

- Outline of the Course
- 10 • Groups + Continuous Groups
 - 12 • Vector Spaces, Inner Product Spaces, Hilbert Spaces
 - 6 • Group Representations
 - 8 • Tensor Analysis

- Groups:
- Definition, Examples, Properties, ~~Proofs~~
 - Multiplication Table, Rearrangement Theorem
 - Conjugacy Classes, Multiplication of classes
 - Subgroups, Cosets, Normal Subgroup, Factor Groups
 - Product of Groups
 - Isomorphism, Homomorphism
 - Permutation Group
 - Rotation Group $O(3)$, $SO(3)$
 - Unitary Groups $SU(1)$, $SU(2)$, $SU(N)$
 - Point Groups and Space groups in Crystals
 - Lorentz & Poincare' Groups

It is important to
With 10 Lectures. achieve some depth and not just definition level.
- 3-Tutorial Sheets -

Books

1. Elements of Group Theory for Physicists, A.W. Joshi, New Age
2. Group Theory in Physics, Wu-Ki Tung, World Scientific
3. Topics in Algebra, I.N. Herstein, Wiley Eastern
4. Linear Algebra, Hoffman - Kunze, PHI
5. Schaum Series, Linear Algebra
6. Linear Algebra and Group Theory for physicists, K.N. Srinivasa Rao, New Age

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$$(c, b) \cdot c = b$$

$$a \cdot (b, c) = b$$

Problem 14/pp 36 Herstein

~~→ Suppose there is no identity \Rightarrow for every $a \in G$ $\exists y(a)$ s.t. $a \cdot y(a) = y(a)$~~

The left cancellation law ($a \cdot u = a \cdot v \Rightarrow u = v$) implies that the multiplication by a is a one to one function on G . Since G is finite, this function is also a bijection $\Rightarrow \exists e \in G$ s.t. $a \cdot e = a \Rightarrow e \cdot e = e$

~~To show that e is unique let $a, b \in G \Rightarrow \exists e_a, e_b$ s.t. $e_a \cdot a = a, e_b \cdot b = b$~~

$$(e_a \cdot a \cdot b) = a \cdot b$$

$$a \cdot e_b = a \quad e_a \cdot a = a$$

$$e_a \cdot a \cdot e_b = a^2$$

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Groups GROUPS

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Definition: A nonempty set G is said to form a group if in G there is defined a binary operation, called multiplication and denoted by \cdot s.t.

1. $a, b \in G$ implies that $a \cdot b \in G$ (closure)
2. $a, b, c \in G$ implies that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. $\exists e \in G$ s.t. $a \cdot e = e \cdot a = a \quad \forall a \in G$
4. For every $a \in G \exists a^{-1} \in G$ s.t. $a a^{-1} = a^{-1} a = e$

Examples 1. $G = \{e\} \quad e \cdot e = e$

2. $G = \{e, a\} \quad a \cdot e = e \cdot a = a, \quad e \cdot e = e, \quad a \cdot a = e$

3. $G = \{1, -1\}$ under usual multiplication

4. $G = \{e, a, b\} \quad e$ is identity, $a \cdot b = e = b \cdot a$, notation $a^2 = b$

Multiplication Table.

5. $G = \mathbb{I} = \text{Set of integers} \quad m \cdot n = m+n$

6. $G = \{e, a, a^2, a^3, \dots, a^{n-1}\} \quad a^m \cdot a^n = a^{(m+n) \bmod n}, \quad \omega^n = 1, \omega = e^{2\pi i/n}$

7. $G = \mathbb{Q} \setminus \{0\}$ under usual multiplication is not a group

8. $G = \mathbb{Q}^*$ Set of Rationals under multiplication.

9. $G = S_3 = A(S)$ where $S = \{a, b, c\}$ ~~set of~~

$A(S)$ is set of all bijections on S : Set of permutations.

10. $G = C_3$ Set of rotation about z axis by 120° .

11. $G = SO(2)$ Set of all rotations about z axis

12. $G = O(2)$ Set of all 2×2 orthogonal matrices

13. $G = O(N)$ Set of all $N \times N$ orthogonal matrices $R^T R = I$

14. $G = U(N)$ Set of all $N \times N$ Unitary matrices $U^\dagger U = I$

Definition: Abelian Groups: A group G is called abelian if $\forall a, b \in G$
 $a \cdot b = b \cdot a$: (commutative)

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Properties: 1. identity is unique : e, f are identities $e = e \cdot f = e \cdot f = f$

2. Inverse is unique : $a \cdot b = e = a \cdot c \Rightarrow (a \cdot b) \cdot c = c \Rightarrow b = c$

3. $(a^{-1})^{-1} = a$: $(a^{-1})(a^{-1})^{-1} = e = a^{-1} \cdot a \Rightarrow (a^{-1})^{-1} = a$ $\text{or } \otimes$

4. $(ab)^{-1} = b^{-1} a^{-1}$

5. $a \cdot u = a \cdot w \Rightarrow u = w$ } Cancellation laws

6. $h \cdot a = w \cdot a \Rightarrow h = w$

Elaboration Let $G = A(S)$ where $S = \{a, b, c\}$

Let $A(S) = \{e, \psi, \psi^2, \sigma, \sigma\psi, \psi\sigma\}$ where $e(\{a, b, c\}) = \{a, b, c\}$

Multiplication table

	e	ψ	ψ^2	σ	$\sigma\psi$	$\psi\sigma$
e	e	ψ	ψ^2	σ	$\sigma\psi$	$\psi\sigma$
ψ	ψ	ψ^2	e	$\psi\sigma$	σ	$\sigma\psi$
ψ^2	ψ^2	e	ψ	$\sigma\psi$	$\psi\sigma$	σ
σ	σ	$\sigma\psi$	$\psi\sigma$	e	ψ	ψ^2
$\sigma\psi$	$\sigma\psi$	$\psi\sigma$	σ	ψ^2	e	ψ
$\psi\sigma$	$\psi\sigma$	σ	$\sigma\psi$	ψ	ψ^2	e

$$\phi(\{a, b, c\}) = \{b, c, a\}$$

$$\sigma(\{a, b, c\}) = \{a, c, b\}$$

$$\psi^2(\{a, b, c\}) = \{c, a, b\}$$

$$\sigma\sigma\psi(\{a, b, c\}) = \{c, b, a\}$$

$$\psi\sigma\sigma(\{a, b, c\}) = \{b, a, c\}$$

defining Equations

$$\psi^3 = e = \sigma^2 = e$$

$$\psi\sigma = \sigma\psi^2$$

$$\psi\sigma\psi = \sigma\psi^2\psi = \sigma$$

$$\psi^2\sigma = \psi\sigma\psi^2 = \sigma\psi$$

$$\sigma\psi^2\sigma = \psi\sigma\sigma = \psi$$

$$\sigma\psi\psi\sigma\psi = \psi^2$$

$$\psi\sigma\psi = \sigma$$

$$\psi^2\sigma\psi = \psi\sigma$$

$$\psi\sigma\psi^2 = \sigma\psi$$

Conjugacy classes $C(e) = \{e\}$

$$C(\psi) = \{\psi, \psi^2\}$$

$$C(\sigma) = \{\sigma, \psi\sigma, \sigma\psi\}$$

$$C(e) = \{e\}$$

$$e \cdot e = e$$

$$e \cdot \psi = \psi \quad \psi \cdot e = \psi \quad e \cdot \psi^2 = \psi^2 \quad \psi^2 \cdot e = \psi^2$$

$$e \cdot \sigma = \sigma \quad \sigma \cdot e = \sigma \quad e \cdot \psi\sigma = \psi\sigma \quad \psi\sigma \cdot e = \psi\sigma$$

$$e \cdot \sigma\psi = \sigma\psi \quad \sigma\psi \cdot e = \sigma\psi \quad e \cdot \psi\sigma\psi = \psi\sigma\psi$$

$$e \cdot \psi\sigma\sigma = \psi\sigma\sigma \quad \psi\sigma\sigma \cdot e = \psi\sigma\sigma$$

$$e \cdot \psi\sigma\psi = \psi\sigma\psi$$

$$e \cdot \psi\sigma\psi^2 = \psi\sigma\psi^2 \quad \psi\sigma\psi^2 \cdot e = \psi\sigma\psi^2$$

$$e \cdot \psi\sigma\psi^2\sigma = \psi\sigma\psi^2\sigma \quad \psi\sigma\psi^2\sigma \cdot e = \psi\sigma\psi^2\sigma$$

$$e \cdot \psi\sigma\psi^2\sigma\psi = \psi\sigma\psi^2\sigma\psi \quad \psi\sigma\psi^2\sigma\psi \cdot e = \psi\sigma\psi^2\sigma\psi$$

$$e \cdot \psi\sigma\psi^2\sigma\psi\sigma = \psi\sigma\psi^2\sigma\psi\sigma \quad \psi\sigma\psi^2\sigma\psi\sigma \cdot e = \psi\sigma\psi^2\sigma\psi\sigma$$

Rearrangement theorem: By Cancellation Laws it is clear that no row or column ^{of multiplication} can have an element appearing twice.

Conjugacy Classes CONJUGACY CLASSES

Definition: $a, b \in G$, b is conjugate to a if $\exists c \in G$ s.t.
 $b = c^{-1}ac$.

Theorem: Conjugacy is an equivalence relation

$$a \sim a, a \sim b \Rightarrow b \sim a, \\ a \sim b, b \sim c \Rightarrow a \sim c$$

Equivalence classes for conjugacy are called conjugate classes.

$$C(a) = \{x \in G \mid x \text{ is conjugate to } a\}$$

Conjugate classes are disjoint (Property of equivalence Relation)

Examples: 1. $G = \{e, a\}$, $a^2 = e$

$$C(e) = \{e\} \quad C(a) = \{a\}$$

$$2. G = \{e, a, a^2\} \quad C(e) = \{e\}, C(a) = \{a\}, C(a^2) = \{a^2\}$$

3. ~~$G = \mathbb{Z}$~~ $A(S)$ See Above.

$$4. C_{4v} : (E), (C_4, C_4^3), (C_4^2), (m_x, m_y), (\sigma_h, \sigma_v)$$



Proposed

Multiplication of classes:

$$\text{Definition: } C_1 C_2 = (A_1 B_1, A_1 B_2, \dots, A_m B_n)$$

$$\Rightarrow C_1 C_2 \text{ contains whole classes.} = \sum a_{ijk} C_k$$

Example: 1. $G = A(S)$, $S = \{a, b, c\}$

$$C(\psi) \cdot C(\sigma) = (\psi\sigma, \psi^2\sigma, \psi^3\sigma, \psi\sigma\psi, \psi^2\sigma\psi, \psi^3\sigma\psi)$$

$$= (\psi\sigma, \sigma\psi, \sigma\psi, \sigma, \sigma, \psi\sigma)$$

$$= 2 \cdot C(\sigma)$$

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Subgroups: ~~A~~ SUBGROUPS

Definition: Let G be a group. A subset H is called a subgroup if H is group by itself.

2. Find a formula

for the inverse of a 2x2 matrix, if it exists, in terms of its entries.

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$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

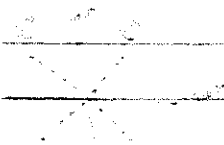
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & bq+ds \\ cr+ds & dr+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Therefore, } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \iff \begin{pmatrix} ap+br & bq+ds \\ cr+ds & dr+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\iff \begin{pmatrix} ap+br & bq+ds \\ cr+ds & dr+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} ap+br & bq+ds \\ cr+ds & dr+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Therefore, } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \iff \begin{pmatrix} ap+br & bq+ds \\ cr+ds & dr+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is equivalent to:



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\text{if } ad-bc \neq 0$$

$$\text{if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ then } ad-bc = 0$$

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$$\text{if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ then } ad-bc = 0$$

conclusion: matrix is invertible (exists) if and only if $ad-bc \neq 0$

if $ad-bc = 0$, it is not invertible

if $ad-bc = 0$, it is not invertible

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$$ad-bc \neq 0$$

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$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ is inverse of } \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if $ad-bc \neq 0$, then the matrix is invertible

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Theorem: A subset H of a group G is a subgroup if and only if

(i) Multiplication is closed in H

(ii) $a \in H$ implies $a^{-1} \in H$

Proof: It is only required if H is a subgroup (i) and (ii) are obvious
if (i) and (ii) are true then it is only required to show that $e \in H$.
Let $a \in H \Rightarrow a^{-1} \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$

Corollary: If H is a nonempty finite subset of a group G and H is closed under multiplication then H is a subgroup of G

Examples: 1. Trivial Subgroups

2. $G = \mathbb{I}$ set of integers, $m \cdot n = m+n$

$$H_5 = \{ \dots, -10, -5, 0, 5, 10, \dots \}$$

3. S_3 : Permutations: $A(3) = \{ e, \sigma, \psi, \psi^2, \sigma\psi, \psi\sigma \}$

$$\left. \begin{array}{l} \{e, \sigma\}, \{e, \sigma\psi\}, \{e, \psi\sigma\} \\ \{e, \psi, \psi^2\} \end{array} \right\} \text{ subgroups}$$

$\{e, \psi\}$ is not a subgroup.

4. ~~$\mathbb{O}(2)$: Set of all orthogonal~~ G : Set of all 2×2 nonsingular matrices

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid ad \neq 0 \right\}$$

H is subset, multiplication is closed, inverses are included.

$$K = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \text{ is real} \right\}$$

$K \subset H \subset G$, K is subgroup of H and G .

5. other known subgroups of $G \supset O(2) \supset SO(2) \supset C_6 \supset C_3$ etc.

5. Let G be a group, $a \in G$ $H = \{e, a, a^2, \dots\}$

6. let G be \mathbb{C}^\times : nonzero complex nos. $H = \{a+bi \mid a^2+b^2=1\}$

16/1/03 : Cosets:

Definition: If H is a subgroup of a group G then $Ha = \{ha \mid h \in H\}$.

Ha is called a right coset of H in G

Lemma: H is a subgroup of G and $a, b \in G$, Ha & Hb are either disjoint or equal.

Proof: $x \in Ha \Rightarrow$ for some h , $x = ha$ & $x \in Hb \Rightarrow$ for some h' , $x = h'b \Rightarrow h^{-1}h'a = b$

$$y \in Ha \Rightarrow \text{for some } h_2, y = h_2 a = (h_2 h^{-1} h') b \Rightarrow y \in Hb$$

$$2 \times 2 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Example: Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.
 Proof: If H is normal, then $gHg^{-1} = H$ for all $g \in G$. Conversely, if $gHg^{-1} = H$ for all $g \in G$, then H is normal.

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3. Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \right\}$$

4. Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$\text{Example: } a(x\vec{e}_1 + y\vec{e}_2) + b\vec{e}_1$$

5. Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \right\} \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

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Multiplication of Cosets: $H_a \cdot H_b = \{xy \mid x \in H_a, y \in H_b\}$

in Example of S_3 : $\psi H \cdot \psi H = \{\psi\psi, \psi\psi\sigma, \psi\sigma\psi, \psi\sigma\psi\sigma\}$

$$\psi H \cdot \psi H = \{\psi^2, \sigma\psi, \sigma, e\}$$

Example: Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

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Example: Let G be a group and H a subgroup. Then H is a normal subgroup of G if and only if $gHg^{-1} = H$ for all $g \in G$.

Lemma: There is one-one correspondence between two right cosets of H in G

$$f: Ha \rightarrow Hb \quad f(x) = xa^{-1}b$$

$$x, y \in Ha, x \neq y, \quad x = h_1a, \quad y = h_2a \quad h_1 \neq h_2$$

$$f(x) = h_1b \quad f(y) = h_2b \Rightarrow f(x) \neq f(y)$$

Theorem: If G is a finite group and H is a subgroup $\Rightarrow |H|$ divides $|G|$

Example: 1. $G = \mathbb{I}$ under addition

$$H = \{5a \mid a \in \mathbb{I}\} \text{ is a subgroup}$$

$$\{\dots, -10, -5, 0, 5, 10, \dots\}$$

$$H_1 = \{5a+1 \mid a \in \mathbb{I}\}$$

$$\{\dots, -9, -4, 1, 6, 11, \dots\}$$

$$H_2 = \{5a+2 \mid a \in \mathbb{I}\}$$

$$\{\dots, -8, -3, 2, 7, 12, \dots\}$$

\vdots

\vdots

Index of H is 5

$$2. G = S_3 \text{ of ACS} \quad |G| = 6$$

$$\Rightarrow H = \{e, \sigma\} \quad |H| = 2$$

$$H\psi = \{\psi, \sigma\psi\}$$

\neq

$$\psi H = \{\psi, \psi\sigma\}$$

$$H\psi^2 = \{\psi^2, \psi\sigma\}$$

\neq

$$\psi^2 H = \{\psi^2, \sigma\psi\}$$

$$\Rightarrow H = \{e, \psi, \psi^2\}$$

$$H\sigma = \{\sigma, \sigma\psi, \sigma\psi^2\}$$

$$\cong \sigma H = \{\sigma, \sigma\psi, \sigma\psi^2\}$$

No subgroup of order 4 is possible

Normal Subgroup.

NORMAL SUBGROUPS

Definition: A subgroup N of G is said to be normal or invariant subgroup if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Lemma: N is a normal subgroup of G if and only if $gNg^{-1} = N$

Proof: if $gNg^{-1} = N \Rightarrow N$ is normal.

if N is normal $\Rightarrow gNg^{-1} \subset N$

$$\text{let } n \in N \Rightarrow g^{-1}ng \in N \Rightarrow g(g^{-1}ng)g^{-1} \in gNg^{-1}$$

$$\Rightarrow n \in gNg^{-1}$$

Lemma: N is normal subgroup of G iff every right coset of N in G is also a left coset of N in G

Proof: let N be normal, $g \in G$. $x \in Ng \Rightarrow x = ng = g(g^{-1}ng)$

$$= g n' \quad \text{when } n' \in N$$

$$\Rightarrow Ng \subset gN, \text{ similarly } gN \subset Ng \Rightarrow Ng = gN$$

if $Ng = gN$ then $ng = gn' \Rightarrow g^{-1}ng \in N \Rightarrow N$ is normal

$$\det(AB) = \det(A) \cdot \det(B)$$

Proof: If A is a square matrix of order n, then $\det(A) = 0$ if and only if A is singular, i.e., if there exists a non-zero vector x such that $Ax = 0$.

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$$Na = (Na)(Nc) = N(ac)$$

$$= Nb \quad (Nb)(Nc) = N(bc)$$

since $bc \in N(ac)$

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Factor GROUPS

Def: A & B are subsets of G . Define product $AB = \{x \in G \mid x = ab \text{ for some } a \in A \text{ \& } b \in B\}$

Lemma: A subgroup N of G is normal if and only if product is two right cosets of N in G is also a right coset of G .

Proof: if N is normal $\Rightarrow Na = aN \Rightarrow (Na)(Nb) \Rightarrow N(Na)b = Nab$

~~$$\text{say } (Na)(Nb) = Nc \Rightarrow n, a, n', b = c$$~~

$$\text{let } n \in N, a \in G \quad (Na)(Na^{-1}) = N \Rightarrow n a n a^{-1} = n'$$

$$\Rightarrow n a n = n' a$$

$$\Rightarrow a n = (n n') a \Rightarrow a n \subset N a$$

$$\Rightarrow a N = N a.$$

Let G/N be a collection of cosets of a ^{normal} subgroup N of G . And the product of cosets is defined above then, G/N is a group under this multiplication. (check all properties)

$$o(G/N) = o(G)/o(N) = \text{index of } N.$$

Example $G = S_3 = \{e, \psi, \psi^2, \sigma, \sigma\psi, \psi\sigma\}$

$$N = \{e, \psi, \psi^2\} \quad N\sigma = \{\sigma, \sigma\psi, \psi\sigma\}$$

$$G/N = \{N, N\sigma\}$$

HOMOMORPHISMS

Definition: A function f from a group G to \bar{G} is called a homomorphism if for all $a, b \in G$, $f(ab) = f(a)f(b)$

Examples: 1 Trivial homomorphism $f(a) = \bar{e} \quad \forall a \in G$

2. $G = \{1, -1\}$ under multiplication

$$\bar{G} = \{0, 1\} \text{ under } a \cdot b = (a+b) \bmod 2$$

3. $G = \mathbb{I}$ under addition

$$\bar{G} = \{2^a \mid a \in \mathbb{I}\} \text{ under multiplication}$$

4. $G = S_3 = \{e, \psi, \psi^2, \sigma, \sigma\psi, \psi\sigma\}$

$$\bar{G} = \{e, \sigma\} \quad \text{let } f(\sigma^i \psi^j) = \sigma^i$$

5. let $G = \{2 \times 2 \text{ matrices (nonsingular real)}\}$

$$\bar{G} = \mathbb{R}^*$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

6. $G = SO(2)$, $\bar{G} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$

$$f\left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\right) \rightarrow e^{i\theta}$$

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$

$$\begin{aligned} 9. \quad 2000 \times 3 &= 6000 \\ 6000 &= 6000 \end{aligned}$$

$$P_1 \cdot P_2 \cdot P_3 = 1 \times 10^5 \text{ dynes/cm}^2 \quad (\text{from } P_1 = 10^5 \text{ dynes/cm}^2)$$

$$f = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \quad \text{in} \quad (f, \phi) = 0.$$

Figure 6

$\frac{d}{dt} \left(\frac{1}{\rho} \right) = - \frac{1}{\rho^2} \frac{d\rho}{dt}$

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
 $\frac{1}{16} \times \frac{1}{16} = \frac{1}{256}$

$$\frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \frac{1}{4} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \quad \frac{1}{8} = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

3. $C = \{0, 1\}$ and $\varphi = \varphi_1 \vee \varphi_2$ where φ_1 and φ_2 are

March 1971 *Neophilaena* 1971 + 1971

Circumstance	Justified (%)	Not justified (%)
If someone is attacking you	85	15
If someone is threatening you	75	25
If someone is harassing you	65	35
If someone is insulting you	55	45
If someone is annoying you	45	55

1. The first part of the document is a list of names and their corresponding dates. The names are: John Doe, Jane Smith, and Bob Johnson. The dates are: 1/1/2020, 2/1/2020, and 3/1/2020.

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Handwritten: The first part of the book is very good.

THE STATE OF TEXAS, COUNTY OF DALLAS, ss. I, _____, Clerk of the County Court, do hereby certify that the foregoing is a true and correct copy of the original of the same as the same appears from the records of the County Court of said County of Dallas, State of Texas.

1. *Chlorophyll *a** and *Chlorophyll *b** were determined by the method of Arar and Collins (1971) using a Shimadzu 1010 spectrophotometer.

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

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Figure 1. Schematic representation of the experimental design. The study was conducted in two phases. In the first phase, the participants were exposed to a 10-day training period. During this period, they were exposed to a 10-day training period. The second phase was a 10-day testing period. During this period, the participants were exposed to a 10-day testing period. The results of the testing period were compared to the results of the training period.

$$\text{Lösung: } 14 \cdot 10^{10} \text{ Joule} \Rightarrow 14 \cdot 10^{10} \cdot \frac{1}{3600} \cdot \frac{1}{1000} = 3888,89 \text{ kWh} = 3,89 \text{ MWh}$$

1. **Identify the main purpose of the document.** The main purpose of this document is to provide a comprehensive overview of the company's financial performance for the year 2023, including key metrics, trends, and forecasts.

300' 4 1/2" 1" 2" 3" 4" 5" 6" 7" 8" 9" 10" 11" 12" 13" 14" 15" 16" 17" 18" 19" 20" 21" 22" 23" 24" 25" 26" 27" 28" 29" 30" 31" 32" 33" 34" 35" 36" 37" 38" 39" 40" 41" 42" 43" 44" 45" 46" 47" 48" 49" 50" 51" 52" 53" 54" 55" 56" 57" 58" 59" 60" 61" 62" 63" 64" 65" 66" 67" 68" 69" 70" 71" 72" 73" 74" 75" 76" 77" 78" 79" 80" 81" 82" 83" 84" 85" 86" 87" 88" 89" 90" 91" 92" 93" 94" 95" 96" 97" 98" 99" 100" 101" 102" 103" 104" 105" 106" 107" 108" 109" 110" 111" 112" 113" 114" 115" 116" 117" 118" 119" 120" 121" 122" 123" 124" 125" 126" 127" 128" 129" 130" 131" 132" 133" 134" 135" 136" 137" 138" 139" 140" 141" 142" 143" 144" 145" 146" 147" 148" 149" 150" 151" 152" 153" 154" 155" 156" 157" 158" 159" 160" 161" 162" 163" 164" 165" 166" 167" 168" 169" 170" 171" 172" 173" 174" 175" 176" 177" 178" 179" 180" 181" 182" 183" 184" 185" 186" 187" 188" 189" 190" 191" 192" 193" 194" 195" 196" 197" 198" 199" 200" 201" 202" 203" 204" 205" 206" 207" 208" 209" 210" 211" 212" 213" 214" 215" 216" 217" 218" 219" 220" 221" 222" 223" 224" 225" 226" 227" 228" 229" 230" 231" 232" 233" 234" 235" 236" 237" 238" 239" 240" 241" 242" 243" 244" 245" 246" 247" 248" 249" 250" 251" 252" 253" 254" 255" 256" 257" 258" 259" 260" 261" 262" 263" 264" 265" 266" 267" 268" 269" 270" 271" 272" 273" 274" 275" 276" 277" 278" 279" 280" 281" 282" 283" 284" 285" 286" 287" 288" 289" 290" 291" 292" 293" 294" 295" 296" 297" 298" 299" 300" 301" 302" 303" 304" 305" 306" 307" 308" 309" 310" 311" 312" 313" 314" 315" 316" 317" 318" 319" 320" 321" 322" 323" 324" 325" 326" 327" 328" 329" 330" 331" 332" 333" 334" 335" 336" 337" 338" 339" 340" 341" 342" 343" 344" 345" 346" 347" 348" 349" 350" 351" 352" 353" 354" 355" 356" 357" 358" 359" 360" 361" 362" 363" 364" 365" 366" 367" 368" 369" 370" 371" 372" 373" 374" 375" 376" 377" 378" 379" 380" 381" 382" 383" 384" 385" 386" 387" 388" 389" 390" 391" 392" 393" 394" 395" 396" 397" 398" 399" 400" 401" 402" 403" 404" 405" 406" 407" 408" 409" 410" 411" 412" 413" 414" 415" 416" 417" 418" 419" 420" 421" 422" 423" 424" 425" 426" 427" 428" 429" 430" 431" 432" 433" 434" 435" 436" 437" 438" 439" 440" 441" 442" 443" 444" 445" 446" 447" 448" 449" 450" 451" 452" 453" 454" 455" 456" 457" 458" 459" 460" 461" 462" 463" 464" 465" 466" 467" 468" 469" 470" 471" 472" 473" 474" 475" 476" 477" 478" 479" 480" 481" 482" 483" 484" 485" 486" 487" 488" 489" 490" 491" 492" 493" 494" 495" 496" 497" 498" 499" 500" 501" 502" 503" 504" 505" 506" 507" 508" 509" 510" 511" 512" 513" 514" 515" 516" 517" 518" 519" 520" 521" 522" 523" 524" 525" 526" 527" 528" 529" 530" 531" 532" 533" 534" 535" 536" 537" 538" 539" 540" 541" 542" 543" 544" 545" 546" 547" 548" 549" 550" 551" 552" 553" 554" 555" 556" 557" 558" 559" 560" 561" 562" 563" 564" 565" 566" 567" 568" 569" 570" 571" 572" 573" 574" 575" 576" 577" 578" 579" 580" 581" 582" 583" 584" 585" 586" 587" 588" 589" 590" 591" 592" 593" 594" 595" 596" 597" 598" 599" 600" 601" 602" 603" 604" 605" 606" 607" 608" 609" 610" 611" 612" 613" 614" 615" 616" 617" 618" 619" 620" 621" 622" 623" 624" 625" 626" 627" 628" 629" 630" 631" 632" 633" 634" 635" 636" 637" 638" 639" 640" 641" 642" 643" 644" 645" 646" 647" 648" 649" 650" 651" 652" 653" 654" 655" 656" 657" 658" 659" 660" 661" 662" 663" 664" 665" 666" 667" 668" 669" 670" 671" 672" 673" 674" 675" 676" 677" 678" 679" 680" 681" 682" 683" 684" 685" 686" 687" 688" 689" 690" 691" 692" 693" 694" 695" 696" 697" 698" 699" 700" 701" 702" 703" 704" 705" 706" 707" 708" 709" 710" 711" 712" 713" 714" 715" 716" 717" 718" 719" 720" 721" 722" 723" 724" 725" 726" 727" 728" 729" 730" 731" 732" 733" 734" 735" 736" 737" 738" 739" 740" 741" 742" 743" 744" 745" 746" 747" 748" 749" 750" 751" 752" 753" 754" 755" 756" 757" 758" 759" 760" 761" 762" 763" 764" 765" 766" 767" 768" 769" 770" 771" 772" 773" 774" 775" 776" 777" 778" 779" 780" 781" 782" 783" 784" 785" 786" 787" 788" 789" 790" 791" 792" 793" 794" 795" 796" 797" 798" 799" 800" 801" 802" 803" 804" 805" 806" 807" 808" 809" 810" 811" 812" 813" 814" 815" 816" 817" 818" 819" 820" 821" 822" 823" 824" 825" 826" 827" 828" 829" 830" 831" 832" 833" 834" 835" 836" 837" 838"

17

Theorem: If f is a homomorphism of G into \bar{G} then

① $f(e) = \bar{e}$

$f(xe) = f(x) \cdot f(e) = f(x) \cdot \bar{e} \Rightarrow f(e) = \bar{e}$

② $f(x^{-1}) = [f(x)]^{-1}$

$f(xx^{-1}) = f(e) = \bar{e}$

$f(x) \cdot f(x^{-1}) = \bar{e}$

Theorem:

Definition: Kernel. If f is a homomorphism ~~from~~ ^{of} G into \bar{G} , the Kernel K of homomorphism is defined as $K = \{x \in G \mid f(x) = \bar{e}\}$

Theorem: Kernel of a homomorphism of G in \bar{G} is a subgroup (Normal)

Definition: Isomorphism: A homomorphism f of G into \bar{G} is said to be an isomorphism if f is one-to-one

Definition: Two groups G and \bar{G} are said to be isomorphic if there exists an isomorphism of G onto \bar{G} .

5/2

⑥

⑥

Continuous Groups

Example:

a. $C_4 = \{e, \sigma, \sigma^2, \sigma^3\}$

Finite

b. $I = \text{Set of integers}$

Infinite, discrete

c. $G = \{2^i \mid i \in I\}$

under multiplication Infinite, discrete

d. $G = \{T_{ab}: \mathbb{R} \rightarrow \mathbb{R} \mid T_{ab}(x) = ax + b, a \neq 0\}$

under function composition Infinite, Continuous 2 parameter

e. $G = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$

Infinite, Continuous, 1 parameter.

f. $G = O(2), 2 \times 2, \text{ Real orthogonal matrices } (?)$

$$1. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (1)$$

$$2. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (2)$$

$$3. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (3)$$

$$4. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (4)$$

$$5. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (5)$$

$$6. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (6)$$

$$7. \quad \beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \quad \text{det } \beta = 1 \text{ (det } \beta \neq 0 \text{)} \quad (7)$$

$$a^2 + c^2 = 1, b^2 + d^2 = 1, ad + bc = 0, ab + cd = 0$$

$$(1) \quad a^2 + c^2 = 1 \Rightarrow a^2 = 1 - c^2 \Rightarrow a^2 = 1 - \left(-\frac{ad}{b}\right)^2$$

$$ab = -cd \Rightarrow a^2 b^2 = c^2 d^2 \Rightarrow a^2 (1 - d^2) = c^2 d^2 \Rightarrow a^2 = (a^2 + c^2) d^2 \Rightarrow a^2 = d^2 \Rightarrow \boxed{a = \pm d}$$

$$(2) \quad a^2 b^2 = c^2 d^2 \Rightarrow (1 - c^2) b^2 = c^2 d^2 \Rightarrow \boxed{b = \pm c}$$

$$(3) \quad (ad - bc)^2 = a^2 + b^2 = 1 \Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$$

O(2) and SO(2) Groups

Definition: $O(2)$ is set of all 2×2 real, ^{orthogonal} matrices.

$$O^T O = I$$

$SO(2)$ is set of all 2×2 real, orthogonal matrices with $\det = +1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} a^2 + c^2 = 1, \quad ab + cd = 0 \\ b^2 + d^2 = 1, \quad ab + cd = 0 \end{array} \right\} \Rightarrow \frac{a}{c} = -\frac{d}{b} \text{ or } \frac{a}{d} = -\frac{b}{c}$$

$$\Rightarrow (ad - bc) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$$

$$\Rightarrow a = \pm d \text{ and } c = \pm b \text{ and } ad - bc = 1$$

$$\Rightarrow \begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix} \text{ and } a^2 + b^2 = 1$$

$$\Rightarrow a = \cos \theta \quad b = \sin \theta$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \begin{bmatrix} \cos \theta & \sin \theta \\ +\sin \theta & -\cos \theta \end{bmatrix}$$

$$-1 \leq a \leq 1, \quad b = \pm \sqrt{1 - a^2}$$

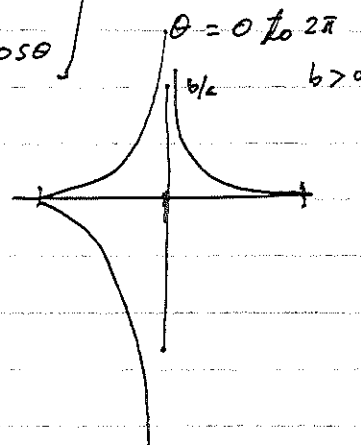
$$\theta = \tan^{-1} \left(\frac{b}{a} \right) \quad -\pi/2 \leq \theta \leq \pi/2 \quad b > 0$$

$$\theta = \tan^{-1} \left(\frac{b}{a} \right) \quad \pi/2 \leq \theta \leq 3\pi/2, \quad b < 0$$

$$\Rightarrow a = \cos \theta$$

$$\Rightarrow b = \sin \theta$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ (-1)^{\text{sgn} a} \sin \theta & (-1)^{\text{sgn} a} \cos \theta \end{bmatrix} \quad \begin{array}{l} \theta \in [0, 2\pi) \\ a \in \{0, 1\} \end{array}$$



$SO(2)$ is a single parameter group.

Parameter Space: Domain in $\mathbb{R}^n \supset D$

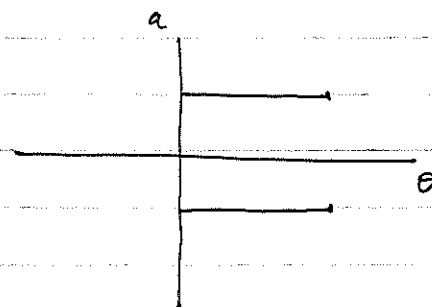
~~Phases~~ GENERATORS AND EXPONENTIAL MAPS

$$F(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\frac{\partial F}{\partial \theta} = \lim_{h \rightarrow 0} \frac{F(\theta+h) - F(\theta)}{h}$$

$$= \lim_{h \rightarrow 0} \begin{bmatrix} \frac{\cos(\theta+h) - \cos \theta}{h} & \frac{\sin(\theta+h) - \sin \theta}{h} \\ -\frac{\sin(\theta+h) - \sin \theta}{h} & \frac{\cos(\theta+h) - \cos \theta}{h} \end{bmatrix} = \begin{bmatrix} -\sin \theta & \cos \theta \\ -\cos \theta & -\sin \theta \end{bmatrix}$$

$$\text{Generator } I = \frac{1}{i} \frac{\partial F}{\partial \theta} \bigg|_{\theta=0} = \frac{1}{i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_y$$



8/2

(7)

1. The first part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

2. The second part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

3. The third part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

4. The fourth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

5. The fifth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

6. The sixth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

7. The seventh part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

8. The eighth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

9. The ninth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

10. The tenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

11. The eleventh part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

12. The twelfth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

13. The thirteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

14. The fourteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

15. The fifteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

16. The sixteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

17. The seventeenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

18. The eighteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

19. The nineteenth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

20. The twentieth part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

21. The twenty-first part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

22. The twenty-second part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

Suppose that $\theta = n\epsilon$ where ϵ is small (infinitesimal)

$$f(\theta) = f(\epsilon) \cdot f(\epsilon) \cdot \dots = [f(\epsilon)]^n$$

since $f(\epsilon) \approx 1 + i\sigma_y \epsilon$

$$\Rightarrow f(\theta) \approx (1 + i\sigma_y \theta/n)^n = \lim_{n \rightarrow \infty} (1 + i\sigma_y \theta/n)^n = \exp(i\sigma_y \theta)$$

$$\Rightarrow f(\theta) = \exp(i\sigma_y \theta)$$

$$\Rightarrow = 1 + i\sigma_y \theta + \frac{i^2 \sigma_y^2 \theta^2}{2} + \dots$$

$$= 1 + i\sigma_y \theta - \frac{1}{2} \theta^2 1 + \frac{i}{6} \sigma_y \theta^3 + \dots$$

$$= 1 \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots \right) + i\sigma_y \left(\theta - \frac{\theta^3}{3!} + \dots \right)$$

$$= 1 \cos \theta + i\sigma_y \sin \theta$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Rotation Group on Functions

let $L = \{ \text{set of all analytic real fn on } \mathbb{R}^2 \}$

When \mathbb{R}^2 there is a rotation of coordinates

Then $f \rightarrow f' : f'(P) = f(P')$

$$\Rightarrow f'(x, y) = f(x', y') = f(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

$$= \exp(i\theta L_2) f(x, y)$$

$$\Rightarrow f' \approx (1 + i\theta L_2) f \quad \text{if } \theta \text{ is small}$$

$$= \exp(i\theta L_2) f \quad L_2 = i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

Ex $f(x, y) = x^2 + y^2$

$$L_2 f(x, y) = i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) f(x, y)$$

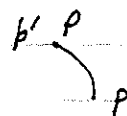
$$= i(2xy - 2yx) f(x, y) = 0$$

$$\Rightarrow \exp(i\theta L_2) f = f$$

Ex $f(x, y) = x \quad L_2 x = i(-y) = -iy \quad L_2^2 x = -i \cdot (-y) = y$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} -i^2 = 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$P = (x, y)$$

$$P' = (x', y')$$

$$f(x) = \dots \quad f'(x) = \dots \quad f''(x) = \dots$$

$$\Rightarrow \text{...}$$

$$\dots$$

$$\dots$$

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...

$$\exp(i\theta L_z) x$$

$$= \left(1 + i\theta L_z - \frac{1}{2}\theta^2 L_z^2 + \frac{i}{3!}\theta^3 L_z^3 - \dots \right) x$$

$$= x \left(1 - \frac{1}{2}\theta^2 + \frac{\theta^4}{4!} \dots \right) + y \left(\theta - \frac{\theta^3}{3!} + \dots \right)$$

$$= x \cos \theta + y \sin \theta$$

8/2

⑧

SO(3) Group and its Generators

Def: SO(3) all 3×3 real orthogonal matrices ~~3x3~~ with $\det = 1$

Requires $A^T A = I$ $A \in SO(3)$

\Rightarrow Since $A^T A$ is symmetric there are six ind. equations

\Rightarrow SO(3) is a three parameter group

Definition: A rotation ~~about an axis (passing thro' origin) \hat{a}~~

A co-ordinate transformation is called $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is called a rotation if it preserves distances and ~~fixes the origin~~ is linear.

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad |T(x) - T(y)| = |x - y| \quad x, y \in \mathbb{R}^3$$

$$T(x) \cdot T(y) = x \cdot y$$

$$\therefore (T^T)^T T y = x \cdot y$$

$$\therefore x^T T^T T y = x \cdot y$$

$$\therefore T^T T = I$$

SO(3) is set of rotations on \mathbb{R}^3 .

A. Axis-Angle Parametrization.

\hat{a} axis of rot: θ angle of rotation

$$\vec{x}' = \vec{OA} + \vec{AD} + \vec{DC}$$

$$\hat{a} \times \hat{x} = \hat{n} \sin \alpha$$

$$\hat{a} \cdot \hat{x} = \cos \alpha$$

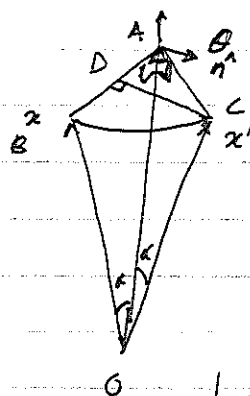
$$\hat{n} \times \hat{a} = \frac{(\hat{a} \times \hat{x}) \times \hat{a}}{\sin \alpha}$$

$$\vec{x}' = \hat{a} x \cos \alpha + \hat{n} \times \hat{a} AC \cos \theta + \hat{n} \cdot \hat{a} C \sin \theta$$

$$= x (\hat{a} \cos \alpha + \hat{n} \times \hat{a} \sin \alpha \cos \theta + \hat{n} \sin \alpha \sin \theta)$$

$$= x (\hat{x} - \hat{n} \times \hat{a} \sin \alpha (1 - \cos \theta) + \hat{a} \times \hat{x} \sin \theta)$$

$$= \vec{x} + \hat{a} \times \vec{x} \sin \theta + (1 - \cos \theta) (\hat{a} \times (\hat{a} \times \vec{x}))$$



$$\begin{aligned} \hat{n} \times \hat{a} &= \frac{(\hat{a} \times \hat{x}) \times \hat{a}}{\sin \alpha} \\ &= \frac{(\hat{a} \cdot \hat{a}) \hat{x} - \hat{a} (\hat{a} \cdot \hat{x})}{\sin \alpha} \\ &= \hat{x} - \hat{a} \cos \alpha \end{aligned}$$

1. $\frac{1}{x^2} = x^{-2}$

2. $\frac{1}{x^3} = x^{-3}$

3. $\frac{1}{x^4} = x^{-4}$

4. $\frac{1}{x^5} = x^{-5}$

5. $\frac{1}{x^6} = x^{-6}$

6. $\frac{1}{x^7} = x^{-7}$

7. $\frac{1}{x^8} = x^{-8}$

8. $\frac{1}{x^9} = x^{-9}$

9. $\frac{1}{x^{10}} = x^{-10}$

10. $\frac{1}{x^{11}} = x^{-11}$

11. $\frac{1}{x^{12}} = x^{-12}$

12. $\frac{1}{x^{13}} = x^{-13}$

13. $\frac{1}{x^{14}} = x^{-14}$

14. $\frac{1}{x^{15}} = x^{-15}$

15. $\frac{1}{x^{16}} = x^{-16}$

16. $\frac{1}{x^{17}} = x^{-17}$

17. $\frac{1}{x^{18}} = x^{-18}$

18. $\frac{1}{x^{19}} = x^{-19}$

19. $\frac{1}{x^{20}} = x^{-20}$

20. $\frac{1}{x^{21}} = x^{-21}$

21. $\frac{1}{x^{22}} = x^{-22}$

22. $\frac{1}{x^{23}} = x^{-23}$

23. $\frac{1}{x^{24}} = x^{-24}$

24. $\frac{1}{x^{25}} = x^{-25}$

25. $\frac{1}{x^{26}} = x^{-26}$

26. $\frac{1}{x^{27}} = x^{-27}$

27. $\frac{1}{x^{28}} = x^{-28}$

28. $\frac{1}{x^{29}} = x^{-29}$

$$\hat{a} \times \vec{x} = (a_2 x_3 - x_2 a_3) \hat{i} + (a_3 x_1 - a_1 x_3) \hat{j} + \hat{k} (a_1 x_2 - x_1 a_2)$$

$$= \begin{bmatrix} a_2 x_3 - a_3 x_2 \\ -a_1 x_3 + a_3 x_1 \\ a_1 x_2 - a_2 x_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ +a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S_a$$

$$\hat{a} \times (\hat{a} \times \vec{x}) = \begin{bmatrix} -a_2^2 - a_3^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & -a_3^2 - a_1^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & -a_2^2 - a_1^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S_a^2$$

~~$$\begin{bmatrix} x' \end{bmatrix} = I[x] + (1 - \cos \theta) S_a[x] +$$~~

$$[x'] = I[x] + \sin \theta S_a[x] + (1 - \cos \theta) \hat{a}^2[x]$$

$$= R(\hat{a}, \theta)[x]$$

$$R(\hat{a}, \theta) = I + \sin \theta S_a + (1 - \cos \theta) S_a^2$$

$$= \begin{bmatrix} a_1^2(1 - \cos \theta) + \cos \theta & a_1 a_2(1 - \cos \theta) - a_3 \sin \theta & a_1 a_3(1 - \cos \theta) + a_2 \sin \theta \\ a_2 a_1(1 - \cos \theta) + a_3 \sin \theta & \dots & \dots \\ a_3 a_1(1 - \cos \theta) - a_2 \sin \theta & \dots & \dots \end{bmatrix}$$

Each Rotation is ~~an~~ $0 \leq \theta \leq \pi$

at π $R(\hat{a}, \pi) = R(-\hat{a}, \pi)$

Parameterization:
Inverse map

$$a_1^2(1 - \cos \theta) + \cos \theta = a_{11}$$

$$a_2^2(1 - \cos \theta) + \cos \theta = a_{22}$$

$$a_3^2(1 - \cos \theta) + \cos \theta = a_{33}$$

$$1 + 2 \cos \theta = \text{Tr}(A)$$

$$\cos \theta = \frac{1}{2} (\text{Tr}(A) - 1)$$

$$2a_1 a_2 (1 - \cos \theta) = 2a_3 \sin \theta = a_{22} - a_{12}$$

$$a_3 = (a_{21} - a_{12}) / 2 \sin \theta$$

$$a_2 = (a_{13} - a_{31}) / 2 \sin \theta$$

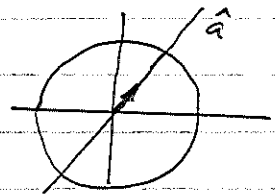
$$a_1 = (a_{32} - a_{23}) / 2 \sin \theta$$

in case of $\pi = \theta$,

$$a_1 = (1 + a_{11}) / \sqrt{2} (1 + a_{11})^{1/2}$$

$$a_2 = a_{12} / \sqrt{2} (1 + a_{11})^{1/2}$$

$$a_3 = a_{13} / \sqrt{2} (1 + a_{11})^{1/2}$$



$\theta \neq 0$ - no axis
 $\theta = \pi$ - Symmetric

$$x^2 = 10^2 - 6^2 = 100 - 36 = 64$$

$$x = \sqrt{64} = 8$$

$$y^2 = 10^2 - 8^2 = 100 - 64 = 36$$

$$y = \sqrt{36} = 6$$

$$x^2 = 10^2 - 6^2 = 100 - 36 = 64$$

$$x = \sqrt{64} = 8$$

$$y^2 = 10^2 - 8^2 = 100 - 64 = 36$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

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$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

$$x = 8, y = 6$$

10/2

Thm: $R(\hat{a}, \theta) = \exp(i S_a \theta)$

$$\vec{a} = \theta \hat{a}$$

(9)

Notice $S_a = \begin{bmatrix} 0 & +a_3 & -a_2 \\ +a_2 & 0 & +a_1 \\ +a_1 & -a_2 & 0 \end{bmatrix} = -S_a$

$$= 1 + i S_a \theta + \frac{-i^2}{2} S_a^2 \theta^2 + \dots$$

$$= 1 + i S_a \left(\theta - \frac{\theta^3}{3} + \dots \right) + S_a^2 \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \dots \right)$$

$$= 1 + \sin \theta S_a + (1 - \cos \theta) S_a^2$$

Generator of rotation about a axis is

$$J_a = \frac{1}{i} S_a$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{bmatrix}$$

$$[L_x, L_y] = i L_z$$

$$[L_y, L_z] = i L_x$$

$$\vdots$$

$$J_y = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

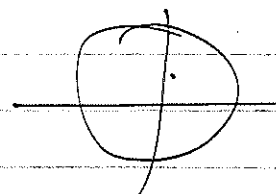
a

$$J_a = a_1 J_x + a_2 J_y + a_3 J_z$$

$$R(\hat{a}, \theta) = \exp[i(a_x J_x + a_y J_y + a_z J_z) \theta] \quad \begin{matrix} (a_x, a_y, a_z) \\ = (a_1 \theta, a_2 \theta, a_3 \theta) \end{matrix}$$

Not Multiplication.

Parametrization



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SU(2) Group

Definition $SU(2) =$ ~~Set~~ Group of all 2×2 unitary matrices.

Require $A \in SU(2) \Rightarrow A^\dagger A = I$

$$\Rightarrow \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I$$

Four equations reduce $U(2)$ to a 4 parameter Group.

General Form

$$A = \begin{bmatrix} \cos \theta e^{i\alpha} & \sin \theta e^{i\gamma} \\ -\sin \theta e^{i(\beta-\gamma)} & \cos \theta e^{i(\beta-\alpha)} \end{bmatrix}$$

det of the matrix $e^{i\beta}$. $|\det A| = 1$

SU(2) elements $\beta = 0$ or $2\pi \Rightarrow \det A = 1$

Identity element $\alpha = \gamma = \theta = 0$

Generators

$$I_0 = \frac{1}{i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \sigma_y$$

$$I_x = \frac{1}{i} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z$$

$$I_y = \frac{1}{i} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \sigma_x$$

Algebra of Generators

$$[\sigma_x, \sigma_y] = i\sigma_z, [\sigma_y, \sigma_z] = i\sigma_x, [\sigma_z, \sigma_x] = i\sigma_y \text{ etc.}$$

There is some connection bet $SO(3)$ and $SU(2)$

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- 10 13/2 1. Definition of Fields \mathbb{R} and \mathbb{C} as examples
 2. Definition of Vector Spaces, Examples
- 11 14/2 2. More Examples, Some basic Properties
 3. Linear Combinations, Linear Independence of a set, Geometric interpretation of Linear independence with examples.
- 12 24/2 4. Span of a set, examples, Basis and Dimension
- 13 25/2 5. Examples of Bases. Polynomials, Finite and infinite, Legendre basis
 • Real fns on lattices
- 4 27/2 6. Co-ordinates, Change of Basis, matrix manipulations.
 7. Subspaces,
- 15 28/2 8. Homomorphism, Kernel, Range, Isomorphism, Isomorphic spaces.
- 16 3/3 9. Inner product, Inner product spaces, Cauchy Schwarz Inequality,
Gram-Schmidt Orthogonalization.
 (GS)
- 17 5/3 10. Metric, Metric spaces, Sequences, Cauchy Sequences, Completeness.
 11. Examples: \mathbb{Q} , \mathbb{R} , \mathbb{R}^n , $C[a,b]$, L^∞ , $L^2[a,b]$,
- 19 7/3 11. Hilbert Space: Formal Definition
 statement: For every IPS, there exists a Hilbert space.
 Cauchy-Schwarz Inequality (should have done earlier)
 Orthonormal Sets: Are linearly independent.
 If $\{x_k\} \subset B$, ~~an~~ is an orthonormal set then if $x \in \text{span}(B)$ then

$$x = \sum_k \langle x, e_k \rangle e_k$$

 and if $x \notin \text{span}(B)$ then $x = y + z$ where $y \in \text{span}(B)$ and $z \perp$
 Infinite Sum: $B = \{e_1, e_2, \dots, e_n, \dots\}$
 let $S_n = \sum_{k=1}^n x_k e_k$
 if Seq. $S_n \rightarrow S$
 then S_n is Cauchy seq. $S \in H$.
 denote $S = \sum_{k=1}^{\infty} x_k e_k$
- $$\begin{aligned} \|S_n - S\| &\rightarrow 0 \\ |\|S_n\| - \|S\|| &\rightarrow 0 \\ \|S_n\| &\rightarrow \|S\| \end{aligned}$$
- Basis (Orthonormal): B is an orthonormal set in H then
 B is called a basis if $\overline{\text{span}(B)} = H$
 Examples: Fourier Series, Legendre, Hermite, Lagrange

1. $\frac{1}{x^2} = x^{-2}$

$$\frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$$

2. $\frac{1}{x^3} = x^{-3}$

$$\frac{d}{dx} x^{-3} = -3x^{-4} = -\frac{3}{x^4}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} + \frac{1}{x^3} \right) = -\frac{2}{x^3} - \frac{3}{x^4}$$

$$= -\frac{2x + 3}{x^4}$$

$$= -\frac{2x^2 + 3x}{x^4}$$

$$= -\frac{2x^2 + 3x}{x^4}$$

$$= -\frac{2x^2 + 3x}{x^4}$$

3. $\frac{1}{x^4} = x^{-4}$

$$\frac{d}{dx} x^{-4} = -4x^{-5} = -\frac{4}{x^5}$$

$$= -\frac{4}{x^5}$$

4. $\frac{1}{x^5} = x^{-5}$

$$\frac{d}{dx} x^{-5} = -5x^{-6} = -\frac{5}{x^6}$$

5. $\frac{1}{x^6} = x^{-6}$

$$\frac{d}{dx} x^{-6} = -6x^{-7} = -\frac{6}{x^7}$$

6. $\frac{1}{x^7} = x^{-7}$

$$\frac{d}{dx} x^{-7} = -7x^{-8} = -\frac{7}{x^8}$$

7. $\frac{1}{x^8} = x^{-8}$

$$\frac{d}{dx} x^{-8} = -8x^{-9} = -\frac{8}{x^9}$$

8. $\frac{1}{x^9} = x^{-9}$

$$\frac{d}{dx} x^{-9} = -9x^{-10} = -\frac{9}{x^{10}}$$

9. $\frac{1}{x^{10}} = x^{-10}$

$$\frac{d}{dx} x^{-10} = -10x^{-11} = -\frac{10}{x^{11}}$$

$$= -\frac{10}{x^{11}}$$

$$= -\frac{10}{x^{11}}$$

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$$= -\frac{10}{x^{11}}$$

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$$= -\frac{10}{x^{11}}$$

$$= -\frac{10}{x^{11}}$$

$$= -\frac{10}{x^{11}}$$

$$= -\frac{10}{x^{11}}$$

Homomorphism

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $B = \{i, j, k\}$

$T(x, y, z) = (x, 0, 0)$. Projection.

Range $(T) = x\text{-axis}$, Kernel $(T) = \{(0, y, z)\}$ yz plane

Not an isomorphism.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex $T: \mathbb{R} \rightarrow \mathbb{R}$ $T(x) = 5x$

$$[T] = [5]$$

Range $(T) = \mathbb{R}$, Kernel $(T) = \{0\}$, Isomorphism

Ex $T: \mathbb{R} \rightarrow \mathbb{R}^2$ $T(x) = (3x, 5x)$

$y' = 5x$ $x' = 3x \Rightarrow 5x' - 3y' = 0$

Range $(T) = \text{st-line}$, Kernel $(T) = \{0\}$, Isomorphism, \mathbb{R} and \mathbb{R}^2 not isomorphic

• Homomorphism maps $0 \rightarrow 0$.

$T(x) = x'$ $T(0) = \xi'$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$x' = T(x) = T(x+0) = T(x) + T(0) = x' + \xi' \Rightarrow \xi' = 0$

• ~~Homomorphism~~ ^{Isomorphism} maps basis of $V \rightarrow$ Basis of $T(V)$

$B = \{e_1, e_2, \dots, e_n\}$ $B' = \{e'_1, e'_2, \dots, e'_n\} = T(B)$

$T(e_i) = e'_i$

$\alpha_i e'_i \neq \alpha_i T(e_i) = T(\alpha_i e_i) = 0$

implicit sum

$\Rightarrow \alpha_i e_i = 0 \Rightarrow \alpha_i = 0 \forall i$

• $x' = \alpha'_i e'_i$ has a preimage $x = \alpha_i e_i$

$T(x) = x' = T(\alpha_i e_i) = \alpha_i T(e_i) = \alpha_i e'_i$

• $F^{(n)}$ is isomorphic $F^{(m)}$ only if $n=m$

• $\dim(V) = n$ V is isomorphic to $F^{(n)}$ in place.

Let B be basis of V $T(e_i) = (0, 0, \dots, 1, \dots)$

• Matrix of Homomorphism $B \subset V$ $B' \subset V'$ $T: V \rightarrow V'$

$T(e_i) = \sum_j T_{ji} e'_j$

$T(\alpha_i e_i) = \sum_j \alpha_i T_{ji} e'_j = \sum_j (\sum_i T_{ji} \alpha_i) e'_j$ $T_{m \times n}$
 $= \alpha'_j e'_j$

$\Rightarrow \begin{bmatrix} \alpha'_j \end{bmatrix} = T \begin{bmatrix} \alpha_i \end{bmatrix}$

GS orthogonalization

1. $\{1, x, x^2, \dots\}$

$$P_0 = 1 \quad P_0 \cdot P_0 = \int dx = 2$$

$$P_1 = x \quad P_1 \cdot P_1 = \int x^2 dx = 2/3$$

$$P_2' = x^2 - \frac{P_2' \cdot P_0}{P_0 \cdot P_0} P_0 - \frac{P_2' \cdot P_1}{P_1 \cdot P_1} P_1 \quad P_2 \cdot P_0 = \int x^2 dx = 2/3$$

$$P_2 \cdot P_1 = 0$$

$$= x^2 - \frac{2/3}{2} P_0$$

$$= x^2 - 1/3 \quad \text{since } P_2'(1) = 2/3$$

$$P_2 = 1/2 (3x^2 - 1)$$

$$P_3' = x^3 - \frac{P_3' \cdot P_1}{P_1 \cdot P_1} P_1 \quad P_3 \cdot P_1 = \int x^4 dx = 2/5$$

$$= x^3 - \frac{2}{5} x \quad P_3'(1) = \frac{3}{5}$$

$$P_3 = \frac{1}{3} (5x^3 - 2x)$$

$$P_4' = x^4 - \frac{P_4' \cdot P_2}{P_2 \cdot P_2} P_2 - \frac{P_4' \cdot P_0}{P_0 \cdot P_0} P_0 \quad P_4' \cdot P_0 = 2/5$$

$$P_4' \cdot P_2 = \frac{1}{2} (3^2/7 - 2/5) = \frac{1}{2} (\frac{30-14}{35})$$

$$= x^4 - \frac{16/70}{2/5} P_2 - \frac{2/5}{2} P_0 \quad P_2 \cdot P_2 = \frac{1}{4} [9 \cdot \frac{2}{5} + 1/2 - 2 \cdot 3 \cdot \frac{2}{3}]$$

$$= x^4 - \frac{4}{7} \cdot \frac{1}{2} (3x^2 - 1) - 1/5 \quad = \frac{1}{4} [\frac{18}{5} - 2] = \frac{2}{5}$$

$$= x^4 - 6/7 x^2 + 2/7 - 1/5$$

$$= x^4 - 6/7 x^2 + 3/35 \quad P_4'(1) = 1 - 6/7 + 3/35 = \frac{35-30+3}{35} = 8/35$$

$$P_4 = \frac{8}{3} \cdot \frac{35}{8} x^4 - \frac{5 \cdot 6}{8} x^2 + 3/8$$

$$= \frac{1}{8} (35x^4 - 30x^2 + 3)$$

Inner Product Spaces

- Introduce IP as generalization of inner product of 3-d vectors
- Definition: V vector space over F . $\phi: V \times V \rightarrow F$ s.t.

$$① \quad u \cdot v = \overline{v \cdot u} \quad u, v, w \in V$$

$$② \quad u \cdot u \geq 0 \text{ and } u \cdot u = 0 \text{ iff } u = 0 \quad \alpha, \beta \in F$$

$$③ \quad (\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w$$

- Ex/ Cartesian product in \mathbb{R}^n

$$\text{Ex/ } \mathbb{R}^2: \quad x \cdot y = 2x_1 y_1 + x_1 y_2 + y_1 x_2 + x_2 y_2$$

Ex/ V : Set of all complex valued fns on $[0, 1]$

$$f \cdot g = \int_0^1 \overline{f(x)} g(x) dx$$

- length of a vector
- Orthogonality of two vectors, Gram-Schmidt process
- Ex V : Set of all Polynomials ≤ 2

- Ex/ V : Set of all polynomials ≤ 2 in x on \mathbb{R}

$$p \cdot q = \int_{-\infty}^{\infty} p(x) q(x) e^{-x^2} dx$$

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = 0 \quad n \text{ odd}$$

- If a set B is orthogonal then B is L.I.

$$= \frac{1}{a^{\frac{n+1}{2}}} \left(\frac{n-1}{2} \right)!$$

- Definition: length: $\|u\|$

$$\text{distance } d(u, v) = \|u - v\|$$

- Cor: $\|\alpha u\| = |\alpha| \|u\|$

- Thm: If $u, v \in V$ then $u \cdot v \leq \|u\| \|v\|$

$$(\lambda u + v) \cdot (\lambda u + v) = \lambda^2 \|u\|^2 + 2\lambda u \cdot v + \|v\|^2 \geq 0 \quad \lambda \text{ arbitrary}$$

$$\Rightarrow \lambda^2 a^2 + 2\lambda b + c \geq 0$$

$$\Rightarrow \frac{1}{a} (\lambda a + b)^2 + (c - b^2/a) \geq 0$$

$$\Rightarrow b^2 \leq ac$$

$$\text{choose } \lambda = -b/a$$

$$\Rightarrow u \cdot v \leq \|u\| \|v\|$$

Cauchy-Schwarz Inequality

- Ex in \mathbb{R}^3 $\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} \leq 1$

- Ex in Set of Polynomials

$$\left| \int_{-1}^1 p(x) q(x) dx \right|^2 \leq \left[\int_{-1}^1 p^2(x) dx \right] \left[\int_{-1}^1 q^2(x) dx \right]$$

1. Stabilität

• Stabilität

• Stabilität

$$\begin{aligned}
 &\Rightarrow \lambda_1 < 0 \text{ und } \lambda_2 < 0 \\
 &\Rightarrow \lambda_1 < 0 \\
 &\Rightarrow \lambda_1(1 + \rho) + (\lambda_2 - \rho\lambda_1) > 0 \\
 &\Rightarrow \lambda_1 \cdot \lambda_2 = 0 \text{ und } \lambda_1 < 0
 \end{aligned}$$

$$(\lambda_1 + \lambda_2) \cdot (1 + \rho) = \lambda_1 \cdot \lambda_2 + \lambda_1 \cdot \lambda_2 + \lambda_1 \cdot \lambda_2 = 0$$

• Stabilität

• Stabilität

$$\lambda_1 + \lambda_2 = 0$$

• Stabilität

• Stabilität

$$\lambda_1 + \lambda_2 = 0$$

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• Stabilität

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$$\lambda_1 + \lambda_2 = 0$$

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20/3/03 L21

Operators on IPS

- Definition, Examples; Polynomials space, \hat{x}, \hat{p} on $C[a,b]$, Simple transformations in \mathbb{R}^2
- Hermitian adjoint, Existence, ^{Finite} Matrix Rep w.r.t. orthonormal basis, Hermitian or Self-Adjoint
- Unitary Transformations, Isomorphism, Matrix Rep w.r.t. orthonormal basis

Figure 1 consists of 12 scatter plots, labeled (a) through (l), arranged in a 3x4 grid. Each plot shows the relationship between the number of species (S) on the y-axis and the number of genera (G) on the x-axis. Both axes are on a logarithmic scale. The data points are represented by small circles. The plots are for the following groups: (a) All species, (b) Plants, (c) Animals, (d) Invertebrates, (e) Vertebrates, (f) Fish, (g) Birds, (h) Mammals, (i) Reptiles, (j) Amphibians, (k) Insects, and (l) Arachnids. The plots show a general positive correlation between S and G, with some groups showing a steeper slope than others.

[illegible]

$$x^2 = \sum_{i=1}^n x_i^2 \Rightarrow x = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{1^2 + 2^2 + 3^2 + \dots + n^2} = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$

[illegible]

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The figure consists of seven scatter plots arranged horizontally, each representing a different socioeconomic indicator. Each plot has 'Number of children per woman' on the x-axis (ranging from 0 to 8) and a specific indicator on the y-axis. All plots show a clear negative correlation.

- (a) Y-axis: Years of schooling completed by women. Range: 0 to 12.
- (b) Y-axis: Mean years of schooling completed by men. Range: 0 to 12.
- (c) Y-axis: Mean years of schooling completed by women. Range: 0 to 12.
- (d) Y-axis: Mean years of schooling completed by both sexes. Range: 0 to 12.
- (e) Y-axis: Mean years of schooling completed by men and women. Range: 0 to 12.
- (f) Y-axis: Mean years of schooling completed by men and women. Range: 0 to 12.
- (g) Y-axis: Mean years of schooling completed by men and women. Range: 0 to 12.

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* If A is an operator on V and B is another operator s.t.

$(A\alpha, \beta) = (\alpha, B\beta)$ then we say B is hermitian adjoint of A
and denote $B \equiv A^\dagger$ (and vice versa)

• If V is finite dimensional then the adjoint always exists, otherwise it may not.

• $\{e_1, e_2, \dots, e_n\}$ be the basis of V . $[A]$ be the matrix of A and $[B]$ of B

then $Ae_i = \sum_j A_{ji} e_j \Rightarrow A_{ji} = \langle e_j, Ae_i \rangle$

$$B_{ji} = \langle Be_i, e_j \rangle = \overline{\langle e_j, Be_i \rangle}$$

$$\overline{B_{ji}} = \langle e_i, Be_j \rangle = A_{ji}$$

$$[B] = [A^\dagger]^T$$

• If $A = A^\dagger$ then we say operator is hermitian

Operators on ~~Q[X]~~ $\mathbb{R}[X]$. IPS.

- Definition of an operator: A LT from $V \rightarrow V$ is called an operator on V

- Examples (a) Rotation on \mathbb{R}^2 , matrix rep.

(b) Set of Polynomials of $\deg \leq 2$, \hat{D} operator

(c) Set of all polynomials, \hat{D}, \hat{x} .

(d) $C[0,1]$, $\hat{x}, \hat{D}(?)$.

\Rightarrow

- Isomorphism from $V \rightarrow W$: Is an isomorphism of vector spaces V ^{onto} W that preserves the inner product.

• V & W must have same dimension

• Transforms every orth. Basis of V to orthonormal basis of W .

• Preserves distances.

- Examples: (a) Rotations above

(b) \hat{D} is not an isomorphism in set of polynomials of $\deg \leq 2$

- A unitary operator is an isomorphism of V onto itself.

• If U is unitary operator then $UU^\dagger = U^\dagger U = I$

$$(U\alpha, \beta) = (U\alpha, UU^\dagger\beta) = (\alpha, U^\dagger\beta) \Rightarrow U^{-1} = U^\dagger \Rightarrow \text{required!}$$

• Corresponding matrices also are unitary.

• Set of all $n \times n$ invertible matrices is a group $GL(n)$

• Set of all $n \times n$ unitary operators is also a group $U(n) \subset GL(n)$

- Normal Operators and Spectral theorem.

Group Representations: Let G be a group. V be ~~a vector~~ an IPS. $GL(V)$ be the group of invertible operators on V . Then a homomorphism $\phi: G \rightarrow GL(V)$ is called a representation of group G . V is called representation of V .

If the dimension of V is n , then $GL(n)$ is isomorphic to $GL(V)$

If $\phi: G \rightarrow GL(n)$ then matrix representation of G .

If $\phi: G \rightarrow U(n)$ then unitary matrix representation of G .

Examples: 1. G be any group and V be abstract. Identity Representation.

2. $G = \{1, -1\}$. $V = \mathbb{R}^2$

(a) Identity Representation $\phi_1(1) = I_2$ $\phi_1(-1) = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Unfaithful

(b) $\phi_2(1) = I_2$, $\phi_2(-1) = m_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Faithful Representations

(c) $\phi_3(1) = I_2$, $\phi_3(-1) = R_x = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

(d) $\phi_4(1) = I_2$, $\phi_4(-1) = m_y = \begin{bmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{bmatrix}$

3/1 (d)

(3) $G = \{1, -1\}$, $V =$ Set of all polynomials of $\deg \leq 2$ in x on $[-1, 1]$

let \mathbb{T} be parity operator $\mathbb{T}p(x) = p(-x)$. $\phi(1) = I$, $\phi(-1) = \mathbb{T}$, matrices

1. The first part of the document is a list of names and their corresponding dates. The names are: "John Doe", "Jane Smith", "Bob Johnson", "Alice Brown", "Charlie White", "David Green", "Eve Black", "Frank Gray", "Grace Pink", "Henry Blue", "Ivy Yellow", "Jack Purple", "Karen Red", "Leo Orange", "Mia Silver", "Noah Gold", "Olivia Bronze", "Peter Copper", "Quinn Iron", "Rory Steel", "Sam Tin", "Tina Lead", "Uma Zinc", "Victor Nickel", "Wendy Platinum", "Xavier Silver", "Yara Gold", "Zoe Bronze", "Adam Copper", "Eve Iron", "Frank Steel", "Grace Tin", "Henry Lead", "Ivy Zinc", "Jack Nickel", "Karen Platinum", "Leo Silver", "Mia Gold", "Noah Bronze", "Olivia Copper", "Peter Iron", "Quinn Steel", "Rory Tin", "Tina Lead", "Uma Zinc", "Victor Nickel", "Wendy Platinum", "Xavier Silver", "Yara Gold", "Zoe Bronze". The dates are: "1990", "1991", "1992", "1993", "1994", "1995", "1996", "1997", "1998", "1999", "2000", "2001", "2002", "2003", "2004", "2005", "2006", "2007", "2008", "2009", "2010", "2011", "2012", "2013", "2014", "2015", "2016", "2017", "2018", "2019", "2020", "2021", "2022", "2023", "2024", "2025", "2026", "2027", "2028", "2029", "2030", "2031", "2032", "2033", "2034", "2035", "2036", "2037", "2038", "2039", "2040", "2041", "2042", "2043", "2044", "2045", "2046", "2047", "2048", "2049", "2050", "2051", "2052", "2053", "2054", "2055", "2056", "2057", "2058", "2059", "2060", "2061", "2062", "2063", "2064", "2065", "2066", "2067", "2068", "2069", "2070", "2071", "2072", "2073", "2074", "2075", "2076", "2077", "2078", "2079", "2080", "2081", "2082", "2083", "2084", "2085", "2086", "2087", "2088", "2089", "2090", "2091", "2092", "2093", "2094", "2095", "2096", "2097", "2098", "2099", "2100", "2101", "2102", "2103", "2104", "2105", "2106", "2107", "2108", "2109", "2110", "2111", "2112", 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Figure 1. The effect of the concentration of the solution on the adsorption of the dye. The concentration of the solution was 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.5, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 15.0, 20.0, 30.0, 40.0, 50.0, 60.0, 70.0, 80.0, 90.0, 100.0, 150.0, 200.0, 300.0, 400.0, 500.0, 600.0, 700.0, 800.0, 900.0, 1000.0, 1500.0, 2000.0, 3000.0, 4000.0, 5000.0, 6000.0, 7000.0, 8000.0, 9000.0, 10000.0, 15000.0, 20000.0, 30000.0, 40000.0, 50000.0, 60000.0, 70000.0, 80000.0, 90000.0, 100000.0, 150000.0, 200000.0, 300000.0, 400000.0, 500000.0, 600000.0, 700000.0, 800000.0, 900000.0, 1000000.0, 1500000.0, 2000000.0, 3000000.0, 4000000.0, 5000000.0, 6000000.0, 7000000.0, 8000000.0, 9000000.0, 10000000.0, 15000000.0, 20000000.0, 30000000.0, 40000000.0, 50000000.0, 60000000.0, 70000000.0, 80000000.0, 90000000.0, 100000000.0, 150000000.0, 200000000.0, 300000000.0, 400000000.0, 500000000.0, 600000000.0, 700000000.0, 800000000.0, 900000000.0, 1000000000.0, 1500000000.0, 2000000000.0, 3000000000.0, 4000000000.0, 5000000000.0, 6000000000.0, 7000000000.0, 8000000000.0, 9000000000.0, 10000000000.0, 15000000000.0, 20000000000.0, 30000000000.0, 40000000000.0, 50000000000.0, 60000000000.0, 70000000000.0, 80000000000.0, 90000000000.0, 100000000000.0, 150000000000.0, 200000000000.0, 300000000000.0, 400000000000.0, 500000000000.0, 600000000000.0, 700000000000.0, 800000000000.0, 900000000000.0, 1000000000000.0, 1500000000000.0, 2000000000000.0, 3000000000000.0, 4000000000000.0, 5000000000000.0, 6000000000000.0, 7000000000000.0, 8000000000000.0, 9000000000000.0, 10000000000000.0, 15000000000000.0, 20000000000000.0, 30000000000000.0, 40000000000000.0, 50000000000000.0, 60000000000000.0, 70000000000000.0, 80000000000000.0, 90000000000000.0, 100000000000000.0, 150000000000000.0, 200000000000000.0, 300000000000000.0, 400000000000000.0, 500000000000000.0, 600000000000000.0, 700000000000000.0, 800000000000000.0, 900000000000000.0, 1000000000000000.0, 1500000000000000.0, 2000000000000000.0, 3000000000000000.0, 4000000000000000.0, 5000000000000000.0, 6000000000000000.0, 7000000000000000.0, 8000000000000000.0, 9000000000000000.0, 10000000000000000.0, 15000000000000000.0, 20000000000000000.0, 30000000000000000.0, 40000000000000000.0, 50000000000000000.0, 60000000000000000.0, 70000000000000000.0, 80000000000000000.0, 90000000000000000.0, 100000000000000000.0, 150000000000000000.0, 200000000000000000.0, 300000000000000000.0, 400000000000000000.0, 500000000000000000.0, 600000000000000000.0, 700000000000000000.0, 800000000000000000.0, 900000000000000000.0, 1000000000000000000.0, 1500000000000000000.0, 2000000000000000000.0, 3000000000000000000.0, 4000000000000000000.0, 5000000000000000000.0, 6000000000000000000.0, 7000000000000000000.0, 8000000000000000000.0, 9000000000000000000.0, 10000000000000000000.0, 15000000000000000000.0, 20000000000000000000.0, 30000000000000000000.0, 40000000000000000000.0, 50000000000000000000.0, 60000000000000000000.0, 70000000000000000000.0, 80000000000000000000.0, 90000000000000000000.0, 100000000000000000000.0, 150000000000000000000.0, 200000000000000000000.0, 300000000000000000000.0, 400000000000000000000.0, 500000000000000000000.0, 600000000000000000000.0, 700000000000000000000.0, 800000000000000000000.0, 900000000000000000000.0, 1000000000000000000000.0, 1500000000000000000000.0, 2000000000000000000000.0, 3000000000000000000000.0, 4000000000000000000000.0, 5000000000000000000000.0, 6000000000000000000000.0, 7000000000000000000000.0, 8000000000000000000000.0, 9000000000000000000000.0, 10000000000000000000000.0, 15000000000000000000000.0, 20000000000000000000000.0, 30000000000000000000000.0, 40000000000000000000000.0, 50000000000000000000000.0, 60000000000000000000000.0, 70000000000000000000000.0, 80000000000000000000000.0, 90000000000000000000000.0, 100000000000000000000000.0, 150000000000000000000000.0, 200000000000000000000000.0, 300000000000000000000000.0, 400000000000000000000000.0, 500000000000000000000000.0, 600000000000000000000000.0, 700000000000000000000000.0, 800000000000000000000000.0, 900000000000000000000000.0, 10000000

Figure 1. The effect of the concentration of the solution on the adsorption of the dye. The concentration of the solution was 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.5, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 15.0, 20.0, 30.0, 40.0, 50.0, 60.0, 70.0, 80.0, 90.0, 100.0, 150.0, 200.0, 300.0, 400.0, 500.0, 600.0, 700.0, 800.0, 900.0, 1000.0, 1500.0, 2000.0, 3000.0, 4000.0, 5000.0, 6000.0, 7000.0, 8000.0, 9000.0, 10000.0, 15000.0, 20000.0, 30000.0, 40000.0, 50000.0, 60000.0, 70000.0, 80000.0, 90000.0, 100000.0, 150000.0, 200000.0, 300000.0, 400000.0, 500000.0, 600000.0, 700000.0, 800000.0, 900000.0, 1000000.0, 1500000.0, 2000000.0, 3000000.0, 4000000.0, 5000000.0, 6000000.0, 7000000.0, 8000000.0, 9000000.0, 10000000.0, 15000000.0, 20000000.0, 30000000.0, 40000000.0, 50000000.0, 60000000.0, 70000000.0, 80000000.0, 90000000.0, 100000000.0, 150000000.0, 200000000.0, 300000000.0, 400000000.0, 500000000.0, 600000000.0, 700000000.0, 800000000.0, 900000000.0, 1000000000.0, 1500000000.0, 2000000000.0, 3000000000.0, 4000000000.0, 5000000000.0, 6000000000.0, 7000000000.0, 8000000000.0, 9000000000.0, 10000000000.0, 15000000000.0, 20000000000.0, 30000000000.0, 40000000000.0, 50000000000.0, 60000000000.0, 70000000000.0, 80000000000.0, 90000000000.0, 100000000000.0, 150000000000.0, 200000000000.0, 300000000000.0, 400000000000.0, 500000000000.0, 600000000000.0, 700000000000.0, 800000000000.0, 900000000000.0, 1000000000000.0, 1500000000000.0, 2000000000000.0, 3000000000000.0, 4000000000000.0, 5000000000000.0, 6000000000000.0, 7000000000000.0, 8000000000000.0, 9000000000000.0, 10000000000000.0, 15000000000000.0, 20000000000000.0, 30000000000000.0, 40000000000000.0, 50000000000000.0, 60000000000000.0, 70000000000000.0, 80000000000000.0, 90000000000000.0, 100000000000000.0, 150000000000000.0, 200000000000000.0, 300000000000000.0, 400000000000000.0, 500000000000000.0, 600000000000000.0, 700000000000000.0, 800000000000000.0, 900000000000000.0, 1000000000000000.0, 1500000000000000.0, 2000000000000000.0, 3000000000000000.0, 4000000000000000.0, 5000000000000000.0, 6000000000000000.0, 7000000000000000.0, 8000000000000000.0, 9000000000000000.0, 10000000000000000.0, 15000000000000000.0, 20000000000000000.0, 30000000000000000.0, 40000000000000000.0, 50000000000000000.0, 60000000000000000.0, 70000000000000000.0, 80000000000000000.0, 90000000000000000.0, 100000000000000000.0, 150000000000000000.0, 200000000000000000.0, 300000000000000000.0, 400000000000000000.0, 500000000000000000.0, 600000000000000000.0, 700000000000000000.0, 800000000000000000.0, 900000000000000000.0, 1000000000000000000.0, 1500000000000000000.0, 2000000000000000000.0, 3000000000000000000.0, 4000000000000000000.0, 5000000000000000000.0, 6000000000000000000.0, 7000000000000000000.0, 8000000000000000000.0, 9000000000000000000.0, 10000000000000000000.0, 15000000000000000000.0, 20000000000000000000.0, 30000000000000000000.0, 40000000000000000000.0, 50000000000000000000.0, 60000000000000000000.0, 70000000000000000000.0, 80000000000000000000.0, 90000000000000000000.0, 100000000000000000000.0, 150000000000000000000.0, 200000000000000000000.0, 300000000000000000000.0, 400000000000000000000.0, 500000000000000000000.0, 600000000000000000000.0, 700000000000000000000.0, 800000000000000000000.0, 900000000000000000000.0, 1000000000000000000000.0, 1500000000000000000000.0, 2000000000000000000000.0, 3000000000000000000000.0, 4000000000000000000000.0, 5000000000000000000000.0, 6000000000000000000000.0, 7000000000000000000000.0, 8000000000000000000000.0, 9000000000000000000000.0, 10000000000000000000000.0, 15000000000000000000000.0, 20000000000000000000000.0, 30000000000000000000000.0, 40000000000000000000000.0, 50000000000000000000000.0, 60000000000000000000000.0, 70000000000000000000000.0, 80000000000000000000000.0, 90000000000000000000000.0, 100000000000000000000000.0, 150000000000000000000000.0, 200000000000000000000000.0, 300000000000000000000000.0, 400000000000000000000000.0, 500000000000000000000000.0, 600000000000000000000000.0, 700000000000000000000000.0, 800000000000000000000000.0, 900000000000000000000000.0, 10000000

Figure 1. The effect of the concentration of the solution on the adsorption of the dye. The concentration of the solution was varied from 0.05 to 0.25 g/L. The adsorption capacity of the adsorbent was determined by the difference in the concentration of the dye before and after adsorption. The adsorption capacity of the adsorbent was determined by the difference in the concentration of the dye before and after adsorption. The adsorption capacity of the adsorbent was determined by the difference in the concentration of the dye before and after adsorption.

[illegible]

The diagram illustrates the experimental setup. A participant is seated at a table, looking at a screen. On the screen, a 3D model of a hand holding a tool is shown. A red dot on the screen indicates the target location. The participant's hand is positioned near the tool. The setup is used for studying the effects of tool use on reaching behavior.

[illegible]

Figure 1. The 1000 Genomes Project. The 1000 Genomes Project is a large-scale genomics project that aims to create a comprehensive reference of human genetic variation. The project involves sequencing the genomes of 1,000 individuals from diverse populations. The data generated is used to identify common and rare genetic variants across the human population. The project is a collaborative effort involving multiple research institutions and is a key resource for understanding human genetic diversity and its role in disease.

Figure 1. The effect of the number of nodes on the performance of the proposed algorithm. The figure shows the execution time (in seconds) on the y-axis (ranging from 0 to 10) and the number of nodes on the x-axis (ranging from 10 to 100). The execution time increases as the number of nodes increases, with a sharp increase observed between 50 and 100 nodes.

1. The first step in the process of creating a new product is to identify a market need. This involves conducting market research to understand the preferences and behaviors of potential customers. Once a need is identified, the next step is to develop a concept that addresses this need. This concept should be unique and offer a clear value proposition. The third step is to create a prototype, which allows the team to test the concept and gather feedback from potential users. Finally, the product is refined based on this feedback and then launched into the market. Throughout this process, it is crucial to maintain open communication with the target audience and be prepared to iterate on the design as needed.

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2} dx$

(continued)

- 75% der Organe sind angeborene oder erworbene Defekte.
- 25% der Organe sind angeborene oder erworbene Defekte.
- 25% der Organe sind angeborene oder erworbene Defekte.

$$\sin^2 \theta = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4}{\pi^2} \ln^2 \frac{1}{\alpha}}} \right) \approx \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4}{\pi^2} \ln^2 \frac{1}{\alpha}}} \right)$$

- $$x^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 4$$

Figure 1. The chemical structures of the monomers and the copolymers.

-
- Figure 1 is a schematic representation of the experimental design. It shows a sequence of events: a subject is presented with a stimulus (a word), then a response is generated (a word), which is then compared to the stimulus. The comparison leads to a decision (Yes/No), which is then compared to the response. The final outcome is a feedback signal (Yes/No).

1. *Journal of the American Medical Association*, 1997; 277: 1001-1005.

$\frac{1}{\sqrt{\pi}} \left(\frac{1}{x} - \frac{1}{y} \right) \ln \frac{x+y}{x-y}$

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[illegible]

Figure 1 consists of 12 numbered line drawings (a-l) illustrating the stages of a bird's nest. (a) shows a single egg. (b) shows a small chick hatching. (c) shows the chick growing and moving within the nest. (d) shows the chick standing and looking out. (e) shows the chick sitting. (f) shows the chick standing. (g) shows the chick sitting. (h) shows the chick standing. (i) shows the chick sitting. (j) shows the chick standing. (k) shows the chick sitting. (l) shows the chick as a fledgling, leaving the nest.

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- _____

1. 2000年12月1日以前

- Equivalence of Matrix Representations: $\phi_1: G \rightarrow GL(n)$ & $\phi_2: G \rightarrow GL(n)$ are two distinct representations (matrix), then the two representations are called equivalent if \exists matrix of order n , nonsingular S , such that

$$\phi_1(g) = S^{-1} \phi_2(g) S \quad \forall g \in G$$

Example given above:

- Invariant Subspaces under a group

Let W be a subspace of V . Let T be an operator on V . If $Tx \in W$ for all $x \in W$ then we say that W is an invariant subspace under T .

Examples: (a) $T(x, y) = (2x, x+y)$ x axis is invariant

(b) $T(x, y) = (y, x)$ $y=x$ line is invariant

(c) Set of all polynomials of deg ≤ 2 , T parity operator
 then $W: p(x) = a + bx^2$ is invariant under T .

- In a suitable basis the matrix of T can be transformed to

$$\left[\begin{array}{c|c} D^{(1)} & 0 \\ \hline X & D^{(2)} \end{array} \right]$$

- The subspace is called invariant subspace under the group G if all the matrices can be cast in the same form

i.e.
$$\phi(g) = \left[\begin{array}{c|c} D^{(1)}(g) & 0 \\ \hline X(g) & D^{(2)}(g) \end{array} \right] \quad \forall g \in G$$

In this case the representation is said to be reducible.

- Theorem 1. Each finite dimensional ^{matrix} representation is equivalent to a unitary matrix representation.

Proof: Let $G = \{g_1, g_2, \dots, g_m\}$ $\phi: G \rightarrow GL(n)$

$$H = \sum_{g \in G} \phi(g) \phi(g)^\dagger \quad \text{Hermitian} \Rightarrow \text{diagonalizable} \Rightarrow \exists U \text{ unitary}$$

s.t.

$$H_d = U^{-1} H U \quad : \text{diagonal matrix with real entries}$$

$$= \sum_g U^{-1} \phi(g) U U^{-1} \phi(g)^\dagger U$$

$$= \sum_g \phi'(g) \phi'^\dagger(g)$$

$$[H_d]_{kk} = \sum_g \sum_l [\phi'(g)]_{kl} [\phi'(g)]_{kl}^* = \sum_g \sum_l [\phi'(g)]_{kl}^2 \quad \checkmark$$

$$\begin{bmatrix} A & 0 \\ X & B \end{bmatrix}^{\dagger} = \begin{bmatrix} A^{\dagger} & X^{\dagger} \\ 0 & B^{\dagger} \end{bmatrix}$$

↙ This is also the matrix of represent.
In other words, if $\phi(g)$ has the form $\begin{bmatrix} A & 0 \\ X & B \end{bmatrix}$ then $\phi(g^{-1}) = [\phi(g)]^{\dagger} = [\phi(g)]^{\dagger} = \begin{bmatrix} A^{\dagger} & X^{\dagger} \\ 0 & B^{\dagger} \end{bmatrix}$ also

must have the same form $\Rightarrow X = X^{\dagger} = 0$

$X=0 \Rightarrow$ if unitary.

$$V = L_2(\mathbb{R})$$

$$\hat{p} = i\hbar \hat{D} \quad \hat{D} = \frac{d}{dx} \quad (\hat{R}f)(x) = f(-x)$$

$$\mathbb{R}_x : \mathbb{R} \rightarrow \mathbb{R} \Rightarrow \mathbb{R}_x x = -x \quad \mathcal{G} = \{I, \mathbb{R}_x\}$$

$$V = V_e \oplus V_o$$

V_e and V_o are \mathbb{R} invariant under \hat{R} and I

$$H = \hat{p}^2/2m + k \hat{x}^2$$

$$[\hat{R}, H] = 0 \Rightarrow$$

$$T: V_1 \rightarrow V_2$$

(*) If T is an isomorphism $\Rightarrow m \leq n$

if $m < n$ then let $W' = \text{Range}(T)$

for every $x \in W'$, $y \in G$

$$\phi_2(g)x \quad \text{let } Ty = x$$

$$= \phi_2(g)Ty$$

$$= T(\phi(g)y) \in \text{Range}(T)$$

$\Rightarrow W'$ is ~~inv.~~ ~~Subspace~~ V_2 subspace under G which is not possible hence.

$\Rightarrow H_d > 0$ Positive definite Matrix

$$V = U H_d^{-1/2}$$

$$\phi''(g) = V^{-1} \phi V = H_d^{-1/2} U^{-1} \phi U H_d^{-1/2}$$

$$\begin{aligned} \Rightarrow \phi''(g) \phi''^\dagger(g) &= [H_d^{-1/2} U^{-1} \phi'(g) H_d^{1/2}] [H_d^{1/2} \phi'^\dagger(g) H_d^{-1/2}] \\ &= H_d^{-1/2} \phi'(g) H_d \phi'^\dagger(g) H_d^{-1/2} \\ &= \int_{\partial'} H_d^{-1/2} \phi'(g) \phi'(g) \phi'^\dagger(g) \phi^\dagger(g) H_d^{-1/2} \\ &= \int_{\partial'} H_d^{-1/2} H_d H_d^{-1/2} = I \end{aligned}$$

- let ϕ be a unitary representation ~~is~~ and reducible

$$\left[\begin{array}{c|c} \phi_1(g) & 0 \\ \hline x & \phi_2(g) \end{array} \right] \rightarrow \left[\begin{array}{c|c} \phi_1(g) & 0 \\ \hline 0 & \phi_2(g) \end{array} \right] \text{ - must have this form -}$$

$$\phi(g) = \left[\begin{array}{c|c|c} \phi_1 & 0 & 0 \\ \hline 0 & \phi_2 & 0 \\ \hline 0 & 0 & \phi_3 \end{array} \right]$$

Reduction of a representation
in irreducible representations

- Schur lemma

If $\phi_1: G \rightarrow GL(V_1)$ and $\phi_2: G \rightarrow GL(V_2)$ are two m and n dimensional \mathbb{R} -representations, let T be a ~~operator~~ LT $T: V_1 \rightarrow V_2$ s.t.

$$T \phi_1(g) = \phi_2(g) T \quad \forall g \in G$$

then (i) if $m \neq n$ $T = 0$

(ii) if $m = n$ then $T = cI$

Proof: $W = \text{Ker } T \subset V_1$.

$$\text{Ker } T = \{x_1 \mid T x_1 = 0\}$$

$$\text{- if } x_1 \in W \Rightarrow T(\phi_1(g)(x_1)) = \phi_2(g) T x_1 = \phi_2(g)(0) = 0$$

$$\Rightarrow \phi_1(g)(x_1) \in W \Rightarrow W \text{ is inv. under } \phi_1 G.$$

- But ϕ_1 does not have inv. subspaces $\Rightarrow W = \{0\}$ or V_1 itself.

- $\text{Ker } T = \{0\} \Rightarrow T$ is an isomorphism* but $m \neq n$ hence this is not possible

- $\text{Ker } T = V_1 \Rightarrow T = 0$.

if $m = n$ then T is an isomorphism and ~~let~~ $c \neq 0$ let c be

$$\text{any eigenvalue of } T \text{ then } (T - cI) = 0 \Rightarrow T = cI.$$

$$C_{3v}: \{e, \psi, \psi^2, \sigma, \sigma\psi, \sigma\psi^2\}$$

	e	ψ	ψ^2	σ	$\sigma\psi$	$\sigma\psi^2$	
$D^{(0)}$	1	1	1	1	1	1	
$D^{(2)}$	1	1	1	-1	-1	-1	$\sigma\psi = \psi^2\sigma$
$D^{(3)}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
D^4	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	ψ^2 $\cos(120) = -\sin(30)$ $= -\frac{1}{2}$

Change the basis by $S^T = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix}$

Show that

$$S^T D^4(\psi) S = \begin{bmatrix} D_1'(\psi) & & \\ & D_2^{(2)}(\psi) & \\ & & D_2^{(2)}(\psi) \end{bmatrix}$$

$$\begin{aligned} \sin(120) &= +\cos(30) \\ &= \frac{\sqrt{3}}{2} \\ \sin(240) &= 2.5(120)\cos(120) \\ &= -2 \cdot \frac{\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \\ \cos(240) &= -\frac{1}{2} \end{aligned}$$

Character Rep:

	$\{e\}$	$\{\psi, \psi^2\}$	$\{\sigma, \sigma\psi, \sigma\psi^2\}$	
D^1	1	1	1	$\rightarrow 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 6$
D^2	1	1	-1	orth. $1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot (-1) + 3 \cdot 1 \cdot 0 = 0$
D^3	2	-1	0	$\rightarrow 4 + 2 = 6$

$$\text{let } T = \sum_g \phi_2(g) \cup \phi_1(g^{-1})$$

For Finite groups

$$\text{Then } \phi_2(g) T = T \phi_1(g^{-1})$$

let matrix of $\phi_1(g)$ as $D_1(g)$ and $\phi_2(g)$ as $D_2(g)$

Choose matrix U s.t. $U_{pq} = 1$ only on p^{th} row and q^{th} col.

$$[T]_{rs} = \sum_{g'} \sum_{t,u} [D_2(g)]_{rt} U_{tu} [D_1(g^{-1})]_{us}$$

$$= \sum_{g'} [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} \quad p, q \text{ arbitrary}$$

But if $\phi_1 \neq \phi_2$ then $T = 0$

$$\Rightarrow \sum_{g'} [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} = 0$$

$$\Rightarrow \sum_g [D_2(g)]_{rp} = 0 \quad \text{by setting } D_1 = 1$$

Case two when $\phi_1 = \phi_2$ $T = \lambda_{p_2} I$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} = \lambda_{p_2} \delta_{rs}$$

Sum over, put $r=s$ and sum over r .

$$\Rightarrow \sum_g \sum_r [D_2(g)]_{rp} [D_1(g^{-1})]_{qr} = \lambda_{p_2} \cdot n$$

$$\Rightarrow \sum_g [D_1(gg^{-1})]_{rp} = \lambda_{p_2} n$$

$$\Rightarrow \sum_g I_{rp} = \lambda_{p_2} n$$

$$\Rightarrow \delta_{rp} \cdot O_g = \lambda_{p_2} n$$

$$\Rightarrow \lambda_{p_2} = \delta_{rp} \frac{O_g}{n}$$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} = \frac{O_g}{n} \delta_{pq} \delta_{rs}$$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1^*(g)]_{sq} = \frac{O_g}{n} \delta_{pq} \delta_{rs}$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 x}{dt^2} \right) = \frac{1}{2} \frac{d^3 x}{dt^3}$$

$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

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$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

for $x=0$, $\frac{d^2 x}{dt^2} = 0$

$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

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$$\Rightarrow \frac{1}{2} \frac{d^2 x}{dt^2} = \frac{1}{2} \frac{d^2 x}{dt^2}$$

→ Organize

→ Argue that if there are c IRReps then

$$\sum n_i^2 \leq g.$$

→ Character Representations

$$\chi(g) = \sum_k D_{kk}(g) \quad \text{equivalent}$$

if $g' = f^{-1}gf$

$$\text{then } \chi(h) = \chi(g)$$

All characters in equivalence class are equal.

→ Orthogonality of characters

$$\sum_g [D_{pq}^i(g)]_{p_2} [D_{rs}^j(g)]^* = \frac{O_g}{n_i} \delta_{ij} \delta_{p_1} \delta_{q_2}$$

put $p=1$ and $r=s$

$$\sum_g \chi^i(g) \chi^j(g)^* = O_g \delta_{ij}$$

$$\sum_k \sqrt{\frac{n_k}{O_g}} \chi^i(k) \chi^j(k)^* = \delta_{ij}$$

$$\# \text{ IRReps} \leq \# \text{ classes in } G$$

\mathbb{C}^n

→ Reducing the representation

$$\chi(g) = \sum_i a_i \chi^i(g)$$

block diagonal/

$$\sum_g \chi^{(i)*}(g) \chi(g) = \sum_g \sum_i a_i \chi^i(g)^* \chi(g) = a_i \cdot g$$

$$\Rightarrow a_i = \frac{1}{g} \sum_g \chi_g^{(i)*} \chi(g)$$

→ Condition for reducibility

$$\sum_g \chi^*(g) \chi(g) = 0 \Leftrightarrow \text{IRReps.}$$

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = 0 \Rightarrow \text{falsch}$$

→ Integral ist unendlich

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\delta(x) = \sum_{n=0}^{\infty} \delta(x)$$

Summe

→ Integral ist unendlich

$$\delta(x)$$

$$\delta(x) \neq 0 \text{ für } x=0$$

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = 0$$

$$\text{für } x=0 \text{ und } x=1$$

$$\int_{-\infty}^{\infty} \delta(x) \delta(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} \delta(x) dx = \frac{1}{2}$$

→ Integral ist unendlich

$$\delta(x) = \delta(x)$$

→ Integral ist unendlich

$$\delta(x) = \delta(x)$$

$$\delta(x) = \delta(x)$$

→ Integral ist unendlich

$$\delta(x) = \delta(x)$$

→ Integral ist unendlich

→ Integral ist unendlich

$$\phi: G \rightarrow V$$

$$\phi_s(t) = st \quad (\phi_s \circ \phi_{s'})(t)$$

$$\phi_s(\phi_{s'}(t)) = \phi_s(s't) = ss't = \phi_{ss'}(t)$$

Matrix $[D(s)]_{t,u} = 1$

$$D(s)_{t,u} = \delta_{st, su}$$

$$tu^{-1} = s$$

$$t = su$$

	e	ψ^2	ψ	σ	$\sigma\psi$	$\sigma\psi^2$
e	e	ψ^2	ψ	σ	$\sigma\psi$	$\sigma\psi^2$
ψ	ψ	ψ^2	ψ	σ	$\sigma\psi$	$\sigma\psi^2$
ψ^2	ψ^2	ψ	ψ	σ	$\sigma\psi$	$\sigma\psi^2$
σ	σ	$\sigma\psi$	$\sigma\psi^2$	σ	$\sigma\psi$	$\sigma\psi^2$
$\sigma\psi$	$\sigma\psi$	$\sigma\psi^2$	$\sigma\psi$	σ	$\sigma\psi$	$\sigma\psi^2$
$\sigma\psi^2$	$\sigma\psi^2$	$\sigma\psi$	$\sigma\psi^2$	σ	$\sigma\psi$	$\sigma\psi^2$

$$[D(s)D(s')](t)$$

$$\sum_{v'} [D(s)]_{p,v'} [D(s')]_{v',q} = \sum_v \delta_{sv',p} \delta_{s'q,v'}$$

$$= \sum_v \delta_{ss'q,p}$$

$$sv' = p \quad s'q = v'$$

$$= [D(ss')]_{p,q}$$

$$[D(s)]_{t,u} = 1 \text{ if } tu = s$$

$$= 0 \text{ otherwise}$$

Regular Repre., contains all

$$\chi(e) = 0_g \quad \chi(g) = 0$$

$$a_i = \frac{1}{0_g} \sum_g \chi(g) \chi^i_g = \frac{1}{0_g} \cdot 0_g \cdot h_i = h_i$$

$$\sum_u [D(s)]_{t,u} [D(s')]_{u,w}$$

$$= \sum_g \chi(g) \chi^i(g) = 0_g \sum_i h_i^2$$

$$= \sum_u \delta_{s',tu} \cdot \delta_{s,uw}$$

$$0_g = 0_g \sum h_i^2$$

$$tu = s$$

$$uw = s'$$

$$ts^{-1}u^{-1} = s$$

$$\Rightarrow \sum h_i^2 = 0_g$$

✓

→

e	a	b		e	a	b
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	e	e	a	b
			a	a	b	e
			b	b	e	a

$$a^2 = e, \forall a \Rightarrow$$

$$(ab)^2 = (ab)(ab) = e$$

$$\boxed{ab = b^{-1}a^{-1} = ba}$$

e_{2V}

$$\begin{array}{cccc} e & R_{1/2} & \sigma_x & \sigma_y \\ R_{1/2} & e & \sigma_y & \sigma_x \\ \sigma_x & \sigma_y & e & R_{1/2} \\ \sigma_y & \sigma_x & R_{1/2} & e \end{array}$$

$$\begin{array}{l} \text{---} \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

$$e \begin{bmatrix} 1 & 0 & 0 & 0 \\ R_{1/2} & 0 & 1 & 0 \\ \sigma_x & 0 & 0 & 1 \\ \sigma_y & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{1/2} \begin{bmatrix} e & 0 & 1 & 0 & 0 \\ R_{1/2} & 1 & 0 & 0 & 0 \\ \sigma_x & 0 & 0 & 0 & 1 \\ \sigma_y & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sigma_y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_x \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Regular Representation: G : group. $g = O(G)$. Consider a Matrix $(g \times g)$ Repr.

$$s.t. [D(s)]_{t,u} = \delta_{t,su}$$

Example C_{2v} : above

⇒ To show that this is a representation

$$\begin{aligned} \sum_u [D(s)]_{t,u} [D(s')]_{u,w} &= \sum_u \delta_{t,su} \cdot \delta_{u,s'w} \Rightarrow \begin{aligned} t &= su \\ u &= s'w \end{aligned} \\ &= \delta_{t,(ss')w} \\ &= [D(ss')]_{t,w} \Rightarrow t = (ss')w \end{aligned}$$

- The characters of Reg. Rep.

$$\chi_e = g, \chi_s = 0 \quad s \neq e.$$

- Suppose $\chi_g^\alpha \quad \alpha=1, \dots, k$ (# of classes?)

$$\chi_g = \sum_\alpha a_\alpha \chi_g^\alpha$$

$$a_\alpha = \frac{1}{g} \sum_g \chi_g^{\alpha*} \chi_g = \frac{1}{g} \cdot \chi_e^\alpha \cdot \chi_e = \chi_e^\alpha = l_\alpha = \text{dimension of } \alpha^{\text{th}} \text{ IR.}$$

Each l_α dimensional IR, appears in Reg. Rep., l_α times

$$\begin{aligned} \text{since } \chi_g &= l_\alpha \cdot \sum_\alpha a_\alpha \chi_g^\alpha \\ \Rightarrow g &= \sum_\alpha l_\alpha \cdot l_\alpha = \sum_\alpha l_\alpha^2 \end{aligned}$$

$$\Rightarrow \boxed{\sum l_\alpha^2 = g}$$

Key Results

- Every Finite dimensional matrix Repr. is equivalent to a unitary matrix Repr.
- Reduction of Matrix Repr. and inv. Subspaces
- Grand Orth. Theorem

$$\sum_g [D^\alpha(g)]_{rp} [D^\beta(g)]_{sq}^* = \frac{O_g}{l_\alpha} \delta_{\alpha,\beta} \delta_{rs} \delta_{pq}$$

$$\sum l_\alpha^2 \leq O_g. \quad \# \text{ of IRReps}$$

$$\# \text{ of IRs} \leq \# \text{ Classes.}$$

$$\sum_g \chi_g^\alpha (\chi_g^\beta)^* = O_g \delta_{\alpha\beta} \quad (\text{IRs})$$

4. $G = C_4$ $C_1 = \{e\}$, $C_2 = \{R_{\pi/2}\}$, $C_3 = \{R_\pi\}$ $C_4 = \{R_{3\pi/2}\}$

$n_1 = n_2 = n_3 = n_4 = 1$

	C_1	C_2	C_3	C_4
D^1	1	1	1	1
D^2	1	i	-1	-i
D^3	1	-1	1	-1
D^4	1	-i	-1	i

$R_{\pi/2}^4 = 1$ $\chi_{R_{\pi/2}} = \pm 1, \pm i$

• Reducible Representation

$$X_g = \sum_{\alpha} a_{\alpha} X_g^{\alpha} \quad \therefore a_{\alpha} = \frac{1}{o_g} \sum_g (X_g^{\alpha})^* X_g$$

• Criterion of Reducibility

$$\sum_g X_g^* X_g = O_g \Leftrightarrow \text{IR}$$

Examples 1. $G = \{1, -1\}$

$$C_1 = \{1\} \quad C_2 = \{-1\}, \quad n_1^2 + n_2^2 = 2 \Rightarrow n_1 = 1, n_2 = 1$$

	C_1	C_2
D^1	1	1
D^2	1	-1

• Find classes (k)

$$\cdot \sum n_{\alpha}^2 = 2 \Rightarrow n_{\alpha}$$

• Construct k x k table

• Identity Rep.

• For 1d rep, $A^{\alpha} = E$

2. $G = C_3$, $C_1 = \{1\}$, $C_2 = \{R_{2\pi/3}\}$, $C_3 = \{R_{4\pi/3}\}$

$$n_1^2 + n_2^2 + n_3^2 = 3 \Rightarrow n_1 = n_2 = n_3 = 1$$

	C_1	C_2	C_3
D^1	1	1	1
D^2	1	$e^{i2\pi/3}$	$e^{i4\pi/3}$
D^3	1	$e^{i4\pi/3}$	$e^{i2\pi/3}$

$$R_{2\pi/3}^3 = E$$

$$\chi_{R_{2\pi/3}}^3 = 1 \Rightarrow \chi_{R_{2\pi/3}} = e^{i2\pi/3}, e^{i4\pi/3}$$

3. $G = C_{4v} = \{e, R_{\pi/2}, R_{\pi}, R_{3\pi/2}, m_x, m_y, \sigma_x, \sigma_y\}$

$$C_1 = \{e\}, \quad C_2 = \{R_{\pi/2}, R_{3\pi/2}\}, \quad C_3 = R_{\pi}, \quad C_4 = \{m_x, m_y\}, \quad C_5 = \{\sigma_x, \sigma_y\}$$

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8 \quad 1, 1, 1, 1, 2$$

	$C_1(1)$	$C_2(2)$	$C_3(1)$	$C_4(2)$	$C_5(2)$
D^1	1	1	1	1	1
D^2	1	-1	1	-1	1
D^3	1	-1	+1	+1	-1
D^4	1	+1	+1	-1	-1
D^5	2	a=0	b=-2	c=0	d=0

$$R_{\pi/2}^4 = E$$

$$\chi_{R_{\pi/2}} = 1, -1, i, -i$$

$$2a + b + 2c + d + 2 = 0$$

$$-2a + b - 2c + 2d + 2 = 0$$

$$-2a + b + 2c - 2d + 2 = 0$$

$$2a + b - 2c - 2d + 2 = 0$$

$$-4a + b = -4$$

$$4a + b = -4$$

$$a = 0$$

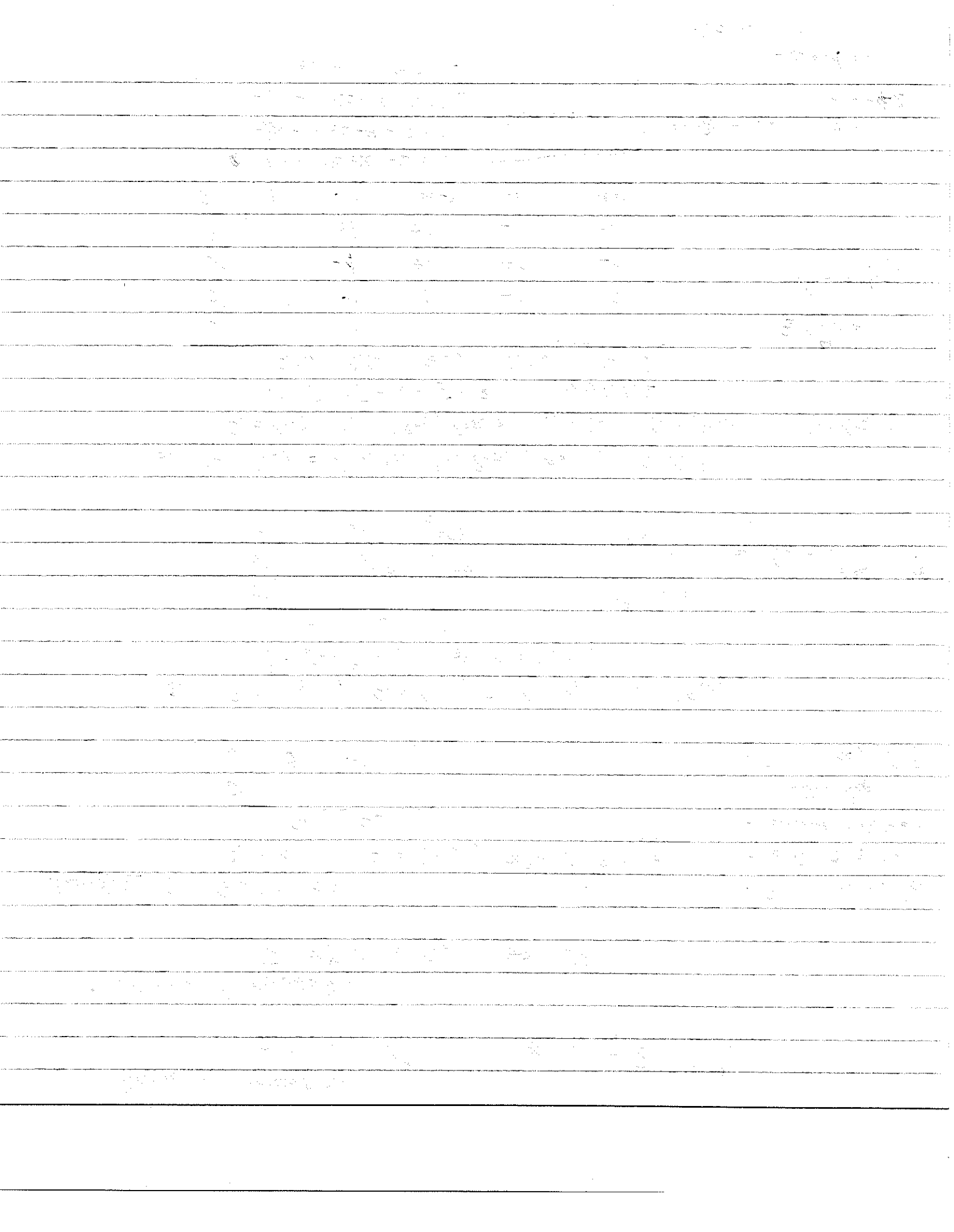
$$b = -4$$

$$2c + 4d = 0$$

$$-2c + 2d = 0$$

$$c + d = 0$$

$$c - d = 0$$



① $P_3[x] = \{1, x, x^2\}$ on $\{-1, 1\}$

$G = \{I, T\}$ $(Tf)(x) = f(-x)$ $(If)(x) = f(x)$

	I	T	
x^1	1	+1	x^1
x^2	1	-1	x^2

D $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\chi_I = 3$

$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\chi_T = 1$

$\chi_G = \sum a_\alpha \chi_\alpha$

$a_1 = \frac{1}{2} (1 \cdot 3 + 1 \cdot 1) = 2$

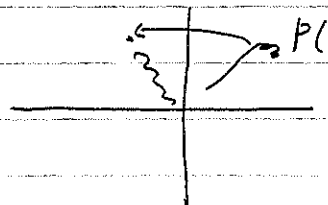
$a_2 = \frac{1}{2} (1 \cdot 3 - 1 \cdot 1) = 1$

② $P_2[x, y] = \{1, x, y, x^2, xy, y^2\}$ 6 dimensional vector space.

$G = \{I, R_\pi\}$ Group of co-ordinate transformation

$GL(P_2[x, y]) \ni U_\pi$

$(U_\pi f)(x, y) = f(R_\pi^{-1}(x, y))$



	before	after
Point	P	P
Co-ordinates	(x, y)	$(x', y') = R_\pi(x, y) = (-x, -y)$
fn.	$f(x, y)$	$f'(x', y') = f(x, y)$ $= f(R_\pi^{-1}(x', y'))$



$(U_\pi f)(x, y) = f(R_\pi^{-1}(x, y)) = f(-x, -y)$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{6 \times 6}$

U_π

$U_\pi(1) = 1$

$U_\pi(x) = -x$

$U_\pi(y) = -y$

$U_\pi(x^2) = x^2$

$U_\pi(xy) = xy$

$U_\pi(y^2) = y^2$

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\chi_I = 6$ $\chi_{U_\pi} = 2 \Rightarrow a_1 = 4$ and $a_2 = 2$

- PC
- PageMaker, office.
- Scanner
- Printer
- Sketch Pens -
- Robing Pen (0.3 \$0.5) \$0.50
- Papers
- Plastic Scales
- HB, B,