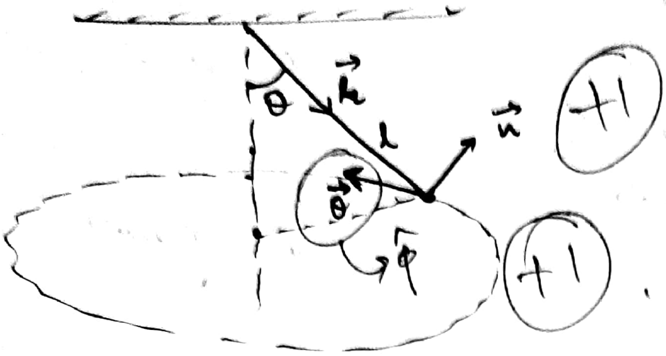
Q1



a) $\vec{L} = \vec{r} \times \vec{p}$

$= r_\perp p \, \hat{n}$

$= l \sin\theta \, mv \, \hat{n}$

$= mvl \sin\theta \, \hat{n}$ +1

b) $\vec{\tau} = \vec{r} \times \vec{F}$

$= \vec{r} \times (m\vec{g} + \vec{T})$

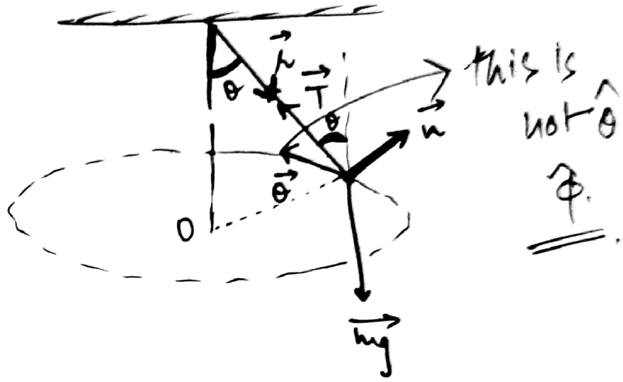
$= \vec{r} \times m\vec{g}$

$= \cancel{l \sin\theta \, mg}$

$= r_\perp mg$

$= mgl\sin\theta \, \hat{\theta}$

+3



$\vec{r} \times \vec{T} = 0$ (they are parallel)

c) $\vec{\tau} = \dfrac{d\vec{L}}{dt} = \dfrac{d}{dt}\left(mvl\sin\theta \, \hat{n}\right)$

$= mvl\sin\theta \, \dfrac{d\hat{n}}{dt}$

$= \cancel{mvl\sin\theta\,(v\hat{\theta})}$

$= \cancel{mv^2 l\sin\theta} \, \hat{\theta}$

+1

$T\cos\theta = mg$

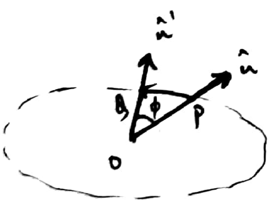
$T\sin\theta = \dfrac{mv^2}{R}$

$\cancel{T\sin}$

$\dfrac{mg}{g}$

$\cancel{mg\tan\theta} = \dfrac{v^2}{Rg}$

$\cancel{v^2 = Rg\tan\theta}$



$\dfrac{d\hat{n}}{dt} = \dfrac{\hat{n}' - \hat{n}}{dt}$

$= \dfrac{\vec{PB}}{dt} = \hat{\theta}$

$= \dfrac{d\phi}{dt} \, \hat{\theta}$

$= \omega\hat{\theta} = \dfrac{v}{l\sin\theta} \, \hat{\theta}$

$$\Rightarrow \quad \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$= \frac{d\hat{n}}{dt} \, mvl\sin\theta$$

$$= mvl\sin\theta \cdot \frac{v}{l\sin\theta} \, \hat{\theta}$$

$$= mv^2 \, \hat{\theta}$$

$$\vec{\tau} = mgl\sin\theta \, \hat{\theta}$$

+1

$v^2 = Rg\tan\theta$

$v^2 = l\cos\theta \cdot g\tan\theta$

$v^2 = lg\sin\theta$

$v^2 = gl\sin\theta$

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Q3 $\quad \frac{\lambda}{r} = -1 + \varepsilon\cos\theta$

a) $\cancel{E = T + V}$

$\cancel{= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}}$

$\cancel{V = \frac{-GMm}{r} = \frac{-k}{r}}$

Let $V = \frac{-Ze^2}{4\pi\varepsilon_0 r} = \frac{-k}{r}$

$L = T - V$

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left(m\dot{r}\right) - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

$$\dot{r}\left(m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2}\right) = 0$$

$$= \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{r}\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}\right) = 0$$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$V = \frac{-k}{r}$

$$\frac{d}{dt}(T+V) = 0$$

$$\frac{dE}{dt} = 0$$

+2

$\therefore$ ~~Energy E~~

$\therefore$ Total Energy E is conserved.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$\frac{dL}{dt} = 0$$

$L = \vec{r} \times \vec{p}$

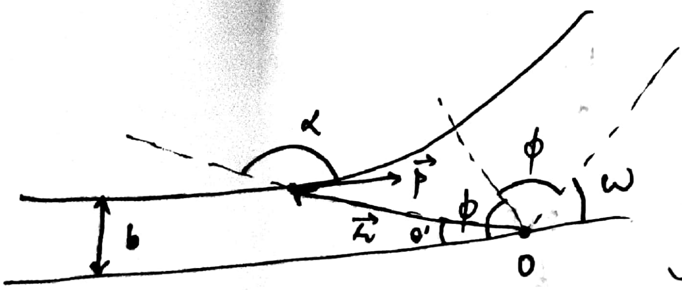
$= \vec{r} \times (m(\dot{\vec{r}} + r\dot{\theta})$

$= mr^2\dot{\theta}$

+2

$\therefore$ Angular momentum L is conserved.

b)



$\vec{L} = \vec{r} \times \vec{p}$

$= rp\sin\alpha$

$= rp\sin\theta'$

$= bp_\infty$

As particle tends to ∞, $= b m v_\infty$

$\theta \to 90'$

$\sin\theta \to 1$

+2

$= mbv_\infty$

c) $\frac{\lambda}{r} = -1 + \varepsilon\cos\theta$

~~When $\theta \to \phi \Rightarrow r \to \infty$~~

~~$\therefore \cos\phi = \frac{1}{\varepsilon}$~~

When $\theta \to \phi \Rightarrow r \to \infty$

$\therefore \cos\phi = 1/\varepsilon$

$L_z = b p_\infty$

~~$\cos\theta =$~~

$$\cos\phi = 1/\varepsilon$$

$$\cos\phi = \frac{1}{\sqrt{1 + \frac{2\lambda E}{k}}}$$

~~$E = \frac{L_z^2}{mk}$~~ $\lambda = \frac{L_z^2}{mk}$

$E = \frac{p_\infty^2}{2m}$

~~$\tan\phi = \frac{2\lambda E}{k} = \frac{2L_z^2}{mk}\cdot\frac{p_\infty^2}{2m}$~~

~~$= \frac{2b^2p_\infty^2 p_\infty^2}{2m^2k}$~~

~~$= \left(\frac{bp_\infty}{m}\right)^2 \frac{1}{k}$~~

$2\phi + \omega = \pi$

~~$\omega = \pi$~~

$\phi = \frac{\pi - \omega}{2}$

$$\tan\phi = \sqrt{\frac{2\lambda E}{k}} = \sqrt{\frac{2E}{k}\cdot\frac{L_z^2}{mk}}$$

$$= \sqrt{\frac{2E}{k}\cdot\frac{b^2p^2}{mk}}$$

$$= \sqrt{\frac{4Eb^2}{k^2}\frac{p^2}{2m}} = \sqrt{\frac{4E^2b^2}{k^2}}$$

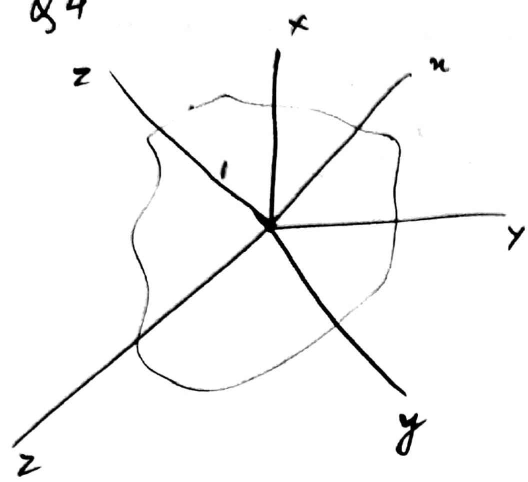
$$\tan\left(\frac{\pi}{2} - \frac{\omega}{2}\right) = \frac{2Eb}{k}$$

+4

$$\cot(\omega/2) = \frac{2bE}{k}$$

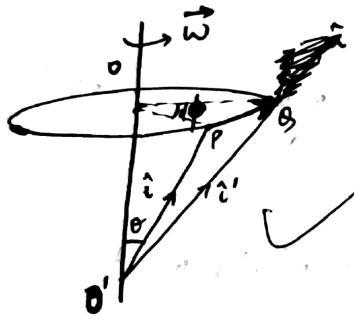
$$b = \frac{k}{2E}\cot(\omega/2)$$

Q4



$$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \frac{dL_x}{dt}\hat{i} + L_x \frac{d\hat{i}}{dt} + \frac{dL_y}{dt}\hat{j} + L_y \frac{d\hat{j}}{dt} + \frac{dL_z}{dt}\hat{k} + L_z \frac{d\hat{k}}{dt}$$



$$\frac{d\hat{i}}{dt} = \frac{\hat{i}' - \hat{i}}{dt} = \frac{PB}{dt} = \frac{\sin\theta \, d\phi}{dt}$$

$$= \omega \sin\theta$$

$$= \vec{\omega} \times \hat{n}_x$$

Similarly, $\frac{d\hat{j}}{dt} = \vec{\omega} \times \vec{n}_y$

L_x & n_x are in same direction.

$$\therefore \left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \frac{dL_x}{dt}\hat{i} + L_x(\vec{\omega} \times \hat{n}_x) + \frac{dL_y}{dt}\hat{j} + L_y(\vec{\omega} \times \hat{n}_y) + \frac{dL_z}{dt}\hat{k} + L_z(\vec{\omega} \times \hat{n}_z)$$

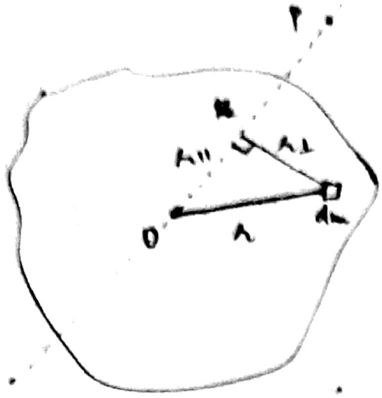
$$= \frac{d}{dt}(L_x \hat{i} + L_y \hat{j} + L_z \hat{k}) + \vec{\omega} \times \vec{L_x} + \vec{\omega} \times \vec{L_y} + \vec{\omega} \times \vec{L_z}$$

$$= \frac{d}{dt}\vec{L} + \vec{\omega} \times (\vec{L_x} + \vec{L_y} + \vec{L_z})$$

$$\left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \left(\frac{d\vec{L}}{dt}\right)_{xyz} + \vec{\omega} \times \vec{L}$$

+10

Q 5



Let n_1, n_2, n_3 be direction cosines of $\vec{OP}$

$n_1^2 + n_2^2 + n_3^2 = 1$

$$I_{OP} = \int dm \cdot r_\perp^2$$

$$= \int dm \left(r^2 - r_{\parallel}^2\right)$$

$$= \int dm \left(x^2+y^2+z^2 - (n_1 x + n_2 y + n_3 z)^2\right)$$

$$= \int dm \Big(x^2+y^2+z^2 - \big(n_1^2x^2 + n_2^2y^2 + n_3^2z^2 + 2n_1n_2xy + 2n_2n_3yz + 2n_1n_3xz\big)\Big)$$

$$= \int dm \Big(x^2(1-n_1^2) + y^2(1-n_2^2) + z^2(1-n_3^2) - 2n_1n_2xy - 2n_2n_3yz - 2n_1n_3xz\Big)$$

$$= \int dm \Big(x^2(n_2^2+n_3^2) + y^2(n_1^2+n_3^2) + z^2(n_1^2+n_2^2) - 2n_1n_2xy - 2n_2n_3yz - 2n_1n_3xz\Big)$$

$$= \int dm \Big(n_1^2(y^2+z^2) + n_2^2(x^2+z^2) + n_3^2(x^2+y^2) - 2n_1n_2xy - 2n_2n_3yz - 2n_1n_3xz\Big)$$

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$$= \int dm\,(y^2+z^2)\,n_1^2 + \int dm\,(x^2+z^2)\,n_2^2 + \int dm\,(x^2+y^2)\,n_3^2$$

$$+ 2n_1n_2\left(-\int dm\, xy\right) + 2n_2n_3\left(-\int dm\, yz\right)$$

$$+ 2n_1n_3\left(-\int dm\, xz\right)$$

$$= \cancel{I_{xx} n_1^2 + I_{yz} n_2^2}$$

$$I_{op} = I_{xx} n_1^2 + I_{yy} n_2^2 + I_{zz} n_3^2 + 2n_1n_2 I_{xy} + 2n_2n_3 I_{yz} + 2n_1n_3 I_{xz}$$

+10

$$I_{op} = I_{xx} n_x^2 + I_{yy} n_y^2 + I_{zz} n_z^2 + 2n_xn_y I_{xy} + 2n_yn_z I_{yz} + 2n_xn_z I_{xz}$$

If we define $\cancel{S_x = \frac{S_x}{\sqrt{I_{op}}},\ y = }\ \frac{S}{\sqrt{I}}$

$$S_x = \frac{n_x}{\sqrt{I_{op}}}, \quad S_y = \frac{n_y}{\sqrt{I_{op}}}, \quad S_z = \frac{n_z}{\sqrt{I_{op}}}$$

Then,

$$I_{xx} S_x^2 + I_{yy} S_y^2 + I_{zz}^2 S_z^2 + 2S_xS_y I_{xy} + 2S_yS_z I_{yz} + 2S_xS_z I_{xz} = 1$$

This is moment of inertia ellipsoid.

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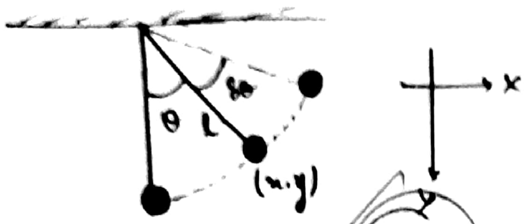
Q2 a) $\frac{d\vec{p}}{dt} = \vec{F} = m\vec{g} + \vec{T}$

+1

b)



$x = l\sin\theta \qquad \delta x = l\cos\theta\,\delta\theta$

$y = l\cos\theta \qquad \delta y = -l\sin\theta\,\delta\theta$

+1

$\dot{x} = l\cos\theta\,\dot{\theta} \qquad \dot{y} = -l\sin\theta\,\dot{\theta}$

$\ddot{x} = -l\sin\theta\,\dot{\theta} + l\cos\theta\,\ddot{\theta} \qquad \ddot{y} = -l\cos\theta\,\dot{\theta} - l\sin\theta\,\ddot{\theta}$ +1

$\vec{T} = -T\cos\theta\,\hat{j} - T\sin\theta\,\hat{i}$

$\vec{g} = g\hat{j}$

+2

By D'Alembert's Principle

$$\Rightarrow \left(-T\sin\theta\,\hat{i} + m\ddot{x}\hat{i}\right)\cdot\vec{\delta x} + \left(-T\cos\theta\,\hat{j} + mg\hat{j} + m\ddot{y}\hat{j}\right)\cdot\vec{\delta y} = 0$$

$$\Rightarrow \left(-T\sin\theta + m(-l\sin\theta\,\dot{\theta} + l\cos\theta\,\ddot{\theta})\right)(l\cos\theta\,\delta\theta)$$
$$+ \left(-T\cos\theta + mg + m(-l\cos\theta\,\dot{\theta} - l\sin\theta\,\ddot{\theta})\right)(-l\sin\theta\,\delta\theta) = 0$$

$$\Rightarrow \left(-\cancel{Tl\sin\theta\cos\theta} - \cancel{ml^2\sin\theta\cos\theta\,\dot{\theta}} + ml^2\cos^2\theta\,\ddot{\theta} + \cancel{Tl\sin\theta\cos\theta} - mgl\sin\theta + \cancel{ml^2\sin\theta\cos\theta\,\dot{\theta}} + ml^2\sin^2\theta\,\ddot{\theta}\right)\delta\theta = 0$$

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$\cancel{\delta\theta \neq 0}$

$$\therefore \quad ml^2\cos^2\theta\,\ddot{\theta} + ml^2\sin^2\theta\,\ddot{\theta} - gl\sin\theta = 0$$

$$\therefore \quad (ml^2\ddot{\theta} - gl\sin\theta)\,\delta\theta = 0$$

+3

d)

$\delta\theta \neq 0$

$$\therefore \quad ml^2\ddot{\theta} - gl\sin\theta = 0$$

$$ml^2\ddot{\theta} = mgl\sin\theta$$

$$l\ddot{\theta} = g\sin\theta.$$

+1

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