CYK/2023/PH201 Mathematical Physics

QUIZ 1

Total Marks: 10 Marks, Duration: 50 Mins
Date: 22 Aug 2023, Tuesday

A

1. [4×1 **Marks**] Answer the following short questions (You can write the answers directly):

   (a) Find $\lim_{z \to e^{i\pi/4}} \frac{z^2}{z^4 + z + 1}$.

   (b) Using the rules of the differentiation, find the derivative of $\sin\left(3z^2 + z\right)$.

   (c) Sketch the map of the unit circle $|z| = 1$ under the transformation $w = e^{i\pi/3}\left(1 + \sqrt{2}e^{i\pi/4}z\right)$.

   (d) Find the numerical value of $\sin^2(2 + 3i)$.

Answers:

(a) $\lim_{z \to e^{i\pi/4}} \frac{z^2}{z^4 + z + 1} = e^{i\pi/4}$.

(b) $\frac{d}{dz}\sin\left(3z^2 + z\right) = (6z + 1)\cos\left(3z^2 + z\right)$

(c) The transformation can be written as $w - e^{i\pi/3} = \sqrt{2}e^{i7\pi/12}z$. Thus, $\left|w - e^{i\pi/3}\right| = \sqrt{2}$. The image of the set is again a circle of radius $\sqrt{2}$ with center at $e^{i\pi/3}$.

(d) $\sin^2(2 + 3i) = 66.4251 - 76.3285i$

2. [**3 Marks**] Determine if the function $u(x, y) = 1 - x(4y + 1)$ is harmonic. If it is harmonic, find the conjugate harmonic function $v(x, y)$ and express $u + iv$ as an analytic function of z.

Answer:
Since $\left(\partial_x^2 + \partial_y^2\right)u = 0$, the function u is harmonic. Since $v_y = u_x$,

$$v_y = -(4y + 1) \implies v(x, y) = -\left(2y^2 + y\right) + g(x).$$

And from $v_x = -u_y$, we get

$$g'(x) = 4x \implies g(x) = 2x^2.$$

And hence $v(x, y) = 2x^2 - 2y^2 - y$. The complex function $u + iv$, will be

$$\begin{aligned} u + iv &= 1 - x(4y + 1) + i\left(2x^2 - 2y^2 - y\right) \\ &= 1 - z + i2z^2. \end{aligned}$$

3. [**3 Marks**]A function $f(z)$ is defined as

$$f(z) = \begin{cases} 2z^2 & \mathrm{Im}(z) > 0 \\ 3\bar{z} & \mathrm{Im}(z) < 0 \end{cases}$$

and C is the anticlockwise circular arc of unit radius with center at $+1$. Write a parametrization for C and using primary definition, find $\int_C f(z)dz$.

Answer:
The whole contour can be written as two contours $C_1 : z = 1 + e^{i\theta}$ with $\theta : -\pi \to 0$ and $C_2 : z = 1 + e^{i\theta}$ with $\theta : 0 \to \pi$. Now, $dz = izd\theta$ for both cases, and

$$\int_{C_1} f(z)dz = \int_{C_1} 3\bar{z}dz = \int_{-\pi}^{0} 3\left(1 + e^{-i\theta}\right)\left(ie^{i\theta}\right)d\theta = 6 + 3i\pi.$$

And

$$\int_{C_2} 2z^2 dz = \int_0^{\pi} 2z^2 \cdot izd\theta = 2i\int_0^{\pi}\left(1 + e^{i\theta}\right)^2 e^{i\theta}d\theta = -\frac{16}{3}$$

Thus,

$$\int_C f(z)dz = \frac{2}{3} + i3\pi.$$