Partial Differential Equations

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Laplace Equation

2.1. Relevance of the Laplace Equation in Physics

The Laplace equation is one of the most important equations in Physics. Some situations are listed below.

(1) **Electrostatics:** Electric potential ϕ satisfies the LE in:
  (a) regions in vaccum without any charges ($\rho = 0$).
  (b) regions in linear (homogeneous, isotropic) media without *free* charges ($\rho_{\text{free}} = 0$).

(2) **Magnetostatics:** Maganetic scalar potentials ϕ_M is defined through $\mathbf{H} = -\nabla\phi_M$ satisfies the LE
  (a) regions in vaccum without any currents ($J = 0$).
  (b) regions in linear (homogeneous, isotropic) media without *free* currents ($J_{\text{free}} = 0$).

(3) **Gravitation:** In free space, gravitational potential satisfies the laplace equation.

(4) **Heat Flow:** Temperature T steady state heat flow Q

(5) **Fluid Flow:** The velocity potential ψ in irrotational motion of an ideal fluid. The velocity potential is defined as $\vec{\mathbf{q}} = -\nabla\psi$ where $\vec{\mathbf{q}}$ is the velocity field.

2.2. Method of Variable Separation

The method of variable separation is one most widely used technique for partial differential equations. The method depends on separability of the differential equation as well as the boundary conditions. The laplace equation is neatly separable.

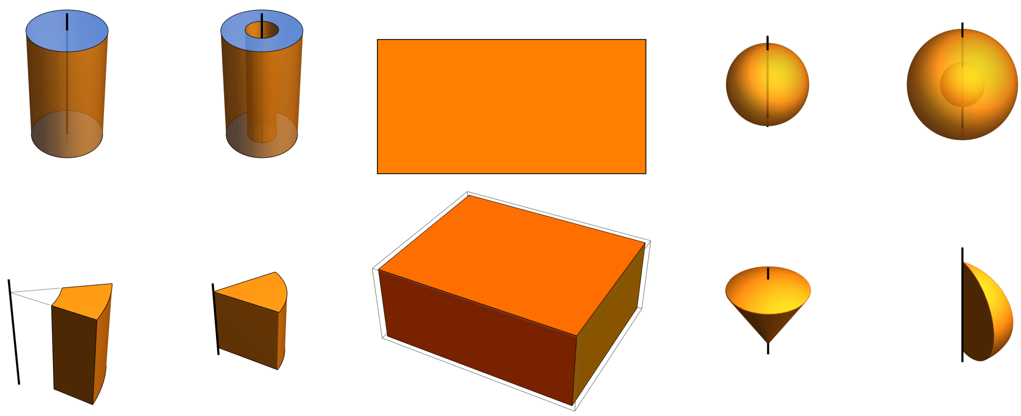
Coordinate System	Separated Equations	Special Functions
Cartesian: $V = X(x)Y(y)Z(z)$ $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	$X'' = -\alpha^2 X;$	$\sin\alpha x, \cos\alpha x$
	$Y'' = -\beta^2 Y;$	$\exp(\pm\alpha x)$
	$Z'' = \left(\alpha^2 + \beta^2\right) Z$	
Cylindrical V= $R(\rho)\,T(\phi)\,Q(z)$ $\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$	$Q'' = k^2 Q$	$\sin m\phi, \cos m\phi$
	$T'' = -m^2 T$	$J_m(k\rho), N_m(k\rho)$
	$\rho^2 R'' + \rho R' + \left(k^2\rho^2 - m^2\right) R = 0$	
Spherical $V = R(r)Y(\theta,\phi)$ $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{-L^2}$	$L^2 Y = l(l+1)Y$	$Y_{lm}(\theta,\phi)$
	$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{l(l+1)}{r^2}R = 0$	$r^l, r^{-(l+1)}$

The figure 2.2.1 shows some representative domains in which variable separation method is successfully employed.

2.3. Cartesian Coordinates (2D and 3D)

Consider a rectangular domain $D = \{(x,y)\,|\,0 < x < a;\ 0 < y < b\}$. We are looking for solutions $V(x,y)$ of the Dirichlet problem

$$\nabla^2 V(x,y) = 0 \qquad (x,y)\in D$$
$$V(0,y) = V(a,y) = V(x,0) = 0 \qquad \forall\, x,y$$
$$V(x,b) = f(x)$$



(A) Cylindrical (B) Cartesian (C) Spherical

FIGURE 2.2.1. Typical domain shapes in different coordinate systems

where $f(x)$ is a given function. Typical application of these involve infinitely long rectangular pipe,s running parallel to the z-axis (from $-\infty$ to ∞). By translational symmetry along z axis, the solution is independent of z and the problem reduces to a 2D problem. The key steps are:

- ▷ Looking for product solutions $V(x, y) = X(x)Y(y)$, the Laplace equation separates into two ODEs.

$$\begin{aligned} \frac{d^2}{dx^2}X &= -\alpha^2 X, \\ \frac{d^2}{dy^2}Y &= \alpha^2 Y. \end{aligned}$$

  The general solutions are $X(x) = A\sin\alpha x + B\cos\alpha x$ and $Y(x) = C\sinh\alpha y + B\cosh\alpha y$.
- ▷ It is important to note that the BC also separate in this case:

$$\begin{aligned} V(0, y) = X(0)\,Y(y) = 0 &\implies X(0) = 0 \\ V(a, y) = X(a)\,Y(y) = 0 &\implies X(a) = 0 \\ V(x, 0) = X(x)\,Y(0) = 0 &\implies Y(0) = 0 \end{aligned}$$

- ▷ Applying BC at $x = 0,\ a$, we get

$$X_m(x) = A_m \sin\left(\frac{m\pi x}{a}\right) \qquad m = 1, 2, \ldots$$

  We discard $m = 0$ (trivial solution) and $m < 0$ solutions since they are linearly dependent on the solutions with $m > 0$.
  With BC at $y = 0$

$$Y_m(x) = C_m \sinh\left(\frac{m\pi}{a}y\right)$$

- ▷ We have got infinitely many solutions $V_m(x, y) = X_m(x)\,Y_m(y)$, which satisfy Laplace equation and BC at three boundaries. Thus the general solution of the original Laplace equation must be linear sum of all these solutions:

$$\Phi(x, y) = \sum_{m=1}^{\infty} V_m(x, y) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi}{a}x\right)\sinh\left(\frac{m\pi}{a}y\right).$$

  The unknowns A_m in the general solution must be determined by the remaining BC.
- ▷ The last BC requires

$$V(x, b) = \sum_m A_m \sinh\left(\frac{m\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right) \qquad \forall x$$

using orthogonality of the harmonics function[1], we can obtain all unknown coefficients

$$A_m = \frac{2}{a\sinh\left(\frac{m\pi}{a}b\right)}\int_0^a V(x,b)\sin\left(\frac{m\pi}{a}x\right)dx$$

We will demonstrate the method with examples.

Example 1. If $V(x,b) = V_0$, then

$$V_0 = \sum_m A_m \sinh\left(\frac{m\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right) \qquad \forall y$$

Then

$$\begin{aligned} A_m &= \frac{2V_0}{m\pi\sinh\left(\frac{m\pi}{a}b\right)}\left(1-(-1)^m\right) \\ &= \begin{cases} \frac{4V_0}{m\pi\sinh\left(\frac{m\pi}{a}b\right)} & \text{odd } m \\ 0 & \text{even } m \end{cases} \end{aligned}$$

Thus the solution is

$$V(x,y) = \sum_{\text{odd } m}\frac{4V_0}{m\pi\sinh\left(\frac{m\pi b}{a}\right)}\sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right)$$

Example 2. If $V(x,b) = x$ then

$$x = \sum_m A_m \sinh\left(\frac{m\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right) \qquad \forall y$$

Then

$$A_m = (-1)^{m+1}\frac{2a}{m\pi\sinh\left(\frac{m\pi}{a}b\right)}$$

Thus the solution is

$$V(x,y) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{2a}{m\pi\sinh\left(\frac{m\pi}{a}b\right)}\sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right)$$

Example 3. If the same problem is posed with BCs $V(0,y) = V(a,y) = 0$, $V(x,0) = V_0$, $V(x,b) = V_0$. The solution is obtained for two separate problems.

- ▷ Let $V_1(x,y)$ be the solution of

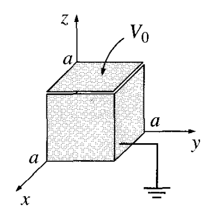
$$\begin{aligned} \nabla^2 V(x,y) = 0 & \qquad (x,y)\in D \\ V(0,y) = V(a,y) = V(x,0) = 0 & \qquad \forall x,y \\ V(x,b) = V_0 & \end{aligned}$$

- ▷ And Let $V_2(x,y)$ be the solution of

$$\begin{aligned} \nabla^2 V(x,y) = 0 & \qquad (x,y)\in D \\ V(0,y) = V(a,y) = V(x,b) = 0 & \qquad \forall x,y \\ V(x,0) = V_0 & \end{aligned}$$

- ▷ Now it can verified that $V(x,y) = V_1(x,y) + V_2(x,y)$ is the solution of the original problem.

Example 4. **Griffiths Problem 3.15** A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (See Fig). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.



[1] $\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx = \frac{a}{2}\delta_{m,n}$

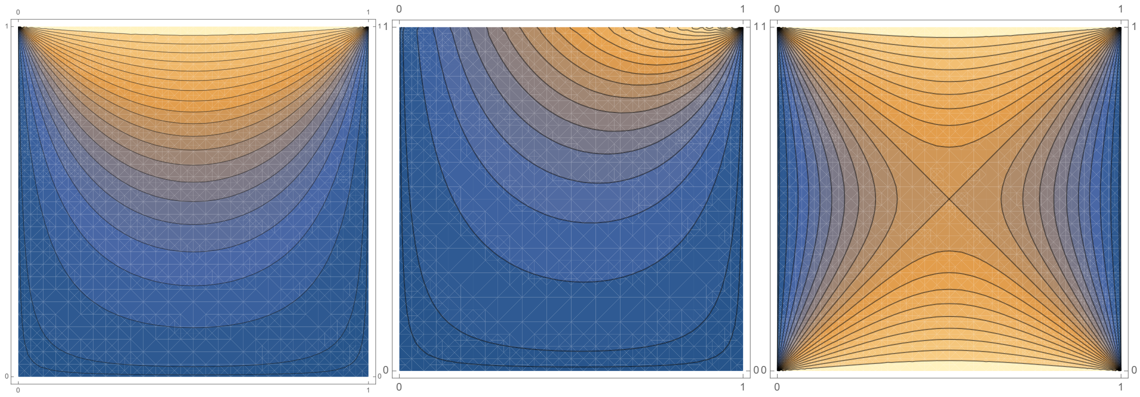


FIGURE 2.3.1. Equipotential Lines for various examples

Using variable separation method with $V(x,y,z) = X(x)Y(y)Z(z)$, we get

$$\begin{aligned}
\frac{d^2}{dx^2}X &= -k_x^2 X \implies X(x) = A\sin k_x x + B\cos k_x x \\
\frac{d^2}{dy^2}Y &= -k_y^2 Y \implies Y(y) = C\sin k_y y + D\cos k_y y \\
\frac{d^2}{dz^2}Y &= k_z^2 Z \implies Z(z) = E\sinh k_z z + F\cosh k_z z.
\end{aligned}$$

where $k_z^2 = k_x^2 + k_y^2$. Thus, the general solution is:

$$\begin{aligned}
V(x,y,z) = &\left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\cos k_y y\right) \\
&\times \left(E\cosh k_z z + F\sinh k_z z\right).
\end{aligned}$$

▷ BC at $x=0$ implies that $A=0$. BC at $x=a$ implies that $k_x = l\pi/a$ for positive integer l.
▷ BC at $y=0$ implies that $C=0$. BC at $y=a$ implies that $k_y = m\pi/a$ for positive integer m.
▷ BC at $z=0$ implies that $E=0$.

Thus,

$$V(x,y,z) = \sum_{lm} B_{lm}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{a}y\sinh\frac{\pi\sqrt{l^2+m^2}}{a}z$$

and the final BC says

$$V(x,y,a) = V_0 = \sum_{lm} B_{lm}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{a}y\sinh\pi\sqrt{l^2+m^2}.$$

Using the fourier trick, we get

$$\begin{aligned}
B_{lm}\cdot\frac{a}{2}\cdot\frac{a}{2}\cdot\sinh\pi\sqrt{l^2+m^2} &= \int_0^a V_0\sin\frac{l\pi}{a}x\sin\frac{m\pi}{a}y\,dx\,dy \\
&= \frac{V_0a^2}{\pi^2 lm}\left(1-(-1)^l\right)\left(1-(-1)^m\right)
\end{aligned}$$

Thus, the final solution is

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{lm=1,3,\ldots}\left(\frac{1}{lm\sinh\pi\sqrt{l^2+m^2}}\right)\sin\frac{l\pi}{a}x\sin\frac{m\pi}{a}y\sinh\frac{\pi\sqrt{l^2+m^2}}{a}z.$$

2.4. Laplace Equation in Plane Polar Coordinates

In plane polar coordinate system, the LE is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}.$$

Let $V(\rho,\phi) = R(r)\,P(\phi)$, then the LE separates into

$$\boxed{P'' - m^2 P = 0} \implies P(\phi) = \begin{cases} A_m \sin(m\phi) + B_m \cos(m\phi) & m \neq 0 \\ A_0\phi + B_0 & m = 0 \end{cases}$$

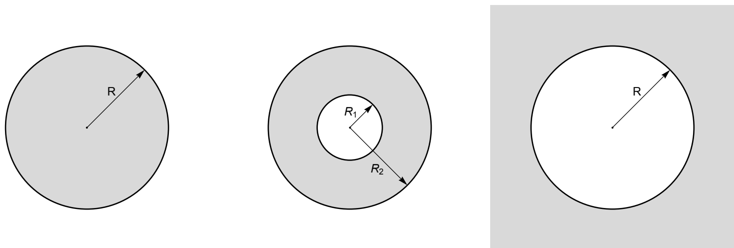
and

$$\boxed{\rho^2 R'' + \rho R' - m^2 R = 0} \implies R(r) = \begin{cases} C_m \rho^m + D_m \rho^{-m} & m \neq 0 \\ C_0 + D_0 \ln \rho & m = 0. \end{cases}$$

Thus, for any value of m (any complex number),

$$V_m(\rho,\phi) = \begin{cases} [C_m\rho^m + D_m\rho^{-m}]\,[A_m \sin(m\phi) + B_m \cos(m\phi)] & m \neq 0 \\ [C_0 + D_0 \ln \rho]\,[A_0\phi + B_0] & m = 0 \end{cases}$$

is a solution of the LE.



Circular and Annular Domains: The three domains shown in the figure are $D_1 = \{(\rho,\phi) \mid \rho \leq R\}$, $D_2 = \{(\rho,\phi) \mid R_1 \leq \rho \leq R_2\}$ and $D_3 = \{(\rho,\phi) \mid \rho \geq R\}$. The BVP will be posed by specifying the values of the solution on the circular boundaries. However, the implied BC in all these cases are:

(1) In all three cases, the solution must be single valued: $V(\rho,\phi+2\pi) = V(\rho,\phi)$. This implies that m must be an integer and also $A_0 = 0$.

(2) In addition, for domain D_1, the solution must be finite at $\rho = 0$: $V(0,\phi) < \infty$. This implies that $D_m = 0$ for all m and also $D_0 = 0$. The genral solution will be

$$V(\rho,\phi) = B_0 + \sum_{m=1}^{\infty} \rho^m \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right).$$

(3) For the domain D_3, the solution must vanish at $\rho \to \infty$: $V(\infty,\phi) = 0$. Thus, the general solution must be

$$V(\rho,\phi) = \sum_{m=1}^{\infty} \rho^{-m} \left(A_m \sin(m\phi) + B_m \cos(m\phi)\right)$$

(4) Since $\rho = 0$ and $\rho = \infty$ are not in the domain D_2, the general solution will be

$$V(\rho,\phi) = [C_0 + D_0 \ln \rho] + \sum_{m=1}^{\infty} \left(C_m\rho^m + D_m\rho^{-m}\right)\left(A_m \sin(m\phi) + B_m \cos(m\phi)\right)$$

The coefficients can be computed using standard Fourier method. For D_1 and D_3,

$$\begin{aligned} A_m &= \frac{1}{\pi R^{\pm m}} \int_0^{2\pi} V(R,\phi) \sin(m\phi)\, d\phi \\ B_m &= \frac{1}{\pi R^{\pm m}} \int_0^{2\pi} V(R,\phi) \cos(m\phi)\, d\phi \qquad m \neq 0 \\ B_0 &= \frac{1}{2\pi} \int_0^{2\pi} V(R,\phi) d\phi \end{aligned}$$

For domain D_2, the same trick gives us, for $m=0$,

$$\begin{aligned} C_0 + D_0 \ln R_1 &= \int_0^{2\pi} V\left(R_1,\theta\right) d\theta \\ C_0 + D_0 \ln R_2 &= \int_0^{2\pi} V\left(R_2,\theta\right) d\theta. \end{aligned}$$

And for $m \neq 0$,

$$\begin{aligned} A_m \pi \left(C_m R_1^m + D_m R_1^{-m}\right) &= \int_0^{2\pi} V(R_1,\phi) \sin\left(m\phi\right) d\phi \\ B_m \pi \left(C_m R_1^m + D_m R_1^{-m}\right) &= \int_0^{2\pi} V(R_1,\phi) \cos\left(m\phi\right) d\phi \\ A_m \pi \left(C_m R_2^m + D_m R_2^{-m}\right) &= \int_0^{2\pi} V(R_2,\phi) \sin\left(m\phi\right) d\phi \\ B_m \pi \left(C_m R_2^m + D_m R_2^{-m}\right) &= \int_0^{2\pi} V(R_2,\phi) \cos\left(m\phi\right) d\phi \end{aligned}$$

Example 5. Find potential inside an infinitely long cylinder (radius a) with walls at fixed potential V_0. Find the potential inside and outside the cylinder.

Clearly the solution must be independent ϕ. Interior problem gives us the constant solution, that is $V\left(\rho,\phi\right) = V_0$. Solution does not exist for the exterior problem if we insist on $V \to 0$ at infinity. However if that constraint is removed then

$$C_0 + D_0 \ln R = V_0.$$

There are two possible solutions $V\left(\rho,\phi\right) = V_0$ or $V\left(\rho,\phi\right) = V_0 \frac{\ln \rho}{\ln R}$.

Example 6. Two halves of a long hollow conducting cylinder of inner radius R are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Find the potential inside.

The general solution by variable separation method is

$$V\left(\rho,\phi\right) = b_0 + \sum_{n=1}^{\infty} \rho^n \left(a_n \sin\left(n\phi\right) + b_n \cos\left(n\phi\right)\right)$$

And

$$\begin{aligned} a_n &= \frac{1}{R^n \pi} \int_0^{2\pi} V(R,\phi) \sin\left(n\phi\right) d\phi \\ b_n &= \frac{1}{R^n \pi} \int_0^{2\pi} V(R,\phi) \cos\left(n\phi\right) d\phi \end{aligned}$$

And the boundary potential is given by

$$V\left(b,\phi\right) = \begin{cases} V_1 & \phi \in \left(0,\pi\right) \\ V_2 & \phi \in \left(-\pi,0\right). \end{cases}$$

Then,

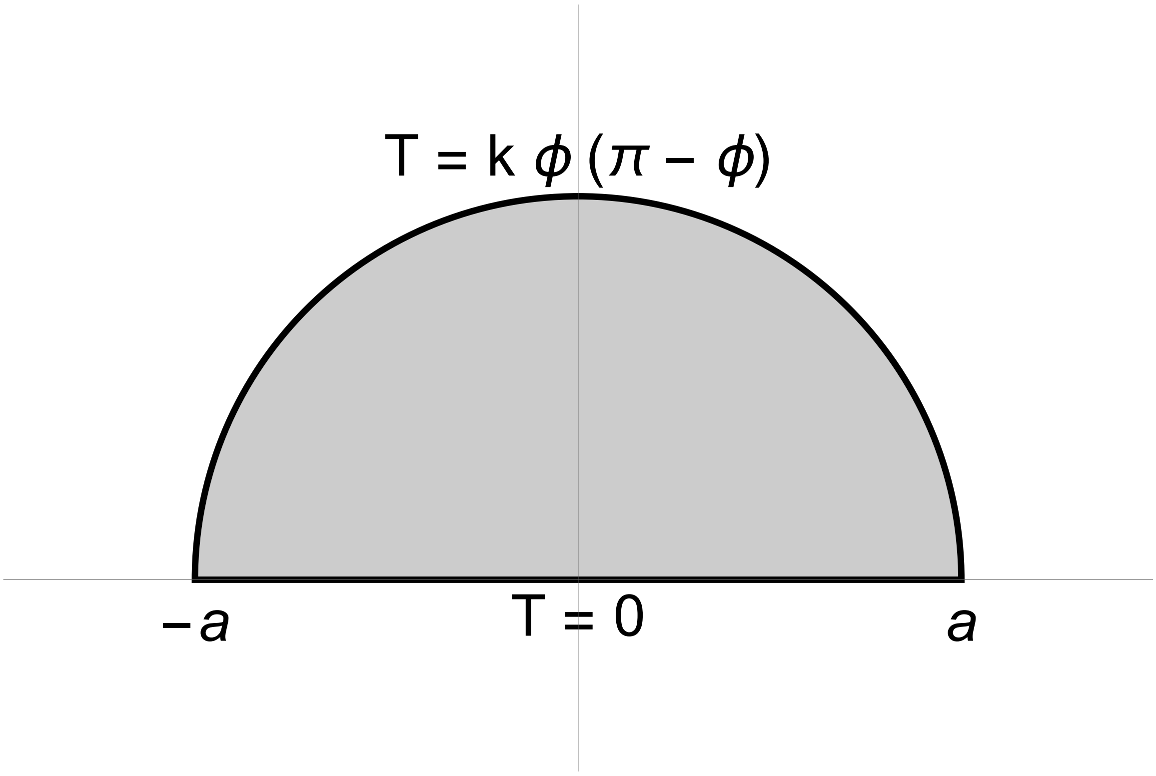
$$a_n = \frac{V_1}{R^n \pi} \int_0^{\pi} \sin\left(n\phi\right) d\phi + \frac{V_2}{R^n \pi} \int_{\pi}^{2\pi} \sin\left(n\phi\right) d\phi = \begin{cases} \frac{2(V_1 - V_2)}{R^n \pi n} & \text{odd } m \\ 0 & \text{even } m \end{cases}$$

and similarly,

$$b_n = \begin{cases} \frac{V_1 + V_2}{2} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Wedge-Shaped Domains.

EXAMPLE 7. A semi-circular plate of radius a has its circumference kept at temperature $T(r=a,\phi) = k\phi(\pi-\phi)$, where r and ϕ are the polar coordinates. The diametric boundary (along the horizontal axis) is kept at zero temperature. k is a positive constant. Let $T(r,\phi)$ be the steady-state temperature distribution on the plate.Sketch isotherms of T.



The solution is of the form

$$T(\rho,\phi) = \begin{cases} [C_m\rho^m + D_m\rho^{-m}][A_m \sin(m\phi) + B_m\cos(m\phi)] & m \neq 0 \\ [C_0 + D_0 \ln\rho][A_0\phi + B_0] & m = 0 \end{cases}$$

but now $\phi \in (0,\pi)$. Since the angle ϕ does not span the whole range of $0 \to 2\pi$, we cannot impose singlevaluedness property on T and consenquently n is *not necessarily an integer.*

- ▷ T must be 0 at $\rho = 0$, thus $C_0 = D_0 = 0$ and $D_m = 0$ for all m.
- ▷ BC at $\phi = 0$ implies that $B_m = 0$ for all m.
- ▷ BC at $\phi = \pi$ implies that $\sin(m\pi) = 0$ thus, m must be an integer.

Now, we can write the linear combination of all solutions to obtain general solution,

$$T(\rho,\phi) = \sum_{m=1}^{\infty} A_m\rho^m \sin(m\phi)$$

The final boundary condition is

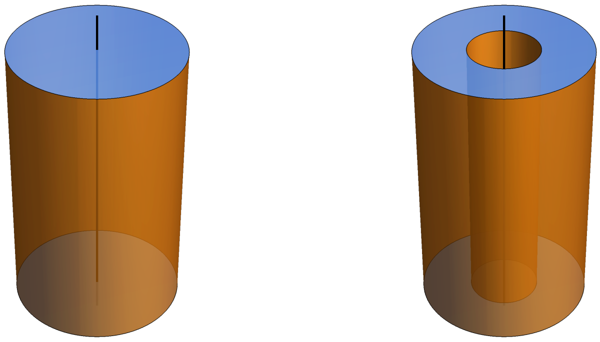
$$k\phi(\pi-\phi) = \sum_{m=1}^{\infty} A_m a^m \sin(m\phi)$$

which gives us

$$\begin{aligned} A_m &= \frac{2k}{\pi}a^{-m}\int_0^{\pi} \phi(\pi-\phi)\sin(m\phi)\,d\phi \\ &= \frac{4k}{\pi}a^{-m}\frac{1-(-1)^m}{m^3} \end{aligned}$$

And the final answer then is

$$T(\rho,\phi) = \frac{8k}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\left(\frac{\rho}{a}\right)^{2m+1}\sin((2m+1)\phi)$$

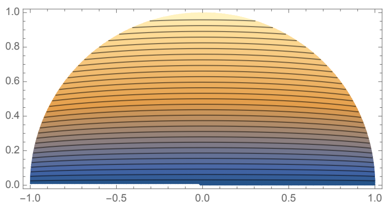


(A) $\rho \geq R$, $z_1 \leq z \leq z_2$ and $0 \leq \phi < 2\pi$

(B) $\rho_1 \leq \rho \leq \rho_2$, $z_1 \leq z \leq z_2$ and $0 \leq \phi < 2\pi$

FIGURE 2.5.1

The isotherm sketches are



2.5. Laplace Equation in Cylindrical/Annular Regions With Caps

▷ These regions are cylindrical (no restriction on ϕ, *ie*, $0 \leq \phi < 2\pi$) but are bounded by $\rho_1 \leq \rho \leq \rho_2$ and $z_1 \leq z \leq z_2$ (at least one of the two z_1 and z_2 is finite).

▷ Separation of variables: If $\psi(\rho, \phi, z) = R(\rho)\, T(\phi)\, Q(z)$,

– ϕ equation:

$$\boxed{T'' - m^2 T = 0} \implies T(\phi) = A \sin(m\phi) + B \cos(m\phi)\,.$$

Singlevaluedness property, $T(\phi + 2\pi) = T(\phi)$ implies that m must be a nonnegative integer.

– z equation:

$$\boxed{Q'' + k^2 Q = 0} \implies Q(z) = Ae^{kz} + Be^{-kz}$$

– ρ equation:

$$\boxed{\rho^2 R'' + \rho R' + \left(k^2\rho^2 - m^2\right) R = 0} \implies R(\rho) = AJ_m(k\rho) + BN_m(k\rho)$$

where J_m and N_m are Bessel and Neumann functions (also called Bessel functions of the first and the second kind).

▷ The product-type solutions (that is $\psi = RTQ$) then are

$$\psi_{km}(\rho, \phi, z) = \left(A_{km} J_m(k\rho) + B_{km} N_m(k\rho)\right)\left(C_{km} \sin(m\phi) + D_{km} \cos(m\phi)\right) \times \left(E_{km} e^{kz} + F_{km} e^{-kz}\right)$$

▷ The general solution to given Dirichlet/Neumann boundary problems (on the regions mentioned above) is

$$\psi(\rho, \phi, z) = \sum_{k,m} \psi_{km}(\rho, \phi, z)$$

EXAMPLE 8. Consider a semi-infinite cylinder which is capped at the bottom. $\rho \leq a$ and $z \geq 0$. The boundary conditions are

$$\psi(\rho = a, \phi, z) = 0$$
$$\psi(\rho, \phi, z = \infty) = 0$$
$$\psi(\rho, \phi, z = 0) = \psi_0(\rho, \phi)$$

where ψ_0 is some given function of ρ and ϕ.

- ▷ $\psi \to 0$ as $z \to \infty$: Coefficients of e^{kz}, $E_{km} = 0$ for all k and m.
- ▷ $\psi < \infty$ at $\rho = 0$: Coefficients of $N_m(k\rho)$, $B_{km} = 0$ for all k and m.

After these two condition, the general solution reduces to

$$\psi(\rho, \phi, z) = \sum e^{-kz} J_m(k\rho)\left(C_{km}\sin(m\phi) + D_{km}\cos(m\phi)\right)$$

with all other coefficients absorbed in C and D

- ▷ $\psi \to 0$ as $\rho = a$: Let χ_{mn} be the n^{th} zero of J_m and $k_{mn} = \chi_{mn}/a$. The boundary condition, $J_m(ka) = 0$ implies that k must be equal to k_{mn} for some n.

Thus, relabling the coeffiecients.

$$\psi(\rho, \phi, z) = \sum_{n=1,m=0}^{\infty} e^{-k_{mn}z} J_m(k_{mn}\rho)\left(C_{nm}\sin(m\phi) + D_{nm}\cos(m\phi)\right)$$

Applying the final condition, $\psi(\rho, \phi, z = 0) = \psi_0(\rho, \phi)$, we get

$$\psi_0(\rho, \phi) = \sum_{n=1,m=0}^{\infty} J_m(k_{mn}\rho)\left(C_{nm}\sin(m\phi) + D_{nm}\cos(m\phi)\right)$$

Using the orthogonality of Bessel and trigonometric functions, and using the fact that $\int_0^a [J_m(k_{mn}\rho)]^2 \rho d\rho = \frac{a^2}{2}[J_{m+1}(\chi_{mn})]^2$ we get

$$\begin{aligned}
C_{nm} &= \frac{2}{\pi a^2 [J_{m+1}(\chi_{mn})]^2} \int_0^a \int_0^{2\pi} \psi_0(\rho, \phi) J_m(k_{mn}\rho)\sin(m\phi)\rho d\phi d\rho \\
D_{nm} &= \frac{2}{\pi a^2 [J_{m+1}(\chi_{mn})]^2} \int_0^a \int_0^{2\pi} \psi_0(\rho, \phi) J_m(k_{mn}\rho)\cos(m\phi)\rho d\phi d\rho \quad m \neq 0 \\
D_{n0} &= \frac{1}{\pi a^2 [J_1(\chi_{0n})]^2} \int_0^a \int_0^{2\pi} \psi_0(\rho, \phi) J_0(k_{0n}\rho)\rho d\phi d\rho \quad m = 0
\end{aligned}$$

If $\psi_0(\rho, \phi) = V_0$, then

$$\begin{aligned}
C_{nm} &= \frac{2V_0}{\pi a^2 [J_{m+1}(\chi_{mn})]^2} \int_0^a J_m(k_{mn}\rho)\rho d\rho \int_0^{2\pi} \sin(m\phi)d\phi = 0 \qquad \forall m, n \\
D_{nm} &= \frac{2V_0}{\pi a^2 [J_{m+1}(\chi_{mn})]^2} \int_0^a J_m(k_{mn}\rho)\rho d\rho \int_0^{2\pi} \cos(m\phi)d\phi = 0 \qquad m \neq 0 \\
&= \frac{2V_0}{a^2 [J_1(\chi_{0n})]^2} \int_0^a J_0(k_{0n}\rho)\rho d\rho \qquad m = 0 \\
&= \frac{2V_0}{\chi_{0n}[J_1(\chi_{0n})]} \qquad m = 0
\end{aligned}$$

using $\int_0^a J_0(k_{0n}\rho)\rho d\rho = \frac{1}{\chi_{0n}} J_1(\chi_{0n})$. thus the final solution is

$$\psi_0(\rho, \phi) = 2V_0 \sum_{n=1}^{\infty} \frac{1}{\chi_{0n}[J_1(\chi_{0n})]} e^{-\chi_{0n}z/a} J_0\left(\chi_{0n}\frac{\rho}{a}\right)$$

2.6. Conformal Mapping

Conformal mapping. 25 March

- **Definition:** A transformation $w = f(z)$ is said to be conformal at z_0, if it is analytic at z_0 and $f'(z_0) \neq 0$.

- **Properties**

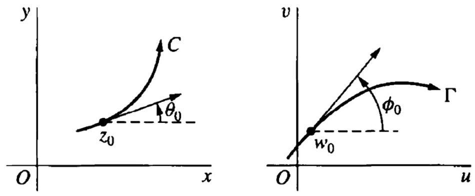
  (a) Conformal mapping preserves angle between curves.

  (b) It scales the shapes by the same factor, thus preserving shapes.

  (c) Conformal mappings have local inverses.

Proofs: (a) Let f be conformal at z_0. Let $f(z_0) = w_0$ and $\arg(f'(z_0)) = \psi_0$. Let $z(t)$ be a curve in z-plane s.t. $z(t_0) = z_0$ and let $w(t) = f(z(t))$.

Now $\frac{dz}{dt}(t_0)$ is the tangent to $z(t)$ curve and $\frac{dw}{dt}(t_0)$ is a tangent to $w(t)$ curve.



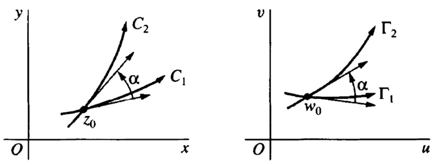
But

$$\frac{dw}{dt}(t_0) = \frac{d}{dt} f(z(t_0)) = \frac{df}{dz}(z_0) \cdot \frac{dz}{dt}(t_0).$$

$$\Rightarrow \quad \mathrm{Arg}(w'(t_0)) = \mathrm{Arg} f'(z_0) + \mathrm{Arg}\, z'(t_0).$$

$$\Rightarrow \quad \phi_0 = \psi_0 + \theta_0$$

If there is another curve passing thro' z_0, $\quad \phi_1 = \psi_0 + \theta_1$



$$\Rightarrow \quad \phi_1 - \phi_0 = \theta_1 - \theta_0$$

(b) $|f(z) - f(z_0)| \doteq |f'(z_0)|\,|z - z_0|$ Lengths are scaled in same way in all directions.

as opposed to $|f(\vec{r}) - f(\vec{r}_0)| \doteq |\nabla f(r_0) \cdot (\bar{r} - \bar{r}_0)|$ in real analysis.

(c) $f'(z_0) \neq 0 \Rightarrow u(x,y)$ and $v(x,y)$ are transformation

$$J(z_0) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 = |f'(z_0)|^2$$

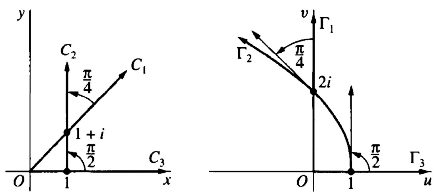
Thus f is one-one mapping in some nbd of z_0.

Ex $f(z) = w = z^2 = x^2 - y^2 + 2ixy$

$C_1: \; x = y = t$ $\qquad \Gamma_1: \; u = 0 \quad v = 2t^2$

$C_2: \; x = 1 \quad y = t$ $\qquad \Gamma_2: \; u = 1 - t^2 \quad v = 2t$

$z_0 = 1 + i$ $\qquad w_0 = 2i$



$\text{Arg}(C_1) = \pi/4$ $\qquad \hat{n}_1 = \partial\Gamma_1/\partial t = 4it \qquad \text{Arg}(\Gamma_1) = \pi/2$

$\text{Arg}(C_2) = \pi/2$ $\qquad \hat{n}_2 = \partial\Gamma_2/\partial t = -2t + 2i \qquad \text{Arg}(\Gamma_2) = \tan^{-1}(-1)$

$= 3\pi/4.$

- Harmonic Functions.

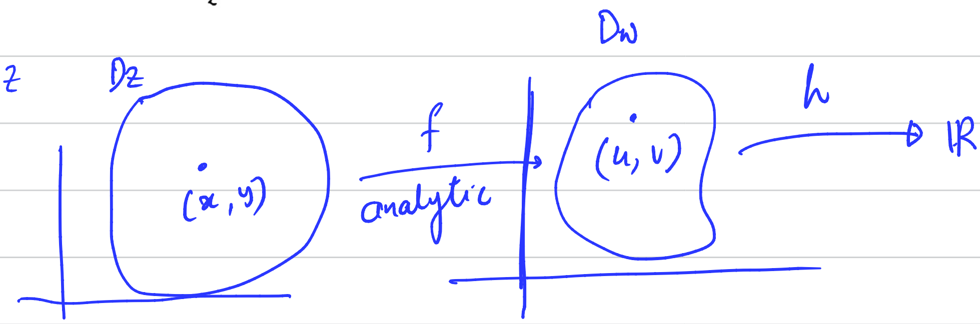
Theorem. *Suppose that an analytic function*

(1) $$w = f(z) = u(x, y) + iv(x, y)$$

maps a domain D_z in the z plane onto a domain D_w in the w plane. If $h(u, v)$ is a harmonic function defined on D_w, then the function

(2) $$H(x, y) = h[u(x, y), v(x, y)]$$

is harmonic in D_z.



$f(z) = u(x,y) + iv(x,y)$

$H: D_z \to \mathbb{R}$ s.t. $H(x,y) = h(u(x,y), v(x,y))$.

given that h is harmonic $\Rightarrow h_{uu} + h_{vv} = 0$

and f is analytic $\Rightarrow u_x = v_y$ and $u_y = -v_x$, $u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

Now $$H_x = \frac{\partial H}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial h}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial h}{\partial v}$$

$$H_{xx} = \partial^2 u/\partial x^2 \; \partial h/\partial u + \partial v/\partial x^2 \; \partial h/\partial v + u_x^2 h_{uu} + 2u_x v_x h_{uv} + v_x^2 h_{vv}$$

$$H_{yy} = u_{yy} \; \partial h/\partial u + v_{yy} \; \partial h/\partial v + u_y^2 h_{uu} + 2u_y v_y h_{uv} + v_y^2 h_{vv}$$

$\Rightarrow H_{xx} + H_{yy} = 0.$

- **Boundary Conditions:**

Thm: Suppose $f(z) = w$ is conformal on a curve C in D_z and Γ be its image. Along Γ a function $h(u,v)$ satisfies

$$h = h_0 \text{ or } dh/dn = 0$$

Then $H(x,y) = h(u(x,y), v(x,y))$ satisfies

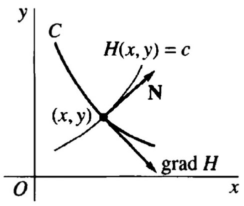
$$H = h_0 \text{ or } dH/dn = 0 \quad \text{on } C.$$

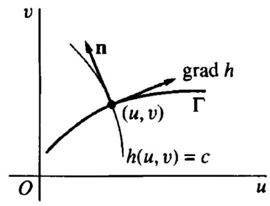
Proof: First part $h = h_0 \Rightarrow H = h_0$ on C.

For the second part,

$$\partial h/\partial n = \nabla h \cdot \hat{n} = 0$$

$\hat{n} \perp \Gamma$. Let $\hat{N} \perp C$. To prove $\nabla H \cdot \hat{N} = 0$





- In D_w, $\nabla h \perp \Gamma'$ where $h = c$ on Γ'.
- In D_z, C' be the preimage of Γ'.
- Then on C', $H = C$.
- $\nabla H \perp C'$
- Now $\angle$ bet C and C' $= \angle$ bet Γ and Γ' $= \pi/2$

$$\Rightarrow \nabla H \perp \hat{N} \Rightarrow \nabla H \cdot \hat{N} = \partial H/\partial \hat{N} = 0.$$

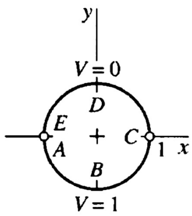
Applications:

Potentials in cylindrical space: Translational symmetry in z s.t. V is ind of z and is only function of x, y.

Example 1.

A cylinder of radius 1 unit. BC

$$V(1,\theta) = \begin{cases} 0 & \theta \in [0,\pi) \\ 1 & \theta \in (-\pi, 0). \end{cases}$$



Find potential inside.

$$V_{xx} + V_{yy} = 0 \quad \forall \; r < 1.$$

Consider a transformation:

$$w = i\,\frac{1-z}{1+z} \qquad\qquad z = \frac{i-w}{i+w}$$

Now $z=e^{i\theta} \Rightarrow w = i\,(1-z)(1+\bar z)\,/\,(1+z\bar z+z+\bar z)$

$$= i\,(1-z\bar z+\bar z-z)\,/\,(1+z\bar z+z+\bar z) \quad -②$$

$$= i\cdot(-2i\sin\theta)/(2+2\cos\theta)$$

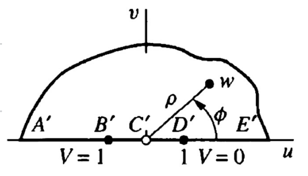
$$= \tan\theta/2$$

as $\theta : -\pi \to \pi$ $\quad W : -\infty$ to ∞ along x axis

and $z = re^{i\theta} \Rightarrow$

$$w = \frac{i\,(1-r^2-2ir\sin\theta)}{1+r^2+2r\cos\theta} = \frac{2r\sin\theta + i(1-r^2)}{1+r^2+2r\cos\theta}$$

that is $\operatorname{Im} w \geq 0$



Thus we find soln.

$$V_{uu} + V_{vv} = 0$$

$V(u,0) = 0$ and $V(-u,0) = 1 \qquad u>0$.

$$V = \frac{\phi}{\pi} \qquad \phi : 0 \to \pi \qquad \nabla^2 = \frac{1}{\rho_w}\frac{\partial}{\partial\rho_w}\rho_w\frac{\partial}{\partial\rho_w} + \frac{1}{\rho_w^2}\frac{\partial^2}{\partial\phi^2}$$

$$\therefore\; V(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{v}{u}\right) = \frac{1}{\pi}\tan\left(\frac{1-x^2-y^2}{2y}\right)$$

Möbius Transformations

- Linear transformation : Translate, Rotate and scaling.

$$W = Az + B = \mathrm{Arg}(A) \cdot |A|\, z + B$$

Step 1 : $Z = |A|\, z$: Scale about origin

Step 2 : $W = \mathrm{Arg}(A) \cdot Z$: Rotate about origin

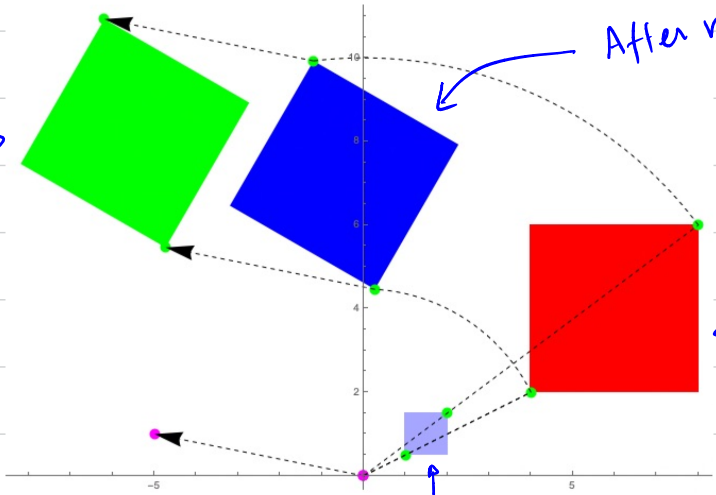
Step 3 : $\omega = W + B$: Translate

- Rotate about $z_0 \Rightarrow \omega = A(z - z_0) + z_0$ $\quad A = e^{i\alpha}$
- Fixed pt : $\omega = Az + B$ leaves $z = B/(1-A)$ invariant if $A \neq 1$.

Example $\omega = Az + B$ $\quad A = 4e^{i\pi/3}$ $\quad B = (5 + i)$

After rotation by $\pi/3$

After translation.



After scaling by 4 times

Original

- Mapping by $1/z$, Extended complex plane $= \mathbb{C} \cup \{\infty\}$, $1/0 = \infty$, $1/\infty = 0$.

$\omega = 1/z$, $\omega = u + iv$ and $z = x + iy$, A, B, C and D real nos.

Circle/Line $A(x^2+y^2) + Bx + Cy + D = 0$ $\quad B^2 + C^2 \geq 4AD$

maps to $D(u^2+v^2) + Bu - Cv + A = 0$

	z plane	w plane
$A \neq 0, D \neq 0$	Circle not thro' origin	Circle not thro' origin
$A = 0, D \neq 0$	Line not thro' origin	Circle thro' origin
$A \neq 0, D = 0$	Circle thro' origin	Line not thro' origin.
$A = 0, D = 0$	Line thro' origin	Line thro' origin

- Mobius Transformations

- $$w = \frac{az+b}{cz+d} \qquad ad-bc \neq 0, \quad a,b,c,d \text{ Complex.}$$

$$= \frac{a}{c} + \frac{(bc-ad)}{c} \frac{1}{cz+d}$$

$Z = cz+d$

$W = 1/Z$

$w = a/c + \frac{(bc-ad)}{c} \cdot W$

- Maps "circles and lines" to "Circles and lines"
- Mapping is one-one and onto extended plane.
- Implicit form $\underbrace{z_1, z_2, z_3}_{\text{distinct}}$ map to $\underbrace{w_1, w_2, w_3}_{\text{distinct}}$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

- Mapping of upper half plane to a circle

$$W = e^{i\alpha} \frac{z-z_0}{z-\bar{z}_0} \qquad \operatorname{Im}(z_0) > 0$$

$$|W| = \frac{|z-z_0|}{|z-\bar{z}_0|} < 1 \quad \text{if } \operatorname{Im}(z) > 0$$

$$= 1 \quad \text{if } \operatorname{Im}(z) = 0$$

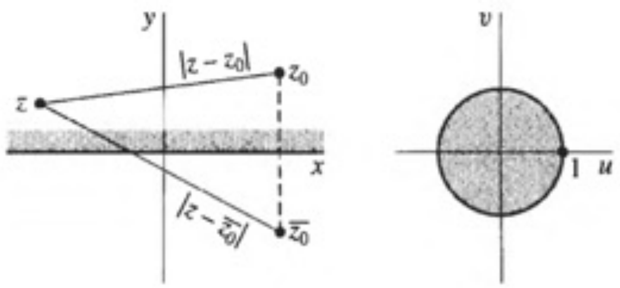


FIGURE 108
$w = e^{i\alpha}\dfrac{z-z_0}{z-\bar{z}_0}$ $(\operatorname{Im} z_0 > 0)$.

Example 2:

$D_z = \{ z \,/\, \mathrm{Im}(z) > 0 \}$

Boundary D_z : x axis

Now $T(x,0) = 1 \qquad |x| < 1$

$\qquad\qquad\quad = 0 \qquad |x| > 1$

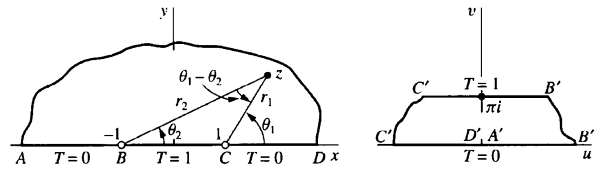


FIGURE 135

$w = \log \frac{z-1}{z+1} \left(\frac{r_1}{r_2} > 0, -\frac{\pi}{2} < \theta_1 - \theta_2 < \frac{3\pi}{2} \right).$

Consider a mapping

$$w = \log \frac{(z-1)}{(z+1)} = \ln\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2)$$

where $z - 1 = r_1 e^{i\theta_1}$ and

$\qquad z + 1 = r_2 e^{i\theta_2}$ $\qquad -\frac{\pi}{2} \leq (\theta_1 - \theta_2) \leq \frac{3\pi}{2}$

On line CD $\quad r_1/r_2 < 1$ and $\theta_1 = \theta_2 = 0.$

Thus $\quad w = \ln(r_1/r_2) \to -\infty \quad$ as $r_1 \to 0$ near C

$\qquad\qquad\qquad\qquad\quad \to 0 \quad$ as $r_1/r_2 \to 1$ near D

The boundaries are shown in the figure.

The new problem in u-v plane is to find h s.t.

$$h_{uu} + h_{vv} = 0 \qquad \text{with} \quad \left.\begin{aligned} h(u,0) &= 0 \\ \text{and } h(u,\pi) &= 1 \end{aligned}\right\} \forall u.$$

since we have bc ind of u, assume that h is fn of v alone, then

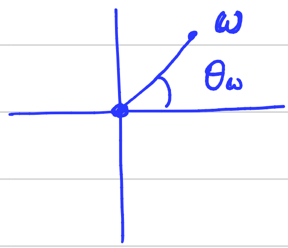
$$h(u,0) = \frac{v}{\pi}. \quad \checkmark$$

$\therefore$

$$T(x,y) = \frac{1}{\pi} \tan^{-1}\left(\frac{2y}{x^2+y^2-1} \right)$$

Problems at the end of section 105. Brown & Churchill

Q1. $h(u,v) = 1/\pi \tan^{-1}(v/u) = \theta_\omega/\pi$



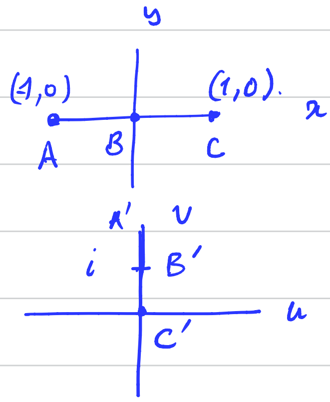
h is harmonic and bounded.

Now $g(u,v) = \frac{1}{\pi}\theta_\omega + \mathrm{Re}(Ae^{\omega})$

$= \frac{1}{\pi}\theta_\omega + e^{u}\sin\theta$

Note that $g(u,0) = h(u,0)$ since $\sin\theta|_{\theta=0} = 0$
and g is harmonic. Thus g is sol^n of the same problem with bc at $\theta=0$ but not at $|z|\to\infty$ (unbounded since $e^u \to \infty$ as $u\to\infty$).

Q2. The mapping: $w = i(1-z)/(1+z)$



for $z = (x,0)$, $x>0$ (Line BC)

$w = i(1-x)/(1+x)$

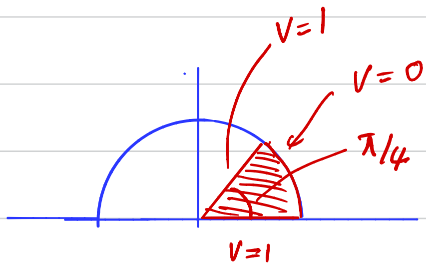
$\Rightarrow u = 0 \qquad 0 \le v \le 1$

for $z = (-x,0)$, $x>0$ (line AB).

$w = i(1+x)/(1-x)$

$\Rightarrow u = 0 \qquad 1 \le v \le \infty$

Q3. Now z^4 will map this wedge to the half disc of problem 2.

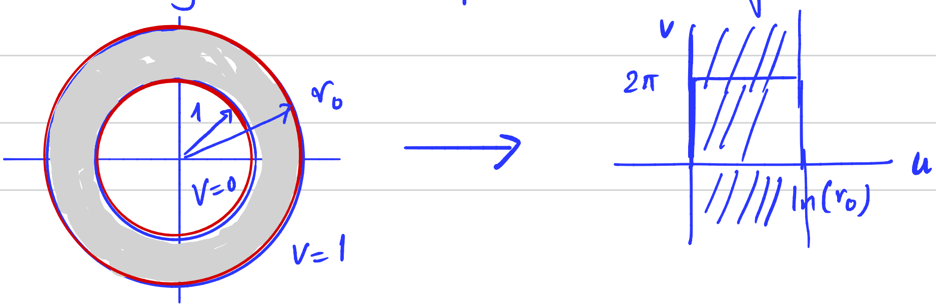


$w = z^4 \Rightarrow u = x^4 + y^4 - 10x^2y^2$

$v = 6xy(x^2-y^2)$

And for half disc the sol^n is $V = \frac{2}{\pi}\tan^{-1}\left(\frac{1-u^2-v^2}{2v}\right)$

Q4. The log function maps annular region



Now since periodicity is ok in this problem, the sol^{n} can be given by $V = \dfrac{u}{\ln r_0} = \dfrac{\ln r}{\ln r_0} = \dfrac{\ln(x^2+y^2)}{2\ln r_0}$.

Q5. The mapping $w = \ln\dfrac{(a-z)}{(a+z)}$ maps this region to a horizontal strip (the temp example above).

Q6. Note that in temp. example above, $V = \tfrac{1}{2}$ on $x^2+y^2=1$ curve! Then, clearly sol^{n} to this problem can be obtained as

$$V = \frac{2}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

Q7. Again refer to the temp example.

$z = -\tanh\frac{w}{2}$

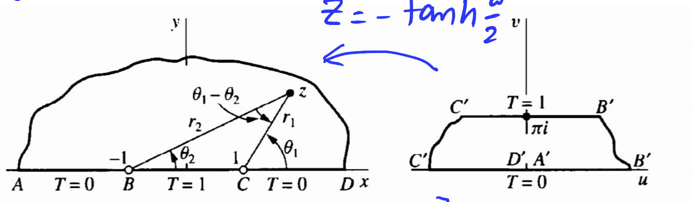
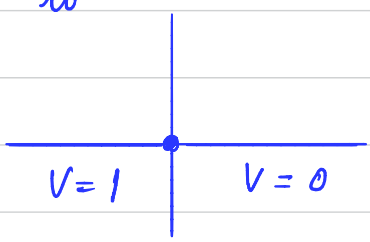


FIGURE 135
$w = \log\dfrac{z-1}{z+1}\quad\left(\dfrac{r_1}{r_2} > 0,\ -\dfrac{\pi}{2} < \theta_1-\theta_2 < \dfrac{3\pi}{2}\right).$

Now, we can use the $z = -\tanh^{-1} w/2$ to map the present problem to



8. Look at this transformation, we need inverse of this to get the answer.

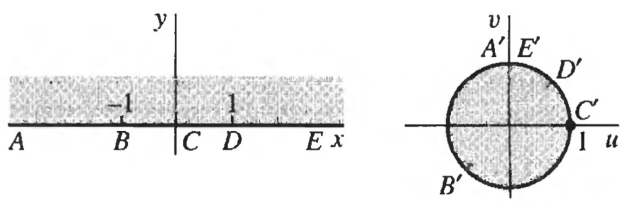


FIGURE 110
$w = \dfrac{iz+\exp(i\pi/4)}{z+\exp(i\pi/4)}.$

CHAPTER 3

Wave Equation

3.1. Wave Equation in 1D

▷ Examples: Strings, long air columns, telegraph equation etc.

3.2. Wave Equation in Cartesian Coordinates (2D)

▷ Typical example for 2D wave equation: Vibrations of thin membranes (drums)
▷ The DE is

$$c^2\nabla^2 u(\mathbf{r},t) = \frac{\partial^2}{\partial t^2} u(\mathbf{r},t) \tag{3.2.1}$$

where $\mathbf{r}$ is a vector in 2D. The first step is separation of spatial and temporal parts. Let $u(\mathbf{r},t) = \psi(\mathbf{r})\,T(t)$. This leads to

$$\begin{gathered}\ddot{T} + \omega^2 T = 0 \\ \nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = 0\end{gathered} \tag{3.2.2}$$

where $\omega = ck$. The second equation is called Helmholtz equation.
▷ Helmholtz equation is separable in cartesian coordinates, If $\psi(x,y) = X(x)Y(y)$, then

$$\begin{aligned} X'' + k_x^2 X = 0 &\implies X = Ce^{ik_x x} \\ Y'' + k_y^2 Y = 0 &\implies Y = Ce^{ik_y y}\end{aligned}$$

leading to the elementary solutions called plane wave solutions

$$u(\mathbf{r},t) = Ce^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

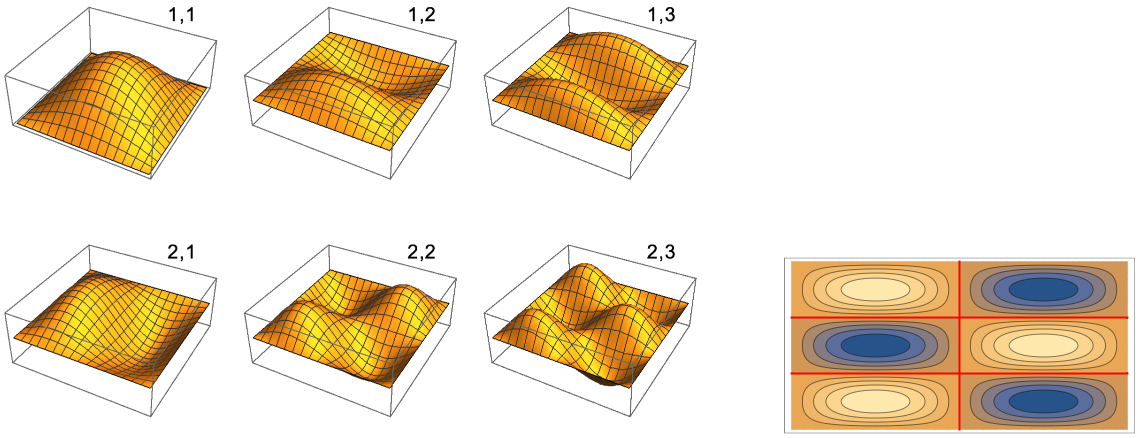
where $\mathbf{k} = (k_x, k_y)$ is called wave vector, $k = |\mathbf{k}|$ and $\omega = ck$. (ω is always taken as positive, where as $\mathbf{k}$ is any vector).
▷ Consider a Dirichlet BC problem on a rectangle, $0 \le x \le a$ and $0 \le y \le b$, with

$$\begin{aligned} u(0,y,t) = u(a,y,t) = 0 \qquad &\forall y, t \\ u(x,0,t) = u(x,b,t) = 0 \qquad &\forall x, t\end{aligned}$$

▷ The Helmholtz equation can be further separated in cartesian coordinates as $\psi(x,y) = X(x)Y(y)$. And applying bc, we get normal modes of vibrations

$$\psi_{mn}(x,y) = A_{mn}\sin(k_{x,n}x)\sin(k_{y,m}y)$$

where $k_{x,n} = n\pi/a$ and $k_{y,m} = m\pi/b$ with $m,n = 1,2,\ldots$. And the frequencies of these normal modes are $\omega_{nm} = ck_{nm} = c\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$.
▷ Some normal modes are shown here. The nodal lines are also shown.



▷ Initial-boundary value problem: The full statement of the problem is

$$\begin{aligned}
c^2\nabla^2 u(\mathbf{r},t) &= \frac{\partial^2}{\partial t^2}u(\mathbf{r},t)\\
u(0,y,t) = u(a,y,t) &= 0 \qquad \forall y,t\\
u(x,0,t) = u(x,b,t) &= 0 \qquad \forall x,t\\
u(x,y,0) &= f(x,y) \qquad \forall x,y\\
\frac{d}{dt}u(x,y,0) &= g(x,y) \qquad \forall x,y.
\end{aligned}$$

The general solution (after the first two boundary conditions) is

$$u(x,y,t) = \sum \left(A_{mn}\sin\omega_{mn}t + B_{mn}\cos\omega_{mn}t\right)\sin(k_{x,n}x)\sin(k_{y,m}y)$$

Now, A_{mn} and B_{mn} can be found from the last two conditions.

3.3. Wave Equation in Plane Polar Coordinates (2D)

▷ Let us begin with Helmholtz equation. Let $\psi(\rho,\phi) = R(\rho)Q(\phi)$. Then

$$\frac{1}{\psi}\left(\nabla^2 + k^2\right)\psi = \frac{1}{R}\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} + k^2R\right) + \frac{1}{\rho^2}\underbrace{\left(\frac{1}{Q}\frac{d^2Q}{d\phi^2}\right)}_{=-m^2} = 0$$

▷ The separated equations are

$$\begin{aligned}
\frac{d^2Q}{d\phi^2} + m^2Q &= 0\\
\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left(k^2\rho^2 - m^2\right)R &= 0
\end{aligned}$$

▷ The elementary solutions are circular waves. Assuming that $\phi \in [0,2\pi)$, single valued ness condition implies that m must be an integer. And assuming that the solutions must be finite at origin, we get

$$u(\rho,\phi,t) = \left[AH_m^{(1)}(k\rho) + BH_m^{(2)}(k\rho)\right]\sin(m\phi+\alpha)\,e^{i\omega t}$$

The Henkel functions have an asymptotic form

$$\begin{aligned}
H_m^{(1)}(k\rho) &= J_m(k\rho) + iN_m(k\rho) \sim \sqrt{\frac{2}{\pi k\rho}}e^{i\left(k\rho-(2m+1)\frac{\pi}{4}\right)}\\
H_m^{(2)}(k\rho) &= J_m(k\rho) - iN_m(k\rho) \sim \sqrt{\frac{2}{\pi k\rho}}e^{-i\left(k\rho-(2m+1)\frac{\pi}{4}\right)}
\end{aligned}$$

which makes these into travelling circular waves, that is,

$$u \sim \frac{1}{\sqrt{\rho}}e^{i(\pm k\rho-\omega t)}$$

▷ Consider a boundary value problem:

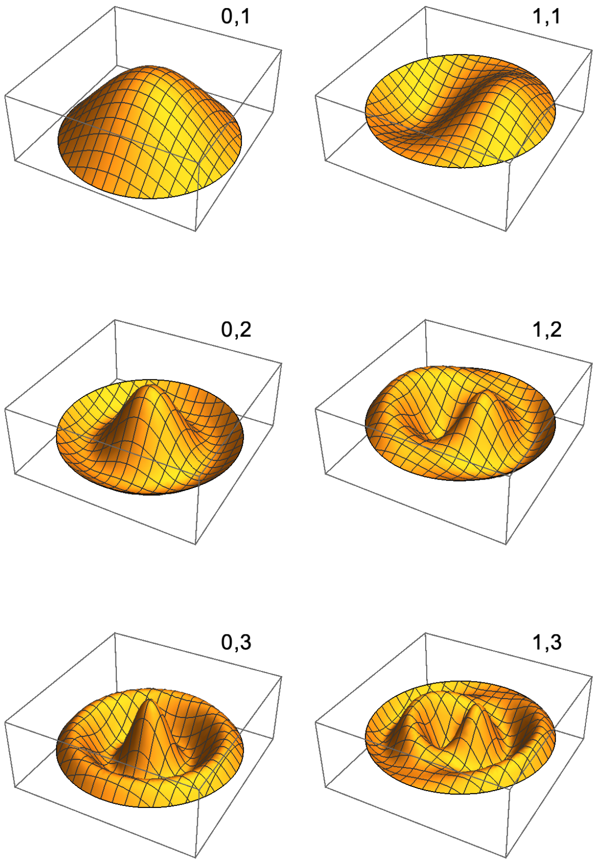
$$u(a, \phi, t) = 0 \qquad \forall t, \phi$$

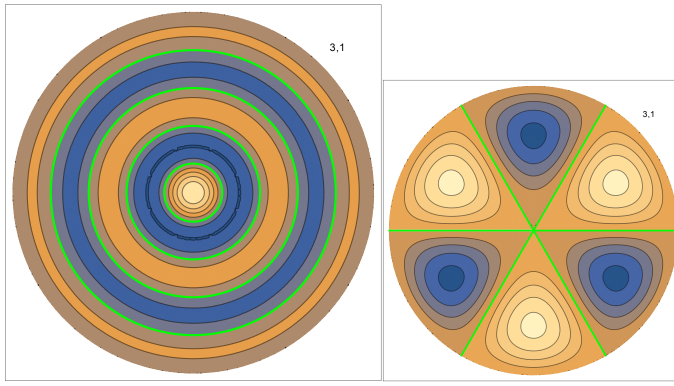
This implies that $\psi(a, \phi) = 0$ for all ϕ and $R(a) = 0$. The solutions to the radial equation are bessel functions, we get

$$\psi_{mn}(\rho, \phi) = J_m(k_{mn}r)\left(A_{mn}\cos m\phi + B_{mn}\sin m\phi\right) \qquad m = 0, 1, \ldots.$$

Here, $k_{mn} = \chi_{mn}/a$ where x_{mn} is n^{th} zero of J_m. The frequencies of these modes are $\omega_{mn} = k_{mn}$.

▷ Some normal modes are shown here:





3.4. Wave Equation in Spherical Coordinates

▷ Again we start with the Helmholtz equation

$$c^2\nabla^2\psi(r, \theta, \phi) + k^2\psi(r, \theta, \phi) = 0.$$

The Laplacian operator in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{-L^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{r^2}$$

▷ Using variable separation as $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} - \frac{1}{r^2}\left(\frac{1}{Y}L^2 Y\right) + k^2 = 0.$$

This gives

$$L^2 Y = l\left(l+1\right)Y$$
$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{l\left(l+1\right)}{r^2}R + k^2 R = 0$$

▷ Special case of $k = 0$ we have Laplace equation. The solutions to the radial equation are simple $R(r) = r^l$ or $r^{-(l+1)}$.

3.4.1. Spherial Harmonics.

▷ $L^2 Y = l\left(l+1\right)Y$ separates to, $Y = P_l^m\left(\theta\right)Q\left(\phi\right)$

$$\frac{d^2}{d\phi^2}Q + m^2 Q = 0$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P_l^m - \left(\frac{m^2}{\sin^2\theta} + l\left(l+1\right)\right)P_l^m = 0$$

▷ Since $Q\left(\phi\right) = Q\left(\phi + 2\pi\right)$, m must be an integer.
▷ When $m = 0$, $Q = A\phi + B$, however, $Q(0) = Q(2\pi)$ implies that Q is a constant function. The second equation becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P_l - l\left(l+1\right)P_l = 0$$

and is put in a familiar form by substituting $x = \cos\theta$, and $d/d\theta = -\sin\theta d/dx$:

$$\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}P_l - l\left(l+1\right)P_l = 0$$
$$\left(1-x^2\right)\frac{d^2}{dx^2}P_l - 2x\frac{d}{dx}P_l - l\left(l+1\right)P_l = 0.$$

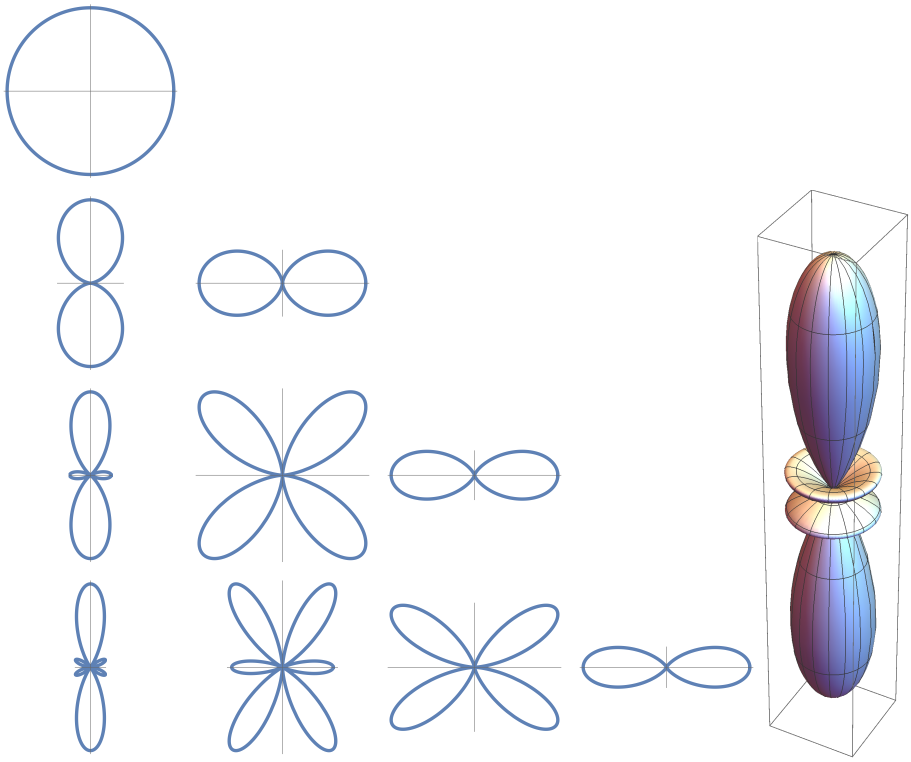
This is the Legendre equation. When solved by Frobenius method, we get two linearly independent solutions. These solutions diverge at $x = \pm 1$, except when the l is a non-negative integer. When l is an integer, one of the two solutions is a polynomials of degree l. These polynomials are called as Legendre Polynomials.
▷ When, $m \neq 0$,clearly, $Q = \exp\left(\pm im\phi\right)$. The second equation has two linearly independent solutions. Again, the well behaved solutions are possible only when $l = 0, 1, 2, \ldots$ and $-l \leq m \leq l$. The well behaved solutions are called as associated Legendre Polynomials and are denoted as $P_l^m\left(\theta\right)$.
▷ Thus,

$$Y_{lm}\left(\theta,\phi\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m\left(\theta\right)e^{im\phi} \qquad m \geq 0$$
$$Y_{l,-m}\left(\theta,\phi\right) = \left(-1\right)^m Y_{lm}^*\left(\theta,\phi\right)$$

▷ First few spherical harmonics:

$l\downarrow\backslash m\rightarrow$	0	1	2
0	$\frac{1}{\sqrt{4\pi}}$		
1	$\sqrt{\frac{3}{4\pi}}\cos\theta$	$-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta$	
2	$\sqrt{\frac{5}{16\pi}}\left(3\cos^2(\theta)-1\right)$	$,-\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta$	$\sqrt{\frac{15}{32\pi}}e^{2i\phi}\sin^2\theta$

▷ Visualization: Polar plots of $\left|Y_{lm}\right|^2$. The squared amplitude is proportional to the energy density in elastic waves. 3D polar plot of $\left|Y_{30}\right|^2$ is shown on the right.



▷ Properties:

(1) Orthogonality:

$$\int_S Y_{l'm'}(\theta,\phi)\, Y_{lm}(\theta,\phi)\, d\Omega = \delta_{l'l}\delta_{m'm}$$

(2) Completeness-I:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

(3) Completeness-II: If $g(\theta,\phi)$ (well behaved?) is any arbitrary function, then

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\phi)$$

and

$$A_{lm} = \int_S Y^*_{lm}(\theta,\phi)\, g(\theta,\phi)\, d\Omega.$$

(4) Addition theorem: If $\mathbf{r} \equiv (r,\theta,\phi)$ and $\mathbf{r}' \equiv (r',\theta',\phi')$ and γ is the angle between two vectors then,

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

and[1]

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

3.4.2. Spherical Bessel functions.

▷ The radial equation

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + k^2R = 0$$

[1]Generating function: $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_l t^l P_l(x)$

▷ It can be rewritten by substitution $u(r) = \sqrt{r}R(r)$:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left[k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right]u = 0$$

This is the Bessel equation and hence the solutions to the radial equation is

$$R_l(r) = \frac{1}{\sqrt{r}}\left(AJ_{l+1/2}\left(kr\right) + BN_{l+1/2}\left(kr\right)\right)$$

▷ Spherical Bessel functions are defined as

$$j_l\left(x\right) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}\left(x\right) = \left(-x\right)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\sin x}{x}\right)$$
$$n_l\left(x\right) = \sqrt{\frac{\pi}{2x}}N_{l+1/2}\left(x\right) = -\left(-x\right)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\cos x}{x}\right)$$
$$h_l^{(1,2)}\left(x\right) = j_l\left(x\right) \pm in_l\left(x\right)$$

▷ First few spherical Bessel functions:

$l\downarrow$	j_l	n_l	$h_l^{(1,2)}$
0	$\frac{\sin x}{x}$	$-\frac{\cos x}{x}$	$\pm\frac{e^{\pm ix}}{ix}$
1	$\frac{\sin x}{x^2} - \frac{\cos x}{x}$	$-\frac{\cos x}{x^2} - \frac{\sin x}{x}$	$\mp\frac{e^{\pm ix}}{x}\left(1 \pm \frac{i}{x}\right)$

▷ Limiting behaviour:

$$\begin{aligned} j_l &\sim \frac{1}{(2l+1)!!}x^l \qquad x\to 0 \\ &\sim \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) \qquad x\to\infty \\ n_l &\sim -\frac{(2l-1)!!}{x^{l+1}} \qquad x\to 0 \\ &\sim -\frac{1}{x}\cos\left(x-\frac{l\pi}{2}\right) \qquad x\to\infty \\ h_l^{(1,2)} &\sim (-i)^{l+1}\frac{e^{\pm ix}}{x} \end{aligned}$$

▷ Orthogonality:

$$\int_0^1 j_m\left(\chi_{mn}r\right)j_m\left(\chi_{mp}r\right)r^2dr = \frac{\delta_{np}}{2}\left[J_{m+1}\left(\chi_{mn}\right)\right]^2$$

3.4.3. Putting it all together.

▷ Thus, elementary solutions are spherical waves given by

$$\begin{aligned} u_{klm}(\mathbf{r},t) &= Y_{lm}\left(\theta,\phi\right)\left(Ah_l^{(1)}\left(kr\right) + Bh_l^{(2)}\left(kr\right)\right)e^{-i\omega t} \\ &\to Y_{lm}\left(\theta,\phi\right)\left(A\frac{1}{r}e^{i(kr-\omega t)} + B\frac{1}{r}e^{-i(kr+\omega t)}\right) \qquad r\to\infty \end{aligned}$$

Here, $r = |\mathbf{r}|$ and $\omega = ck$.

▷ Consider a boundary value problem with $u\left(a,\theta,\phi,t\right) = 0$ for all θ,ϕ and t. The product solutions are

$$\psi(r,\theta,\phi) = Y_{lm}\left(\theta,\phi\right)\left(A_{lm}j_l\left(kr\right) + B_{lm}n_l\left(kr\right)\right).$$

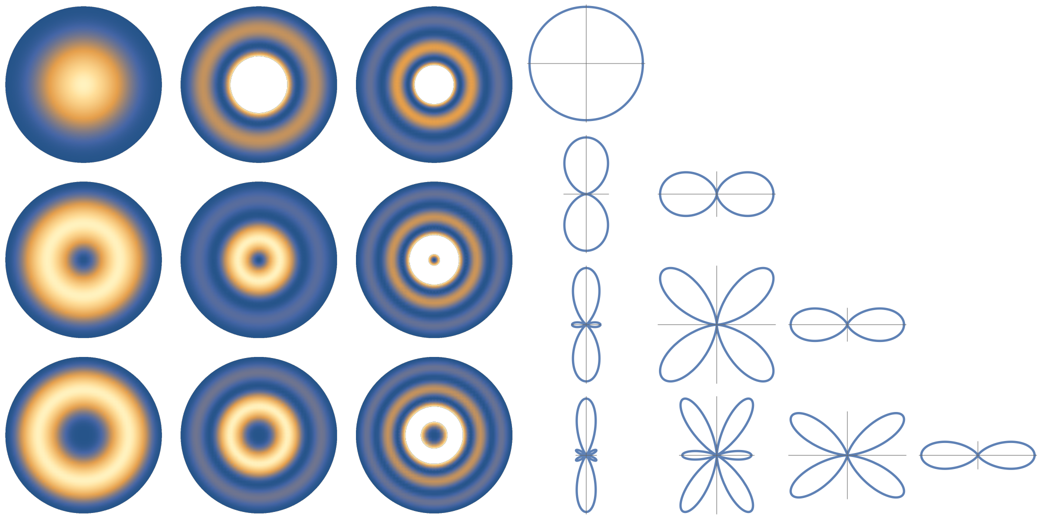
We want the solutions to be finite at $r = 0$, hence $B_{lm} = 0$. And the boundary condition at $r = a$ implies that k must be equal to one of the

$$k_{ln} = \frac{\chi_{ln}}{a} \qquad n = 1, 2, \ldots$$

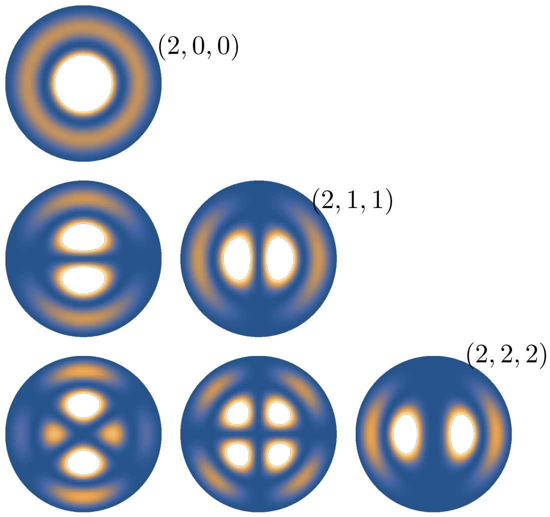
where χ_{ln} is a n^{th} zero of j_l. Thus, the normal modes are

$$u_{nlm}(r,\theta,\phi,t) = A_{nlm}Y_{lm}\left(\theta,\phi\right)j_l\left(k_{ln}r\right)e^{-i\omega_{ln}t}$$

▷ Visualizing the energy density:



(A) Density Plots of $|j_l(k_{ln}r)|^2$ (B) Polar Plots of $|Y_{lm}(\theta,\phi)|^2$



(C) Density plots of the normal modes

FIGURE 3.4.1. Density plots of the normal modes are obatained by multiplying the two plots above, that is$|\psi_{nlm}(r,\theta,\phi)|^2$.

CHAPTER 4

Green's Functions: Inhomogenous Equations

4.1. Introduction and Motivation

Let $\mathcal{L}_x$ be a *linear* differential operator (in x) defined on a domain with specified initial conditions or boundary conditions. The Green's function G is the solution of the equation

$$\mathcal{L}_x G\left(x, x'\right) = \delta\left(x - x'\right)$$

where δ is Dirac's delta function.

Claim 9. The solution of

$$\mathcal{L}_x y(x) = f(x)$$

is

$$y(x) = \int G\left(x, x'\right) f\left(x'\right) dx'. \tag{4.1.1}$$

Proof. Operate by $\mathcal{L}$ on (4.1.1),

$$\begin{aligned}\mathcal{L}_x y(x) &= \int \mathcal{L}_x G\left(x, x'\right) f\left(x'\right) dx' \\ &= \int \delta\left(x - x'\right) f\left(x'\right) dx' = f(x)\end{aligned}$$

□

Thus, at this point the motivation is restricted to solving non-homogeneous PDEs.

4.2. Some Examples

Example 10. Let us find Green's function for $\mathcal{L}_t = \frac{d}{dt}$ and use it to solve $\frac{dy(t)}{dt} = f(t) = t$ on domain $[0, 1\}$. To find $G\left(t, t'\right)$ such that

$$\frac{d}{dt} G\left(t, t'\right) = \delta\left(t - t'\right),$$

first note that for a given t', when $t \neq t'$, the differential equation reduces to $dG/dt = 0$, which implies that

$$G\left(t, t'\right) = \begin{cases} a & t < t' \\ b & t > t' \end{cases}$$

where a and b are some constants. Now,

$$\begin{aligned} \int_{t'-\epsilon}^{t'+\epsilon} \frac{d}{dt} G\left(t, t'\right) dt &= \int_{t'-\epsilon}^{t'+\epsilon} \delta\left(t - t'\right) dt \\ \Longrightarrow \quad G\left(t' + \epsilon, t'\right) - G\left(t' - \epsilon, t'\right) &= 1 \implies b = a + 1. \end{aligned}$$

Thus,

$$\begin{aligned} G\left(t, t'\right) &= \begin{cases} a & t < t' \\ a + 1 & t > t' \end{cases} \\ &= a + \Theta\left(t - t'\right). \end{aligned}$$

where Θ is a heaviside function. Now, the solution to inhomogeneous DE can be found as

$$\begin{aligned} y(t) &= \int_0^1 G\left(t,t'\right) f\left(t'\right) dt' \\ &= \int_0^1 \left(a + \Theta\left(t-t'\right)\right) t' dt' \\ &= \frac{a}{2} + \int_0^t t' dt' = \frac{a}{2} + \frac{t^2}{2} \end{aligned}$$

EXAMPLE 11. Let us look at the forced oscillator as an initial value problem. Let $\mathcal{L}_t = \frac{d^2}{dt^2} + \omega_0^2$. The DE is

$$\begin{aligned} \mathcal{L}_t y(t) &= \frac{d^2 y(t)}{dt^2} + \omega_0^2 y(t) = F \sin \omega t \qquad t \geq 0, \\ y(0) &= \frac{dy}{dt}(0) = 0. \end{aligned}$$

To find the Green's function, we solve

$$\mathcal{L}_t G\left(t,t'\right) = \delta\left(t-t'\right).$$

Using the trick from the previous example, we get

$$G\left(t,t'\right) = \begin{cases} A \sin \omega_0 t + B \cos \omega_0 t & t < t' \\ C \sin \omega_0 t + D \cos \omega_0 t & t > t' \end{cases}$$

Apply the initial conditions to the green's function, that is, $G(0,t') = dG\left(0,t'\right)/dt = 0$. This gives

$$G\left(t,t'\right) = \begin{cases} 0 & t < t' \\ C \sin \omega_0 t + D \cos \omega_0 t & t > t' \end{cases}$$

Now, because this is second ordered DE, G must be continous at t' and dG/dt discontinuous at t',

$$\begin{aligned} \lim_{\epsilon \to 0} \left[G(t'+\epsilon,t') - G(t'-\epsilon,t')\right] &= 0 \\ \lim_{\epsilon \to 0} \left[\frac{d}{dt} G(t'+\epsilon,t') - \frac{d}{dt} G(t'-\epsilon,t')\right] &= 1. \end{aligned}$$

That is

$$\begin{aligned} C \sin \omega_0 t' + D \cos \omega_0 t' &= 0 \\ \omega_0 C \cos \omega_0 t' - \omega_0 D \sin \omega_0 t' &= 1 \end{aligned}$$

and

$$C = \frac{1}{\omega_0} \cos \omega_0 t', \quad D = -\frac{1}{\omega_0} \sin \omega_0 t'.$$

Finally,

$$\begin{aligned} G\left(t,t'\right) &= \begin{cases} 0 & t < t' \\ \frac{1}{\omega_0} \sin \omega_0 \left(t-t'\right) & t > t' \end{cases} \\ &= \frac{1}{\omega_0} \sin \omega_0 \left(t-t'\right) \Theta\left(t-t'\right). \end{aligned}$$

The required solution is

$$\begin{aligned} y(t) &= \int_0^\infty G\left(t,t'\right) f\left(t'\right) dt' \\ &= \frac{F}{\omega_0} \int_0^\infty \sin \omega_0 \left(t-t'\right) \Theta\left(t-t'\right) \sin \omega t' \, dt' \\ &= \frac{F}{\omega_0} \int_0^t \sin \omega_0 \left(t-t'\right) \sin \omega t' \, dt' \\ &= \frac{F}{\left(\omega_0^2 - \omega^2\right)} \left[-\frac{\omega}{\omega_0} \sin \omega_0 t + \sin \omega t\right] \end{aligned}$$

Check:

$$\begin{aligned}\ddot{y}+\omega_0^2 y &= \frac{F}{\left(\omega_0^2-\omega^2\right)}\left[\omega\omega_0\sin\omega_0 t-\omega^2\sin\omega t\right]\\&+\frac{F}{\left(\omega_0^2-\omega^2\right)}\left[-\omega\omega_0\sin\omega_0 t+\omega_0^2\sin\omega t\right]\\&=F\sin\omega t \qquad \checkmark\\y(0)&=0 \qquad \checkmark\\\dot{y(0)}&=\frac{F}{\left(\omega_0^2-\omega^2\right)}\left[-\omega\cos\omega_0 t+\omega\cos\omega t\right]_{t=0}=0 \qquad \checkmark\end{aligned}$$

EXAMPLE 12. Let us look at the first example as a boundary value problem. Let $\mathcal{L}_t=\frac{d^2}{dt^2}+\omega_0^2$ on $[0,a]$. Find a GF such that $G(0,t')=G(a,t')=0$.
Answer:

$$G\left(t,t'\right)=\frac{\sin\left(\omega_0 t_<\right)\sin\left(\omega_0\left(a-t_>\right)\right)}{\omega_0\sin\left(\omega_0 a\right)}$$

Exercises.

(1) Find a GF for operator $\mathcal{L}_t=\frac{d}{dt}+\gamma$, $0\le t$ with initial condition $G(0,t')=0$.
[**Ans:** $G(t,t')=e^{-\gamma(t-t')}\Theta(t-t')$]

(2) Find a GF for operator $\mathcal{L}_x=\frac{d}{dx^2}$, $0\le x\le a$ such that $G(0,x')=G(a,x')=0$.
[**Ans:** $G(x,x')=x_<(a-x_>)/a$]

(3) Find a GF for operator $\mathcal{L}_x=\frac{d}{dx^2}$, $0\le x\le a$ such that $G(0,x')=G(a,x')=0$. Since $G(x,x')$ is a continuous function of x, that vanishes at the boundaries, we can expand G as

$$G\left(x,x'\right)=\sum_{n=1}^{\infty}A_n\left(x'\right)\left[\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right]$$

Find the coefficients by $A_n(x')$. [Hint: Remember the completeness condition

$$\sum_{n=1}^{\infty}\frac{2}{a}\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{n\pi x}{a}\right)=\delta\left(x-x'\right)$$

] [**Ans:** $G(x,x')=-\sum_{n=1}^{\infty}\frac{2a}{n^2\pi^2}\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$]

(4) Consider the operator $\mathcal{L}_t=\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}+\omega_0^2$ where $t\ge 0$ and $\omega_0>\gamma>0$. Find a GF such that $G(0,t')=\frac{d}{dt}G(0,t')=0$.
[**Ans:** $G(t,t')=\frac{1}{\sqrt{\omega_0^2-\gamma^2}}e^{-\gamma(t-t')}\sin\left(\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right)\Theta(t-t')$]

4.3. Green's Function for Laplace Operator

▷ Laplace operator $\mathcal{L}=\nabla^2$.
  - Laplace Equation $\mathcal{L}u(\mathbf{r})=0$ (homogeneous)
  - Poisson Equation $\mathcal{L}u(\mathbf{r})=f(\mathbf{r})$ (non-homogeneous). In ED, $f(\mathbf{r})=-\rho/\epsilon_0$.

▷ The Green's functions $G(\mathbf{r},\mathbf{r}')$ for Laplace operator are defined as[1]

$$\nabla^2 G\left(\mathbf{r},\mathbf{r}'\right)=-4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right).$$

▷ We already know one GF, that is $1/\left|\mathbf{r}-\mathbf{r}'\right|$ since

$$\nabla^2\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=-4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right).$$

Thus, if $\nabla^2 H(\mathbf{r},\mathbf{r}')=0$ for some $H(\mathbf{r},\mathbf{r}')+1/\left|\mathbf{r}-\mathbf{r}'\right|$ is also a GF for Laplace operator.

[1]Factor of -4π instead of 1 is just convenience/convention.

▷ Let us see how to use GF to solve the Poisson/Laplace equation with boundary conditions. Let ϕ be a solution of

$$\nabla^2\phi(\mathbf{r}) = f(\mathbf{r}) \qquad \mathbf{r} \in V$$

and let S be the boundary surface of V.

We will convert this into an integral equation for ϕ. To do this we will use the Green's identity[2]

$$\begin{aligned}\int_v \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dv &= \int_v \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)\, dv \\ &= \oint_s (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}ds\end{aligned}$$

$$\int_v \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dv = \oint_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds \tag{4.3.1}$$

This equation is true for any ψ. Let us choose $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$. Using the fact $\nabla^2\phi(\mathbf{r}) = f(\mathbf{r})$ and $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$, we get

$$-4\pi\phi(\mathbf{r}') - \int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})\, dv = \oint_s \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right) ds$$

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, dv' - \frac{1}{4\pi}\oint_s \left(\phi\frac{\partial G}{\partial n'} - G\frac{\partial\phi}{\partial n'}\right) ds'$$

If we find appropriate G, (f is given), we still need ϕ and $\partial\phi/\partial n$ on the surface S to compute ϕ in V.

▷ For Dirichlet problem, we only know ϕ on S. If we find G such that $G(\mathbf{r}, \mathbf{r}') = 0$ on S. Then,

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') - \frac{1}{4\pi}\oint_s \phi\frac{\partial G}{\partial n'} ds'.$$

▷ For Neumann problem, we only know $\partial\phi/\partial n$ on S. If we find G such that $\partial G/\partial n = 4\pi/S$ on S. Then,

$$\phi(\mathbf{r}) = \langle\phi\rangle_s - \frac{1}{4\pi}\int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') + \frac{1}{4\pi}\oint_s G\frac{\partial\phi}{\partial n'} ds'.$$

4.3.1. Some Examples.

Example 13. **Free Space Green's Functions:** Let $f(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0$. Let V be the whole space, then S is at infinity. If $\rho(\mathbf{r})$ is localized charge distribution then $\phi \to 0$ on S, that is as $r \to \infty$. The required GF is $1/|\mathbf{r} - \mathbf{r}'|$. The solution is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}.$$

Example 14. **Green's Function for Half Space:** Consider upper half of the space V ($z > 0$), bounded by surface $z = 0$ (xy-plane). If a Dirichlet problem is posed as follows:

$$\begin{aligned}\nabla^2\Phi(\mathbf{r}) &= 0 \qquad \mathbf{r} \in V \\ \Phi(x, y, 0) &= g(x, y) \qquad \text{on } S.\end{aligned}$$

Since there is no charge in the upper half space, the solution will be

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\oint_s \Phi\frac{\partial G}{\partial n'} ds'$$

where $G = 0$ on S. It is easy to check that the GF

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\left|\mathbf{r} - \left(\mathbf{r}' - 2\left(\mathbf{r}'\cdot\hat{\mathbf{k}}\right)\hat{\mathbf{k}}\right)\right|}$$

[2] $\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$

is indeed the required GF. Note that $G(\mathbf{r}, \mathbf{r}') = 0$ when $\mathbf{r}$ or $\mathbf{r}'$ is on xy plane. Also, note that $\left(\mathbf{r}' - 2\left(\mathbf{r}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}}\right)$ is the image of $\mathbf{r}'$ in xy plane. Thus

$$\begin{aligned} \phi(\mathbf{r}) &= -\frac{1}{4\pi}\oint_s g(x', y')\frac{\partial}{\partial z'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\left|\mathbf{r} - \left(\mathbf{r}' - 2\left(\mathbf{r}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}}\right)\right|}\right) ds' \\ &= -\frac{1}{4\pi}\oint_s g(x', y')\left(\frac{z}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{z}{\left|\mathbf{r} - \left(\mathbf{r}' - 2\left(\mathbf{r}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}}\right)\right|^3}\right) ds'(?) \end{aligned}$$

EXAMPLE 15. A point charge above the grounded conducting xy plane. The same GF as in previous example will work. Then, $f(r) = -\delta(\mathbf{r} - d\hat{k})/\epsilon_0$

$$\begin{aligned} \Phi(\mathbf{r}) &= -\frac{1}{4\pi}\int_v G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, dv' \\ &= \frac{1}{4\pi\epsilon_0}\int_v G(\mathbf{r}, \mathbf{r}')\,\delta(\mathbf{r} - d\hat{k}) dv' \\ &= \frac{1}{4\pi\epsilon_0} G\left(\mathbf{r}, d\hat{k}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left|\mathbf{r} - d\hat{k}\right|} - \frac{1}{\left|\mathbf{r} + d\hat{k}\right|}\right). \end{aligned}$$

This is the answer we know from method of images.

EXAMPLE 16. **Green's Function for Sperical Volume:** Consider a hollow sphere of radius a. The potential Φ on the surface of the sphere is given as $\Phi(R, \theta, \phi) = V(\theta, \phi)$. We want to find a Green's that vanishes on the surface of the sphere. The suggestion (again, motivated by method of images, image of value) is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{\left|\mathbf{r} - \left(\frac{R^2}{r'^2}\right)\mathbf{r}'\right|}.$$

Remember, $\mathbf{r}$ and $\mathbf{r}'$ are inside the sphere and $\left(\frac{R^2}{r'^2}\right)\mathbf{r}'$ is outside. Thus,

$$\begin{aligned} \nabla^2 G(\mathbf{r}, \mathbf{r}') &= \nabla^2\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{\left|\mathbf{r} - \left(\frac{R^2}{r'^2}\right)\mathbf{r}'\right|}\right]. \\ &= -4\pi\delta(\mathbf{r} - \mathbf{r}') + 0 \end{aligned}$$

and $G(\mathbf{r}, \mathbf{r}') = 0$ if either $\mathbf{r}$ or $\mathbf{r}'$ is on the surface of the sphere. If $\mathbf{r} = (r, \theta, \phi)$ and $\mathbf{r}' = (r', \theta', \phi')$ then

$$G = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{R^2} + R^2 - 2rr'\cos\gamma}}$$

with $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. Now,

$$\left.\frac{\partial}{\partial r'} G(\mathbf{r}, \mathbf{r}')\right|_{r'=a} = -\frac{R^2 - r^2}{a(r^2 + a^2 - 2ra\cos\gamma)}$$

This gives us the final solution to the Dirichlet problem

$$\begin{aligned} \Phi(r, \theta, \phi) &= -\frac{1}{4\pi}\oint_S \Phi\frac{\partial G}{\partial n'} ds' \\ &= \frac{1}{4\pi a}\oint_S \frac{(R^2 - r^2)\, V(\theta', \phi')}{a(r^2 + a^2 - 2ra\cos\gamma)}\sin\theta' d\theta' d\phi' \end{aligned}$$

4.3.2. Some Techniques to Find Green's Functions.

Fourier Transform. Now, another powerful technique to obtain GF is Fourier transform. Let

$$G\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{\left(2\pi\right)^{3}}\int g\left(\mathbf{k}\right)e^{i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}d^{3}k$$

We also know that

$$\delta\left(\mathbf{r}-\mathbf{r}'\right)=\frac{1}{\left(2\pi\right)^{3}}\int e^{i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}d^{3}k$$

Using these two in the definition of the GF

$$\nabla^{2}G\left(\mathbf{r},\mathbf{r}'\right)=-4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

$$\frac{1}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}\left(ik\right)^{2}g\left(\mathbf{k}\right)e^{i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}d^{3}k=\frac{-4\pi}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}e^{i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}d^{3}k$$

$$g\left(\mathbf{k}\right)=\frac{-4\pi}{-k^{2}}.$$

Then

$$G\left(\mathbf{r},\mathbf{r}'\right)=\frac{4\pi}{\left(2\pi\right)^{3}}\int\frac{e^{i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}}{k^{2}}d^{3}k.$$

To perform this integral, choose k_z along $(\mathbf{r}-\mathbf{r}')$. Let $R=|(\mathbf{r}-\mathbf{r}')|$ and $\mathbf{k}=(k,\theta,\phi)$ in spherical coordinates. Then

$$\begin{aligned}G\left(\mathbf{r},\mathbf{r}'\right)&=\frac{4\pi}{\left(2\pi\right)^{3}}\int\frac{e^{ikR\cos\theta}}{k^{2}}k^{2}\sin\theta d\theta d\phi dk\\&=\frac{4\pi}{\left(2\pi\right)^{3}}\int\frac{e^{ikR\cos\theta}}{k^{2}}k^{2}\sin\theta d\theta d\phi dk\\&=\frac{4\pi}{\left(2\pi\right)^{2}}\int_{0}^{\infty}\frac{e^{ikR}-e^{-ikR}}{ikR}dk=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin kR}{kR}dk\\&=\frac{1}{R}=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\end{aligned}$$

Eigenfunction Exapansion.

Example 17. **1D Laplace Equation:** Find a GF for operator $\mathcal{L}_x=\frac{d^2}{dx^2}$, $0\le x\le a$ such that $G(0,x')=G(a,x')=0$. ($\mathcal{L}_xG=\delta(x-x')$). Since $G(x,x')$ is a continuous function of x, that vanishes at the boundaries, we can expand G as

$$G\left(x,x'\right)=\sum_{n=1}^{\infty}A_{n}\left(x'\right)\left[\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right].$$

To find the coefficients by $A_n(x')$, operate by $\mathcal{L}_x=\frac{d^2}{dx^2}$,

$$\nabla^{2}G\left(x,x'\right)=\sum_{n=1}^{\infty}A_{n}\left(x'\right)\nabla^{2}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right]$$

$$\implies\delta\left(x-x'\right)=-\sum_{n=1}^{\infty}\frac{n^{2}\pi^{2}}{a^{2}}A_{n}\left(x'\right)\left[\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right]$$

Using fourier trick, we get

$$A_{n}\left(x'\right)=-\frac{a^{2}}{n^{2}\pi^{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x'}{a}\right)\right]$$

Thus,

$$G\left(x,x'\right)=-\sum_{n=1}^{\infty}\frac{2a}{n^{2}\pi^{2}}\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

[We could use the completeness condition

$$\sum_{n=1}^{\infty}\frac{2}{a}\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{n\pi x}{a}\right)=\delta\left(x-x'\right)$$

and just compare coefficients.

EXAMPLE 18. **2D Laplace Equation (Jackson 2.15):** The Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0=x\leq 1, 0\leq y\leq 1$, has an expansion in x

$$G\left(x,x'\right)=\sum_{n=1}^{\infty}A_n\left(y;x',y'\right)\sin\left(n\pi x\right).$$

Using the same idea from previous example

$$\begin{aligned}\nabla^2 G\left(x,x'\right)&=\sum_{n=1}^{\infty}\nabla^2\left[\sqrt{2}\sin\left(n\pi x\right)\right]A_n\left(y;x',y'\right)\\ \Longrightarrow\ -4\pi\delta\left(x-x'\right)\delta\left(y-y'\right)&=\sum_{n=1}^{\infty}\left[\sqrt{2}\sin\left(n\pi x\right)\right]\left(-n^2\pi^2+\frac{\partial^2}{\partial y^2}\right)A_n\left(y;x',y'\right)\\ \Longrightarrow\ -4\pi\left[\sqrt{2}\sin\left(n\pi x'\right)\right]\delta\left(y-y'\right)&=\left(-n^2\pi^2+\frac{\partial^2}{\partial y^2}\right)A_n\left(y;x',y'\right)\end{aligned}$$

Putting $A_n\left(y;x',y'\right)=g_n\left(y,y'\right)\left[\sqrt{2}\sin\left(n\pi x'\right)\right]$, we get

$$\left(-\frac{n^2\pi^2}{a^2}+\frac{\partial^2}{\partial y^2}\right)g_n\left(y,y'\right)=-4\pi\delta\left(y-y'\right).$$

Also, $G(0,y;x',y')=0$ for all y,y' and x' implies that $g_n(0,y)=0$ and simillarly, $g_n(1,y)=0$. Now, g_n is just a Green's function in one variable, y. The solutions can be obtained by the methods used in the first section to get

$$G\left(x,y;x',y'\right)=\sum_{n=1}^{\infty}\frac{8}{n\sinh\left(n\pi\right)}\sin\left(n\pi x\right)\sin\left(n\pi x'\right)\sinh\left(n\pi y_<\right)\sinh\left(n\pi\left(1-y_>\right)\right).$$

EXAMPLE 19. **(Jackson 2.16)** A two-dimensional potential exists on a unit square area $(0\leq x\leq 1, 0\leq y\leq 1)$ bounded by "surfaces" held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z). Using the Green function of the previous example, (the rhs is $-1/\epsilon_0$)

$$\begin{aligned}\Phi\left(x,y\right)&=\frac{1}{4\pi\epsilon_0}\int_0^1\int_0^1 dx'dy'G\left(x,y;x',y'\right)\\ &=\frac{1}{4\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{8}{n\sinh\left(n\pi\right)}\sin\left(n\pi x\right)\int_0^1 dx'\sin\left(n\pi x'\right)\\ &\quad\times\left[\int_0^y dy'\sinh\left(n\pi y'\right)\sinh\left(n\pi\left(1-y\right)\right)+\int_y^1 dy'\sinh\left(n\pi y\right)\sinh\left(n\pi\left(1-y'\right)\right)\right]\\ &=\frac{4}{\pi^3\epsilon_0}\sum_{n=0}^{\infty}\frac{\sin\left[\left(2m+1\right)\pi x\right]}{\left(2m+1\right)^3}\left\{1-\frac{\cosh\left[\left(2m+1\right)\pi\left(y-\frac{1}{2}\right)\right]}{\cosh\left[\left(2m+1\right)\frac{\pi}{2}\right]}\right\}\end{aligned}$$

4.4. Green's Function for Helmholtz Operator

- ▷ Helmholtz operator $\mathcal{L}=\nabla^2+k^2$.
- ▷ The Green's functions $G\left(\mathbf{r},\mathbf{r}'\right)$ for Laplace operator are defined as

$$\nabla^2 G\left(\mathbf{r},\mathbf{r}'\right)=-\delta\left(\mathbf{r}-\mathbf{r}'\right).$$

To obtain free space GF we will use Fourier transform. Let

$$G\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{(2\pi)^3}\int g\left(\mathbf{q}\right)e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^3q$$

We also know that

$$\delta\left(\mathbf{r}-\mathbf{r}'\right)=\frac{1}{(2\pi)^3}\int e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^3q$$

Using these two in the definition of the GF

$$\left(\nabla^2+k^2\right)G\left(\mathbf{r},\mathbf{r}'\right)=-\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left(k^2-q^2\right)g\left(\mathbf{q}\right)e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^3q=\frac{-}{(2\pi)^3}\int_{-\infty}^{\infty}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^3q$$

$$g\left(\mathbf{q}\right)=\frac{-1}{k^2-q^2}.$$

Then

$$G\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{(2\pi)^3}\int\frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{q^2-k^2}d^3q.$$

To perform this integral, choose q_z along $(\mathbf{r}-\mathbf{r}')$. Let $R=|(\mathbf{r}-\mathbf{r}')|$ and $\mathbf{q}=(q,\theta,\phi)$ in spherical coordinates. Then

$$\begin{aligned}G\left(\mathbf{r},\mathbf{r}'\right)&=\frac{1}{(2\pi)^3}\int\frac{e^{iqR\cos\theta}}{q^2-k^2}q^2\sin\theta d\theta d\phi dq\\&=\frac{1}{(2\pi)^2\,iR}\int_0^{\infty}\frac{e^{iqR}-e^{-iqR}}{q^2-k^2}qdq\\&=\frac{1}{8\pi^2iR}\int_{-\infty}^{\infty}\frac{e^{iqR}-e^{-iqR}}{q^2-k^2}qdq\end{aligned}$$

The integral can be computed using method of residues

$$\int_{-\infty}^{\infty}\frac{e^{iqR}}{q^2-k^2}qdq=\frac{\pi i}{2}\left(e^{ikR}+e^{-ikR}\right)$$

CHAPTER 5

Appedix

5.1. A: Bessel Functions

$$
\begin{aligned}
J_0(0) &= 1, \\
J_\nu(x) &= 0 \quad (\text{if } \nu > 0), \\
J_{-n}(x) &= (-1)^n J_n(x), \\
\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] &= -x^{-\nu} J_{\nu+1}(x), \\
\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] &= x^{\nu} J_{\nu-1}(x), \\
\frac{d}{dx}\left[J_\nu(x)\right] &= \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right], \\
x J_{\nu+1}(x) &= 2\nu J_\nu(x) - x J_{\nu-1}(x), \\
\int x^{-\nu} J_{\nu+1}(x)\, dx &= -x^{-\nu} J_\nu(x) + C, \\
\int x^{\nu} J_{\nu-1}(x)\, dx &= x^{\nu} J_\nu(x) + C.
\end{aligned}
$$