Syllabus

PH402: Mathematical Physics II (2-1-0-6)

Tensors, inner and outer products, contraction, symmetric and antisymmetric tensors, metric tensor, covariant and contravariant derivatives;Complex Analysis: Functions, derivatives, Cauchy-Riemann conditions, analytic and harmonic functions, contour integrals, Cauchy-Goursat Theorem Cauchy integral formula; Series: convergence, Taylor series, Laurent series, singularities, residue theorem, applications of residue theorem, conformal mapping and application;Group Theory: Groups, subgroups, conjugacy classes, cosets, invariant subgroups, factor groups, kernels, continuous groups, Lie groups, generators, SO(2) and SO(3),SU(2), crystallographic point groups.

Texts:

1. J. Brown and R.V.Churchill, *Complex Variables and Applications*, McGraw-Hill, 8th Edition (2008)
2. A.W.Joshi, *Elements of Group Theory*, New Age Int. (2008)
3. A.W.Joshi, *Matrices and Tensors in Physics*, 3rd Edition, New Age Int. (2005)

References:

1. M.L.Boas, *Mathematical Methods in Physical Sciences*, John Wiley & Sons (2005)
2. G.B.Arfken, H.J.Weber and F.E. Harris, *Mathematical Methods for Physicists*, Seventh Edition, Academic Press (2012)
3. M. Hamermesh, *Group Theory and Its Applications to Physical Problems*, Dover (1989)

1. Continuous Groups:

Examples:

1. $C_4 = \{e, \sigma, \sigma^2, \sigma^3\}$ Finite ; $O(C_4) = 4$.

2. $I = $ (Set of integers, +) Infinite ; Discrete

   $\boxed{\text{One-one mapping with set of int}}$

3. $G = \{2^i \mid i \in I\}$ under multiplication

   Infinite ; Discrete.

4. $G = \{e^{i\theta} / \ \theta \in [0, 2\pi)\}$ under multiplication.

   $G \subset \mathbb{C}^2$, $e^{i\theta} : [0,2\pi) \in \mathbb{R} \to G$, is one-one and cont, differentiable.

5. $G = SO(2)$. $R^T R = I$ and $\det(R) = 1$.

   $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $R : [0, 2\pi) \subset \mathbb{R} \to SO(2)$.

   Mapping is one-one, continuous (Meaning) and diffentiable (Meaning). 1 parameter Group.

6. $G = \{ T_{ab} : \mathbb{R} \to \mathbb{R} \mid T_{ab}(x) = ax + b , \ a \neq 0 \}$

   $g : U \in \mathbb{R}^2 \to G$ $g(a,b) = T_{ab}$

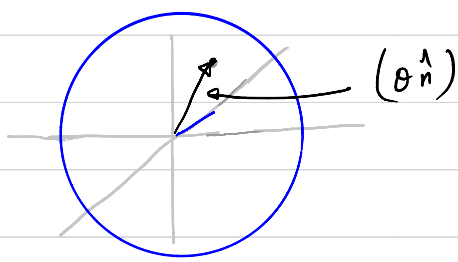
   $T_{ab} \Leftrightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ $x \Leftrightarrow \begin{pmatrix} x \\ 1 \end{pmatrix}$: Put a matrix norm on G

   g is One-one , cont, diff. 2 parameter

7. $G = SO(3)$: $R(\hat{n}, \theta)$: three parameters 2 for direction of $\hat{n}$, 1 for θ

   $R(\hat{n}, \theta) = I + \sin\theta \, S_n + (1 - \cos\theta) S_n^2$

   $S_n = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$



$U = \{ \bar{x} \in \mathbb{R}^3 / \ |\bar{x}| \leq \pi \}$ $R(\bar{x}) = R(\hat{n}, \theta)$.

Map is Cont., differentiable : 3 parameters.

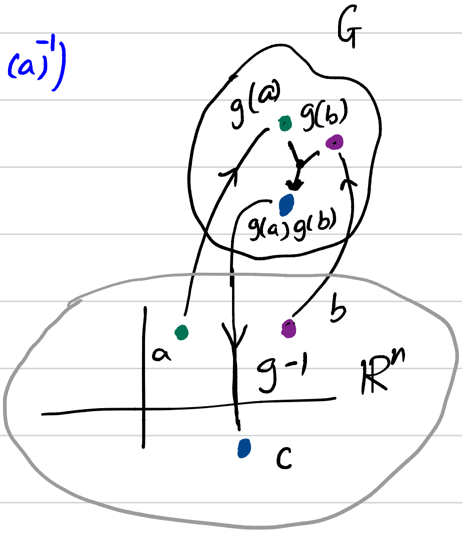
Definition: G be a group with a one-one mapping g with $U \subset \mathbb{R}^n$. Let $f : U \times U \to U$ s.t.

$$f(a,b) = g^{-1}(g(a)\cdot g(b)) \qquad \text{if } g(a)\cdot g(b) = g(c)$$
$$= c$$

and $h : U \to U$ s.t. $h(a) = g^{-1}(g(a)^{-1})$

If f and h are continuous f^n then G is called n parameter continuous group.

When f and h are differentiable then the group is called Lie group.



Alternate definition: A group G is called a Lie group if it is also a smooth differentiable manifold.

Note: In physics, almost all groups are matrix groups. The definitions given above are very abstract and are designed to encompass a larger class of groups, which are less relevant in physics. What we will do in remaining sections is to learn a class of groups called Matrix Lie Groups.

2. Matrix Lie Groups.

- $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$: Set of all $n\times n$ matrices with real/complex elements.
  - Forms a vector space with matrix addition (Isomorphic to R^{n^2})
  - Is closed under matrix multiplication.
  - Is normed (Same as norm on $\mathbb{R}^{n^2}$)
  - Notion of limits, continuity and differentiability of matrix-valued fns is defined. (This would be eventually same as element-wise limits, cont and differentiability)
- $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$: Subset of $M_n(\mathbb{R})/M_n(\mathbb{C})$ containing all invertible matrices.
  - Forms a group under matrix multiplication.

DEFINITION: Any <u>closed</u> subgroup G in $GL(n)$ is called a **Matrix Lie Group**. Closed means that every sequence in G, that converges in $GL(n)$, also converges in G.

Examples:
- $GL(n)$: A invertible
- $SL(n)$: $\det(A) = 1$
- $O(n)$: A is orthogonal
- $SO(n)$: A is orthogonal and $\det(A) = +1$.
- $U(n)$: A is unitary
- $SU(n)$: A unitary and $\det(A) = 1$.
- $Sp(n)$: Sympletic matrices $\quad AJA^T = J, \; J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$
- $E(n)$: Euclidean Groups
- Lorentz and Poincaré Groups.

DEFINITION: A Matrix Lie group G is said to be **compact** if G is closed in $M_n(R)/M_n(C)$ and is bounded.

Note:
- G by def is closed in $GL(n)$, Now it is closed in M_n.
- **Bounded**: $\exists\, C$, a constant s.t.

$$|A_{ij}| < C \quad \forall\, i,j$$

Examples: $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, are compact
$GL(n)$, $SL(n)$, $E(n)$, Lorentz group are not compact.

DEFINITION: A matrix Lie group G is connected if $\forall A, B \in G$ $\exists$ cont path $A(t)$, $a \le t \le b$, s.t. $A(a) = A$ and $A(b) = B$.

Note: A group that is not connected can be decomposed into several connected pieces called components.

- For connected group, if every closed path can be continuously shrunk to a point, then it is called simply connected.

Examples:
- $SO(n)$, $U(n)$, $SU(n)$ Connected
- Not connected

$O(n)$	2 components	
$SO(n;1)$	Lorentz group	2 components
$O(n;1)$	– " –	4 components

3 Exponential Maps and Lie Algebras

- Matrix Exponential ($M_n \to M_n$)

$$e^A = \sum_{n=0}^{\infty} \frac{A^m}{m!} \qquad A^0 = I$$

This is a continuous fn. of A.

- Properties:

1. $e^0 = I$ $\quad 0$: null matrix
2. $(e^A)^* = e^{A^*}$
3. $(e^A)^{-1} = e^{-A}$, $\quad e^A$: Always invertible.
4. $e^{(\alpha+\beta)A} = e^{\alpha A} \cdot e^{\alpha B}$
5. If $[A,B] = 0 \Rightarrow e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$
6. If C is invertible then $e^{CAC^{-1}} = Ce^AC^{-1}$
7. $\|e^A\| \le e^{\|A\|}$.
8. Lie Product Formula $\quad e^{A+B} = \lim_{n\to\infty} \left(e^{A/n} e^{B/n}\right)^n$
9. $\det(e^A) = e^{tr(A)}$
10. $\operatorname{Log} A = \sum_1^\infty (-1)^{m+1} (A-I)^m/m \qquad \|A-I\| < 1$

If $\|A-I\| < 1$ then $e^{\operatorname{Log} A} = A$. $\quad |\quad$ $\|A-I\|<1$ and $\|A\| < \log 2$ $\quad \log e^A = A$.

<u>DEFINITION</u>: Let G be a matrix Lie group. The Lie algebra of G, denoted by $\mathfrak{g}$, is

$$\mathfrak{g} = \{X \in M_n \mid e^{tX} \in G \;\; \forall\, t \in \mathbb{R}\}$$

<u>Examples</u>: (a) $gl(n,\mathbb{R}) = M_n(\mathbb{R})$ $\quad$ LA of $GL(n,\mathbb{R})$

(b) $SL(n,\mathbb{R})$: $\det(A) = 1$ $\qquad \det\{e^{tX}\} = e^{t \cdot trX} = 1$

$\Rightarrow sl(n,\mathbb{R})$: traceless matrices. $\qquad \Rightarrow t \cdot tr(X) = 0 \Rightarrow tr(X) = 0$

(c) $SU(n)$, $U(n)$ $\qquad (e^A)^{-1} = e^{A^\dagger} = e^{-A}$

$u(n)$: antihermitian matrices $\qquad \Rightarrow A^\dagger = -A$

$su(n)$: traceless antihermitian.

(d) $SO(n)$, $O(n)$

$o(n)$: antisymmetric matrices, traceless

$so(n)$: traceless antisymmetric matrices

Theorem: Let G be a matrix Lie group with Lie algebra g. If $X, Y \in g$ then

(i) $AXA^{-1} \in g \quad \forall A \in G$ $\qquad e^{tAXA^{-1}} = Ae^{tX}A^{-1} \in G$

(ii) $sX \in g \quad \forall s \in \mathbb{R}$

(iii) $X+Y \in g$

(iv) $XY - YX \in g$

Proof: (iii) $e^{(X+Y)t} = \lim_{m\to\infty} (\underbrace{e^{tX/n} e^{tY/n}}_{\in G})^n$

Thus $(e^{tx/n} e^{ty/n})^n \in G$ and G is closed

Note: Lie algebra is a vector space with a bilinear product $[X, Y] = XY - YX$. Here $[x, Y]$ is

(i) bilinear

(ii) Skew symmetric

(iii) Jacobi Identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Def: If $A : \mathbb{R} \to G$ is a continuous map s.t.

(i) $A(0) = I$

(2) $A(t) \cdot A(s) = A(t+s) \quad \forall t, s \in \mathbb{R}$

Then $\{A(t) / t \in \mathbb{R}\}$ is said to be one parameter subgroup.

Thm: $\forall X \in g$, $\{e^{tX} \mid t \in \mathbb{R}\}$ is a one-parameter subgroup of G.

4. Examples

4.1. O(2) and SO(2)

- SO(2) is compact, connected but not simply connected
  O(2) is compact, two connected components, one of which is SO(2).

- Parametrization

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \pm\sin\theta & \pm\cos\theta \end{pmatrix} \qquad \begin{array}{l} + : \det = +1 \\ - : \det = -1 \end{array}$$

- Lie Algebra : $(e^X)^T = (e^X)^{-1} \Rightarrow X^T = -X \Rightarrow$ antisymmetric

$$\det(e^X) = 1 \Rightarrow tr(X) = 0$$

$$so(2) = \left\{ \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} \Big/ t \in \mathbb{R} \right\}$$

one dimensional vector space with

$$B = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

Infinitesimal Generators of G.

4.2 SO(3)

- Parametrization

$$g = I + \sin\theta \cdot S_a + (1-\cos\theta) S_a^2 \; ; \quad S_a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

$$= \exp(\theta S_a)$$

$a_1^2 + a_2^2 + a_3^2 = 1$

$0 \leq \theta \leq \pi$

- Compact, connected (doubly connected)
  Two points can be connected in two different ways.

- Lie Algebra
  so(3) = all traceless antisymmetric matrices

generators : $\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right.$

4.3 SU(2)

- Parametrization

$$X = \begin{pmatrix} e^{i\xi}\cos\theta & -e^{+i\eta}\sin\theta \\ e^{-i\eta}\sin\theta & e^{-i\xi}\cos\theta \end{pmatrix} \qquad \begin{array}{l} 0 < \theta \leq \pi/2 \\ 0 \leq \xi, \eta \leq 2\pi \end{array}$$

- Compact and connected.

- Lie algebra is given by traceless antihermitian matrices

with basis given by

$$\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$J_k = i\sigma_k \Rightarrow [J_i, J_j] = i\epsilon_{ijk} J_k.$$

Physicist convention.

4.4. Lorentz Group: SO(1,3)

⇒ Matrices that keep

$$X^T g X = g \qquad g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- Contains $O(3)$ $\quad \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & A & \\ 0 & & & \end{array}\right), \quad A^T A = I$

- Contains boosts $\quad \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad$ std textbook Lorentz transformation.

⇒ Not compact, 4 connected components each containing

I, T, P and TP $\qquad T = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

⇒ Component containing I, $SO^+(1,3)$

⇒ Lie Algebra ⇒ $g L^{-1} g = L^T$, if $L = e^X$ then

$$-gXg = X^T$$

Then

$$X = \begin{pmatrix} 0 & a & b & c \\ a & 0 & -d & e \\ b & d & 0 & -f \\ c & -e & f & 0 \end{pmatrix} \quad \text{six dimensional.}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ etc.} \qquad k_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix} \text{ etc.}$$

- Commutation

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

$$[K_i, K_j] = -\epsilon_{ijk} J_k$$

$$[J_i, K_j] = \epsilon_{ijk} K_k.$$