Sol. tut 3

(1) Azimuthal Symmetry.

$\therefore$ $a \le r \le b$, $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$ —— (1)

$r \ge b$ $V(r,\theta) = \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l \cos\theta$ —— (2)

Boundary conditions:

i V continuous at $r = a$

ii $V(r=a) = V_0$

iii ~~V cons~~ metallic sphere $\therefore$ ~~E_0 at $r=a$, $\left.\frac{\partial V}{\partial r}\right|_{r=a} = 0$~~

$\therefore$ ~~$\left.\frac{\partial V(r\le a)}{\partial r}\right|_{r=a} = 0$~~

IV V continuous at $r = b$

V $-\epsilon_0 \left[\left.\frac{\partial V}{\partial r}\right|_{r\ge b} - \left.\frac{\partial V}{\partial r}\right|_{r\le b} \right]_{r=b} = k\cos\theta.$

After implementing the boundary condition as discussed in the class, The only non zero coefficients are

$B_0 = aV_0$ | $A_1 = \frac{k}{3\epsilon_0}$

$C_0 = aV_0$ | $B_1 = -a^3 k / 3\epsilon_0$

$C_1 = (b^3 - a^3)\frac{k}{3\epsilon_0}$

On substituting in eq. (1) & (2)

$$\boxed{\begin{aligned} a \le r \le b, \quad V(r,\theta) &= \frac{aV_0}{r} + \frac{kr}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta \\ r \ge b \quad V(r,\theta) &= \frac{aV_0}{r} + \frac{k(b^3 - a^3)}{3\epsilon_0 r^2}\cos\theta \end{aligned}}$$

First terms in both the regions is due to the given condition of potential V_0 ~~at the metallic~~ on of the conducting sphere.

The second term is potential due to the surface charge density at $r = b$.

(Superposition principle also holds)

Sol. t problem (1) continued.

~~due to the surface charge density at $r=b$~~

surface charge density on the conducting sphere

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a}$$

$$= -\epsilon_0 \left[\frac{-aV_0}{r^2} + \frac{p}{3\epsilon_0}\left(1 - \frac{2a^3}{a^3}\right)\cos\theta \right]_{r=a}$$

$$= \frac{\epsilon_0 V_0}{a} - p\cos\theta$$

First term is the surface charge density on the ~~metal~~ conducting sphere due to its potential V_0.
The second term is due to the induced charge density which is generated due to the electric field at $r=a$ (because of the $p\cos\theta$ at $r=b$)

(Note the sign of induced surface charge density at $r=a$)

3

② Azimuthal symmetry.

~~$r \leq R$~~

$r \leq R$, $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ —— ①

$r \geq R$ $V(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l \cos\theta$. —— ②

List all the boundary conditions, as usual, additional

at $r = R$, $V(\theta) = k\cos(3\theta)$

expanding $\cos(3\theta)$

$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

Recall $P_1(\cos\theta) = \cos\theta$

$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

$\Rightarrow \cos(3\theta) = \frac{1}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$

$\therefore V(\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$

$\Rightarrow$ only non zero coefficient: $l = 1 \,\&\, 3$.

apply all the boundary conditions on ① & ② and solving

$$* \ \sigma = -\epsilon_0\left[\frac{\partial V_{r\geq R}}{\partial r} - \frac{\partial V_{r\leq R}}{\partial r}\right]_{r=R}$$

$A_1 = -\frac{3k}{5R}$, $A_3 = 8k/5R^3$

$B_1 = -\frac{3kR^2}{5}$ $B_3 = \frac{8kR^4}{5}$

$$\therefore \boxed{r \leq R, \ V(r,\theta) = \frac{4kr^3}{5R^3}(5\cos^3\theta - 3\cos\theta) - \frac{3kr}{5R}\cos\theta}$$

$$\boxed{r \geq R, \ V(r,\theta) = \frac{4kR^4}{5r^4}(5\cos^3\theta - 3\cos\theta) - \frac{3kR^2}{5r^2}\cos\theta}$$

and $\boxed{\sigma(\text{at } r=R) = \frac{\epsilon_0 k}{5R}\left[140\cos^3\theta - 93\cos\theta\right]}$ *

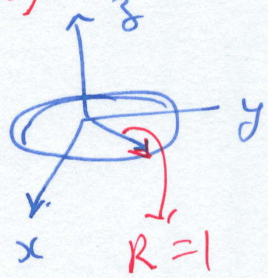
(4)

(3) , then $\theta = \frac{\pi}{2}$, only r and φ dependence.

(Laplace's eq.: $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial V}{\partial \varphi} = 0$

$(\theta = \pi/2)$

here $r \leq R$.



$\therefore V(r,\varphi) = \sum r^n \left(A_n e^{in\varphi} + B_n e^{-in\varphi}\right)$

~~$(r = R\sin\theta)$, $\theta = \pi/2$~~

at $r = R\sin\theta$, $\theta = \pi/2 \Rightarrow$

$V(R,\theta) = 2\cos(5\varphi)$ ——— (1)

$\Rightarrow A_n = B_n$, $n = 5$, for $n \neq 5$, all the coeff. are zero.

$\therefore V(r,\varphi) = r^5 A_5 \cos 5\varphi$

at $V(r = R\sin\theta, \theta = \pi/2, \varphi) = 2\cos 5\varphi = R^5 A_5 \cos 5\varphi$

$\Rightarrow A_5 = \frac{2}{R^5}$

$\therefore V(r,\varphi) = \frac{2}{R^5} r^5 \cos 5\varphi$

$$\boxed{V(r,\varphi) = 2r^5 \cos 5\varphi \quad (\because R = 1)}$$

Alternatively start with

$V(r,\varphi) = \sum r^n (A_n \cos n\varphi + B_n \sin n\varphi)$

for eq. (1), $n = 5$, $B_n = 0$

Remaining calculations are same as above.

(4) $\vec{E} = y\hat{i} + x\hat{j}$ (5)

$$V = -\int \vec{E}\cdot d\vec{l} + C, \quad C \to 0$$

$$= -\int (y\hat{i} + x\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \cancel{-2xdy} -2xy$$

as the surface is spherical, ∴ converting into polar coordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi$$

$$\therefore V = -2r^2\sin^2\theta\sin 2\varphi$$

$$= -2r^2\sin^2\theta\left(e^{2i\varphi} - e^{-2i\varphi}\right) \quad \text{--- (1) --- (1)}$$

~~in the neighbourhood of the sphere~~, general form of the pot.

$$V = \sum\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm} \quad \text{--- (2) (2)}$$

at $r = R$, $V = 0$

$$\therefore \cancel{A_{lm}} = B_{lm} = -A_{lm}R^{2l+1} \quad \text{--- (3) --- (3)}$$

for $r \to \infty$, $\sum(A_{lm}r^l)Y_{lm}(\theta,\varphi) = -2r^2\sin^2\theta\left(e^{i2\varphi} - e^{-i2\varphi}\right)$ --- (4) (4)

for (1), $m = 2, \& -2$, recalling

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\varphi} \quad \text{--- (4)}$$

$$Y_{2,-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\varphi} \quad \text{--- (5) (5)}$$

(5)

$\Rightarrow l = 2$, $\cancel{B_l}$ ∴ $B_l = 0$, $A_l = 0$ for $l \neq 2$

~~On comparing (1) & (2) & (3) eq~~ from equations (1) to (5)

$$\boxed{V(r,\theta) = -\left(r^2 - \frac{R^5}{r^3}\right)\sin^2\theta\sin\varphi\cos\varphi}$$

surface charge density $\sigma(\theta,\varphi) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$

$$\boxed{\sigma(\theta,\varphi) = 5\epsilon_0 R\sin^2\theta\sin\varphi\cos\varphi}$$