Tutorial 2: Contour Integrals

1. For each of the following smooth curves give an admissible parametrization that is consistent with the indicated direction.
   (a) The line segment from $z = 1 + i$ to $z = -1 - 3i$.
   (b) the circle $|z - 2i| = 4$ traversed once in the clockwise direction starting from the point $z = -2i$.
   (c) the segment of the parabola $y = x^2$. from point $(1, 1)$ to $(3, 9)$.

2. Show that if m and n are integers,
$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when} \quad m \neq n \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

3. A semicircular contour is given by two separate parametrization
$$\begin{aligned} z_1(t) &= 2e^{it} \qquad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right) \\ z_2(\tau) &= \sqrt{4 - \tau^2} + i\tau \qquad (-2 \leq \tau \leq 2). \end{aligned}$$
   (a) Find the length of the contour using each parametrization.
   (b) Find a function $t = \phi(\tau)$ such that $z_2(\tau) = z_1(\phi(\tau))$.

4. Let C be the perimeter of a square with vertices at $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order. Compute following integrals using primary definition:
   (a) $\int_C e^z dz$;
   (b) $\int_C \bar{z}^2 dz$.

5. Practice line integrals in real plane.
   (a) Evaluate $\int_{(0,1)}^{(2,5)} ((3x + y)\, dx + (2y - x)\, dy)$ along (a) the curve $y = x^2 + 1$, (b) the straight line joining the two limit points.
   (b) Evaluate $\oint ((x + 2y)\, dx + (y - 2x)\, dy)$ around the ellipse C defined by $x = 4\cos\theta$ and $y = 3\sin\theta$, $0 \leq \theta < 2\pi$.

6. Evaluate $\int_C (x - 2xyi)\, dz$ over the contour $C : z = t + it^2, 0 \leq t \leq 1$.

7. Evaluate $\int_C \overline{z}^2\, dz$ around the circles (a) $|z| = 1$, (b) $|z - 1| = 1$.

8. Verify Green's Theorem in the plane for $\int_C (x^2 - 2xy)\, dx + (y^2 - x^3 y)\, dy$ where C is a square with vertices at $(0, 0), (2, 0), (2, 2)$ and $(0, 2)$.

9. Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is the ellipse $|z - 3i| + |z + 3i| = 20$.

10. Show that
    (a) $\left|\int_C \frac{dz}{z^2 - 1}\right| \leq \frac{3\pi}{4}$ if $C : |z| = 3$.
    (b) $\left|\int_C \frac{e^{3z}}{1 + e^z} dz\right| \leq \frac{2\pi e^{3R}}{e^R - 1}$ if C is a vertical line segment from $z = R\,(> 0)$ to $z = R + 2\pi i$.

11. Use antiderivatives to evaluate following integrals:
    (a) $\int_i^{i/2} e^{\pi z} dz$;

(b) $\int_0^{\pi+2i} \cos(z/2)\, dz$

(c) $\int_1^3 (z-2)^3\, dz$.

12. Use antiderivatives to show that

$$\int \frac{dz}{z^2-a^2} = \frac{1}{2a} \log\left(\frac{z-a}{z+a}\right) + c_1 = \frac{1}{a} \coth^{-1}\left(\frac{z}{a}\right) + c_2$$

13. Let C be a square with vertices on $z = \pm 2 \pm 2i$. Evaluate following integrals using Cauchy integral formula to evaluate following integrals:

(a) $\int_C \frac{e^{-z}dz}{z-(\pi i/2)}$;

(b) $\int_C \frac{\cos z}{z(z^2+8)} dz$;

(c) $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz \qquad (-2 < x_0 < 2)$.

14. Show that $\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2+1} dz = \sin t$ if $t(>0)$ is a real constant and $C : |z| = 3$.

15. Prove Cauchy's Inequality which states that if $f(z)$ is analytic on and inside of a circle of radius R and center a, then

$$\left|f^{(n)}(a)\right| \leq \frac{M \cdot n!}{R^n}$$

where M is a maximum of $|f(z)|$ on the circle.