CYK/2023/PH201 Mathematical Physics

Mid-Semester Exam

Total Marks: 30; **Duration:** 2 Hours (9AM-11AM); **Date:** 18 Sept 2023, Monday

1. [6×2 **Marks**] Answer the following questions: (Write steps and reasons in **brief**. Writing just final answers will not be awarded marks.)

   (a) Find and sketch the image of the straight line $y = x + 1$ under the transformation $w = \frac{1}{z}$.

   (b) The branch cut of $\ln z$ is chosen along the radial line making an angle of 120 degrees with the positive x axis. If $\ln 1 = 0$, then what are the values of $\ln(i)$, $\ln(1)$ and $\ln(-1)$?

   (c) Show that if a function $f(z) = u + iv$ is analytic at z, level curves of u and v passing through z are orthogonal.

   (d) At which points the function $f(z) = \bar{z}^2$, analytic?

   (e) If $C : |z| = R$ is a positively oriented circular contour, compute

   $$\oint_C \frac{e^z}{\left(z - \frac{\pi i}{4} a\right)^2} dz \qquad (R > a)\,.$$

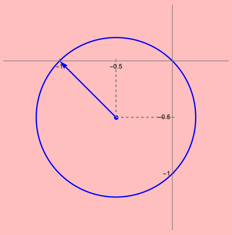
   (f) Discuss and classify the singularities of $\frac{1}{\sin(\pi/z)}$.

Answers:

(a) Since $z = 1/w$ implies $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$. Substituting in $y = x + 1$ gives

$$\left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$$

which is a circle with center at $-(1+i)/2$ and passing through origin.



(b) In this case, $-4\pi/3 < \theta < 2\pi/3$. And hence, $\ln i = i\pi/2$, $\ln(-1) = -\pi i$ and $\ln(-i) = -i\pi/2$. (Typing error in QP, marks awarded as per QP)

(c) The angle between two curves is given by the angles between the normals to the curves. The normal to $u = c$ is $\nabla u = (u_x, u_y)$ and similarly, the normal to $v = c'$ is $\nabla v = (v_x, v_y)$. If the angle between the curves is ϕ then

$$\cos\phi = \frac{\nabla u \cdot \nabla v}{|\nabla u|\,|\nabla v|} = \frac{1}{|\nabla u|\,|\nabla v|}\left(u_x v_x + u_y v_y\right) = 0$$

The last step follows from CR conditions, $v_y = u_x$ and $v_x = -u_y$.

(d) Here $u = x^2 - y^2$ and $v = -2xy$. The CR condition are valid only at $x = y = 0$ and the function is differentiable at $z = 0$. But, it is not analytic since it is not differentiable in any deleted-nbd of $z = 0$.

(e) Clearly, $R > a$ implies $R > a\pi/4$. Using Cauchy integral formula, we get

$$\oint_C \frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2} dz = \frac{2\pi i}{1!}\,\frac{d}{dz}e^z\bigg|_{z=i\pi a/4} = 2\pi i e^{i\pi a/4}.$$

(f) The denominator has a simple zero at $z = 1/n$, where n is any integer. Thus, the given function has isolated simple poles at $z = 1/n$. The function also has a singular point at $z = 0$ but this singularity is not isolated since every nbd of $z = 0$ contains infinitely many singularities of the function.

2. [2×3 **Marks**] Answer the following questions:

(a) [3] The Euler numbers E_n are defined by the power series $\dfrac{1}{\cosh z} = \displaystyle\sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$. What is the radius of convergence for this series? Compute E_0 to E_4.

(b) [3] Given the function $f(z) = \dfrac{z}{(z-2)(z+i)}$, expand the function in a series about $z_0 = 0$, in the regions (i) $|z| < 1$ (ii) $1 < |z| < 2$ and (iii) $|z| > 2$.

Answers:

(a) From the definition, it is clear that $E_n = \frac{d^n}{dz^n}\text{sech}(0)$. The table of derivatives

n	$\frac{d^n}{dz^n}\text{sech}(z)$	E_n
0	$\text{sech}(z)$	1
1	$\tanh(z)(-\text{sech}(z))$	0
2	$\tanh^2(z)\text{sech}(z) - \text{sech}^3(z)$	-1
3	$5\tanh(z)\text{sech}^3(z) - \tanh^3(z)\text{sech}(z)$	0
4	$5\text{sech}^5(z) - 18\tanh^2(z)\text{sech}^3(z) + \tanh^4(z)\text{sech}(z)$	5

(b) Note that

$$\frac{z}{(z-2)(z+i)} = \frac{1}{2+i}\left[\frac{2}{(z-2)} + \frac{i}{(z+i)}\right].$$

i. Given $|z| < 1$ implies that $|z/2| < 1$ and $|z/i| < 1$. Thus,

$$\begin{aligned}\frac{1}{2+i}\left[\frac{2}{(z-2)} + \frac{i}{(z+i)}\right] &= \frac{1}{2+i}\left[\frac{-1}{(1-z/2)} + \frac{1}{(1+z/i)}\right] \\ &= \frac{1}{2+i}\sum_{n=0}^{\infty}\left[-\frac{1}{2^n} + i^n\right] z^n\end{aligned}$$

ii. Given $1 < |z| < 2$ implies that $|z/2| < 1$ and $|z/i| > 1$. Thus,

$$\begin{aligned}\frac{1}{2+i}\left[\frac{2}{(z-2)} + \frac{i}{(z+i)}\right] &= \frac{1}{2+i}\left[\frac{-1}{(1-z/2)} + \frac{i}{z(1+i/z)}\right] \\ &= \frac{1}{2+i}\left[-\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{z}\right)^{n+1}\right]\end{aligned}$$

iii. Given $2 < |z|$ implies that $|z/2| > 1$ and $|z/i| > 1$. Thus,

$$\begin{aligned}\frac{1}{2+i}\left[\frac{2}{(z-2)} + \frac{i}{(z+i)}\right] &= \frac{1}{2+i}\left[\frac{2}{z(1-2/z)} + \frac{i}{z(1+i/z)}\right] \\ &= \frac{1}{2+i}\left[\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{z}\right)^{n+1}\right]\end{aligned}$$

3. [3×4 **Marks**] Using the method of residues, answer the following questions: (**sketch contours and show contributions from each segment of contours explicitly**)

(a) Compute

$$\text{pv}\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$$

Answer:
Let$f(z) = \dfrac{1}{z^2+2z+2}$. Let $C = C_1 + C_R$ be the contour as shown in the figure. Note that

$$\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{R\to\infty}\int_{C_1} f(z)\,dz.$$

The integrand has simple poles at $(-1 \pm i)$ of which $-1+i$ is inside the contour C. Thus,

$$\int_C f(z)\,dz = 2\pi i \text{Res} f(-1+i)$$
$$\lim_{R\to\infty}\int_{C_1} f(z)\,dz = 2\pi i \text{Res} f(-1+i) - \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Now,

$$\text{Res} f(-1+i) = \frac{1}{2i}$$

Also,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{1}{R^2}\pi R \sim \frac{1}{R} \to 0 \quad R\to\infty$$

Finally,

$$\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$$

(b) Compute

$$\int_{-\infty}^{\infty} \frac{\cos kx}{(x+b)^2+a^2}dx$$

where $a > 0,\ k > 0$ and b is a real number.

Answer:
Let$f(z) = \dfrac{e^{ikz}}{(z+b)^2+a^2}$. Let $C = C_1 + C_R$ be the contour as shown in the figure where $R^2 > a^2+b^2$. Note that

$$\int_{-\infty}^{\infty} \frac{\exp(ikx)}{(x+b)^2+a^2}dx = \lim_{R\to\infty}\int_{C_1} f(z)\,dz.$$

The integrand has simple poles at $-b \pm ia$ of which $-b+ia$ is inside the contour C. Thus,

$$\int_C f(z)\,dz = 2\pi i \text{Res} f(-b+ia)$$
$$\lim_{R\to\infty}\int_{C_1} f(z)\,dz = 2\pi i \text{Res} f(-b+ia) - \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Now,

$$\text{Res} f(-b+ia) = \frac{\exp(ik(-b+ia))}{-b+ia-(-b-ia)} = e^{-ka}\frac{e^{-ikb}}{2ia}$$

Also, by Jordan lemma,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

since $\left|\dfrac{1}{(z+b)^2+a^2}\right| \sim \dfrac{1}{R^2} \to 0$ as $R \to \infty$. Finally,

$$\int_{-\infty}^{\infty} \frac{\exp(ikx)}{(x+b)^2+a^2}dx = \frac{\pi}{a}e^{-ka}e^{-ikb}$$

and the required integral is

$$\frac{\pi}{a}e^{-ka}\cos kb$$

(c) Compute

$$\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{a^2-2a\cos\theta+1}; \qquad -1 < a < 1.$$

Answer:
The integrand even function of theta hence the required integral is 1/2 of

$$\begin{aligned}\frac{1}{a}\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{\left(1+a^2\right)/a-2\cos\theta} &= \frac{1}{a}\int_0^{2\pi}\frac{\left(2\cos^2\theta-1\right)}{\left(1+a^2\right)/a-2\cos\theta}d\theta \\ &= \frac{i}{2a}\oint_{|z|=1}\frac{\left(\left(z^2+1\right)^2-2z^2\right)}{z^2-\frac{1}{a}\left(1+a^2\right)z+1}\frac{dz}{z^2}.\end{aligned}$$

The integrand has simple poles at a and $1/a$ and a pole of order 2 at 0. Thus

$$I=-\frac{\pi}{2az^2}\left[\frac{1+a^2}{a}+\frac{a^4+1}{a\left(a^2-1\right)}\right]=\frac{\pi a^2}{1-a^2}$$