PH 203

Classical Mechanics

Quiz 1 & 2 - 10% + 10%

Mid-sem - 30%

End-Sem - 50%

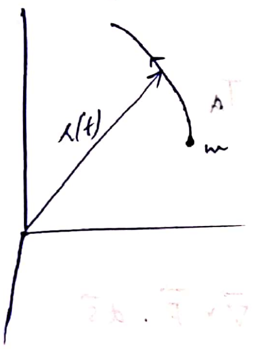
Hamilton's Principle

$$S = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} (T - V) \, dt$$

S is min. or max. for motion b/w t_1 & t_2

$$\delta S = 0$$

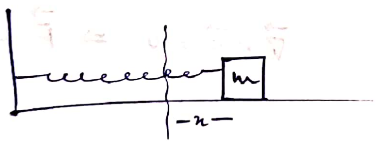
Momentum of a single particle



$$\frac{d\vec{p}}{dt} = \vec{F}$$

$$m \frac{d\vec{v}}{dt} = \vec{F}$$

$$m \frac{d^2\vec{r}}{dt^2} = \vec{F}$$



$$F = -kx$$

$$m \frac{d^2x}{dt^2} = -kx$$

$\boxed{\text{H.W.}}$

Solve

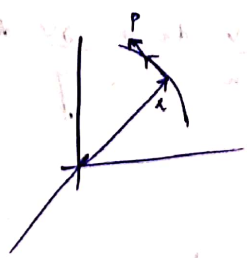
$$m \frac{d^2x}{dt^2} = -kx$$

$$x(0) = 0$$

$$\dot{x}(0) = v_0$$

$\frac{d\vec{p}}{dt} = \vec{F}$, if $\vec{F} = 0$, $\vec{p}$ = constant,

$$\vec{L} = \vec{r} \times \vec{p}$$



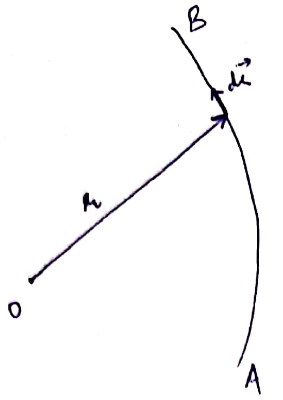
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= m\vec{v} \times \vec{v} + \vec{r} \times \vec{F}$$

$$= \vec{r} \times \vec{F}$$

$$= \vec{\tau}$$



$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$$

$$= \int_{A}^{B} \frac{d\vec{p}}{dt}\cdot d\vec{r}$$

$$= m\int_A^B \frac{d\vec{v}}{dt}\cdot v\,d\vec{t}$$

$$= \frac{m}{2}\int_A^B \frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt$$

$$= \frac{m}{2}\int_A^B d(\vec{v}\cdot\vec{v})$$

$$= \frac{m}{2}\int_A^B d(v^2)$$

$$= \frac{m}{2}\left[v_B^2 - v_A^2\right]$$

$$W_{AB} = T_B - T_A$$

$\boxed{\text{H.W.}}$

Verify WE Theorem for free fall of particle

Verify eqn of motion by integration

W_{AB} is independent of path AB

$$W_{AB} = 0 = \oint \vec{F}\cdot d\vec{r} = \int \vec{\nabla}\times\vec{F}\cdot d\vec{s}$$

$$\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F} = -\vec{\nabla}V$$

$$\int \vec{F}\cdot d\vec{r} = -\int \nabla\vec{V}\cdot d\vec{r}$$

$$\nabla\vec{V} = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

$$\nabla\vec{V}\cdot d\vec{r} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

$$V_P = V(x,y,z)$$

$$V_Q = V(x+dx,\, y+dy,\, z+dz)$$

$$= V(x,y,z) + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \dots$$

$$dV = V_Q - V_P = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$-\int \nabla \vec{V} \cdot \vec{dr} = -\int dV = -[V_B - V_A]$$

$$W_{AB} = V_A - V_B \quad (\text{For conservative force})$$

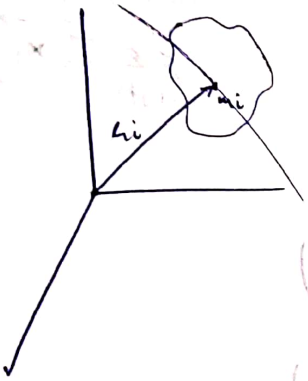
By WE Theorem

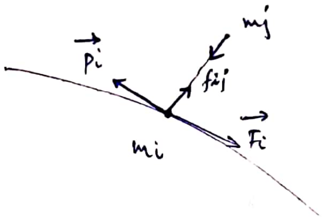
$$T_B - T_A = V_A - V_B$$

$$V_A + T_A = \cancel{V_A}\ V_B + T_B$$

$$E_A = E_B \quad (\text{Conservative force})$$

Multiparticle System





For the i^{th} particle

$$\frac{d\vec{p_i}}{dt} = \cancel{\vec{F_i}^{ext} + \sum \vec{f_{ij}}}$$

$$\vec{F_i}^{ext} + \sum_{j\neq i} \vec{f_{ij}}$$

$$\vec{P}_{tot} = \sum_i \vec{p_i}$$

$$\frac{d\vec{P}_{tot}}{dt} = \sum_i \frac{d\vec{p_i}}{dt}$$

$$= \sum_i \left(\vec{F_i}^{ext} + \sum_{j\neq i} \vec{f_{ij}} \right)$$

$$= \cancel{\sum_i F_{ext}}$$

$$= \sum_i F_i^{ext} + \sum_i \sum_{j\neq i} \vec{f_{ij}}$$

If there are 2 particles

$$\sum_{i=1}^{2} \sum_{j\neq i} \vec{f_{ij}} = \sum_{j\neq 1} \vec{f_{1j}} + \sum_{j\neq 2} f_{2j}$$

$$= \vec{f_{12}} + \vec{f_{21}}$$

$\boxed{\text{H.W.}}$

Expand $\sum_{i=1}^{3} \sum_{j=1}^{3} \vec{f_{ij}}$

Newton's 3^{rd} Law $f_{ij} + f_{ji} = 0$

$$\therefore \frac{d\vec{P}_{total}}{dt} = \sum F_i^{ext} \quad \left(\because \sum_i \sum_{j\neq i} \vec{f_{ij}} = 0 \right)$$

$$\frac{d\vec{P}_{Total}}{dt} = \vec{F}_{tot}^{ext}$$

$$\vec P_{Total} = \sum_i \vec p_i = \sum m_i \vec v_i$$
$$= \frac{d}{dt} \sum m_i \vec r_i$$
$$= \frac{d}{dt}\left(M \vec R_{CM}\right)$$
$$= M \vec V_{CM}$$

$$\vec R_{CM} = \frac{m_1 \vec r_1 + m_2 \vec r_2 + \dots + m_N \vec r_N}{m_1 + m_2 + \dots + m_N}$$
$$= \frac{\sum_i m_i r_i}{\sum_i m_i}$$
$$\Rightarrow \sum m_i r_i = M \vec R_{CM}$$

$$\frac{d\vec P_{Total}}{dt} = \frac{d}{dt}\left(M \vec V_{CM}\right) = \vec F_{ext}$$
$$= M \frac{d}{dt}\left(\vec V_{CM}\right) = \vec F^{\,ext}_{Total}$$
$$M \vec a_{CM} = \vec F^{\,ext}_{Total}$$

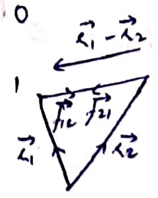
$\boxed{\text{H.W.}}$

How many terms
$$\sum_{i=1}^{3} \sum_{j \neq i} \vec r_i \times \vec f_{ij}$$

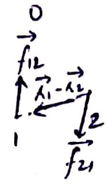
$$\vec L_{Total} = \sum_i \vec r_i \times \vec p_i$$
$$\frac{d}{dt} \vec L_{Total} = \sum_i \left(\frac{d}{dt} \vec r_i \times \vec p_i + \frac{d}{dt} \vec p_i \times \vec r_i \right)$$
$$= \sum_i \left(\cancelto{0}{\vec v_i \times m_i \vec v_i} + \vec r_i \times \frac{d\vec p_i}{dt} \right)$$
$$= \sum_i \left(\vec r_i \times \left(\vec F_{ext} + \sum_{j \neq i} \vec F_{ij} \right)\right)$$
$$= \sum_i r_i \times \vec F_{ext} + \left(\sum_i \sum_{j \neq i} \vec r_i \times \vec f_{ij} \right)$$

$$\frac{d}{dt}\left(\vec L_{Total}\right) = \sum_i \vec r_i \times \vec F_{ext}$$
$$= \vec \tau^{\,ext}_{Total}$$

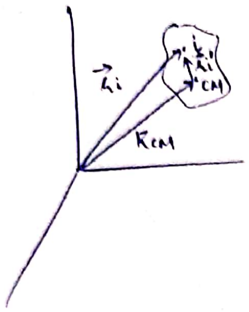
$$\sum_i \sum_{j \neq i} \vec r_i \times \vec f_{ij}$$
$$= \vec r_1 \times \vec f_{12} + \vec r_2 \times \vec f_{21}$$
$$= \vec r_1 \times \vec f_{12} - \vec r_2 \times \vec f_{12}$$
$$= \left(\vec r_1 - \vec r_2\right) \times \vec f_{12}$$
$$= 0$$



strong form of 3rd law



weak form of 3rd law



$$L_{Total} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\vec{r}_i = \vec{R}_{CM} + \vec{r}_i\,'$$

$$\vec{v}_i = \vec{v}_{CM} + \vec{v}_i\,'$$

$$\vec{L}_{Total} = \sum_i (\vec{R}_{CM} + \vec{r}_i\,') \times m_i(\vec{v}_{CM} + \vec{v}_i\,')$$

$$= \sum_i m_i \left[\vec{R}_{CM} \times \vec{v}_{CM} + \vec{R}_{CM} \times \vec{v}_i\,' + \vec{r}_i\,' \times \vec{v}_{CM} + \vec{r}_i\,' \times \vec{v}_i\,' \right]$$

$$\sum m_i r_i = M R_{CM} + \sum m_i r_i'$$

$$\cancel{M R_{CM}} = \cancel{M R_{CM}} + \sum m_i r_i'$$

$$\sum m_i r_i' = 0$$

$$= M \vec{R}_{CM} \times \vec{v}_{CM} + \cancel{\vec{R}_{CM} \times \sum m_i \vec{v}_i\,'} \to 0 + \cancel{(\sum m_i \vec{r}_i\,') \times \vec{v}_{CM}} \to 0 + \sum m_i \vec{r}_i\,' \times \vec{v}_i\,'$$

$$\vec{L}_{Total} = M \vec{R}_{CM} \times \vec{v}_{CM} + \sum m_i \vec{r}_i\,' \times \vec{v}_i\,'$$

= ~~Ang mom + about CM~~ = Ang. mom of CM + Ang. mom about CM

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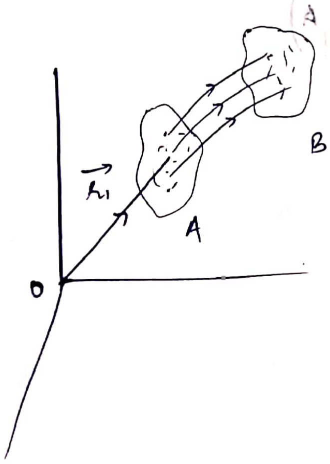


$$W_{AB} = \sum_{i=1}^{N} \int \vec{F}_i \cdot d\vec{r}_i$$

$$\vec{F}_i = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{f}_{ij}$$

$$W_{AB} = \sum_{i=1}^{N} \int \left(\vec{F}_i^{ext} + \sum_{j\neq i} f_{ij}\right) \cdot d\vec{r}_i$$

$$= \int \sum_{i=1}^{N} \vec{F}_i^{ext} \cdot d\vec{r}_i + \int \sum_i \sum_{j \neq i} \vec{f}_{ij} \cdot d\vec{r}_i$$

$$\vec{F}_i^{ext} = -\vec{\nabla}_i V_i$$

$$\nabla_i V_i = \hat{i} \frac{\partial V_i}{\partial x_i} + \hat{j} \frac{\partial V_i}{\partial y_i} + \hat{k} \frac{\partial V_i}{\partial z_i}$$

$$\nabla_i V_i \cdot d r_i = \frac{\partial V_i}{\partial x_i} dx_i + \frac{\partial V_i}{\partial y_i} dy_i + \frac{\partial V}{\partial z_i} dz_i$$

$$= dV_i$$

$$= -V_i(x_i, y_i, z_i)$$

1^{st} term $= -\int \sum_i \vec{\nabla}_i V_i \cdot d\vec{r}_i$

$$= \cancel{-\int \sum} \quad - \sum \int_A^B dV_i$$

$$= -\sum \left(V_i^B - V_i^A \right)$$

2^{nd} term $\quad \vec{f}_{ij} = -\vec{\nabla}_{ij} V_{ij}$

$$\vec{\nabla}_{ij} V_{ij} = \hat{i} \frac{\partial V_{ij}}{\partial x_{ij}} + \hat{j} \frac{\partial V_{ij}}{\partial y_{ij}} + \hat{k} \frac{\partial V_{ij}}{\partial z_{ij}} \qquad x_{ij} = x_i - x_j$$

$$d\vec{r}_{ij} = \hat{i}\, dx_{ij} + \hat{j}\, dy_{ij} + \hat{k}\, dz_{ij}$$

$$\vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij} = \frac{\partial V_{ij}}{\partial x_{ij}} dx_{ij} + \frac{\partial V_{ij}}{\partial y_{ij}} dy_{ij} + \frac{\partial V_{ij}}{\partial z_{ij}} dz_{ij}$$

$$= dV_{ij}$$

$$-\int \vec{f}_{ij} \cdot d\vec{r}_{ij} = -\int_A^B dV_{ij} = V_{ij}^A - V_{ij}^B$$

$$W_{AB} = -\sum_i \left(V_i^B - V_i^A \right) - \sum_i \sum_{j \neq i} \left(V_{ij}^B - V_{ij}^A \right)$$

$$W_{AB} = \sum_{i=1}^N \int \vec{F}_i \cdot d\vec{r}_i$$

$$F_i = \frac{d\vec{p}_i}{dt}$$

$$= \sum_i \int \frac{d\vec{p}_i}{dt} \cdot d\vec{r}_i$$

$$= \sum_i \int m_i \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i \, dt$$

$$= \sum_i \int m_i \frac{d}{dt} \left(\frac{1}{2} \vec{v}_i \cdot \vec{v}_i \right) dt$$

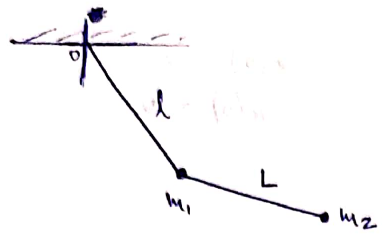
$$= \sum_i \int m_i \cdot d\left(\frac{1}{2} v_i^2 \right)$$

$$= \sum_i \frac{1}{2} m_i v_i^2 \Big|_A^B = \cancel{\sum_i \left(\frac{1}{2} m_i v_{Bi}^2 - \right)}$$

$$= \sum_i \left(\frac{1}{2} m_i v_i^{2B} - \frac{1}{2} m_i v_i^{2A} \right)$$

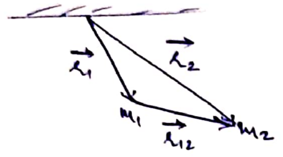
Scanned with OKEN Scanner

$$W_{AB} = T_B - T_A$$



$$\frac{d\vec{p}_1}{dt} = \vec{T} + m_1\vec{g} + \vec{f}_{12}$$

$$\frac{d\vec{p}_2}{dt} = m_2\vec{g} + \vec{f}_{21}$$



$$\vec{p}_r = m_r\vec{v}_r = m_r\dot{\vec{r}}_r$$

$$m_1\ddot{\vec{r}}_1 = \vec{T} + m_1\vec{g} + \vec{f}_{12}$$

$$m_2\ddot{\vec{r}}_2 = m_2\vec{g} + \vec{f}_{21}$$

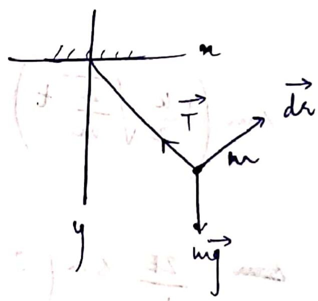
$\vec{f}_{12}$ & $\vec{f}_{21}$ $\Rightarrow$ forces of constraint

$$m\ddot{\vec{r}} = m\vec{g} + \vec{T}$$

$$m\ddot{x} = T_x$$

$$m\ddot{y} = mg + T_y$$

$$(m\ddot{\vec{r}} - m\vec{g} - \vec{T}) \cdot d\vec{r} = 0$$



$\boxed{\text{H.W}}$. Sol^n $\Rightarrow$ $m\ddot{x} = -kx$

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\frac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}k\frac{d}{dt}(x^2) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = C = E$$

$$\frac{dx}{dt} = \dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}kx^2}$$

$$\int \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}} = \pm\sqrt{\frac{2}{m}}\int dt$$

$$\frac{1}{k/2}\int \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \pm\sqrt{\frac{2}{m}}\int dt$$

$$\frac{1}{k/2}\sin^{-1}\left(\frac{kx}{2E}\right) = \pm\sqrt{\frac{2}{m}}\,t$$

$x(0) = 0$
$\dot{x}(0) = v_0$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}kx^2}$$

$t = 0$, ~~$x=0$~~, $\dot{x} = v_0$

$$v_0 = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}kx^2}$$

$$\frac{1}{2}mv_0^2 = E - \frac{1}{2}kx^2$$

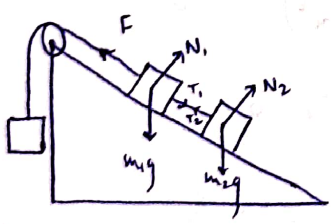
$$\frac{1}{2}kx^2 = E - \frac{1}{2}mv_0^2$$

$$E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx^2$$

$$\frac{kx}{2E} = \sin\left(\pm\frac{k}{2}\sqrt{\frac{2}{m}}\,t\right)$$

$$x = \cancel{\sin}\ \frac{2E}{k}\sin\left(\pm\frac{k}{2}\sqrt{\frac{2}{m}}\,t\right)$$

$$x = \frac{2}{k}\left(\frac{1}{2}mv_0^2 + \frac{1}{2}kx^2\right)\cdot\sin\left(\pm\frac{k}{2}\sqrt{\frac{2}{m}}\,t\right)$$



2 eq^n's $\Rightarrow$ $m_1\ddot{\vec{r}}_1 = m_1\vec{g} + \vec{N}_1 + \vec{T}_1 + \vec{F}$

2 eq^n's $\Rightarrow$ $m_2\ddot{\vec{r}}_2 = m_2\vec{g} + \vec{N}_2 + \vec{T}_2$

4 eq^n's Total

D'alembert

Constraints $\Rightarrow$ $\vec{T}_1 = -\vec{T}_2$ 2eq^n's

$\vec{T}_1$ & $\vec{T}_2$ are along line joining them

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \text{const}$$ (distance b/w particles same)

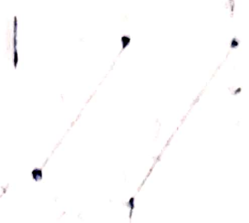
$y_1 = 0$

2 eqn

$y_2 = 0$

$$\vec{0} = -m_1\ddot{\vec{r}}_1 + m_1\vec{g} + \vec{N}_1 + \vec{T}_1 + \vec{F}$$

$$\vec{0} = (-m_1\ddot{\vec{r}}_1 + m_1\vec{g} + \vec{N}_1 + \vec{T}_1 + \vec{F})\cdot\delta\vec{r}_1$$



$$\vec{0} = (-m_2\ddot{\vec{r}}_2 + m_2\vec{g} + \vec{N}_2 + \vec{T}_2)\cdot\delta\vec{r}_2$$

$$\delta\vec{r}_1 = \delta\vec{r}_2$$

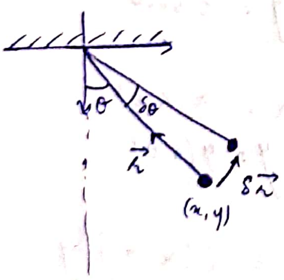
$$\delta\vec{r}_1\left(-m_1\ddot{\vec{r}}_1 - m_2\ddot{\vec{r}}_2 + m_1\vec{g} + m_2\vec{g} + \underbrace{\vec{T}_1 + \vec{T}_2}_{0} + \vec{F}\right) = 0$$

$$\delta\vec{r}_1 = \hat{i}\,\delta x_1 + \hat{j}\,\underset{\substack{\downarrow\\0}}{\delta y_1}$$

$$0 = (-m_1\ddot{x}_1 - m_2\ddot{x}_2 + m_1 g_x + m_2 g_x + F_x)\cdot\delta x_1 = 0$$

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = (m_1+m_2)g_x - F$$

$$\ddot{x}_{CM} = \frac{m_1\ddot{x}_1 + m_2\ddot{x}_2}{m_1+m_2} = g_x - \frac{F}{m_1+m_2}$$



$$m\ddot{\vec{r}} = m\vec{g} + \vec{T}$$

$$\vec{0} = -m\ddot{\vec{r}} + m\vec{g} + \vec{T}$$

$$\vec{0} = (-m\ddot{\vec{r}} + m\vec{g} + \vec{T})\cdot\delta\vec{r}$$

$$\delta\vec{r} = \delta x\,\hat{i} + \delta y\,\hat{j} = \hat{i}(l\cos\theta\,\delta\theta) + \hat{j}(l\sin\theta\,\delta\theta)$$

$$(-m\ddot{x} + mg_x + T_x)\cdot\delta x + (-m\ddot{y} + mg_y + T_y)\cdot\delta y = \vec{0}$$

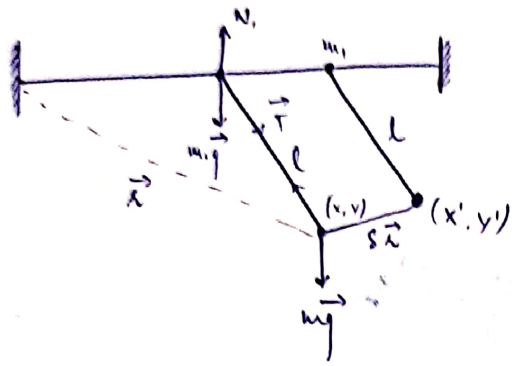
$x = l\sin\theta$ $\quad$ $y = l\cos\theta$

$\dot{x} = l\cos\theta\,\dot{\theta}$ $\quad$ $\dot{y} = -l\sin\theta\,\dot{\theta}$

$$\ddot{x} = l\cos\theta\,\ddot{\theta} - l\sin\theta\,\dot{\theta}^2 \qquad \ddot{y} = -l\sin\theta\,\ddot{\theta} - l\cos\theta\,\dot{\theta}^2$$

Final eq^n $\Rightarrow$ $l\ddot{\theta} + g\sin\theta = 0$ (we only have to deal with one variable)

Sliding Pendulum

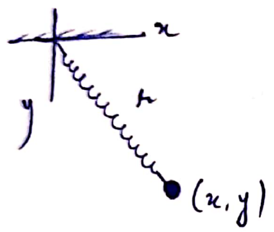


$$m\ddot{\vec{r}} = m\vec{g} + \vec{T}$$

$$m_1\ddot{\vec{r}}_1 = \vec{T}_1 + m_1\vec{g} + \vec{N}_1$$

$$(-m\ddot{\vec{r}} + m\vec{g} + \vec{T})\cdot\delta\vec{r}$$

$$+ (-m_1\ddot{\vec{r}}_1 + \vec{T} + \vec{N}_1 + m\vec{g})\cdot\delta\vec{r}_1 = 0$$



$(x,y) \Leftarrow$ inconvenient

$(r,\theta) \Leftarrow$ convenient

$$m\ddot{\vec{r}} = \vec{F} = m\vec{g} + \vec{F}_s$$

$$\vec{g} = g\hat{j}$$

$$\vec{F}_s = -k(r-l)\hat{r} \qquad \hat{r} = \hat{i}x + \hat{j}y$$

$x = r\cos\theta$

$y = r\sin\theta$

$$m\ddot{x}\hat{i} + m\ddot{y}\hat{j} = F_x\hat{i} + F_y\hat{j}$$

d'Alembert principle

$x = x(r,\theta)$

$y = y(r,\theta)$

$$m\ddot{\vec{r}}\cdot\delta\vec{r} = \vec{F}\cdot\delta\vec{r}$$

$$\delta\vec{r} = \hat{i}\,\delta x + \hat{j}\,\delta y$$

$$m\ddot{x}\,\delta x + m\ddot{y}\,\delta y = F_x\,\delta x + F_y\,\delta y$$

$$(m\ddot{x} - F_x)\,\delta x + (m\ddot{y} - F_y)\,\delta y = 0$$

$$m\ddot{x} - F_x = 0 \qquad m\ddot{y} - F_y = 0$$

$$\delta x = \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta$$

$$\delta y = \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta$$

$$\vec{F}\cdot\delta\vec{r} = F_x\,\delta x + F_y\,\delta y$$

$$= F_x\left(\frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta\right) + F_y\left(\frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta\right)$$

$$= \left(F_x\frac{\partial x}{\partial r} + F_y\frac{\partial y}{\partial r}\right)\delta r + \left(F_x\frac{\partial x}{\partial \theta} + F_y\frac{\partial y}{\partial \theta}\right)\delta\theta$$

$$= \left(F\cdot\frac{\partial \vec{r}}{\partial r}\right)\delta r + \left(\vec{F}\cdot\frac{\partial r}{\partial \theta}\right)\delta\theta \quad \text{RHS,}$$

$$= Q_r\cdot\delta r + Q_\theta\cdot\delta\theta$$

LHS $\left(m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial r}\right)\delta r + \left(m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial r}\right)\delta\theta$

LHS = RHS $\Rightarrow \left(m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial r} - Q_r\right)\delta r + \left(m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial\theta} - Q_\theta\right)\delta\theta = 0$

$$m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial r} = Q_r \qquad m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial\theta} = Q_\theta$$

$r = q_1$ & $Q_r = Q_1$

$\theta = q_2$ $\quad Q_\theta = Q_2$

$$m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_i} = Q_i, \quad i = 1,2$$

$$T = \frac{1}{2}m(\dot{\vec{r}}\cdot\dot{\vec{r}}) \qquad \vec{r} = \vec{r}(q_1, q_2)$$

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2$$

$$\frac{\partial T}{\partial\dot{q}_i} = m\,\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_i}$$

$$\dot{\vec{r}} = \sum_{k=1,2}\frac{\partial\vec{r}}{\partial q_k}\dot{q}_k$$

$$\frac{\partial T}{\partial\dot{q}_i} = m\dot{\vec{r}}\frac{\partial\vec{r}}{\partial q_i}$$

$$\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_i} = \sum_{k=1,2}\frac{\partial\vec{r}}{\partial q_k}\cdot\frac{\partial\dot{q}_k}{\partial\dot{q}_i} = \frac{\partial\vec{r}}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) = m\ddot{\vec{r}}\frac{\partial\vec{r}}{\partial q_i} + m\dot{\vec{r}}\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial q_i}\right)$$

$$\Rightarrow \frac{\partial\dot{\vec{r}}}{\partial\dot{q}_i} = \frac{\partial\vec{r}}{\partial q_i}$$

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = m\ddot{\vec{r}}\frac{\partial \vec{r}}{\partial q_i} + m\dot{\vec{r}}\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_i}\right)$$

$$\vec{r} = \vec{r}(q_1, q_2)$$

$$\frac{\partial r}{\partial q_i} = \frac{\partial r}{\partial q_i}(q_1, q_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = m\ddot{\vec{r}}\frac{\partial \vec{r}}{\partial q_i} + \cancel{m\dot{r}\frac{d\vec{r}}{dt}}\; m\dot{\vec{r}}\frac{\partial \dot{\vec{r}}}{\partial q_i}$$

$$\Downarrow\ \partial T/\partial q_i$$

$$\cancel{\frac{d}{dt}\left(\frac{\partial r}{\partial q_i}\right)} = \cancel{\ }$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

$$f = f(q_1, q_2)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2$$

$$= \sum \frac{\partial f}{\partial q_k}\dot{q}_k$$

$$Q_i = \vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i}$$

$$\therefore \frac{d}{dt}\left(\cancel{\ }\frac{\partial r}{\partial q_i}\right) = \sum_k \frac{\partial^2 \vec{r}}{\partial q_i \partial q_k}\cdot\dot{q}_k$$

Conservative $\Rightarrow$

$$\vec{F} = -\vec{\nabla}V$$

$$f = \vec{r} \Rightarrow \dot{\vec{r}} = \sum_k \frac{\partial r}{\partial q_k}\cdot\dot{q}_k$$

$$Q_i = -\vec{\nabla}V\cdot\frac{\partial \vec{r}}{\partial q_i}$$

$$\frac{\partial \dot{\vec{r}}}{\partial q_i} = \sum_k \frac{\partial^2 r}{\partial q_i \partial q_k}\cdot\dot{q}_k$$

$$= -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)\cdot\frac{\partial}{\partial q_i}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial r}{\partial q_i}\right) = \frac{\partial \dot{\vec{r}}}{\partial q_i}$$

$$Q_i = -\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_i}\right)$$

$$V(x, y) \rightarrow V(q_1, q_2)$$

$$\frac{\partial V}{\partial q_i} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_i}$$

$$Q_i = \frac{-\partial V}{\partial q_i}$$

&

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{-\partial V}{\partial q_i}$$ (For conservative force)

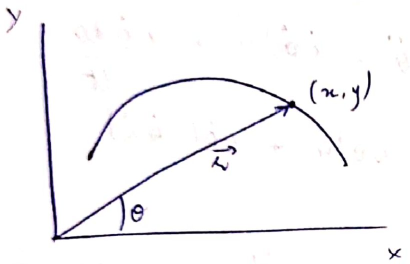
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_i}(T-V) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Rightarrow \text{Euler Langrange Eq}^n$$

Projectile Motion



$$m\ddot{\vec{r}} = \vec{F} = m\vec{g}$$

$$\left.\begin{aligned} m\ddot{x} &= 0 = F_x \\ m\ddot{y} &= -mg = F_y \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= F_x \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= F_y \end{aligned}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{dp_x}{dt} = m\ddot{x}$$

$$p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y} \Rightarrow \frac{dp_y}{dt} = m\ddot{y}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{dp_r}{dt} = m\ddot{r}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \frac{dp_\theta}{dt} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$Q_r = \vec{F}\cdot\frac{\partial \vec{r}}{\partial r} = \vec{F}\cdot\hat{r} = -F\sin\theta \qquad Q_\theta = \vec{F}\cdot\frac{\partial \vec{r}}{\partial \theta} = r\vec{F}\cdot\hat{\theta} = -rF\cos\theta$$

$$\frac{d\hat{r}}{d\theta} = \frac{\vec{r}' - \vec{r}}{d\theta} = \frac{r\, d\theta\, \hat{\theta}}{d\theta} = r\hat{\theta}$$

Aliter $\quad m\ddot{\vec{r}} = \vec{F}$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

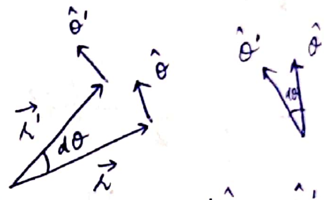
$$\frac{d\hat{r}}{dt} = \frac{\vec{r}' - \vec{r}}{dt} = \frac{d\theta\,\hat{\theta}}{dt} = \dot{\theta}\hat{\theta}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r})$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

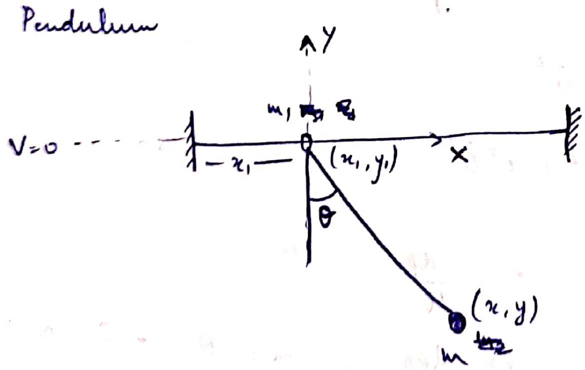


$$d\hat{\theta} = \hat{\theta}' - \hat{\theta} = -d\theta\,\hat{r}$$

$$m(\ddot{r} - r\dot{\theta}^2) = F_r$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = F_\theta$$

Sliding Pendulum



DOF = 2

No. of co-ordinates = 4 (x, y, x_1, y_1)

No. of constraints = 2

$$\left(\begin{array}{l} y_1 = 0 \\ \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = l \ \text{const.} \end{array} \right)$$

DOF = 4 − 2 = 2

$x = x_1 + l\sin\theta$ (Co-ordinate Transformation Eqⁿ)

$y = l\cos\theta$

$x_1 = x_1$

$y_1 = 0$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_1 \dot{x}_1^2$$

$$V = -mgy$$

$$L = T - V$$

$\dot{x} = \dot{x}_1 + l\cos\theta\,\dot{\theta}$

$\dot{y} = -l\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2} m\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}\, l\cos\theta + l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2\right) + \frac{1}{2} m_1 \dot{x}_1^2$$

$$V = -mgl\cos\theta$$

Eqⁿ of motions ⇒ Euler - Langrange Eqⁿ

$$T = \frac{1}{2}(m+m_1)\dot{x}_1^2 + ml\dot{x}_1\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2$$

$$\frac{\partial L}{\partial \dot{x}_1} = (m+m_1)\dot{x}_1 + ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) \Rightarrow (m+m_1)\ddot{x}_1 + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)$$

$$\frac{\partial L}{\partial x_1} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$= (m+m_1)\ddot{x}_1 + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}_1\cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\left(\ddot{x}_1\cos\theta - \dot{x}_1\dot{\theta}\sin\theta\right) + ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}_1\dot{\theta}\sin\theta - mgl\sin\theta$$

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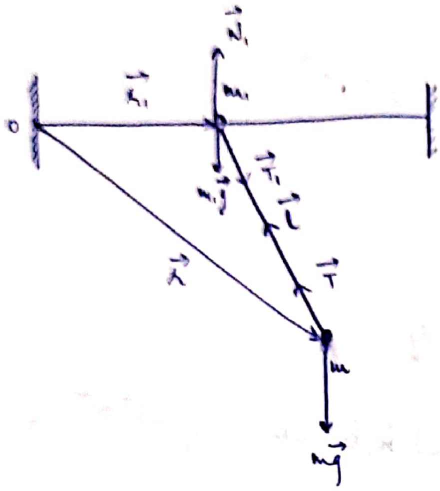
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml\left(\ddot{x}_1 \cos\theta - \dot{x}_1\dot\theta \sin\theta\right) + ml^2\ddot\theta + ml\dot{x}_1\dot\theta\sin\theta + mgl\sin\theta = 0$$

$$= ml\ddot{x}_1\cos\theta + mgl\sin\theta = 0$$

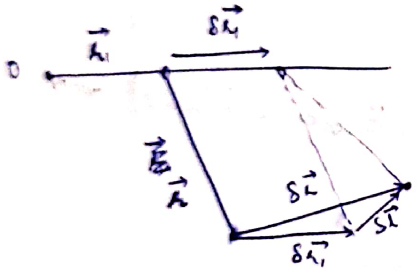
$$= ml\left(\ddot{x}_1\cos\theta + g\sin\theta\right) = 0$$



$$m\ddot{\vec{r}} = m\vec{g} + \vec{T}$$
$$m_1\ddot{\vec{r}} = m_1\vec{g} + \vec{T}_1 + \vec{N}_1$$

$$\left(-m\ddot{\vec{r}} + m\vec{g} + \vec{T}\right)\cdot\delta\vec{r} + \left(-m_1\ddot{\vec{r}}_1 + m_1\vec{g} + \vec{T}_1 + \vec{N}_1\right)\cdot\delta\vec{r}_1 = 0$$

$$\vec{r} = \vec{r}_1 + \vec{l} \qquad m_1\vec{g} = -\vec{N}_1$$
$$\delta\vec{r} = \delta\vec{r}_1 + \delta\vec{l} \qquad \vec{T} + \vec{T}_1 = 0$$



$$\left(-m\ddot{\vec{r}} + m\vec{g} + \cancel{\vec{T}} - m_1\ddot{\vec{r}}_1 + \cancel{\vec{T}_1}\right)\cdot\delta\vec{r}_1$$
$$+ \left(-m\ddot{\vec{r}} + m\vec{g} + \vec{T}\right)\cdot\delta\vec{l} = 0$$

$$\vec{g}\cdot\delta\vec{r}_1 = 0$$

$$\left(-m\ddot{\vec{r}} - m_1\ddot{\vec{r}}_1\right)\cdot\delta\vec{r}_1 + \left(-m\ddot{\vec{r}} + m\vec{g} + \vec{T}\right)\cdot\delta\vec{l} = 0$$
$$\vec{T}\cdot\delta\vec{l} = 0$$

$$\vec{r} = \vec{r}_1 + \vec{l} = \hat{i}x_1 + \hat{i}\,l\sin\theta + \hat{j}\,l\cos\theta$$
$$\dot{\vec{r}} = \hat{i}\dot{x}_1 + \hat{i}\,l\cos\theta\,\dot\theta - \hat{j}\,l\sin\theta\,\dot\theta$$
$$\ddot{\vec{r}} = \hat{i}\ddot{x}_1 + \hat{i}\,l\left(\ddot\theta\cos\theta - \dot\theta^2\sin\theta\right) - \hat{j}\,l\left(\sin\theta\,\ddot\theta + \dot\theta^2\cos\theta\right)$$

$$\vec{\delta l} = \hat{i}\,\delta l\cos\theta - \hat{j}\,\delta l\sin\theta$$

$$\ddot{\vec{x}}\cdot\vec{\delta l} = \ddot{x}_1\,\delta l\cos\theta + l\,\delta l(\ddot{\theta}\cos^2\theta - \cancel{\dot{\theta}^2\sin\theta\cos\theta}) + l\,\delta l(\ddot{\theta}\sin^2\theta + \cancel{\dot{\theta}^2\sin\theta\cos\theta})$$

$$= \ddot{x}_1\,\delta l\cos\theta + l\,\delta l\,\ddot{\theta}$$

$$\delta\vec{x}_1 = \hat{i}\,\delta x_1$$

$$\delta l = l\,\delta\theta$$

$$\vec{g} = g\hat{j}$$

$$\vec{g}\cdot\vec{\delta l} = -g\,\delta l\sin\theta$$

$$-(m\ddot{\vec{x}} + m_1\ddot{\vec{x}}_1)\delta x_1 - m(\ddot{x}_1\,\delta l\cos\theta + l\,\delta l\,\ddot{\theta} + g\,\delta l\sin\theta) = 0$$

$$-(m\ddot{\vec{x}} + m_1\ddot{\vec{x}}_1)\delta x_1 - m(\ddot{x}_1\cos\theta + l\ddot{\theta} + g\sin\theta)l\,\delta\theta = 0$$

$$m\ddot{\vec{x}} + m_1\ddot{\vec{x}}_1 = 0 \qquad \ddot{x}_1\cos\theta + l\ddot{\theta} + g\sin\theta = 0$$

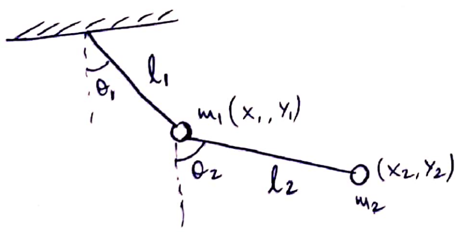
$$\ddot{X} = \frac{m\ddot{x} + m_1\ddot{x}_1}{m + m_1}$$

$$\ddot{X} = 0$$

$\boxed{\text{H.W.}}$

Construct Langrangian for sliding pendulum
Find eqn of motion

Double Pendulum



$\boxed{\text{H.W.}}$

D'Alembert's principle for double pendulum.

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$V = -m_1 g y_1 - m_2 g y_2$$

$$x_1 = l_1 \sin\theta_1 \qquad x_2 = l_2 \sin\theta_2 + l_1 \sin\theta_1$$

$$y_1 = l_1 \cos\theta_1 \qquad y_2 = l_2 \cos\theta_2 + l_1 \cos\theta_1$$

$$\dot{x}_1 = l_1 \cos\theta_1 \, \dot{\theta}_1 \qquad \dot{x}_2 = l_2 \cos\theta_2 \, \dot{\theta}_2 + l_1 \cos\theta_1 \, \dot{\theta}_1$$

$$\dot{y}_1 = -l_1 \sin\theta_1 \, \dot{\theta}_1 \qquad \dot{y}_2 = -l_2 \sin\theta_2 \, \dot{\theta}_2 - l_1 \sin\theta_1 \, \dot{\theta}_1$$

$$L = T - V$$

$$= \frac{1}{2} m_1 \left(l_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2 \right)$$

$$+ \frac{1}{2} m_2 \Big(l_2^2 \cos^2\theta_2 \, \dot{\theta}_2^2 + l_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2$$

$$+ 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \, \dot{\theta}_1 \dot{\theta}_2$$

$$+ l_2^2 \sin^2\theta_2 \, \dot{\theta}_2^2 + l_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2$$

$$+ 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \, \dot{\theta}_1 \dot{\theta}_2 \Big)$$

$$+ \cancel{m_1 g(-l_1 \sin\theta_1 \dot{\theta}_1) + m g(-l_2 \sin\theta_2}$$

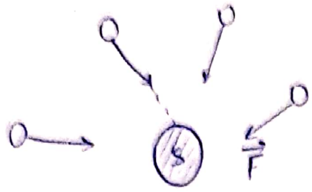
$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_2 \cos\theta_2 + l_1 \cos\theta_1)$$

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right)$$

$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

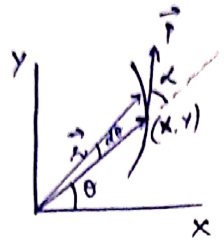
Central Force

A force directed to a fixed point.



$$\vec{F} = -G\frac{Mm}{r^2}\hat{r}$$

$$\vec{L} = \vec{r}\times\vec{p} = rp\sin\alpha\,\hat{z} = rp_\perp\hat{z} = mr^2\dot{\theta}\,\hat{z}$$



$$v_\perp = r\frac{d\theta}{dt} = r\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2), \quad V = -G\frac{Mm}{r} = \frac{-k}{r}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\dot{x}^2+\dot{y}^2 = \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 - \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}} + \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

$$L = L(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = L(r, \cancel{\theta}, \dot{r}, \dot{\theta}, \cancel{t})$$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ Energy $\Rightarrow$ Conserved

$$\cancel{\frac{d}{dt}\frac{\partial L}{\partial \theta}} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - \cancel{\ } mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot\theta\right) = 0 \implies mr^2\dot\theta = \text{constant} = L_z$$

$$m\ddot r - mr\left(\frac{L_z}{mr^2}\right)^2 + \frac{k}{r^2} = 0$$

$$m\ddot r - \frac{L_z^2}{mr^3} + \frac{k}{r^2} = 0$$

2nd order ODE

$$\dot r\left[m\ddot r - \frac{L_z^2}{mr^3} + \frac{k}{r^2}\right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}m\dot r^2 + \frac{L_z^2}{2mr^2} - \frac{k}{r}\right] = 0$$

$$\frac{1}{2}m\dot r^2 + \frac{L_z^2}{2mr^2} - \frac{k}{r} = \text{const.}$$

$$E = T + V$$

$$= \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2\right) - \frac{k}{r}$$

$$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\left(\frac{L_z}{mr^2}\right)^2 - \frac{k}{r}$$

$$E = \frac{1}{2}m\dot r^2 + \frac{L_z^2}{2mr^2} - \frac{k}{r} = \text{const.}$$

$$\frac{1}{2}m\dot r^2 = E - \frac{L_z^2}{2mr^2} + \frac{k}{r}$$

$$V_{eff}(r) = \frac{-k}{r} + \frac{L_z^2}{2mr^2}$$

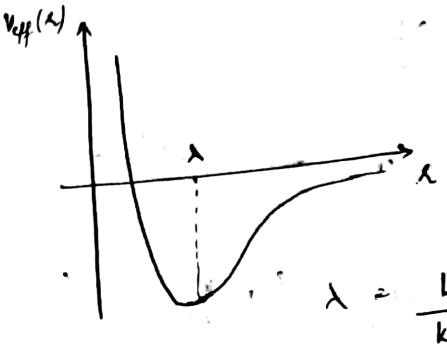
$$= E - V_{eff}(r)$$

$$\frac{dr}{dt} = \dot r = \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{eff}(r)}$$

$$\frac{d\theta}{dt} = \dot\theta = \frac{L_z}{mr^2}$$

$$\frac{dr}{d\theta} = \pm\sqrt{\frac{2}{m}}\,\frac{mr^2}{L_z}\sqrt{E - V_{eff}(r)}$$

$$V_{eff}(r) = \frac{-k}{r} + \frac{L_z^2}{2mr^2}$$



$$\lambda = \frac{L_z^2}{km}$$

$$\frac{dV_{eff}(r)}{dr} = 0$$

$$\frac{k}{r^2} - \frac{L_z^2}{mr^3} = 0$$

$$\frac{k}{\lambda^2} = \frac{L_z^2}{m\lambda^3}$$

$$\lambda = \frac{L_z^2}{mk}$$

$$\frac{dr}{d\theta} = \pm\sqrt{\frac{2}{m}}\,\frac{mr^2}{L_z}\sqrt{E + \frac{k}{r} - \frac{L_z^2}{2mr^2}}$$

Let $\frac{\lambda}{r} = x$, (dimensionless)

$$dr = \frac{-\lambda}{x^2}\,dx$$

$$\cancel{\frac{-\lambda}{x^2}}\frac{dx}{d\theta} = \pm\sqrt{\frac{2}{m}}\,\frac{m}{L_z}\,\frac{\cancel{\lambda^2}\lambda}{\cancel{x^2}}\sqrt{E + \frac{k}{\lambda}x - \frac{L_z^2}{2m}\,\frac{x^2}{\lambda^2}}$$

$$\cancel{dx}\quad \frac{-dx}{d\theta} = \pm\sqrt{\frac{2}{m}}\,\frac{m}{\sqrt{\lambda k m}}\,\lambda\sqrt{E + \frac{kx}{\lambda} - \frac{\lambda k \cancel{m}}{2\cancel{m}}\,\frac{x^2}{\lambda^2}}$$

$$\frac{-dx}{d\theta} = \pm\sqrt{\frac{2\lambda}{k}}\sqrt{E + \frac{kx}{\lambda} - \frac{k}{2\lambda}x^2}$$

$$\frac{-dx}{d\theta} = \pm\sqrt{\frac{2\lambda E}{k} + 2x - x^2}$$

$$\frac{-dx}{d\theta} = \pm\sqrt{\left(1 + \frac{2\lambda E}{k}\right) - \left(1 - 2x + x^2\right)}$$

$$\frac{-dx}{d\theta} = \pm\sqrt{\varepsilon^2 - (x-1)^2}$$

$$x - 1 = \varepsilon\cos\phi$$

$$dx = -\varepsilon\sin\phi\, d\phi$$

$$\varepsilon\sin\phi\frac{d\phi}{d\theta} = \pm\sqrt{\varepsilon^2 - \varepsilon^2\cos^2\phi}$$

$$\varepsilon\sin\phi\frac{d\phi}{d\theta} = \pm\varepsilon\sin\phi$$

$$d\phi = \pm d\theta$$

$$\phi = \pm\theta + c$$

$$\cos^{-1}\left(\frac{x-1}{\varepsilon}\right) = \pm(\theta + c)$$

$$\frac{x-1}{\varepsilon} = \cos(\pm\theta + c)$$

$$x = 1 + \varepsilon\cos(\pm\theta + c)$$

ε = eccentricity

λ = semi-latus recta

$$\frac{\lambda}{r} = 1 + \varepsilon\cos\theta$$

$$\varepsilon = \sqrt{1 + \frac{2\lambda E}{k}}$$

$$\lambda = \frac{L_z^2}{mk}$$

$E < 0, \ \varepsilon < 1$ ellipse

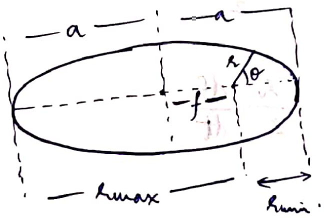
$E = 0, \ \varepsilon = 1$ parabola

$E > 0, \ \varepsilon > 1$ hyperbola

Case I $\quad E < 0 \quad \varepsilon < 1$

$$\frac{\lambda}{r_{min}} = 1 + \varepsilon\cos 0^\circ = 1 + \varepsilon \Rightarrow r_{min} = \frac{\lambda}{1+\varepsilon} = a(1-\varepsilon)$$

$$\frac{\lambda}{r_{max}} = 1 + \varepsilon\cos 180^\circ = 1 - \varepsilon \Rightarrow r_{max} = \frac{\lambda}{1-\varepsilon} = a(1+\varepsilon)$$



$$2a = r_{max} + r_{min}$$

$$2a = \frac{\lambda}{1-\varepsilon} + \frac{\lambda}{1+\varepsilon}$$

$$\lambda = a(1-\varepsilon^2)$$

$\lambda \Rightarrow$ semi-latus rectum $\qquad f = a\varepsilon$

$$a = \frac{\lambda}{1-\varepsilon^2} \qquad b = a\sqrt{1-\varepsilon^2}$$

$$b = \frac{\lambda}{\sqrt{1-\varepsilon^2}}$$

$$a = \frac{\lambda}{1-\varepsilon^2} = \frac{k}{2|E|}$$

$$\varepsilon^2 = 1 + \frac{2\lambda E}{k}$$

$$1 - \varepsilon^2 = \frac{-2\lambda E}{k} = \frac{2\lambda|E|}{k}$$

$$\lambda = \frac{L_z^2}{mk}$$

$$b = \frac{\lambda}{\sqrt{1-\varepsilon^2}} = \lambda\sqrt{\frac{k}{2\lambda E}} = \sqrt{\lambda}\sqrt{\frac{k}{2|E|}} = \frac{L_z}{\sqrt{2m|E|}}.$$

$$\vec{L} = \text{constant}$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \cancel{\vec{v} \times \vec{p}}^{\,0} + \vec{r} \times \vec{F} \qquad \vec{F} \propto -\hat{r}$$

$$= \vec{r} \times \hat{r}$$

$$= 0$$

$$L_z = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}$$

$$dA = \frac{1}{2} r (r\,d\theta)$$

$$= \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$$

$$L_z = 2m\frac{dA}{dt}$$

$$L_z\int_0^T dt = 2m\int_0^A dA$$

$$L_z \cdot T = 2mA$$

$$T = \frac{2mA}{L_z}$$

$$T = \frac{2m\pi ab}{L_z}$$

$$T = 2m\pi \frac{k}{2|E|} \cdot \frac{L_z}{\sqrt{2m|E|}} \cdot \frac{1}{L_z}$$

$$T = \frac{\pi}{2} k \frac{\sqrt{2m}}{|E|^{3/2}}$$

$E \geqslant 0$, $\varepsilon \geqslant 1$

$$\frac{\lambda}{r} = +1 + \varepsilon \cos\theta$$

$$\frac{\lambda}{r_{min}} = +1 + \varepsilon \cos 0^\circ = +1 + \varepsilon$$

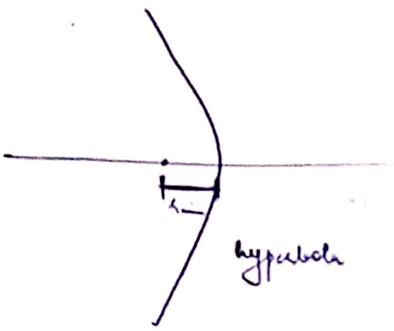
$$\frac{\lambda}{r_{max}} = +1 + \varepsilon \cos 180^\circ = +1 - \varepsilon < 0$$

$$r_{max} \not< 0$$

$$\therefore r_{max} \to \infty$$

$$0 = 1 + \varepsilon \cos\theta_{max}$$

$$\cos\theta_{max} = -1/\varepsilon$$



$$r_{min} = \frac{\lambda}{\varepsilon + 1} = \frac{\lambda}{2}$$

(hyperbola) (parabola)

$$\vec{F} = \frac{z_1 z_2 e^2}{4\pi\varepsilon_0 r^2} \hat{r}$$

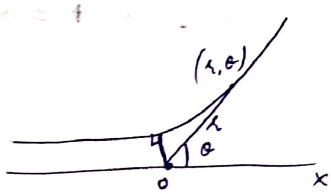
$$-\nabla\vec{V} = \vec{F}$$

$$-\frac{dV}{dr} \cancel{\hat{r}} = \frac{k}{r^2} \hat{r}$$

$$-V = k\int \frac{dr}{r^2}$$

$$-V = \frac{-k}{r} + c$$

$$V = \frac{k}{r}$$



$$L = T - V$$
$$= \frac{1}{2} m(\dot{r}^2 + \dot{\theta}^2) - \frac{k}{r}$$

$$mr^2\dot{\theta} = L_z$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + \frac{L_z^2}{m^2 r^2}\right) - \frac{k}{r}$$

$$E = \frac{1}{2} m\dot{r}^2 + \frac{L_z^2}{2mr^2} + \frac{k}{r} \qquad (\because E = T + V)$$

$$\dot{r} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{L_z^2}{2mr^2} - \frac{k}{r}}$$

$$\dot{\theta} = \frac{L_z}{mr^2}$$

$$\frac{dr}{d\theta} = \pm\frac{mr^2}{L_z}\sqrt{\frac{2}{m}}\sqrt{E - \frac{L_z^2}{2mr^2} - \frac{k}{r}}$$

$$\lambda = \frac{L_z^2}{mk}$$

$$\frac{\lambda}{r} = x$$

$$\frac{dx}{d\theta} = \sqrt{\left(1 + \frac{2\lambda E}{k}\right) - \left(x^2 + 2x + 1\right)}$$

$$\frac{dx}{d\theta} = \pm\sqrt{\varepsilon^2 - (x+1)^2}$$

$$x + 1 = \varepsilon\cos\psi$$

$$dx = -\varepsilon\sin\psi\, d\psi$$

$$-\varepsilon\sin\psi\frac{d\psi}{d\theta} = \pm\varepsilon\sin\psi$$

$$\frac{d\psi}{d\theta} = \pm 1$$

$$\psi = \pm\theta + c$$

$$\psi = \pm(\theta + c)$$

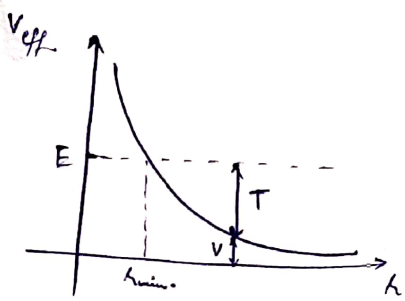
$$\cos\psi = \cos\theta$$

$$\frac{x+1}{\varepsilon} = \cos\theta$$

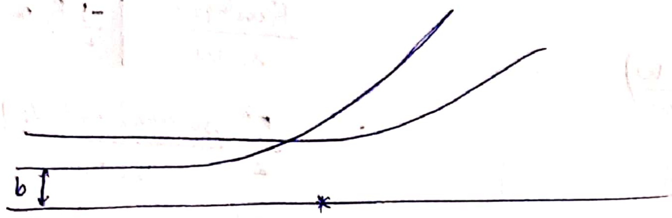
$$x = -1 + \varepsilon\cos\theta$$

$$E = \frac{1}{2} m\dot{r}^2 + \frac{L_z^2}{2mr^2} + \frac{k}{r}$$

$$V_{eff} = \frac{L_z^2}{2mr^2} + \frac{k}{r}$$



b ⇒ Impact parameter



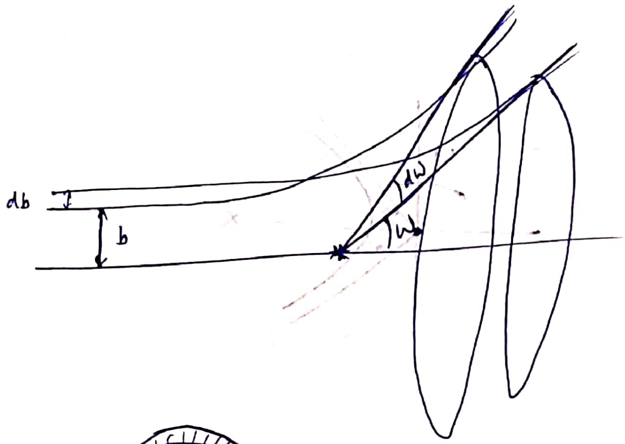
$d\dot{N}$ = No. of deflected particles per unit area per sec.

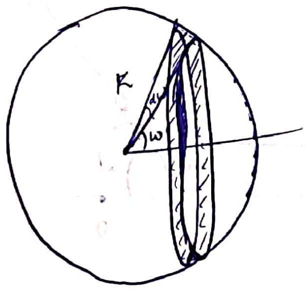
I = No. of particles per unit area per sec.

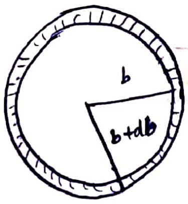
$$d\dot{N} \propto I$$

$$d\dot{N} = d\sigma\, I$$

$d\sigma$ ⇒ differential cross section







$$d\dot{N} = I\,dA$$

$$d\sigma\, I = I\,dA$$

$$d\sigma = dA$$

$$d\sigma = 2\pi b\,db$$

$d\sigma$ ⇒ Differential cross-section

$$dA_{ribbon} = 2\pi R \sin\omega\, R\,d\omega$$

$$= 2\pi R^2 \sin\omega\, d\omega$$

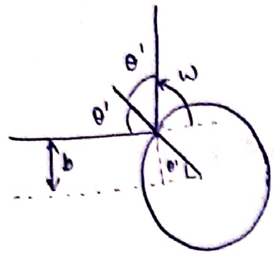
$$d\Omega = \frac{\text{Area of ribbon}}{R^2}$$

$$= \frac{2\pi R \sin\omega \cdot R\,d\omega}{R^2}$$

$$d\Omega = 2\pi \sin\omega\, d\omega$$

$$\cancel{\frac{d\sigma}{d\Omega}} = \cancel{\frac{b\,db}{\sin\omega\, d\omega}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\omega}\left|\frac{db}{d\omega}\right|$$



E.g. Scattering from hard ball

$b = \cancel{R} R \sin\theta'$

$\theta' = \frac{\pi - W}{2}$

$b = R \sin\left(\frac{\pi - W}{2}\right)$

$b = R\cos(W/2)$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin W}\left|\frac{db}{dW}\right|$$

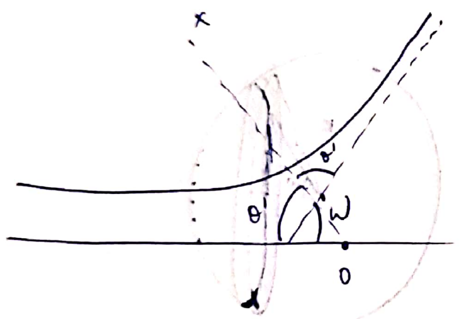
$$= \frac{R\cos(W/2)}{\sin W} \cdot \left|\frac{-1}{2} R \sin(W/2)\right|$$

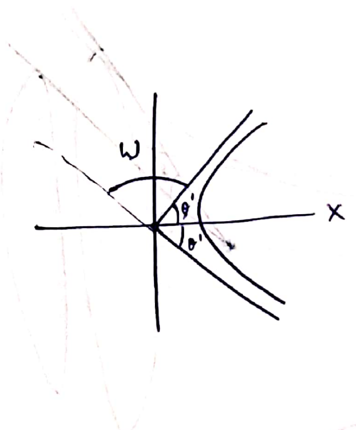
$$= \frac{R^2}{2}\frac{\sin(W/2)\cos(W/2)}{\sin W}$$

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$

$\cancel{d\sigma}$

$$\sigma = \int d\sigma = \frac{R^2}{4}\int d\Omega = \frac{R^2}{4} \cdot 4\pi = \pi R^2$$





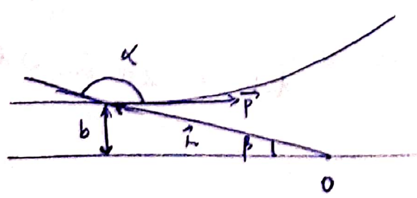
$$\frac{\lambda}{r} = -1 + \varepsilon\cos\theta'$$

$$\frac{\lambda}{\infty} = -1 + \varepsilon\cos\theta'$$

$$\cos\theta' = 1/\varepsilon$$

$$\varepsilon = \sqrt{1 + \frac{2\lambda E}{k}}$$

$$\lambda = \frac{L_z^2}{mk}$$



$$|L| = |\vec{r} \times \vec{p}|$$

$$= rp\sin\alpha$$

$$= rp\sin(\pi - \beta)$$

$$= rp\sin\beta$$

$$= b p_\infty$$

$$\lambda = \frac{L_z^2}{mk} = \frac{b^2 p_\infty^2}{mk} \qquad \varepsilon = \sqrt{1 + \left(\frac{2bE}{k}\right)^2}$$

$$E = \frac{p_\infty^2}{2m} \qquad \cos\theta' = \frac{1}{\varepsilon} = \frac{1}{\sqrt{1+\left(\frac{2bE}{k}\right)^2}}$$

$$\lambda = \frac{2b^2E}{k} \qquad \tan\theta' = \frac{2bE}{k}$$

$$2\theta' + \omega = \pi \qquad \tan\left(\frac{\pi - \omega}{2}\right) = \frac{2bE}{k}$$

$$\theta' = \frac{\pi - \omega}{2} \qquad \cot(\omega/2) = \frac{2bE}{k}$$

$$b = \frac{k}{2E}\cot(\omega/2)$$

$$\frac{db}{d\omega} = \frac{-k}{4E}\csc^2(\omega/2)$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\omega}\left|\frac{db}{d\omega}\right|$$

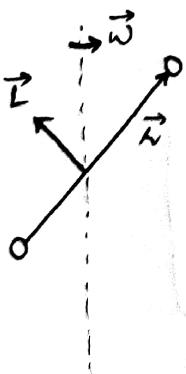
$$= \frac{\frac{k}{2E}\cot(\omega/2)}{\sin\omega}\left|\frac{-k}{4E}\csc^2(\omega/2)\right|$$

$$= \frac{k^2}{8E^2}\,\frac{\cos(\omega/2)}{\sin(\omega/2)}\,\frac{1}{\sin^2(\omega/2)}\cdot\frac{1}{2\sin(\omega/2)\cos(\omega/2)}$$

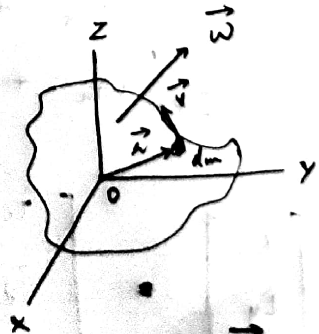
$$= \frac{k^2}{8E^2}\cdot\frac{1}{2\cdot\sin^4(\omega/2)}$$

$$= \frac{k^2}{16E^2}\cdot\frac{1}{\sin^4(\omega/2)}$$

Rigid Bodies



$\vec{L} \nparallel \vec{\omega}$



$$\vec{dp} = dm\,\vec{v}$$

$$\vec{dL} = \vec{r} \times \vec{dp} = dm\,(\vec{r} \times \vec{v})$$

$$\vec{L} = \int \vec{dL} = \int dm\,(\vec{r} \times \vec{v})$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{L} = \int dm \; \vec{r} \times (\vec{\omega} \times \vec{r})$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \hat{i}\left[(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z\right]$$
$$+ \hat{j}\left[-xy\,\omega_x + (z^2+x^2)\omega_y - yz\,\omega_z\right]$$
$$+ \hat{k}\left[-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\omega_z\right]$$

$$\vec{L} = \hat{i}\Big[\omega_x \underbrace{\int dm(y^2+z^2)}_{I_{xx}} - \omega_y \underbrace{\int dm\, xy}_{-I_{xy}} - \omega_z \underbrace{\int dm\, xz}_{-I_{xz}}\Big]$$
$$+ \hat{j}\left[\quad\right]$$
$$+ \hat{k}\left[\quad\right]$$

$$\vec{L} = \hat{i}\left[I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right]$$
$$+ \hat{j}\left[I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right]$$
$$+ \hat{k}\left[I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right]$$

$I_{xx}, I_{yy}, I_{zz} \Rightarrow$ Moments of inertia

$$\left.\begin{matrix} I_{xy} \\ I_{xz} \\ I_{yz} \end{matrix}\right\}$$ Products of inertia

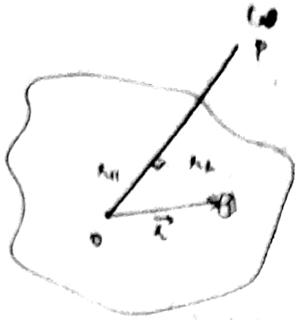
$$I_{xx} = \int dm(x^2+y^2)$$
$$I_{xy} = -\int dm\, xy = I_{yx}$$
$$I_{xz} = -\int dm\, xz = I_{zx}$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

MOI matrix

Moment of Inertia



$$I_{OP} = \int dm\, n_\perp^2$$

$$= \int dm \left(r^2 - n_{11}^2 \right)$$

Let n_1, n_2, n_3 be DC's of OP

$$\hat{n} = i n_1 + j n_2 + k n_3$$

$\cos\alpha \quad \cos\beta \quad \cos\gamma$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$$n_{11} = \vec{r} \cdot \hat{n}$$
$$= x n_1 + y n_2 + z n_3$$

$$I_{OP} = \int dm \left[x^2 + y^2 + z^2 - (x n_1 + y n_2 + z n_3)^2 \right]$$

$$I_{OP} = \int dm \left[x^2 + y^2 + z^2 - (x^2 n_1^2 + y^2 n_2^2 + z^2 n_3^2 + 2xy n_1 n_2 + 2yz n_2 n_3 + 2xz n_1 n_3) \right]$$

$$= \int dm \left[x^2(1 - n_1^2) + y^2(1 - n_2^2) + z^2(1 - n_3^2) - 2xy n_1 n_2 - 2yz n_2 n_3 - 2xz n_1 n_3 \right]$$

$$= \int dm \left[(n_2^2 + n_3^2) x^2 + (n_1^2 + n_3^2) y^2 + (n_1^2 + n_2^2) z^2 - 2xy n_1 n_2 - 2yz n_2 n_3 - 2xz n_1 n_3 \right]$$

$$= \int dm \left[(y^2 + z^2) n_1^2 + (x^2 + z^2) n_2^2 + (x^2 + y^2) n_3^2 - 2xy n_1 n_2 - 2yz n_2 n_3 - 2xz n_1 n_3 \right]$$

$$\rho_x = \frac{n_1}{\sqrt{I_{OP}}}$$

$$\rho_y = \frac{n_2}{\sqrt{I_{OP}}}$$

$$\rho_z = \frac{n_3}{\sqrt{I_{OP}}}$$

$$I_{OP} = I_{xx} n_1^2 + I_{yy} n_2^2 + I_{zz} n_3^2 + 2 I_{xy} n_1 n_2 + 2 I_{yz} n_2 n_3 + 2 I_{xz} n_1 n_3$$

$$I_{xx} \rho_x^2 + I_{yy} \rho_y^2 + I_{zz} \rho_z^2 + 2 I_{xy} \rho_x \rho_y + 2 I_{yz} \rho_y \rho_z + 2 I_{zx} \rho_x \rho_z = 1$$

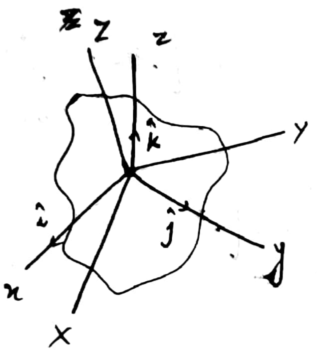
MOI Ellipsoid

Principle axes $\Rightarrow$ $I_{xy} = 0 = I_{xz}$

MI matrix becomes diagonal.

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$ (w.r.t. an inertial frame)

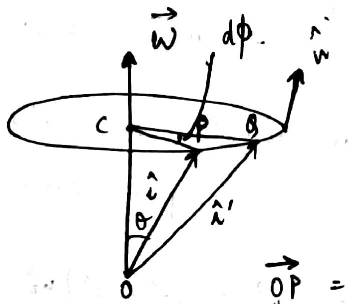


XYZ ⇒ space fixed (inertial)

xyz ⇒ body fixed

$$\vec{L} = \hat{i} L_x + \hat{j} L_y + \hat{k} L_z$$

$$\left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \hat{i}\frac{dL_x}{dt} + L_x \frac{d\hat{i}}{dt} + \hat{j}\frac{dL_y}{dt} + L_y\frac{d\hat{j}}{dt} + \hat{k}\frac{dL_z}{dt} + L_z\frac{d\hat{k}}{dt}$$



$\vec{OP} = \hat{i}$

$\vec{OQ} = \hat{i}'$

$$\frac{d\hat{i}}{dt} = \frac{\hat{i}' - \hat{i}}{dt} = \frac{\vec{PQ}}{dt}$$

$CP = OP \sin\theta$

$PQ = CP\, d\phi$

$= OP \sin\theta\, d\phi$

$= 1 \cdot \sin\theta \cdot d\phi$

$$= \frac{\sin\theta\, d\phi}{dt}\hat{n}$$

$$= \omega \sin\theta\, \hat{n}$$

$$\vec{\omega} \times \hat{i} = \omega \sin\theta\, \hat{n}$$

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \hat{i}\frac{dL_x}{dt} + L_x(\vec{\omega}\times\hat{i}) + \hat{j}\frac{dL_y}{dt} + L_y(\vec{\omega}\times\hat{j}) + \hat{k}\frac{dL_z}{dt} + L_z(\vec{\omega}\times\hat{k})$$

$$= \left(\frac{d\vec{L}}{dt}\right)_{xyz} + \vec{\omega} \times (\hat{i}L_x + \hat{j}L_y + \hat{k}L_z)$$

$$\left(\frac{d\vec{L}}{dt}\right)_{XYZ} = \left(\frac{d\vec{L}}{dt}\right)_{xyz} + \vec{\omega} \times \vec{L}$$

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ L_x & L_y & L_z \end{vmatrix} = \hat{i}(\omega_y L_z - \omega_z L_y) + \hat{j}(\omega_z L_x - \omega_x L_z) + \hat{k}(\omega_x L_y - \omega_y L_x)$$

$$\tau_x = \frac{dL_x}{dt} + (\omega_y L_z - \omega_z L_y)$$

$$\tau_y = \frac{dL_y}{dt} + (\omega_z L_x - \omega_x L_z)$$

$$\tau_z = \frac{dL_z}{dt} + (\omega_x L_y - \omega_y L_x)$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\tau_x = (I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z) + \omega_y(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)$$

$$(\dot{I}_{xx} = \dot{I}_{xy} = \dot{I}_{xz} = \dot{I}_{yy} \ldots = 0)$$

$\tau_y = \ldots$

$\tau_z = \ldots$

If choose principle axes

$$\tau_1 = I_1\dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2$$

$$\tau_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)$$

$$\tau_1 = I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3)$$

Similarly,

$$\tau_2 = I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1)$$

$$\tau_3 = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2)$$

Euler's Equations