Tutorial 8: Group Theory

Group Theory: Properties, Equivalence Classes, Subgroups, Quotient Group.

1. Verify that each of the following sets is a group with given *group product*. Which groups are abelian?
   (a) The set of all non-zero rationals under multiplication.
   (b) The set of all complex numbers of unit magnitude under multiplication.
   (c) The set of all complex roots of the equation $z^n = 1$.
   (d) The set $\{1, 2, \dots, p-1\}$ under multiplication modulo (p), where p is a prime number.
   (e) The set of following matrices under matrix multiplication:
   $$\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\begin{bmatrix}0&-1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&-1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}-1&0\\0&1\end{bmatrix},\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\right\}$$
   (f) The set of functions under function composition:
   $$\begin{aligned} f_1(x) &= x, & f_2(x) &= 1-x, & f_3(x) &= x/(x-1), \\ f_4(x) &= 1/x, & f_5(x) &= 1/(1-x), & f_6(x) &= (x-1)/x, \end{aligned}$$
   (g) The set $\{T_{ab} \mid a, b \in \mathbb{R},\ a \neq 0\}$of all linear transformations on $\mathbb{R}$, such that $T_{ab}(x) = ax + b$.
   (h) Set of all isometries on $\mathbb{R}^2$ that set of all linear transformations that leave the Euclidean norm invariant.
   (i) The set of all real 2×2 matrices $\begin{bmatrix}a&b\\0&d\end{bmatrix}$, where $ad \neq 0$.

2. Does the set of the three matrices
   $$E = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \quad A = \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}, \quad B = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$
   form a group under matrix multiplication? If not, add a minimum number of matrices to complete the group. Prepare the multiplication table.

3. Generate the matrix group (under matrix multiplication) which contains the two elements $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $\begin{bmatrix}0&i\\i&0\end{bmatrix}$. What is the order of the group? Is it abelian?

4. Dihedral group (D_n) is a group generated by two elements A and B subject to relations $A^2 = B^n = (AB)^2 = I$. What is the order of this group? Write down the elements of D_4.

5. Find the subgroup of the symmetric (permutation) group S_4, which leaves the polynomial $x_1x_2 + x_3 + x_4$ invariant.

6. An element a of a group G is said to be conjugate to $b \in G$ if $a = g^{-1}bg$ for some $g \in G$ and is denoted by $a \sim b$.
   (a) Show that the conjugacy relation is an equivalence relation.
   (b) Show that all elements of a class have same order. (For $a \in G$, minimal n such that $a^n = e$ is called the order of a)
   (c) Show that in an abelian group, every element is a class by itself.
   (d) Show that a normal subgroup contains complete classes.

7. Find all conjugacy classes and subgroups of the following groups.

(a) The set of following matrices under matrix multiplication:

$$\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\right\}$$

(b) The group generated by two elements $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & i\\i & 0\end{bmatrix}$.

(c) Group generated by σ and ψ with constraints $\sigma^2 = \psi^3 = e$ and $\sigma\psi = \psi^2\sigma$.

8. Show that every subgroup of index 2 is a normal subgroup.

9. Show that the dihedral group D_4 is homomorphic to the group $\mathbb{Z}_2 = \{1, -1\}$ under multiplication.

10. Show that the group of all positive real numbers under multiplication is isomorphic to the set of real numbers under addition. (Hint: Mapping is logarithm function)

11. Construct a homomorphism of the group S_3 onto $\mathbb{Z}_2$. What is the kernel of the homomorphism? Find the factor group of S_3 that is isomorphic to $\mathbb{Z}_2$.
Note: The permutation group, S_n, is a group of all permutations of n symbols (which we will take as integers from 1 to n). Each element is denoted by $a = \begin{pmatrix}1 & 2 & \cdots & n-1 & n\\a_1 & a_2 & \cdots & a_{n-1} & a_n\end{pmatrix}$ or just by $a = (a_1, \ldots, a_n)$ where a_1 to a_n are permutations of 1 to n. The product is given by $a \circ b = (a_{b_1}, a_{b_2}, \ldots, a_{b_n})$. For example:

$$\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 1 & 2 & 4\end{pmatrix}$$

And $\mathcal{O}(S_n) = n!$.

12. Let N be a normal subgroup of G. Show that G is homomorphic to G/N.