① Sol $u(x,t) = X(x) \cdot T(t)$

$X(x) = A\sin(kx) + B\cos(kx)$

$T(t) = C\sin(\omega t) + D\cos(\omega t)$

Applying Boundary conditi.

- $u(0,t) = 0$ $\Rightarrow$ $\boxed{B = 0}$
- $u(a,t) = 0$, $ka = n\pi$, $\boxed{k = \frac{n\pi}{a}}$

So, finally.

$$u(x,t) = \sum \sin\left(\frac{n\pi x}{a}\right)\left[C_n \sin(\omega t) + D_n \cos(\omega t)\right]$$

$$u(x,0) \Rightarrow f(x) = \sum \sin\left(\frac{n\pi x}{a}\right)[D_n]$$

$$\frac{\partial}{\partial t}u(x,0) \Rightarrow g(x) = \sum \sin\left(\frac{n\pi x}{a}\right)[C_n]$$

(a) $a = \pi$, $f(x) = \sin 3x$, $g(x) = 4$

Now, $\therefore$ $\sin 3x = \sum \sin\left(\frac{n\pi x}{a}\right) D_n$

~~clearly on compari~~

D_3

$\cancel{\sin(3x)}$

$$\int_0^a \sin(3x)\sin\left(\frac{n\pi x}{a}\right)dx = \int_0^a \sin\left(\frac{n\pi x}{a}\right)^2 \cdot D_n \cdot dx$$

$$= \frac{a}{2} D_n$$

$$D_n = \frac{2}{a}\int_0^a \sin(3x)\sin\left(\frac{n\pi x}{a}\right) \cdot dx$$

at $g(x) = 4 = \sum \sin\left(\frac{n\pi x}{a}\right) C_n$

So, $C_n = \frac{2}{a}\int \sin\left(\frac{n\pi x}{a}\right) g(x) \cdot dx$

②

$u = u_0\left[\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta\right]$

(2) $u(x,t) = X(x)\,T(t)$

Now

$$u(x,t) = \sum \left(A \sin kx + B\cos kx\right)\cdot\left(C \sin\omega t + D\cos\omega t\right)$$

$x=0$ $x=a$

B.C. $u(0,t) = 0$

$u(a,t) = 0$

$\dot{u}(x,0) = 0$

$u(x,0) = u_0 \sin^3\left(\frac{\pi x}{a}\right)$

Applying B.C :-

Now $u(0,t) \Rightarrow \boxed{B = 0}$

$u(a,t) \Rightarrow ka = n\pi$

$\boxed{k = \frac{n\pi}{a}}$

So, $u(x,t) = \sum_{n=1}^{\infty} A \sin\left(\frac{n\pi x}{a}\right)\left[C \sin\omega t + D\cos\omega t\right]$

$\dot{u}(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[\omega C\cos\omega t - \omega D \sin\omega t\right]$

$\dot{u}(x,0) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[\omega C - 0\right]$

$\Rightarrow \boxed{C = 0}$

So, $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) D_n \cos(\omega t)$

Now, for $u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) D_n$

$u_0 \sin^3\left(\frac{\pi x}{a}\right) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) D_n$

$u_0\left[-\frac{1}{4}\sin\left(\frac{3\pi x}{a}\right) + \frac{3}{4}\sin\left(\frac{\pi x}{a}\right)\right] = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) D_n$

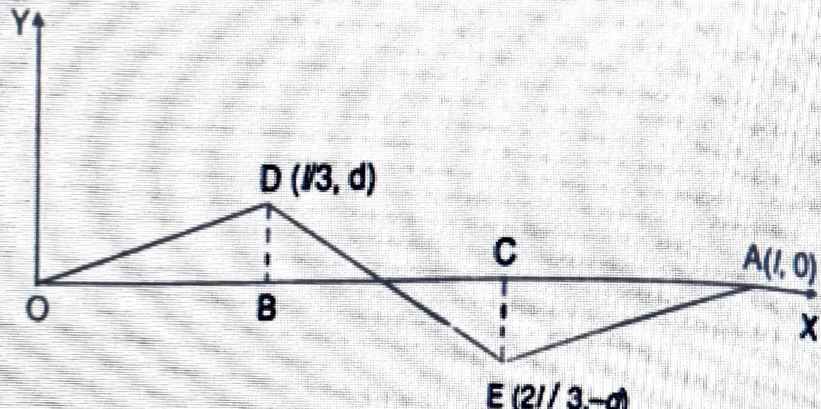
on Comparison.

$\boxed{D_3 = -\frac{u_0}{4}}$ $\boxed{D_1 = \frac{3u_0}{4}}$ or else $\boxed{D_n = 0}$

Example 3

The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string is always remains at rest.

Solution.



$$BD = CE = d.$$

The displacement $y(x, t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{--(1)}$$

The boundary conditions here are

$$y(0, t) = 0 \qquad \text{for } t \geq 0 \qquad \text{...(i)}$$

$$y(l, t) = 0 \qquad \text{for } t \geq 0 \qquad \text{...(ii)}$$

and

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \qquad \text{for } 0 \leq x \leq l \qquad \text{...(iii)}$$

To find the initial position of the string, we require the equation of $ODEA$.

The equation of OD is $\quad y = \dfrac{d}{l/3}x = \dfrac{3dx}{l}$.

The equation of DE is $y - d = -\dfrac{d}{(l/6)}(x - l/3)$

i.e.,

$$y = \frac{3d}{l}(l - 2x).$$

The equation of EA is $\quad y = \dfrac{3d}{l}(x - l)$.

The fourth initial condition is

$$y(x, 0) = \begin{cases} \dfrac{3dx}{l} & \text{for } 0 \leq x \leq l/3 \\ \dfrac{3d}{l}(l - 2x) & \text{for } \dfrac{l}{3} \leq x \leq \dfrac{2l}{3} \\ \dfrac{3d}{l}(x - l) & \text{for } \dfrac{2l}{3} \leq x \leq l \end{cases} \qquad \text{———— (iv)}$$

Solving (1) and selecting the suitable solution and using the boundary conditions (i), (ii) and (iii) as in example 2, we get

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

Using the initial condition (iv) we get,

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y(x,0) = \frac{3dx}{l} \text{ for } 0 \le x \le l/3$$

$$= \frac{3d}{l}(l-2x), \text{ for } \frac{l}{3} \le x \le \frac{2l}{3}$$

$$= \frac{3d}{l}(x-l), \text{ for } \frac{2l}{3} \le x \le l.$$

Finding Fourier sine series of $y(x, 0)$ in $(0, l)$ we get in the usual

way $y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

$$\therefore \qquad B_n = b_n = \frac{2}{l}\int_0^l y(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$\therefore \qquad B_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3dx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3d}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l} \frac{3d}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6d}{l^2}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/3}$$

$$+ \frac{6d}{l^2}\left[(l-2x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-2)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l}^{2l/3}$$

$$+\frac{6d}{l^2}\left[(x-l)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{2l/3}^{l}$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}+\cos n\pi\cdot\sin\frac{n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1+(-1)^n\right]$$

$=0$ if n is odd.

$$=\frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3}\text{ if } n \text{ is even.}$$

Hence,

$$y(x,t)=\frac{36d}{\pi^2}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$\text{i.e., } y(x,t)=\frac{9d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cdot\cos\frac{2n\pi at}{l}.$$

By putting $x=l/2$, we get the displacement of the midpoint.

$\therefore\ y\left(\frac{l}{2},t\right)=0$, since $\sin\frac{2n\pi x}{l}$ becomes $\sin n\pi=0$ when $x=l/2$.