Tutorial 5: Laplace Equation

Variable Separation in 3D, Conformal Mapping

1. Find the steady-state temperature distribution in solid cylinder of height h and radius a if the top and the curved surface are held at 0° and the base at 100°.

Answer:
The given BCs are

$$\psi(a,\phi,z)=0, \quad \psi(\rho,\phi,h)=0, \quad \psi(\rho,\phi,0)=T_0=100.$$

In addition, there are implicit conditions that $\psi(\rho,\phi,z)$ is always finite. The solution of form

$$\psi(\rho,\phi,z)=\sum_{m=0,n=1}^{\infty}\sinh\left(k_{mn}(h-z)\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right)$$

where $k_{mn}=\chi_{mn}/a$ (χ_{mn} is n^{th} zero of J_m), satisfies all conditions except the one at $z=0$. Applying this condition, we get

$$T_0=\sum_{mn}\sinh\left(k_{mn}h\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right).$$

From the orthogonality of the trigonometric functions, we can immediately set $C_{mn}=0$ for all m,n and $D_{mn}=0$ for all m,n except $m=0$. Thus,

$$\rho=\sum_{n}\sinh\left(k_{0n}h\right)D_{0n}J_0\left(k_{0n}\rho\right).$$

Using $\int_0^a\left[J_m\left(k_{mn}\rho\right)\right]^2\rho d\rho=\frac{a^2}{2}\left[J_{m+1}\left(\chi_{mn}\right)\right]^2$,

$$\begin{aligned}D_{0n}&=\frac{2T_0}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\sinh\left(k_{0n}h\right)}\int_0^a\rho J_0\left(\chi_{0n}\rho/a\right)d\rho=\frac{2}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\sinh\left(k_{0n}h\right)}\left(\frac{a}{\chi_{0n}}\right)J_1\left(\chi_{0n}\right)\\&=\frac{2T_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\sinh\left(k_{0n}h\right)}\end{aligned}$$

Finally,

$$\psi(\rho,\phi,z)=\sum_{n=1}^{\infty}\frac{2T_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\sinh\left(k_{0n}h\right)}\sinh\left(k_{0n}(h-z)\right)J_0\left(k_{0n}\rho\right).$$

2. Find the steady-state temperature distribution in a solid semi-infinite cylinder (bounded by $\rho=a$ and $z=0$) if the boundary temperatures are $T=0$ at $\rho=a$ and $T=\rho\sin\phi$ at $z=0$. Hints: This problem is similar to the one we did in the class except the last integral. Look at the recursion relations to integrate integrands with Bessel functions.

Answer:
This problem is similar to the one discussed in the class. The given BCs are

$$\psi(a,\phi,z)=0 \quad \psi(\rho,\phi,0)=\rho\sin\phi.$$

In addition, there are implicit conditions that $\psi(\rho,\phi,z)$ is always finite and $\psi(\rho,\phi,z)\to 0$ as $z\to\infty$. The solution of form

$$\psi(\rho,\phi,z)=\sum_{m=0,n=1}^{\infty} e^{-k_{mn}z} J_m(k_{mn}\rho)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right)$$

where $k_{mn}=\chi_{mn}/a$ (χ_{mn} is n^{th} zero of J_m), satisfies all conditions except the one at $z=0$. Applying this condition, we get

$$\rho\sin\phi=\sum_{mn} J_m(k_{mn}\rho)\left(C_{mn}\sin(m\phi)+D_{mn}\cos(m\phi)\right).$$

From the orthogonality of the trigonometric functions, we can immediately set $D_{mn}=0$ for all m,n and $C_{mn}=0$ for all m,n except $m=1$. Thus,

$$\rho=\sum_{n} C_{1n}J_1(k_{1n}\rho).$$

Using $\int_0^a [J_m(k_{mn}\rho)]^2\rho d\rho=\frac{a^2}{2}[J_{m+1}(\chi_{mn})]^2$,

$$\begin{aligned} C_{1n} &= \frac{2}{a^2(J_2(\chi_{1n}))^2}\int_0^a \rho^2 J_1(\chi_{1n}\rho/a)\,d\rho=\frac{2}{a^2(J_2(\chi_{1n}))^2}\left(\frac{a}{\chi_{1n}}\right)J_2(\chi_{1n}) \\ &= \frac{2a}{\chi_{1n}J_2(\chi_{1n})} \end{aligned}$$

The last step uses the recurrence relation

$$\frac{d}{dx}\left(x^p J_p(x)\right)=x^p J_{p-1}(x).$$

Finally,

$$\psi(\rho,\phi,z)=\sum_{n=1}^{\infty}\frac{2}{a\chi_{1n}J_2(\chi_{1n})}e^{-k_{1n}z}J_1(k_{1n}\rho)\sin(\phi)$$

3. Find the steady-state, bounded temperature distribution in the interior of a solid cylinder of radius a and height h, given that the temperature of the curved lateral surface is kept at zero, the base is *insulated*, and the top is kept at constant temperature u_0.

Answer:
The given BCs are

$$\psi\left(a,\phi,z\right)=0,\quad \psi\left(\rho,\phi,h\right)=u_0,\quad \frac{d\psi}{dz}\left(\rho,\phi,0\right)=0.$$

In addition, there are implicit conditions that $\psi\left(\rho,\phi,z\right)$ is always finite. The solution for z-equation is

$$Q(z)=C\sinh\left(kz\right)+D\cosh\left(kz\right).$$

Applying the condition $dQ/dz=0$ implies that $C=0$. Thus, the solution ψ is of the form

$$\psi\left(\rho,\phi,z\right)=\sum_{m=0,n=1}^{\infty}\cosh\left(k_{mn}z\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin\left(m\phi\right)+D_{mn}\cos\left(m\phi\right)\right)$$

where $k_{mn}=\chi_{mn}/a$ (χ_{mn} is n^{th} zero of J_m), satisfies all conditions except the one at $z=h$. Applying this condition, we get

$$u_0=\sum_{mn}\cosh\left(k_{mn}h\right)J_m\left(k_{mn}\rho\right)\left(C_{mn}\sin\left(m\phi\right)+D_{mn}\cos\left(m\phi\right)\right).$$

From the orthogonality of the trigonometric functions, we can immediately set $C_{mn}=0$ for all m,n and $D_{mn}=0$ for all m,n except $m=0$. Thus,

$$\rho=\sum_{n}\cosh\left(k_{0n}h\right)D_{0n}J_0\left(k_{0n}\rho\right).$$

Using $\int_0^a\left[J_m\left(k_{mn}\rho\right)\right]^2\rho d\rho=\frac{a^2}{2}\left[J_{m+1}\left(\chi_{mn}\right)\right]^2$,

$$\begin{aligned}D_{0n}&=\frac{2u_0}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\cosh\left(k_{0n}h\right)}\int_0^a\rho J_0\left(\chi_{0n}\rho/a\right)d\rho=\frac{2u_0}{a^2\left(J_1\left(\chi_{0n}\right)\right)^2\cosh\left(k_{0n}h\right)}\left(\frac{a}{\chi_{0n}}\right)J_1\left(\chi_{0n}\right)\\&=\frac{2u_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\cosh\left(k_{0n}h\right)}\end{aligned}$$

Finally,

$$\psi\left(\rho,\phi,z\right)=\sum_{n=1}^{\infty}\frac{2u_0}{a\chi_{0n}J_1\left(\chi_{0n}\right)\cosh\left(k_{0n}h\right)}\cosh\left(k_{0n}z\right)J_0\left(k_{0n}\rho\right).$$

4. Discuss the image of the circle $|z-2|=1$ and its interior under the following transformations:

   (a) $w=z-2i$;

   (b) $w=3iz$;

   (c) $w=\dfrac{z-2}{z-1}$;

   (d) $w=\dfrac{z-4}{z-3}$;

   (e) $w=1/z$.

Answer:

(a) $|z-2|=1 \implies |w-(2-2i)|=1$. Maps to the circle with center at $2-2i$.

(b) $|z-2|=1 \implies |w/3i-2|=1 \implies |w-6i|=3$. Maps to the circle with radius 3 and center at $6i$.

(c) Let us write,

$$w=1-W, \qquad W=\frac{1}{Z}, \qquad Z=z-1$$

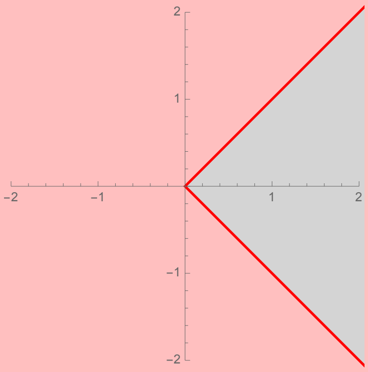
- ▷ $|z-2|=1$ maps to $|Z-1|=1$, a circle which passes through origin (in Z plane).
- ▷ $|Z-1|=1$ maps to a vertical line passing through $(1/2,0)$ in W plane.
- ▷ Finally maps to the same line in w plane.

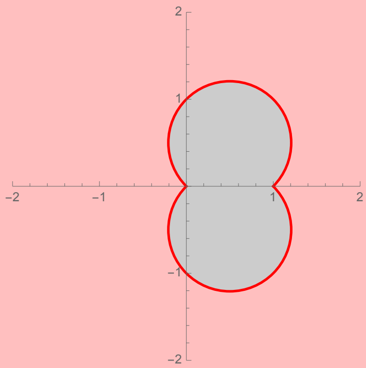
(d) Another way: Let $z_1=1,\ z_2=3$ and $z_3=2+i$. The images are $w_1=3/2$, $w_2=\infty$ and $w_3=(i-2)/(i-1)=\frac{1}{2}(3+i)$. Clearly this is a line passing through w_1 and w_3.

(e) We know that $1/z$ maps circles not passing through origin to circles. If $z_1=1,\ z_2=3$ and $z_3=2+i$, then $w_1=1$, $w_2=1/3$ and $w_3=(2-i)/5$. This is a circle of radius $1/3$ centered at $(2/3,0)$.

5. What is the image of the sector $-\pi/4<\arg z<\pi/4$ under the mapping $w=z/(z-1)$?

Answer:
Notice that $w(0)=0$ and $w(\infty)=1$. $w(1\pm i)=1\mp i$. Thus the center of circle lies at $\frac{1}{2}(1\mp i)$ with radius $\frac{1}{\sqrt{2}}$. The image is shown below.





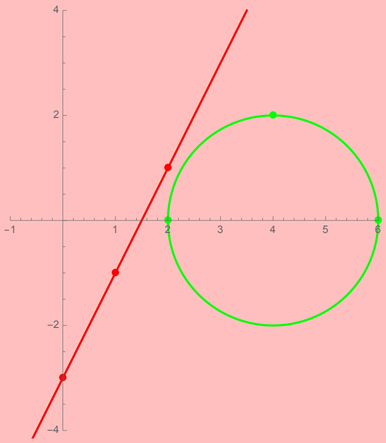
6. Write an equation defining a Möbius transformation that maps the half-plane below the line $y=2x-3$ onto the interior of the circle $|w-4|=2$. Repeat for the exterior of this circle.

Answer:
The implicit formula for a Möbius transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Line passes through $(0,-3)$, $(1,-1)$ and $(2,1)$. Let us map these three points to $2, 4+2i$ and 6. Then, using implicit form, we get

$$w = \frac{(4-2i)(z+(1+2i))}{z+1}$$



7. Two points z_1 and z_2 are said to be symmetric with respect to a circle (line) C if every straight line or circle passing through z_1 and z_2 intersects C orthogonally.

   (a) Show that if C is a line then it must be a perpendicular bisector of the line segment joining z_1 and z_2.

   (b) Show that if C is a circle then z_1 and z_2 lie on some radius of the circle C and if the radius of the circle is R and if distances of z_1 and z_2 from the center of the circle are a and d then $R^2 = ad$.

   (c) Show that if the center of the circle at z_0 in above question, then the same conditions can be put as

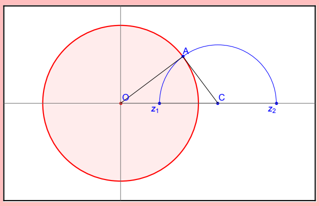
$$\begin{aligned} \arg(z_1 - z_0) &= \arg(z_2 - z_0) \\ |z_1 - z_0| &= \frac{R^2}{|z_2 - z_1|} \end{aligned}$$

   (d) Prove following theorem:
   (Symmetry Principle) Let C_z be a line or a circle in z-plane, and let $w = f(z)$ be any Möbius transformation.. Then two points z_1 and z_2 are symmetric with respect to C_z if and only if their images $w_1 = f(z_1)$ and $w_2 = f(z_2)$ are symmetric with respect to the image C_w of C_z under f.

Answer:

(a) If line C is $\perp$ to any circle passing through z_1 and z_2 then the center of such circle must be on C. The result follows immediately.

(b) Look at the geometry shown in the figure. Clearly,



$$\begin{aligned} OA^2 + AC^2 &= OC^2 \\ R^2 + \left(\frac{d-a}{2}\right)^2 &= \left(\frac{d+a}{2}\right)^2 \\ \implies \quad R^2 &= ad. \end{aligned}$$

(c) The two points are on the radius implies that $\arg(z_1 - z_0) = \arg(z_2 - z_0)$. And the second condition is $|z_1 - z_0| = a$ and $|z_2 - z_0| = d$.

(d) The fact that the MT f is conformal, preserves the definition of the symmetry.

8. Find a point symmetric to $4 - 3i$ with respect to each of the following circles:

   (a) $|z| = 1$;

   (b) $|z - 1| = 1$;

   (c) $|z - 1| = 2$.

Answer:
Let $z_1 = 4 - 3i$.

(a) Since the center of the circle is the origin, the symmetric point z_2 is along the same radial line from the origin and hence $z_2 = \gamma z_1$ where γ is a positive real number. And from the second condition, $|z_2| = 1/|z_1|$. Thus,

$$\gamma |z_1| = \frac{1}{|z_1|} \implies \gamma = \frac{1}{25}.$$

Thus, $z_2 = (4 - 3i)/25$.

(b) Here, $R = 1$ and $z_0 = 1$. Now, $\arg(z_2 - z_0) = \arg(z_1 - z_0) = -\pi/4$ and $|z_2 - z_0| = 1/|z_1 - z_0| = 1/3\sqrt{2}$. Thus,

$$z_2 - z_0 = \frac{1}{3\sqrt{2}} e^{-i\pi/4} = \frac{1}{6}(1 - i)$$
$$z_2 = \frac{1}{6}(7 - i)$$

(c) Left for you.

9. By Completing following steps prove that any two non intersecting circles C_1 and C_2 there always exist two distinct points z_1 and z_2 that are symmetric with respect to C_1 and C_2 *simultaneously.*

   (a) Argue that there exists a Möbius transformation that maps C_1 onto the real axis and C_2 onto some circle C of the form $|w - \lambda i| = R$ with λ real and $R < |\lambda|$.

   (b) Show that w_1 and w_2 are symmetric with respect $\mathbf{R}$ and C if and only if

$$w_2 = \overline{w_1} \text{ and } w_2 = \frac{R^2}{\overline{w_1} + \lambda i} + \lambda i.$$

   Solve this pair of equations to obtain

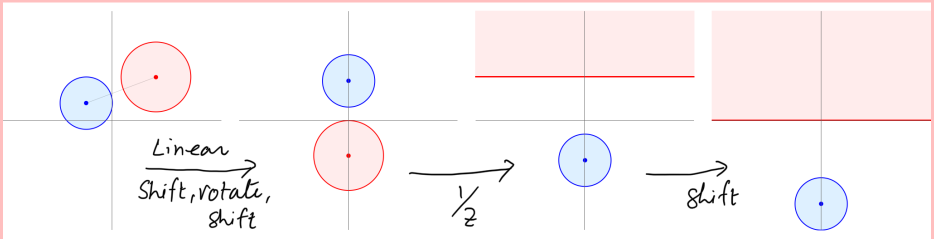
$$w_1 = i\sqrt{\lambda^2 - R^2} \text{ and } w_2 = -i\sqrt{\lambda^2 - R^2}$$

   as simultaneous symmetric points.

   (c) Use the symmetry principle to conclude that there are points z_1 and z_2 are symmetric with respect to both C_1 and C_2.

Answer:

(a) One can perform simple elementary transformations to achieve this:



We know that the combination of these operations is a linear fractional transform. Can you visualize a similar sequence of operations if the blue circle (smaller) is completely inside the red circle (larger)? Clearly from figure, R (radius of the blue circle) is smaller than $|\lambda|$ (distance of the center of the blue circle from the origin).

(b) The first part is easy, since w_1 and w_2 are symmetric with respect to the real axis, $w_2 = \bar{w}_1$. Without loss of generality, let us assume that w_2 in upper half plane. If the same w_1 and w_2 are symmetric wrt the circle, these two must be on imaginary axis. Then, using earlier result, $a = -(|w_1| + \lambda)$ and $d = |w_2| - \lambda$ and thus,

$$|w_2| - \lambda = \frac{R^2}{-(|w_1| + \lambda)}$$

$$i\,|w_2| = \frac{-iR^2}{|w_1| + \lambda} + \lambda i$$

$$w_2 = \frac{R^2}{|w_1|\,i + \lambda i} + \lambda i = \frac{R^2}{\bar{w}_1 + \lambda i} + \lambda i$$

(c) Now, we pull w_1 and w_2 back in z-plane to z_1 and z_2. Clearly, these two points are simultaneously symmetric wrt C_1 and C_2.

10. Use the results of previous problem to show that for any two non-intersecting circles C_1 and C_2 there exists a Möbius transformation that maps C_1 and C_2 onto *concentric circles*. Hint: Map z_1 to origin and z_2 to infinity.

Answer:
Consider a circle and two points z_1 and z_2 symmetric wrt the cirle. The LFT

$$w = \lambda \frac{z - z_1}{z - z_2}$$

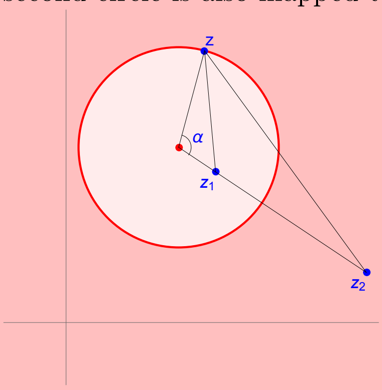
maps z_1 to origin and z_2 to ∞. From figure it is clear that

$$|z - z_1| = \sqrt{R^2 + a^2 - 2Ra\cos\alpha}$$

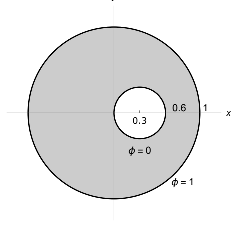
$$|z - z_2| = \sqrt{R^2 + (R^2/a)^2 - 2R\,(R^2/a)\cos\alpha}$$

$$= \frac{R}{a}\sqrt{a^2 + R^2 - 2Ra\cos\alpha} = \frac{R}{a}\,|z - z_1|$$

Thus, the circle is mapped to a circle with radius $\lambda a/R$ and center at origin. In the same way, the second circle is also mapped to a circle with center at origin.



11. Find the function ϕ that is harmonic in the shaded region depicted in the figure and takes values 0 on the inner circle and 1 on the outer circle. This is a cylindrical capacitor with nonconcentric cylinders.



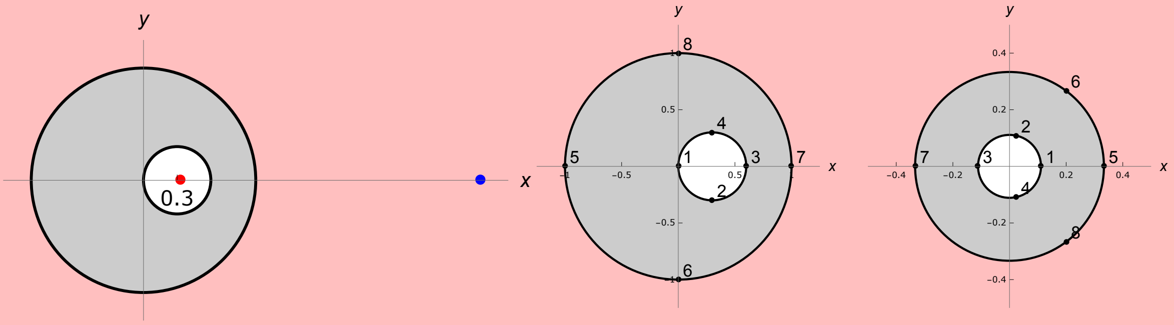
Answer:
Clearly z_1 and z_2 are on the real axis. Then by solving

$$z_1 - 0.3 = \frac{0.3^2}{z_2 - 0.3}$$
$$z_1 = \frac{1^2}{z_2},$$

we get $z_1 = 1/3$ and $z_2 = 3$. The required transformation is

$$w = \frac{z - z_1}{z - z_2}$$

which maps outer circle to a circle of radius $1/3$ and inner circle to a circle $(1/3 - 3/10)/3/10 = 1/9$.

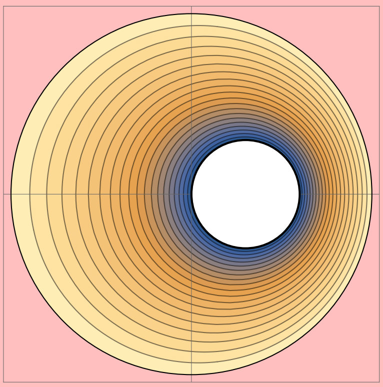


The mapped points have been shown in the figure on the right. The solution to the Laplace equation in the w-plane is $\phi = A + B \ln |w|$. Applying the boundary conditions,

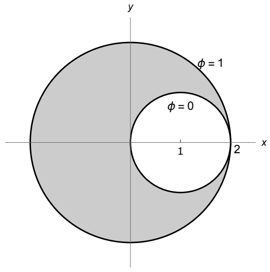
$$\phi\left(1/9, \theta_w\right) = 0 \implies A - 2B\ln 3 = 0$$
$$\phi\left(1/3, \theta_w\right) = 1 \implies A - B\ln 3 = 1.$$

Thus, $B = 1/\ln 3$ and $A = 2$. Thus, $\phi = 2 + \frac{\ln|w|}{\ln 3}$. Thus

$$\phi\left(x, y\right) = 2 + \frac{1}{\ln 3} \ln \frac{|z - z_1|}{|z - z_2|}$$
$$= 2 + \frac{1}{2\ln 3} \ln \frac{\left(x - z_1\right)^2 + y^2}{\left(x - z_2\right)^2 + y^2}$$



12. Find electrostatic potential in the shaded region in the Figure.

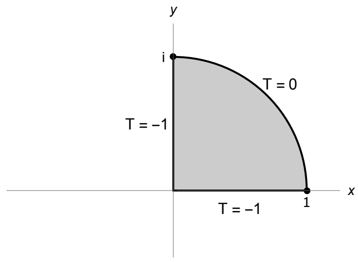


Answer:
Since these two circles meet at $z = 2$, a simple suggestion would be to map $z = 2$ to infinity. If we shift the origin by $Z = z - 2$, these would be two circles passing through origin and we know that $1/Z$ maps these circles to lines not passing through. Thus, the suggestion for mapping is

$$w = \frac{1}{Z} = \frac{1}{z-2}.$$

The remaining part is left for you to complete.

13. Find steady state temp in the shaded region in the Figure.



Answer:
Suggestion: $w = z^2$. After this transformation, compare the new boundary value problme with the first example in the notes.