)} \quad \text{if } m \,\&\, n \text{ are odd}$$

$$= 0 \quad \text{otherwise}$$

Cylindrical Co-ordinates

(ρ, ϕ, z)

$0 \le \phi \le 2\pi$

$R_1 \le \rho \le R_2$

$z_1 \le z \le z_2$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right)V + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}V + \frac{\partial^2 V}{\partial z^2} = 0$$

$V = R(\rho)\, T(\phi)\, Q(z)$

$T'' = -m^2 T \Rightarrow T = A\sin(m\phi) + B\cos(m\phi)$ $m = 0, 1, 2$

$Q'' = k^2 Q \Rightarrow Q = A\sinh(kz) + B\cosh(kz)$

$\rho^2 R''(\rho) + \rho R'(\rho) + (k^2\rho^2 - m^2) R(\rho) = 0$ (Bessel DE)

$x = k\rho$

$x^2 R''(x) + x R'(x) + (x^2 - m^2) R(x) = 0$

Series solⁿ by Frobenius method

$J_m(x) = J_{-m}(x) \cdot (-1)^m$

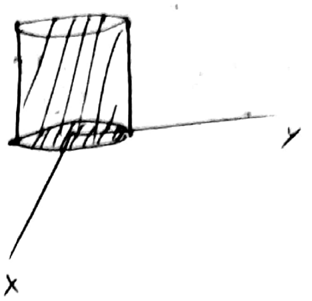
$N_m(x) \Rightarrow$ Neumann

Orthogonality $$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) \cdot J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right) \cdot \rho\, d\rho = 0$$

Normalization $$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2 d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu m})\right]^2$$

$$\Psi_{k,m}(\rho, \phi, z) = (A \sin(m\phi) + B\cos(m\phi)) \cdot (C J_m(k\rho) + D N_m(k\rho)) \cdot (E e^{kz} + F e^{-kz})$$

Example : $\nabla^2 \Psi(\rho, \phi, z) = 0$



$\Psi(a, \phi, z) = 0 \Rightarrow k = \frac{X_{m,n}}{a}$

$\Psi(\rho, \phi, z \to \infty) = 0$

$\Psi(\rho, \phi, z = 0) = \Psi_0(\rho, \phi)$

$\Psi(\rho \to 0, \phi, z) < \infty$

$\cancel{D = 0 \ (\because \Psi(a, \phi, z) = 0)}$

$D = 0 \quad (\because \Psi(a, \phi, z) = 0)$

$E = 0 \quad (\because \Psi(\rho, \phi, z \to \infty) = 0)$

$$\Psi(\rho, \phi, z) = \sum_{m,n} e^{\frac{-X_{m,n} \cdot z}{a}} \cdot J_m\left(\frac{X_{m,n} \cdot \rho}{a}\right)\left(A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)\right)$$

$$\Psi_0(\rho, \phi) = \sum_{m,n} J_m\left(\frac{X_{m,n} \cdot \rho}{a}\right)\left(A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)\right)$$

$$\int_0^a \int_0^{2\pi} \Psi_0(\rho, \phi) \cdot J_p\left(\frac{X_{p,q} \cdot \rho}{a}\right) \cdot \sin(p\phi) \cdot \rho \, d\rho \, d\phi$$

$$= \int \sum J_m\left(\frac{X_{m,n} \cdot \rho}{a}\right) \cdot J_p\left(\frac{X_{p,q} \cdot \rho}{a}\right) \rho \, d\rho \left(A_{mn} \int_0^{2\pi} \sin(m\phi) \sin(p\phi) \, d\phi + B_{mn} \int_0^{2\pi} \cos(m\phi) \sin(p\phi) \, d\phi \right)$$

$$= \int \sum J_m\left(\frac{X_{m,n} \cdot \rho}{a}\right) \cdot J_p\left(\frac{X_{p,q} \cdot \rho}{a}\right) \cdot \rho \, d\rho \left(A_{mn} \cdot \delta_{m,p} \cdot \pi + 0 \right)$$

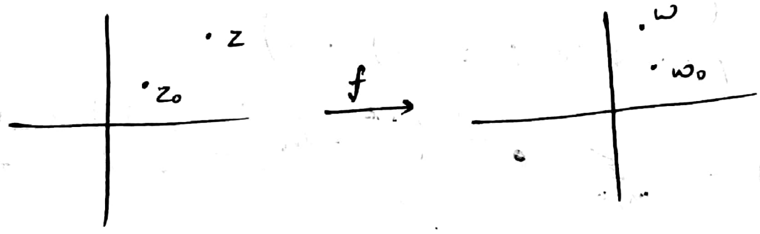
$$= \int \sum J_p\left(\frac{X_{p,q} \cdot \rho}{a}\right) \cdot J_p\left(\frac{X_{p,q} \cdot \rho}{a}\right) \cdot \rho \, d\rho \, (A_{mn} \cdot \pi)$$

$$= \frac{a^2}{2}\left[J_{p+1}(x_{p,q}) \right]^2 \cdot A_{p,q} \cdot \pi \qquad p \neq 0$$

$$\Rightarrow \quad A_{p,q} = \frac{2}{a^2 \pi \left[J_{p+1}(x_{p,q}) \right]^2} \int_0^a \int_0^{2\pi} \cancel{\psi(\rho,\phi)}\; \psi_0(\rho,\phi) \cdot J_p\left(\frac{x_{p,q} \cdot \rho}{a} \right) \cdot \sin(p\phi) \cdot \rho \, d\rho \, d\phi$$

Conformal Mapping :

A transformation $w = f(t)$ is conformal at z_0 if f is analytic at z_0 & $f'(z_0) \neq 0$



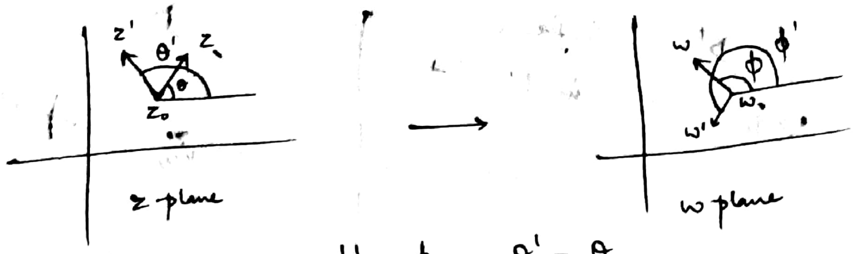
z-plane w-plane

$$w - w_0 = f'(z_0) \cdot (z - z_0)$$

$$= (u_x + i v_x)\left[(x - x_0) + i(y - y_0) \right]$$

$$\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix} = \underbrace{\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}}_{|f'(z_0)| = u_x^2 + v_x^2 \neq 0} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Orthogonal



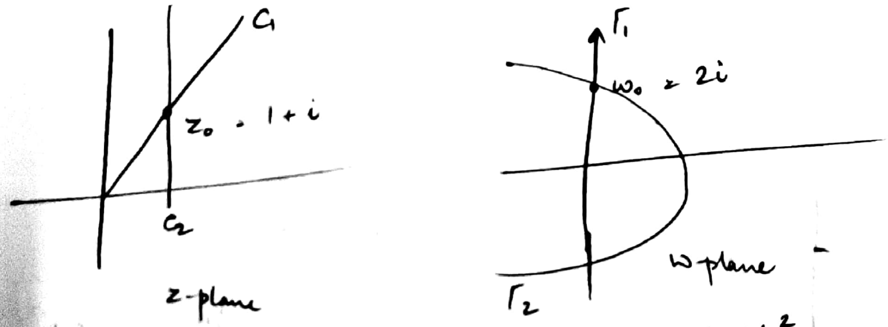
$$\phi' - \phi = \theta' - \theta$$

$$|w - w_0| = |f'(z_0)|\,|z - z_0|$$

Properties $\Rightarrow$ Angle preserving
Simple scaling
Inverse exists

Example $\quad f(z) = z^2 = x^2 - y^2 + 2ixy$

$$u = x^2 - y^2 \qquad v = 2xy$$



$C_1 : x = y = t \qquad z_0 = C_1(1)$

$C_2 : x = 1,\ y = t$

$\Gamma_1 : 2t^2$

$\Gamma_2 : \underbrace{1 - t^2}_{u} + \underbrace{2t}_{v} i$

$$\frac{d\Gamma_2}{dt} = -2t + 2i$$

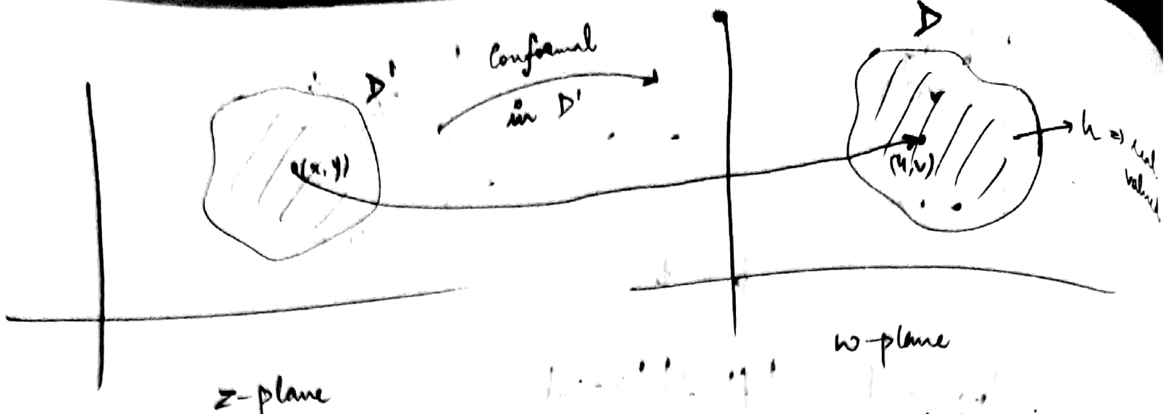
$z(t), \quad z_0 = z(t_0)$

$$\frac{dz}{dt}(t_0)$$

$w(t) = f(z(t)), \quad \frac{dw}{dt}(t_0)$

$$\frac{dw}{dt} = \frac{df}{dz}(z_0) \cdot \frac{dz}{dt}(t_0)$$

$$\left|\frac{dw}{dt}\right| = \left|\frac{df}{dz}\right| \cdot \left|\frac{dz}{dt}\right| \qquad \operatorname{Arg}\left(\frac{dw}{dt}\right) = \operatorname{Arg}\left(\frac{df}{dz}\right) + \operatorname{Arg}\left(\frac{dz}{dt}\right)$$



Let h be harmonic

$$\nabla^2 h(u,v) = 0$$

$$\Rightarrow \frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} = 0$$

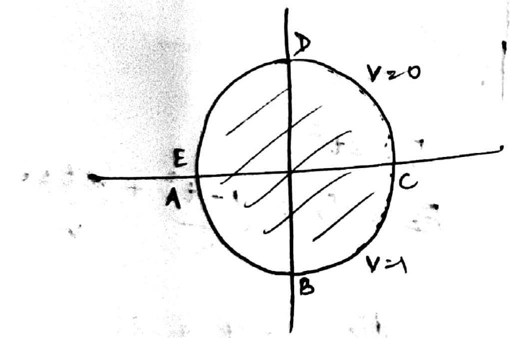
$$f(D') = D$$

$$H(x,y) = h\left(u(x,y),\, v(x,y)\right)$$

$\Rightarrow$ H is harmonic

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$$

Example



V = 0 above
V = 1 below

$$z = re^{i\theta}, \quad r < 1$$

$$w = i\,\frac{1-z}{1+z}$$

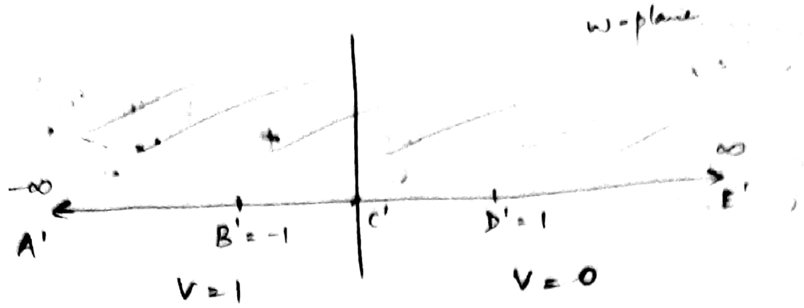
$$= \frac{i(1-z)(1+\bar{z})}{(1+z)(1+\bar{z})}$$

$$= \frac{i\left(1 - z + \bar{z} - z\bar{z}\right)}{\left(1 + z + \bar{z} + z\bar{z}\right)}$$

$$= \frac{i\left(1 - r^2 - 2ir\sin\theta\right)}{1 + r^2 + 2r\cos\theta}$$

$$= \frac{2r\sin\theta + i(1-r^2)}{1 + r^2 + 2r\cos\theta}$$

$r = 1 \Rightarrow w = \tan(\theta/2)$



$$\nabla^2 u(u,v) = 0$$

$$v > 0$$

$$u(u,0) = 1 \qquad u < 0$$

$$u(u,0) = 0 \qquad u > 0$$

$$u(\rho,\phi) = (A\phi + B)(C + D\ln\rho) + (C'\rho^{m} + D'\rho^{-m})(A'\sin m\phi + B'\cos m\phi)$$

$$u(\rho\to\infty,\phi) < \infty \Rightarrow C' = D' = 0$$

$$u(\rho\to 0,\phi) < \infty \Rightarrow D = 0$$

$$u(\rho,\phi) = A\phi + B$$

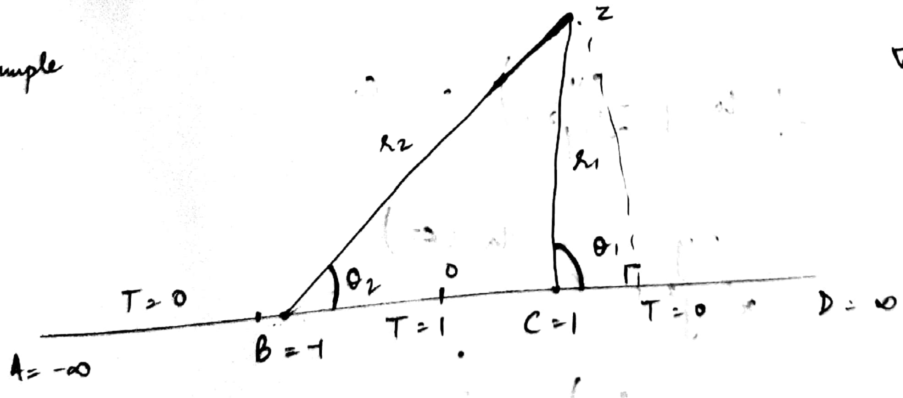
$$u = \frac{\phi}{\pi} = \frac{1}{\pi}\tan^{-1}\left(\frac{v}{u}\right)$$

$$u(r,\theta) = \frac{1}{\pi}\tan^{-1}\left(\frac{1-r^2}{2r\sin\theta}\right)$$

$$u(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{1-(x^2+y^2)}{2y}\right)$$

Example

$$\nabla^2 T = 0$$



$$w = \log\left(\frac{z-1}{z+1}\right) \qquad \log \text{ branch } (-\pi/2, 3\pi/2)$$

$$z - 1 = r_1 e^{i\theta_1}$$

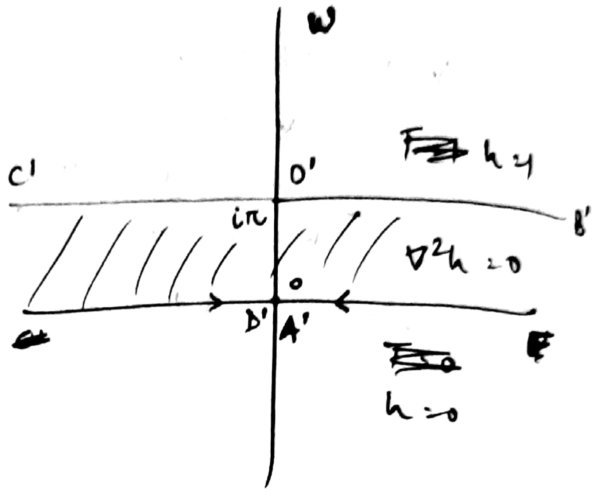
$$z + 1 = r_2 e^{i\theta_2}$$

$$w = \log\left(\frac{z-1}{z+1}\right)$$

$$= \log \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \underbrace{\log \frac{r_1}{r_2}}_{u} + i\underbrace{(\theta_1 - \theta_2)}_{v}$$

$A \to -\infty \Rightarrow r_1 > r_2$, $\theta_1 = \theta_2$

$B \to -1$ from left side $\Rightarrow r_1 > r_2$, $\theta_1 = \theta_2$



$$h = Av + B$$

$$= v/\pi$$

$$H = \frac{\theta_1 - \theta_2}{\pi}$$

$$H = \frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

Isotherm $\Rightarrow$ $\frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = c$

$$\frac{2y}{x^2+y^2-1} = \tan(c\pi)$$

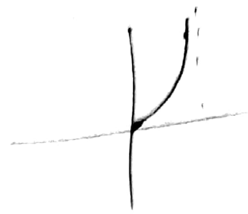
$$\tan(c\pi)\left(x^2+y^2-1\right) = 2y$$

$c = 1/2$

$x^2 + y^2 = 1$

$0 < c < 1/2$

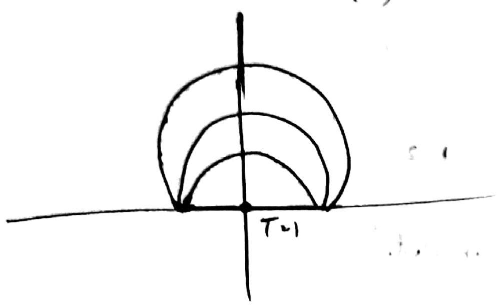
$$x^2 + y^2 - 1 = \frac{2y}{\tan(cx)}$$



$$x^2 + y^2 - 1 - \frac{2y}{\tan(cx)} = 0$$

$0 < c < 1/2$

$0 < \tan(cx) < \infty$



Wave Equation (1-D)

$$\frac{d^2\psi(x,t)}{dx^2} = \frac{1}{c^2}\frac{d^2}{dt^2}\psi(x,t) \qquad 0 \leq x \leq a, \quad 0 \leq t \leq T$$

$$\psi(x,t) = X(x)\cdot T(t)$$

$$\frac{1}{X}X'' = \frac{1}{c^2 T}\ddot{T} = -k^2 \qquad \omega = ck$$

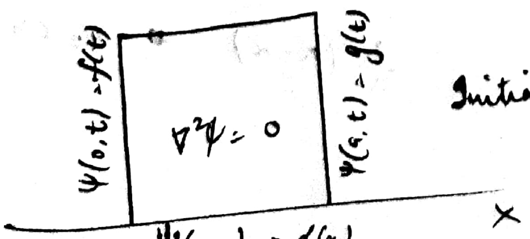
$$X'' + k^2 X = 0 \Rightarrow e^{\pm ikx}$$

$$\ddot{T} + \omega^2 T = 0 \Rightarrow e^{\pm i\omega t}$$

$$\psi(x,t) = e^{i(kx \pm \omega t)} \qquad \text{elementary sol}^n$$

$k \Rightarrow$ wave no.

$\omega \Rightarrow$ frequency

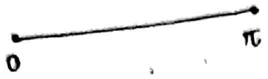


Initial Boundary Condition

$\psi(x,0) = \alpha(x)$

$\dot{\psi}(x,0) = \beta(x)$

Example



$$\psi(0,t) = 0$$
$$\psi(\pi,t) = 0$$
$$\psi(x,0) = f(x)$$
$$\dot{\psi}(x,0) = g(x) = 0$$

$c = 1$

$$X(x) = A\sin(kx) + B\cos(kx)$$
$$X(0) = X(\pi) = 0$$

$$X_n(x) = A\sin(nx) \qquad n = 1, 2, \ldots$$

$$T_n(t) = C\sin(nt) + D\cos(nt)$$

$$\psi_n(x,t) = \sum \sin(nx)\left[A\sin(nt) + B\cos(nt)\right]$$

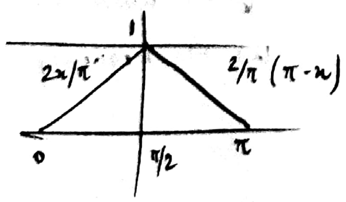
B.C. : $\dot{\psi}(x,0) = 0$

$$A = 0$$

$$\psi(x,t) = \sum_n A_n \sin(nx)\cos(nt)$$

$$\psi(x,0) = f(x) = \sum_n A_n \sin(nx)$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$



$$f(x) = \frac{2x}{\pi} \qquad 0 < x < \pi/2$$
$$\frac{2}{\pi}(\pi - x) \qquad \pi/2 < x < \pi$$

$$A_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{2}{\pi} x \sin(nx)\, dx + \int_{\pi/2}^{\pi} \frac{2}{\pi}(\pi - x)\sin(nx)\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{\pi}\left(\frac{x\cos nx}{n}\Big|_{\pi/2}^{0} + \frac{\cos nx}{n}\Big|_{\pi/2}^{0}\right)\right.$$

$$\left. + 2\cos nx\Big|_{\pi}^{\pi/2} - \frac{2}{\pi}\int_{\pi/2}^{\pi} x \sin(nx)\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{\pi}\left(-\frac{\pi}{2n}\cos(n\pi/2)\right) + \left(\frac{1}{n} - \frac{\cos(n\pi/2)}{n}\right)\right.$$

$$+ 2\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)$$

$$- \frac{2}{\pi}\left[\frac{x\cos nx}{-n}\Big|_{\pi}^{\pi/2} + \frac{\cos nx}{n}\Big|_{\pi}^{\pi/2}\right]$$

Mapping by $1/z$

z-plane	w-plane
circle not thru origin	circle not thru origin
line not thru origin	circle thru origin
circle thru origin	line not thru origin
line thru origin	line thru origin

Mobius Transformation or Linear Fractional Transformation

$$w = \alpha \frac{z - z_1}{z - z_2} \qquad \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Wave Eq^n in 1-D

B.C. : $\psi(0,t) = \psi(a,t) = 0 \quad \forall t$

$$\psi_n(x,t) = \sin\left(\frac{n\pi x}{a}\right)\left(C_n \sin \omega_n t + D_n \cos \omega_n t\right)$$

$$\psi = \sum_n \psi_n \qquad \psi(x,0) = f(x)$$

$$\dot\psi(x,0) = g(x)$$

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Wave Eq^n in 2-D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,y,t)$$

$$\psi(x,y,t) = X(x)\,Y(y)\,T(t)$$

$$X'' = -k_x^2 X$$

$$Y'' = -k_y^2 Y$$

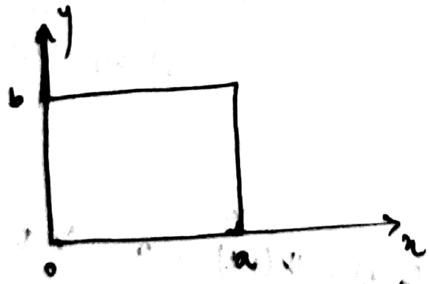
$$\ddot{T} = -\underbrace{(k_x^2 + k_y^2)c^2}_{\omega^2} T$$

$$\psi(x,y,t) = (A\sin k_x x + B\cos k_x x)\cdot(C\sin k_y y + D\cos k_y y)\cdot T$$

$$= e^{i(k_x x + k_y y + \omega t)}$$

$$\vec{k} = (k_x, k_y) \qquad = e^{i(\vec{k}\cdot\vec{x} \pm \omega t)}$$

Example



$$\psi(0,y,t) = \psi(a,y,t) = 0, \quad \forall\, y, t$$

$$\psi(x,0,t) = \psi(x,b,t) = 0 \quad \forall\, x, t$$

$$\psi_{mn}(x, y, t) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \left[A_{mn} \sin \omega_{mn} t + B_{mn} \cos \omega_{mn} t\right]$$

$$\omega_{mn} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \pi c \qquad \begin{matrix} m = 1, 2 \\ n = 1, 2, \end{matrix}$$

Example $\quad \psi(x, y, 0) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$

$\quad \dot{\psi}(x, y, 0) = 0$

$$\therefore \psi_{mn}(x, y, t) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \cdot \cos\left(\frac{(m^2 + n^2)^{1/2} \pi c}{a} t\right)$$

$\psi_{mn}^{1} + \psi_{nm} \Rightarrow$ linear combination of will have same frequency as the original eq^n if $a = b$.

Cylindrical co-ordinates for wave eq^n in 2-D

$$\nabla^2 \psi(\rho, \phi, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(\rho, \phi, t)$$

$$\psi(\rho, \phi, t) = \underbrace{R(\rho) \cdot P(\phi)}_{\Phi(\rho, \phi)} \cdot T(t)$$

$$\left.\begin{aligned} \nabla^2 \Phi(\rho, \phi) &= -k^2 \Phi(\rho, \phi) \\ \ddot{T}(t) &= -\omega^2 T(t) \end{aligned}\right\} \omega = ck$$

$$T(t) = A \sin \omega t + B \cos \omega t$$

Helmholtz $\rightarrow \quad (\nabla^2 + k^2) \Phi(\rho, \phi) = 0$

$$\frac{1}{R} \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \frac{1}{\rho^2 \phi} \frac{\partial^2 P}{\partial \phi^2} + k^2 = 0$$

$$P = C \sin m\phi + D \cos m\phi$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (k^2\rho^2 - m^2) R = 0$$

Bessel eq^n ~~function~~

$m = 0, 1, 2, \ldots$

J_m , N_m

Bessel Neumann

$$\psi(\rho, \phi, t) = \left(E J_m^{(k\rho)} + F N_m\right)\left(C \sin m\phi + D \cos m\phi\right)\left(A \sin \omega t + B \cos \omega t\right)$$

$$E J_m + F N_m = \frac{1}{2}(E - iF)\underbrace{(J_m + iN_m)}_{H_m^{(1)}} + \frac{1}{2}(E + iF)\underbrace{(J_m - iN_m)}_{H_m^{(2)}}$$

Hankel functions

$$H_m^{(1)} \approx e^{\pm ik\rho} \frac{1}{\sqrt{\rho}}$$

Elementary ~~sol~~^{n} $\Rightarrow \frac{1}{\sqrt{\rho}} e^{i(k\rho \pm \omega t)}$

Example

$$\left.\begin{array}{l} \psi(\rho = a, \phi, t) = 0 \\ \psi(\rho \to 0, \phi, t) < \infty \end{array}\right\} \forall\ \phi, t \qquad \begin{array}{l} \psi(\rho, \phi, 0) = f(\rho, \phi) \\ \dot{\psi}(\rho, \phi, 0) = g(\rho, \phi) \end{array}$$

$F = 0$

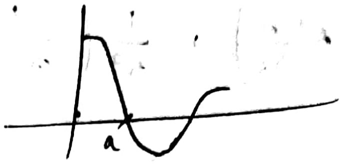
$$k = \frac{X_{ml}}{a} \qquad X_{ml} \Rightarrow l^{th} \text{ zero of } J_m$$

$$\psi_{ml} = J_m\left(\frac{X_{ml}}{a}\rho\right) \sin(m\phi - \alpha)\left(A_{ml} \sin \omega t + B_{ml} \cos \omega t\right)$$

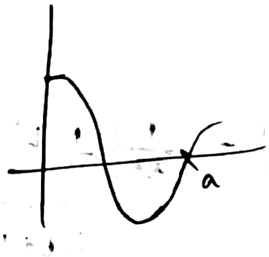
$$\sin(m\phi - \alpha) \Leftarrow C \sin m\phi + D \cos m\phi$$

$$\psi = \sum_{m=0,\ l=1}^{\infty\ \infty} \psi_{ml}$$

$$\Phi(\rho, \phi) = J_0\left(\frac{x_{01}\rho}{a}\right) \qquad m = 0,\ l = 1$$



$m = 0,\ l = 2$



$$\Phi = J_1\left(\frac{x_{11}\rho}{a}\right)\sin\phi \qquad m = 1,\ l = 1$$

Wave Eq^n in 3-D spherical co-ordinates

$$\nabla^2 \Psi(r, \theta, \phi, t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(r, \theta, \phi, t)$$

$$\Psi(r, \theta, \phi, t) = \psi(r, \theta, \phi) \cdot T(t)$$

$$\left.\begin{aligned} T(t) &= A\sin\omega t + B\cos\omega t \\ &= Ae^{i\omega t} + Be^{-i\omega t} \end{aligned}\right\}\ \omega = ck$$

$$\nabla^2\psi(r, \theta, \phi) + k^2\psi(r, \theta, \phi) = 0 \qquad \text{Helmholtz Eq}^n$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi \qquad \text{Schrodinger Eq}^n$$

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

if $V = 0$

$$\nabla^2\psi = -\underbrace{\left(\frac{2mE}{\hbar^2}\right)}_{k^2}\psi = 0$$

~~$\nabla^2 \Psi = k^2 \Psi$~~

$$\nabla^2 \psi = k^2 \psi$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{-L^2}$$

$$\psi(r,\theta,\phi) = R(r)\, Y(\theta,\phi)$$

$$\nabla^2\psi = \frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R\right) - \frac{1}{r^2}\underbrace{\left(\frac{1}{Y}L^2 Y\right)}_{l(l+1)} + k^2 = 0$$

$$= \frac{d}{dr}r^2\frac{d}{dr}R + \left(k^2r^2 - l(l+1)\right)R = 0$$

$$L^2 Y = l(l+1)$$

$$Y(\theta,\phi) = P(\theta)\,Q(\phi)$$

$$\frac{d^2Q}{d\phi^2} = -m^2 Q$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P - \frac{m^2}{\sin^2\theta}P = -l(l+1)P$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

Associated Legendre Eq^n

$m = 0 \Rightarrow$ standard Legendre Eq^n

$x = \cos\theta$

$$(1-x^2)\frac{d^2}{dx^2}P - 2x\frac{dP}{dx} + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P = 0$$

$\Rightarrow m = 0 \qquad Q = A + \cancel{B\phi} \qquad Q(0) = Q(2\pi)$

Sol^n which are finite at $x = \pm 1$

$\Rightarrow l = 0, 1, 2, \ldots$

Legendre polynomials

$P_l(x) = (-1)^l P_l(x)$

$m \neq 0 \qquad Q_m = e^{\pm im\phi}$

P_l^m

$l = 0, 1, 2 \ldots$

$-l \leq m \leq l$

$$Y_{lm}(\theta, \phi) = N_{lm}\, P_l^m(\theta)\, Q_m(\phi)$$

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{11} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta \cdot e^{-i\phi}$$

$$\frac{\partial^2}{\partial\phi^2}Q(\phi) = -m^2 Q(\phi)$$

Take root

$$\Rightarrow -i\frac{\partial}{\partial\phi}Q = mQ$$

$L_z Q = mQ \qquad L_z \Rightarrow -i\frac{\partial}{\partial\phi}$

Orthogonality $\Rightarrow$ $$\int Y_{lm}^{*}(\theta,\phi)\, Y_{l'm'}(\theta,\phi)\, d\Omega = \delta_{l'l}\,\delta_{m'm}$$

$$d\Omega = \sin\theta \; d\theta\, d\phi$$

Completeness - I $$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

Completeness - II
$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\, Y_{lm}(\theta,\phi)$$

$$\& \quad A_{lm} = \int_{s} Y_{lm}^{*}(\theta,\phi)\, g(\theta,\phi)\, d\Omega$$

Spherical Bessel Functions

$$u(r) = \sqrt{r}\, R(r)$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left[k^2 - \frac{(l+1/2)^2}{r^2}\right]u = 0$$

$$u = J_{l+1/2}(kr) \quad \text{or} \quad N_{l+1/2}(kr)$$

$$R = \underbrace{\frac{1}{\sqrt{r}}\, J_{l+1/2}(kr)}_{j_l(kr)} \quad \text{or} \quad \underbrace{\frac{1}{\sqrt{r}}\, N_{l+1/2}(kr)}_{n_l(kr)}$$

Spherical Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right) = \sqrt{\frac{\pi}{2x}}\, J_{l+1/2}(x)$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right) = \sqrt{\frac{\pi}{2x}}\, N_{l+1/2}(x)$$

$$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$$

Orthogonality

$$\int_0 j_m(X_{mn} r)\, j_m(X_{mp} r)\, r^2 dr = \frac{\delta_{np}}{2}\left[j_{m+1}(X_{mn})\right]^2$$

Free Space

$$\Psi(\vec{r}, t) = Y_{lm}(\theta, \phi) \cdot j_l(kr)\ (A \sin \omega t + B \cos \omega t)$$

$$h_l(kr) \qquad A e^{i\omega t} + B e^{-i\omega t}$$

$$\frac{1}{r} e^{ikr} e^{\pm i\omega t} \quad \text{as} \quad r \to \infty$$

Green's Functions

$L(x)$: Linear diff. operator on D

Green's function for L on D

$$L\, G(x, s) = \delta(x-s)$$

$\Rightarrow$ Motivation $\quad Lu = f$

$$u(x) = \int G(x,s)\, f(s)\, ds = \left(\int G(x,s)\, ds\right) f(s)$$

$$L(x)\, u(x) = \int \underbrace{L(x)\, G(x,s)}_{\delta(x-s)}\, f(s)\, ds = f(x)$$

Example

$$L = \frac{d}{dt} \quad \text{on } [0, 1]$$

$$LG(t,s) = \delta(t-s)$$

$t < s$	$t > s$
$LG(t,s) = 0$	$LG(t,s) = 0$
$\frac{dG}{dt} = 0 \Rightarrow G = a$	$G = b$

$$\int_{s-\varepsilon}^{s+\varepsilon} \frac{d}{dt} G(t,s)\, dt = \int_{s-\varepsilon}^{s+\varepsilon} \delta(t-s)\, dt = 1$$

$$G(s+\varepsilon, s) - G(s-\varepsilon, s) = 1$$

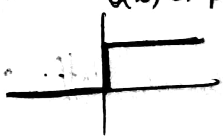
$$b - a = 1$$

$$b = a + 1$$

$$G(t,s) = \begin{cases} a & \text{if } t < s \\ a+1 & \text{if } t > s \end{cases}$$

$G(t,s) = a + \theta(t-s)$

$\theta(x) \Rightarrow$ Heaviside function



Let $Lu = t$

$$\frac{du}{dt} = t \Rightarrow u = c + \frac{t^2}{2}$$

$$u(t) = \int_0^1 G(t,s)\, f(s)\, ds$$

$t > s$

$$= \int_0^t (a+1)\, s\, ds + \int_t^1 a\, s\, ds$$

$$= \frac{(a+1)t^2}{2} + \frac{a}{2} - \frac{at^2}{2}$$

$$u(t) = \frac{a}{2} + \frac{t^2}{2}$$

$$u(t) = c + \frac{t^2}{2}$$

Example

$$L = \frac{d^2}{dt^2} + w_0^2 \qquad Lu = f \qquad t = [0, \infty)$$

$$f(t) := F \sin wt$$

$$u(0) = 0 \qquad \dot{u}(0) = 0$$

$$L\,G(t,s) = \delta(t-s)$$

$t < s$

$$\frac{d^2G}{dt^2} + w_0^2 G = 0$$

$$G(t,s) = A \sin(w_0 t) + B \cos(w_0 t)$$

$t > s$

$$\frac{d^2G}{dt^2} + w_0^2 G = 0$$

$$G(t,s) = C \sin(w_0 t) + D \cos(w_0 t)$$

$$\lim_{\varepsilon \to 0} \left[\dot{G}(s+\varepsilon, s) - \dot{G}(s-\varepsilon, s) + w_0^2 \int_{s-\varepsilon}^{s+\varepsilon} G(t,s)\,dt \right] = \int_{s-\varepsilon}^{s+\varepsilon} \delta(t-s)\,dt = 1$$

G is continuous

$$\dot{G}(s+\varepsilon, s) - \dot{G}(s-\varepsilon, s) + \cancel{w_0^2(0)}^{\,0} = 1$$

$$G(0,s) = 0 \Rightarrow B = 0$$

$$\dot{G}(0,s) = 0 \Rightarrow A = 0$$

$$G(t<s, s) = 0$$

$$G(t>s, s) = C \sin w_0 t + D \cos w_0 t$$

at $t = s$

$$G(t>s, s) = 0 \Rightarrow C \sin w_0 s + D \cos w_0 s = 0$$

$$\frac{dG}{dt}(t>s, s) = 1 \Rightarrow C w_0 \cos w_0 s + D w_0 \sin w_0 s = 0$$

$$C = \frac{1}{W_0} \cos(W_0 s)$$

$$D = \frac{-1}{W_0} \sin(W_0 s)$$

$$G(t>s, s) = \frac{1}{W_0}\left[\sin(W_0 t)\cos(W_0 s) - \cos(W_0 t)\sin(W_0 s)\right]$$

$$= \frac{1}{W_0} \sin(W_0(t-s))$$

$$u(t) = \int_0^\infty G(t,s)\, f(s)\, ds$$

$$= \int_0^t G(t>s)\, f(s)\, ds + \cancel{\int_t^\infty G(t<s)\, f(s)\, ds} \to 0$$

$s < t$ $\qquad\qquad$ $t > s$

$$G(t,s) = 0 \qquad t < s$$

$$= \frac{1}{W_0} \sin(t-s) \qquad t > s$$

Laplace Operator $\Rightarrow$ Green's function

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi\, \delta^3(\vec{r} - \vec{r}')$$

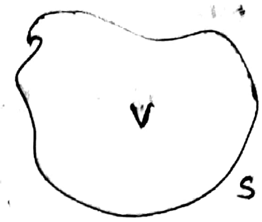
$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\nabla^2 H(\vec{r}, \vec{r}') = 0$$

$$G' = G + H$$

ψ, ϕ

$$\psi \nabla^2 \phi - \phi \nabla^2 \psi$$
$$= \nabla \cdot (\psi \Delta \phi - \phi \Delta \psi)$$



$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, d\tau$$
$$= \int_S (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, ds$$
$$= \int \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds$$

$$\nabla^2 \phi = f(\vec{r}\,')$$
$$\psi(\vec{r}, \vec{r}\,') = G(\vec{r}, \vec{r}\,')$$
$$\nabla^2 \psi = -4\pi\, \delta(\vec{r} - \vec{r}\,')$$

LHS $\Rightarrow \int G(\vec{r}, \vec{r}\,')\, f(\vec{r}\,')\, d\tau' + 4\pi \int \phi(\vec{r})\, \delta(\vec{r} - \vec{r}\,')\, d\tau'$

LHS = RHS $\Rightarrow \phi(\vec{r}) = \frac{-1}{4\pi} \int_V G(\vec{r}, \vec{r}\,')\, f(\vec{r}\,')\, d\tau'$
$$+ \frac{1}{4\pi} \int_S \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds'$$

Dirichlet BC ϕ on S is given

$G(\vec{r}, \vec{r}\,') = 0$ on S

Neumann BC $\frac{\partial \phi}{\partial n}$ on S

$\frac{\partial G}{\partial n} = 0$ on S

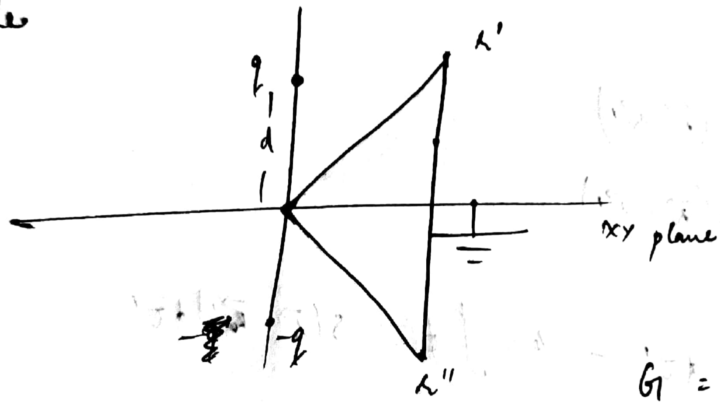
Example

V = Entire Space

$$\left.\begin{array}{l}\phi \to 0 \\ \frac{\partial\phi}{\partial n} \to 0\end{array}\right\} \vec{r} \to \infty,$$

$$\phi(r) = \frac{-1}{4\pi}\int \frac{f(r')}{|r-r'|}\, d\tau'$$

Example



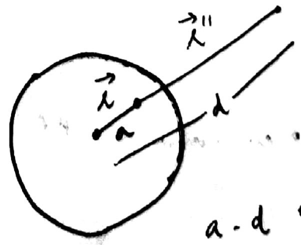
$$G = \frac{1}{|r - d\hat{k}|} - \frac{1}{|r + d\hat{k}|}$$

$$G(r;r') = \frac{1}{|\bar{r}-\bar{r}'|} - \frac{1}{|\bar{r}-\bar{r}'-2(\bar{r}\cdot\hat{n})\hat{k}|}$$

Example

$$V = \{\vec{r} \,|\, |\vec{r}| < R\}$$

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/a}{|\vec{r}-\vec{r}''|}$$



$a \cdot d = R^2$

$q' = \frac{-qR}{a}$

$$\vec{r}'' = \frac{R^2}{a^2}\vec{r}'$$

Example : $V = \mathbb{R}^3$ Fourier Transforms

$$G(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int g(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3k$$

$$\delta^3(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} d^3k$$

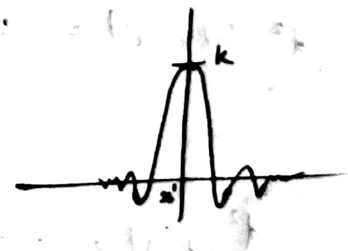
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \qquad \text{IFT}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad \text{FT}$$

$$\delta(x - x') \longrightarrow F(k) = \int_{-\infty}^{\infty} \delta(x-x') e^{-ikx} dx$$

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk$$

$$F(k) = \int_{-K}^{K} e^{ik(x-x')} dk = \frac{\sin(k(x-x'))}{x-x'}$$

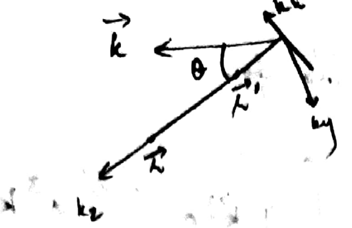


$$\nabla^2 G = \frac{1}{(2\pi)^3} \int g(\vec{k}) (-k^2) e^{i\vec{k}\cdot\vec{r}} d^3k$$

$$= \frac{-4\pi}{(2\pi)^3} \int e^{ik(\vec{r}-\vec{r}')} d^3k$$

$$g(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}'}$$

$$G(\vec{r}, \vec{r}') = \frac{4\pi}{(2\pi)^3} \int \frac{1}{k^2} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} d^3k$$



$$d^3k = k^2 \sin\theta \, d\theta \, d\phi \, dk \quad \text{(in spherical)}$$

$$G(\vec{r}, \vec{r}') = \frac{4\pi}{(2\pi)^3} \iiint \cancel{\frac{1}{k^2}} \, e^{ik|r-r'|\cos\theta} \, \cancel{k^2} \sin\theta \, d\theta \, d\phi \, dk$$

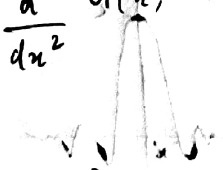
$$= \frac{4\pi}{(2\pi)^3} \iiint \cancel{\ } \, e^{ik|r-r'|t} \, dt \, d\phi \, dk$$

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$$

Eigenfunction Expansion

Example : $L = \dfrac{d^2}{dx^2}$ on $[0, a]$

$$\frac{d^2}{dx^2} G(x, x') = \delta(x - x') \qquad G(0, x') = G(a, x') = 0$$



$$X = \{ f : [0, a] \to \mathbb{R}^2 \}$$

$$\frac{d^2}{dx^2} f(x) = -\omega^2 f(x) \qquad Lf = \lambda f$$

$$u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$f(x) = \sum_n A_n u_n(x) \qquad A_n = \int_0^a f(x) u_n(x) \, dx$$

$$G(x, x') = \sum_n A_n u_n(x)$$

$$\frac{d^2}{dx^2} G(x, x') = \delta(x - x') = \sum_n A_n \left(-\left(\frac{n\pi}{a}\right)^2 u_n x \right)$$

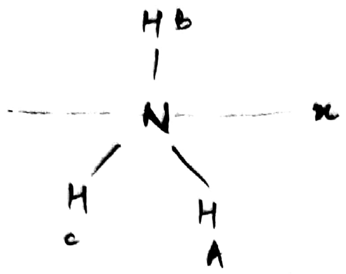
$$\Rightarrow A_n \left(-\frac{n^2\pi^2}{a^2} \right) = \int \delta(x - x') u_n(x) \, dx$$

$$G(x, x') = \sum_n \frac{-a^2}{n^2\pi^2} u_n(x') u_n(x)$$

$D_n \Rightarrow$ Dihedral Group

$D_3 : \{ e, a, a^2, b, ab, a^2b \}$

D_n always contain $2n$ elements



$R_z\left(\frac{2\pi}{3}\right) = a$

$R_z\left(\frac{4\pi}{3}\right) = a^2$

$R_z(2\pi) = a^3 = e$

$M_{yz} = b$

$b^2 = e$

$R_z\left(\frac{2\pi}{3}\right) M_{yz}$

$$\begin{array}{ccc} ABC & \xrightarrow{M} & CBA \\ & & \downarrow R \\ & & ACB \end{array}$$

$$\begin{array}{ccc} ABC & \xrightarrow{R^{-1}} & BCA \\ & & \downarrow M \\ & & ACB \end{array}$$

Permutation Group S_n

$(a \; b \; c \; d)$

$(b \; a \; c \; d)$

$(c \; b \; a \; d)$

$(b \; a \; c \; d) \cdot \begin{pmatrix} \overset{a}{\downarrow} & \overset{b}{\downarrow} & \overset{c}{\downarrow} & \overset{d}{\downarrow} \\ c & b & a & d \end{pmatrix} = (b \; c \; a \; d)$

$S_3 \Rightarrow$

$(a \; b \; c)$

$\left.\begin{matrix} (b \; a \; c) \\ (c \; b \; a) \\ (a \; c \; b) \end{matrix}\right\}$ Transposition

$\left.\begin{matrix} a & (b \; c \; a) \\ a^2 & (c \; a \; b) \end{matrix}\right\}$ Cyclic

$\Rightarrow$ Resembles D_3

Matrix Groups $(M_n, +)$ $M \times N$ matrix

$GL(n, \mathbb{R})$: General linear

$[\;\vdots\;]_{n \times n}$ non-singular

$O(n)$: Orthogonal

$MM^T = I = M^TM$ $\det(M) = \pm 1$

$SO(n)$ $MM^T = I = M^TM$

Special orthogonal $\det(M) = +1$

Transformation Groups Euclidean Group $E(n)$.

Lorentz group $O(1,3)$ etc.

$\sqrt{c^2t^2 - x^2 - y^2 - z^2}$

Subgroups

Ex. Trivial subgroups G; $\{e\}$

Ex. $(\mathbb{Z}, +) = G$ $H = \{5n \mid n \in \mathbb{Z}\}$

$= \{\ldots -5, 0, 5, 10, \ldots\}$

subset H of group G iff

1) multiplication is closed in H.

2) for each $a \in H$, a^{-1} is also in H

If group is finite, then only first condition is sufficient.

Ex. $(\mathbb{Z}, +) \subset (\mathbb{Q}, +) \subset (\mathbb{R}, +) \subset (\mathbb{C}, +)$

Ex. $GL(n) \supset O(n) \supset SO(n)$

Ex. $G = \{e, a, a^2, b, ab, a^2b\}$

$H_1 = \{e, a, a^2\}$ $H_2 = \{e, b\}$ $H_3 = \{e, ab\}$

$H_4 = \{e, a^2b\}$

Ex. $GL(2)$

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle/ ad \neq 0 \right\}$$

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} e & f \\ 0 & g \end{bmatrix} = \begin{bmatrix} ae & af + bg \\ 0 & dg \end{bmatrix}$$

$|H| = aedg \neq 0$

Closure satisfied

$$H^{-1} = \frac{1}{ad} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$$

Cosets

$G > H \quad , g \in G$

$gH = \{ gh / h \in H \}$

$Hg = \{ hg / h \in H \}$

Ex. $H = \{ 5n / n \in \mathbb{Z} \} = 0H = 5H$

$1H = \{ 5n+1 / n \in \mathbb{Z} \} = 6H$

$= \{ \ldots -4, 1, 6, \ldots \}$

$2H = 7H$

5 cosets of subgroup H

Index $= 5$

Ex. $H_1 = \{ e, a, a^2 \}$

$aH_1 = H_1$

$bH_1 = H_1$

$(ab)H_1 = \{ ab, \; aba = b, \; aba^2 = a^2 b \}$

$H_2 = \{e, b\}$

$aH_2 = \{a, ab\}$

$H_2 a = \{a, ba = a^2 b\}$

Theorem 2 right (left) cosets are either disjoint or equal.

Proof $G \supset H$ $a, b \in G$ Ha, Hb

Let $Ha \cap Hb \neq \phi$, $\exists$ $x \in Ha$ $x \in Hb$

$x = h_1 a = h_2 b$

$b = (h_2^{-1} h_1) a$

$b = h_3 a$

$y \in Hb \Rightarrow y = h_4 b$
$= (h_4 h_3) a$
$= h_5 a \Rightarrow y \in Ha$

$y \in Ha \Rightarrow y = h_6 a$
$= (h_6 h_3^{-1}) b$
$= h_7 b \Rightarrow y \in Hb$

This can happen only if both are equal.

$\therefore$ They are either disjoint or equal.

Theorem There is a one-one correspondence b/w 2 right cosets of H in G.

Ha, Hb $f: Ha \longrightarrow Hb$

$f(x) = x a^{-1} b$

if $f(x_1) = f(x_2)$ then $x_1 = x_2$

$x_1 a^{-1} b = x_2 a^{-1} b \Rightarrow x_1 = x_2$

Theorem If G is finite & H is subgroup of G then order of G is integral multiple of order of H.

Normal subgroup

subgroup N of G is normal subgroup if every $g \in G$ & $n \in N$ then $gng^{-1} \in N$

$g = b$ $\quad beb^{-1} = e \quad bab^{-1} = a^2 \quad ba^2b^{-1} = a$

$H_2 = \{e, b\}$ $\quad aea^{-1}$, $aba^{-1} = aba^2 = a^2b$

Theorem N is a normal subgroup in G if & only if $gNg^{-1} = N$

Proof Let N is normal

$N \subset gNg^{-1} \qquad gNg^{-1} \subset N$

$n \in N \qquad n = g(gng^{-1})g^{-1} = gn'g^{-1} \subset gNg^{-1}$

Theorem N is a normal subgroup in G iff every right coset Na is equal to left coset aN in G.

Proof $\Rightarrow$ N is normal

$$x \in Na \Rightarrow x = na$$
$$= a(a^{-1}na)$$
$$= an'$$
$$\Rightarrow x \in aN$$

$$x \in aN \Rightarrow x = an$$
$$= \cancel{n(a^{-1}na)}$$
$$= n(n^{-1}an)$$
$$= na'$$
$$\Rightarrow x \in Na$$

Theorem N is a normal subgroup in G iff product of 2 right cosets of N is a right coset of N.

Proof N is normal

To Prove $(Na)(Nb) = (Nc)$

$$na\,n'b = n\,a\,n'(a^{-1}a)\,b$$
$$= n\,(an'a^{-1})\,ab$$
$$= (nn'')ab$$
$$= n'''ab$$
$$= Nab$$

Factor Groups

A collection of cosets of normal subgroup N in G is a group with product of sets as binary operations. Group is called factor ~~group~~ group & is denoted by G/N.

~~Ne = N, Na, Na₂~~

$G/N = \{ Ne = N, Na_1, Na_2, \dots \}$

$(Na)\cdot(Nb)$ = Product of sets

$D_3 \Rightarrow \quad G/N = \{N, Nb\} \qquad N = \{e, a, a^2\}$

	N	Nb
N	N	Nb
Nb	Nb	N

Homomorphism

Mapping f from a group $(G, \odot)$ to a group $(\bar{G}, \otimes)$ is homomorphism if $\forall\ a, b \in G$

$$f(a \odot b) = f(a) \otimes f(b)$$

Example $\quad G = \{1, -1\}$ multiplication

$\bar{G} = \{0, 1\} \qquad a \otimes b = (a+b) \bmod 2$

$f(1) = 0$

$f(-1) = 0$

	1	-1
1	1	-1
-1	-1	1

	0	1
0	0	1
1	1	0

$g(1) = 0 \quad , \quad g(-1) = 0$

Example $\quad G = (\mathbb{Z}, +)$

$\bar{G} = (2^a, a \in \mathbb{Z})$ multi.

$f(a) = 2^a$

$f(a \odot b) = f(a+b) = 2^{a+b} = 2^a \cdot 2^b$

$= f(a) \otimes f(b)$

Example $\quad G = GL(2)$ $\quad$ 2x2 non-singular matrices

$\bar{G} = (\mathbb{R}^*, \times)$

$f(A) = \det(A)$

Example $\quad G = SO(2) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\bar{G} = \{ e^{i\theta} \mid \theta \in [0, 2\pi) \}$ multiplication

$f(M(\theta)) = e^{i\theta}$

$U^\dagger U = I$

$\bar{\bar{G}} = \{$ 1x1 unitary matrix $\}$

Example $G = \{e, a, a^2, ab, b, a^2b\}$ $O(G) = 6$

Example $G = \{(\mathbb{Z}, +)$ Infinite, discrete

$G = \{2^i, i \in \mathbb{Z}\}, \times$ Mapping of $\mathbb{Z}$ to G

Example $G = \{ e^{i\theta} \mid \theta \in [0, 2\pi) \}, \times$

parameter $\theta \in [0, 2\pi) \subset \mathbb{R}$

Example $SO(2) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = M(\theta)$

$\theta : [0, 2\pi)$

Example

$T_{ab} : \mathbb{R} \to \mathbb{R}$

$T_{ab}(x) = ax + b$

$G = \{ T_{ab} \mid a \neq 0 \}$, $T_{ab} \circ T_{cd}(x)$

$= T_{ab}(cx + d)$

$= a(cx + d) + b$

$= acx + ad + b$

$T_{1,0} = e$ $= T_{ac, ad+b}$

$T_{ab}^{-1} = T_{\frac{1}{a}, \frac{-b}{a}}$

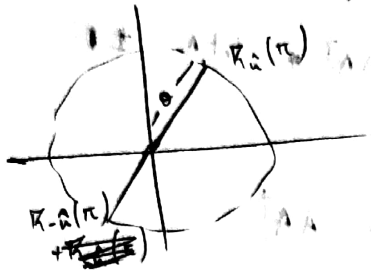
Example $SO(3) \Rightarrow$ Rotations in 3-D

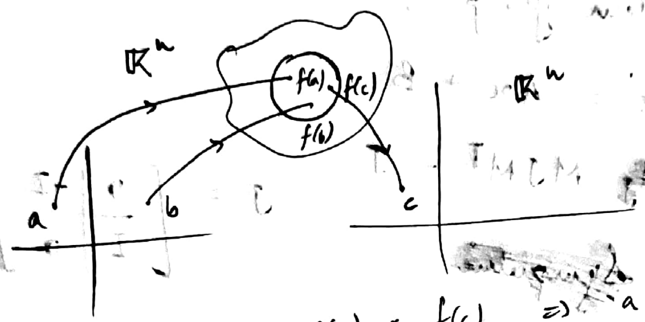
$R_{\hat{n}}(\theta) = I + \sin\theta\, S_{\hat{n}} + (1 - \cos\theta) S_{\hat{n}}^2$

$S_{\hat{n}} = \begin{bmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{bmatrix}$

$\theta : 0 \to \pi$

Sphere of radius θ





$f(a) \cdot f(b) = f(c) \Rightarrow a \cdot b \to c$

$a \to a^{-1}$

If continuous $\Rightarrow$ continuous groups

differentiable $\Rightarrow$ Lie groups

Matrix Lie Groups

$M_n(\mathbb{R})$, $M_n(\mathbb{C})$ $n \times n$ Matrices

Vector space, $+$ dimension $= n^2 \Rightarrow \mathbb{R}^{n^2}$

Normed $\|A\|$

$$\begin{bmatrix} 3 \times 3 \end{bmatrix} \begin{bmatrix} 3 \times 1 \end{bmatrix}$$
$$A \qquad x$$

Closed under multiplication

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Limits, ~~Continuous~~ Continuity, Differentiability

$GL(n)$ – Invertible matrices
group under multiplication

$G \subset GL(n)$ is called a Lie group

if it is closed in $GL(n)$
$\to$ (closed in Cauchy sequence)

E.g.

$GL(n)$ = Invertible

$SL(n) = \det(A) = +1$

$O(n) = A^T A = I = AA^T, \; \det A = \pm 1$

$SO(n) = \det A = +1$

$U(n) = A^\dagger A = I = AA^\dagger$

$SU(n) = \det(A) = +1$

$E(n)$ = Euclidean group, Translations

$$x' = Ax + B$$

$Sp(n) = MJM^T = J$ (Symplectic) $\quad J = \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right]$

~~Symplectic~~

$SO(3,1) \Rightarrow$ Lorentz Group

$$x' = \gamma(x - \beta t)$$
$$t' = \gamma(t - \beta x)$$
$$y' = y$$
$$z' = z$$

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

Poincare Group

Compact group if it is bounded & is closed in M_n

$$A = \{a_{ij}\} \text{ if } |a_{ij}| < L \;\; \forall \, i,j$$

Compact Example : $O(n)$, $SO(n)$, $SU(n)$

Non-Compact Example $GL(n)$, $E(n)$, Lorentz group

Connectedness

Connected if $f(t)$ $\quad t : 0 \to 1$

such that $f(0) = A$

$f(1) = B \quad \forall\ A, B = G$

- Not Connected $\Rightarrow$ Components – Connected region

~~So~~

Example $\Rightarrow$ $SO(3) \Rightarrow$ Connected

$O(3) \Rightarrow$ Two components

$\cancel{SO(3,1)}$

Lorentz group $\Rightarrow$ four components