Q1. I. Continuous-time Fourier series II. Discrete-time Fourier series III. Continuous-time Fourier transform IV. Discrete-time Fourier transform

CTFS, CTFT

CTFT

CTFT, CTFS*

DTFS, DTFT

DTFT

DTFT, DTFS*

*Because these two signals are aperiodic, we know that they do not possess a Fourier series. However, since they are both finite duration, the Fourier series can be used to express a periodic signal that is formed by periodically replicating the finite-duration signal.

Q2 (a)

Because of the discrete nature of a discrete-time signal, the time/frequency scaling property does not hold. A result that closely parallels this property but does hold

for discrete-time signals can be developed. Define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{otherwise} \end{cases}$$

$x_{(k)}[n]$ is a "slowed-down" version of $x[n]$ with zeros interspersed. By analysis in the frequency domain,

$$X_{(k)}(\Omega) = X(k\Omega),$$

which indicates that $X_{(k)}(\Omega)$ is compressed in the frequency domain.

Q2(b)

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \\ &= \sum_{m=-\infty}^{\infty} \alpha^{n-m}\, u\,[n-m]\beta^m u[m] \\ &= \sum_{m=0}^{n} \alpha^{n-m}\beta^m, \qquad n > 0, \end{aligned}$$

$$\begin{aligned} y[n] &= \alpha^n \sum_{m=0}^{n} \left(\frac{\beta}{\alpha}\right)^m = \alpha^n \left[\frac{1-(\beta/\alpha)^{n+1}}{1-(\beta/\alpha)}\right] \\ &= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, \qquad n \geq 0, \\ y[n] &= 0, \qquad n < 0 \end{aligned}$$

Q3

We are given an LTI system with impulse response

$$h[n] = \frac{\sin(\pi n/3)}{\pi n}$$

(a) We know from duality that $H(\Omega)$ is a pulse sequence that is periodic with period 2π. Suppose we assume this and adjust the parameters of the pulse so that

$$\frac{1}{2\pi}\int H(\Omega)e^{j\Omega n}\,d\Omega = h[n]$$

Let a be the pulse amplitude and let $2W$ be the pulse width. Then

$$\frac{a}{2\pi}\int_{-W}^{W} e^{j\Omega n}\,d\Omega = \frac{a}{2\pi}\left(\frac{e^{j\Omega W} - e^{-j\Omega W}}{jn}\right)$$

$$= \frac{a}{2\pi}\,\frac{2\sin Wn}{n},$$

so $a = 1$ and $W = \pi/3$, as indicated in Figure S11.9-1.

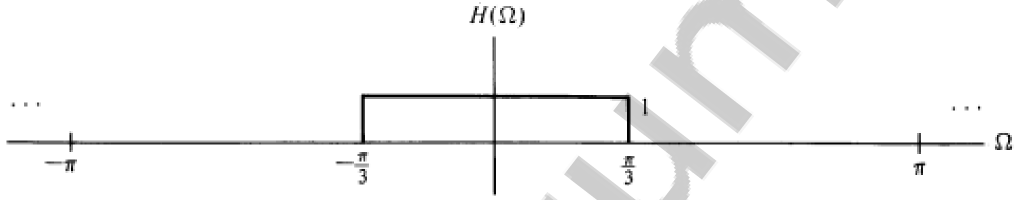

Figure S11.9-1

(b) We know that

$$\cos\frac{3\pi}{4}n \overset{\mathcal{F}}{\longleftrightarrow} \pi\left[\delta\left(\Omega - \frac{3\pi}{4}\right) + \delta\left(\Omega + \frac{3\pi}{4}\right)\right],$$

periodically repeated, and that multiplication by $(-1)^n$ shifts the periodic spectrum by π, so the spectrum $Y(\Omega)$ is as shown in Figure S11.9-2.

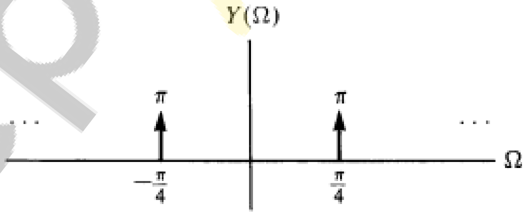

Figure S11.9-2

From Figures S11.9-1 and S11.9-2, we can see that

$$Y(\Omega) = H(\Omega)X(\Omega) = X(\Omega)$$

Therefore,

$$y[n] = x[n] = (-1)^n \cos\frac{3\pi}{4}n = \cos\frac{\pi n}{4}$$

Q4.

$$Y(\Omega) = 2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega}$$

(a) (i) The system is linear because if

$$x[n] = ax_1[n] + bx_2[n],$$

then

$$y[n] = ay_1[n] + by_2[n],$$

where $y_1[n]$ is obtained from $x_1[n]$ via the given transfer function. The similar result applies for $y_2[n]$.

(ii) The system is time-varying by the following argument.
If $x[n] \to y[n]$, does $x[n-1] \to y[n-1]$?

$$x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega}X(\Omega)$$

The corresponding $Y(\Omega)$ is

$$2e^{j\Omega}X(\Omega) + e^{-j\Omega}X(\Omega)e^{-j\Omega} + je^{-j\Omega}X(\Omega) - e^{-j\Omega}\frac{dX(\Omega)}{d\Omega}$$

$$\neq e^{-j\Omega}\left[2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega}\right]$$

(iii) If $x[n] = \delta[n]$, $X(\Omega) = 1$. Then

$$Y(\Omega) = 2 + e^{-j\Omega},$$
$$y[n] = 2\delta[n] + \delta[n-1]$$

Q5.

(a) $\tilde{x}_1[n] = 1 + \sin\left(\frac{2\pi n}{10}\right)$

To find the period of $\tilde{x}_1[n]$, we set $\tilde{x}_1[n] = \tilde{x}_1[n + N]$ and determine N. Thus

$$1 + \sin\left(\frac{2\pi n}{10}\right) = 1 + \sin\left[\frac{2\pi}{10}(n + N)\right]$$
$$= 1 + \sin\left(\frac{2\pi}{10}n + \frac{2\pi}{10}N\right)$$

Since

$$\sin\left(\frac{2\pi}{10}n + 2\pi\right) = \sin\left(\frac{2\pi}{10}n\right),$$

the period of $\tilde{x}_1[n]$ is 10. Similarly, setting $\tilde{x}_2[n] = \tilde{x}_2[n + N]$, we have

$$1 + \sin\left(\frac{20\pi}{12}n + \frac{\pi}{2}\right) = 1 + \sin\left[\frac{20\pi}{12}\left(n + N\right) + \frac{\pi}{2}\right]$$
$$= 1 + \sin\left(\frac{20\pi}{12}n + \frac{\pi}{2} + \frac{20\pi}{12}N\right)$$

Hence, for $\frac{20}{12}\pi N$ to be an integer multiple of 2π, N must be 6.

(b) $\tilde{x}_1[n] = 1 + \sin\left(\frac{2\pi n}{10}\right)$

Using Euler's relation, we have

$$x_1[n] = 1 + \frac{1}{2j}e^{j(2\pi/10)n} - \frac{1}{2j}e^{-j(2\pi/10)n} \qquad \text{(S10.2-1)}$$

Note that the Fourier synthesis equation is given by

$$\tilde{x}_1[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

where $N = 10$. Hence, by inspection of eq. (S10.2-1), we see that

$$a_0 = 1, \qquad a_{1-1} = \frac{-1}{2j},$$
$$a_{11} = \frac{1}{2j}, \qquad \text{and}$$
$$a_{1k} = 0, \qquad 2 \le k \le 8,$$
$$-8 \le k \le -2$$

Similarly,

$$\tilde{x}_2[n] = 1 + \frac{1}{2j}e^{j(\pi/2)}e^{j(20\pi/12)n} - \frac{1}{2j}e^{-j(\pi/2)}e^{-j(20\pi/12)n}$$

Therefore, $N = 12$.

$$a_{20} = 1, \quad a_{2-1} = -\frac{e^{-j(\pi/2)}}{2j} = \frac{1}{2}, \quad a_{21} = \frac{1}{2j}e^{j(\pi/2)} = \frac{1}{2}, \quad \text{and}$$
$$a_{2\pm2}, \ldots, a_{2\pm11} = 0$$

(c) The sequence a_{1k} is periodic with period 10 and a_{2k} is periodic with period 12.

Q6.

(a)

$$\begin{aligned}\tilde{w}[n] &= \tilde{x}[n] + \tilde{y}[n],\\ \tilde{w}[n+NM] &= \tilde{x}[n+NM] + \tilde{y}[n+NM]\\ &= \tilde{x}[n] + \tilde{y}[n]\\ &= \tilde{w}[n]\end{aligned}$$

Hence, $\tilde{w}[n]$ is periodic with period NM.

(b)

$$\begin{aligned}c_k &= \frac{1}{NM}\sum_{n=0}^{NM-1} \tilde{w}[n]e^{-jk(2\pi/NM)n} = \frac{1}{NM}\sum_{n=0}^{NM-1} [\tilde{x}[n] + \tilde{y}[n]]e^{-jk(2\pi/NM)n}\\ &= \frac{1}{NM}\sum_{n=0}^{NM-1} \tilde{x}[n]e^{-jk(2\pi/NM)n} + \frac{1}{NM}\sum_{n=0}^{NM-1} \tilde{y}[n]e^{-jk(2\pi/NM)n}\\ &= \frac{1}{NM}\sum_{n=0}^{N-1} \tilde{x}[n]\sum_{l=0}^{M-1} e^{-jk(2\pi/NM)(n+lN)} + \frac{1}{NM}\sum_{n=0}^{M-1} \tilde{y}[n]\sum_{l=0}^{N-1} e^{jk(2\pi/NM)(n+lM)}\\ &= \begin{cases}\frac{1}{N}a_{k/M} + \frac{1}{M}b_{k/N}, & \text{for } k \text{ a multiple of } M \text{ and } N,\\ \frac{1}{N}a_{k/M}, & \text{for } k \text{ a multiple of } M,\\ \frac{1}{M}b_{k/N}, & \text{for } k \text{ a multiple of } N,\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

Q7.

The signal $x(t) = \cos(\omega_0 t + \theta)$, where $\omega_0 = 2\pi f_0$, can be written as

$$x(t) = \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

and the spectrum of $x(t)$ is given by

$$X(\omega) = \pi e^{j\theta}\delta(\omega - \omega_0) + \pi e^{-j\theta}\delta(\omega + \omega_0)$$

The spectrum of $p(t)$ is given by

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore, the spectrum of $x_p(t)$ is

$$X_p(\omega) = \frac{1}{2\pi}\left(\frac{2\pi^2}{T}\right)\left[\sum_{k=-\infty}^{\infty} e^{j\theta}\delta\left(\omega - \frac{2\pi k}{T} - \omega_0\right) + e^{-j\theta}\delta\left(\omega - \frac{2\pi k}{T} + \omega_0\right)\right]$$

and the spectrum of $X_r(\omega)$ is given by

$$X_r(\omega) = H(\omega)X_p(\omega)$$

(a) $\omega_0 = 2\pi \times 250, \quad \theta = \frac{\pi}{4}, \quad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 250)$$
$$+ e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 250)]$$

Hence, only the $k = 0$ term is passed by the filter:

$$X_r(\omega) = \pi[e^{j\theta}\delta(\omega - 2\pi \times 250) + e^{-j\theta}\delta(\omega + 2\pi \times 250)]$$

and

$$\begin{aligned} x_r(t) &= \frac{1}{2}e^{j\theta}e^{j2\pi \times 250t} + \frac{1}{2}e^{-j\theta}e^{-j2\pi \times 250t} \\ &= \cos(2\pi \times 250t + \theta) \\ &= \cos\left(2\pi \times 250t + \frac{\pi}{4}\right) \end{aligned}$$

(b) $\omega_0 = 2\pi \times 750$ Hz, $T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T} \sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 750) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 750)]$$

Only the $k = \pm 1$ term has nonzero contribution:

$$X_r(\omega) = \frac{\pi}{T}[e^{j\theta}\delta(\omega + 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 250)]$$

Hence,

$$x_r(t) = \cos(2\pi \times 250t - \theta) = \cos\left(2\pi \times 250t - \frac{\pi}{2}\right)$$

(c) $\omega_0 = 2\pi \times 500$, $\theta = \frac{\pi}{2}$, $T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T} \sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 500) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 500)]$$

Since $H(\omega) = 0$ at $\omega = 2\pi \times 500$, the output is zero: $x_r(t) = 0$.

Q8.

Let $\omega = 2\pi f$. Then $d\omega = 2\pi df$, and

$$x(t) = \frac{1}{2\pi} \int_{f=-\infty}^{\infty} X(2\pi f) e^{j2\pi ft} 2\pi \, df = \int_{f=-\infty}^{\infty} X_a(f) e^{j2\pi ft} \, df$$

Thus, there is no factor of 2π in the inverse relation.

(b) Comparing

$$X_b(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-jvt} \, dt \qquad \text{and} \qquad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt,$$

we see that

$$X_b(v) = \frac{1}{\sqrt{2\pi}} X(\omega)\Big|_{\omega = v} \qquad \text{or} \qquad X(\omega) = \sqrt{2\pi}\, X_b(\omega)$$

The inverse transform relation for $X(\omega)$ is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{2\pi}\, X_b(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_b(v) e^{jvt} \, dv,$$

where we have substituted v for ω. Thus, the factor of $1/2\pi$ has been distributed among the forward and inverse transforms.