Tutorial 2: Contour Integrals

1. For each of the following smooth curves give an admissible parametrization that is consistent with the indicated direction.

   (a) The line segment from $z = 1 + i$ to $z = -1 - 3i$.

   (b) the circle $|z - 2i| = 4$ traversed once in the clockwise direction starting from the point $z = -2i$.

   (c) the segment of the parabola $y = x^2$. from point $(1, 1)$ to $(3, 9)$.

Answer:

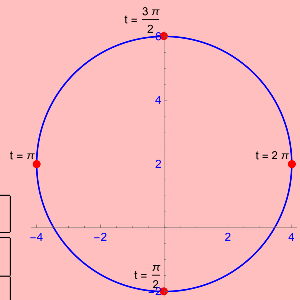
(a) Let $z_1 = 1 + i$ and $z_2 = -1 - 3i$. Then the line segment from z_1 to z_2 is given by

$$z(t) = z_1 + (z_2 - z_1)\,t, \qquad t : 0 \to 1$$

(b) It is given that the center is at $z_0 = 2i$ and the radius $R = 4$. The parametrization is

$$\begin{aligned} z(t) &= z_0 + Re^{-it}, \qquad t : \pi/2 \to 5\pi/2 \\ &= 2i + 4e^{-it}. \end{aligned}$$

The negative sign in the exponential ensures that the orientation is clockwise. Just to check:



t	$z(t)$
$\pi/2$	$2i + 4e^{-i\pi/2} = 2i + 4(-i) = -2i$
π	$2i + 4e^{-i\pi} = 2i + 4(-1) = 2i - 4$
$3\pi/2$	$2i + 4e^{-i3\pi/2} = 2i + 4(i) = 6i$
2π	$2i + 4e^{-i\pi} = 2i + 4(1) = 2i + 4$

2. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when} \quad m \neq n \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

Answer:
If $m = n$, then the integrand is 1 and hence the integral is 2π. When $m \neq n$ then

$$\begin{aligned} \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta &= \int_0^{2\pi} \cos\left((m-n)\,\theta\right) d\theta + i \int_0^{2\pi} \sin\left((m-n)\,\theta\right) d\theta \\ &= 0 + i0 = 0. \end{aligned}$$

3. A semicircular contour is given by two separate parametrization

$$\begin{aligned} z_1(t) &= 2e^{it} \qquad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right) \\ z_2(\tau) &= \sqrt{4 - \tau^2} + i\tau \qquad (-2 \leq \tau \leq 2)\,. \end{aligned}$$

   (a) Find the length of the contour using each parametrization.

(b) Find a function $t = \phi(\tau)$ such that $z_2(\tau) = z_1(\phi(\tau))$.

Answer:

(a) $z_1'(t) = 2ie^{it}$. Then, $|z_1'(t)| = 2$. Thus, the length is given by

$$L = \int_{-\pi/2}^{\pi/2} \left|z_1'(t)\right| dt = 2\pi.$$

$z_2'(\tau) = \frac{-\tau}{\sqrt{4-\tau^2}} + i$. Then, $|z_2'(\tau)| = \frac{2}{\sqrt{4-\tau^2}}$ and

$$L = \int_{-2}^{2} \left|z_2'(\tau)\right| d\tau = \int_{-2}^{2} \frac{2}{\sqrt{4-\tau^2}} d\tau$$
$$= 2\sin^{-1}\frac{\tau}{2}\Big|_{-2}^{2} = 2\pi$$

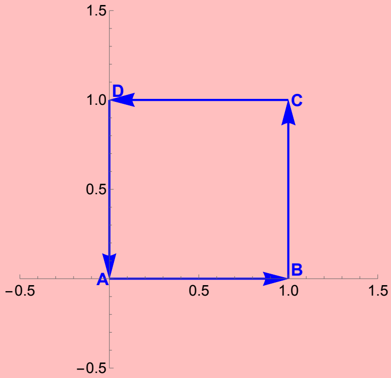
(b) Comparing imaginary part of z_1 and z_2, we get $2\sin t = \tau$. Thus, $\phi(\tau) = \sin^{-1}(\tau/2)$.

4. Let C be the perimeter of a square with vertices at $z = 0$, $z = 1$, $z = 1+i$ and $z = i$ traversed once in that order. Compute following integrals using primary definition:

(a) $\int_C e^z dz$;

(b) $\int_C \bar{z}^2 dz$.

Solution:



(a) See the same procedure as in part (b):

$$\int_C e^z dz = 0$$

(b) Let $A = 0$, $B = 1$, $C = 1 + i$ and $D = i$.

▷ Then along AB, $z(t) = t$ where $t : 0 \to 1$. Then, $z'(t) = 1$ and

$$\int_{AB} \bar{z}^2 dz = \int_0^1 t^2 dt = \frac{1}{3}$$

▷ Along BC, $z(t) = 1 + it$ where $t : 0 \to 1$. Then $z'(t) = i$ and

$$\int_{BC} \bar{z}^2 dz = \int_0^1 (1 - it)^2 \cdot i \cdot dt = 1 + \frac{2}{3}i$$

▷ Along CD, $z(t) = 1 - t + i$ where $t : 0 \to 1$. Then $z'(t) = -1$ and

$$\int_{CD} \bar{z}^2 dz = \int_0^1 (1 - t - i)^2 \cdot (-1) \cdot dt = \frac{2}{3} + i$$

▷ Along DA, $z(t) = i(1 - t)$ where $t : 0 \to 1$. Then $z'(t) = -i$ and

$$\int_{BC} \bar{z}^2 dz = \int_0^1 (-i)^2 (1 - t)^2 \cdot (-i) \cdot dt = \frac{1}{3}i$$

Thus, the net integral around the square is $2 + 2i$.

5. Practice line integrals in real plane.

(a) Evaluate $\int_{(0,1)}^{(2,5)} ((3x + y)\, dx + (2y - x)\, dy)$ along (a) the curve $y = x^2 + 1$, (b) the straight line joining the two limit points.

(b) Evaluate $\oint ((x + 2y)\, dx + (y - 2x)\, dy)$ around the ellipse C defined by $x = 4\cos\theta$ and $y = 3\sin\theta$, $0 \le \theta < 2\pi$.

Answer:

(a) For the curve $y = x^2 + 1$, we can use x as the parameter of the curve. Then, $dy = 2xdx$. Thus,

$$\begin{aligned}
&\int_{(0,1)}^{(2,5)} ((3x + y)\, dx + (2y - x)\, dy) \\
=&\int_0^2 \left((3x + x^2 + 1)\, dx + \left(2(x^2 + 1) - x\right) 2xdx\right) \\
=&32 - \frac{8}{3}
\end{aligned}$$

Along straight line, $y = 2x + 1$ and using x as the parameter $(dy = 2dx)$, we get

$$\begin{aligned}
&\int_{(0,1)}^{(2,5)} ((3x + y)\, dx + (2y - x)\, dy) \\
=&\int_0^2 ((5x + 1)\, dx + (2(2x + 1) - x)\, 2dx) \\
=&32
\end{aligned}$$

Why are these two integrals not equal?

(b) Left for you. Answer: -48π.

6. Evaluate $\int_C (x-2xyi)\,dz$ over the contour $C: z=t+it^2, 0\le t\le 1$.

Answer:
The parametrization for the contour is $x(t)=t$ and $y(t)=t^2$. And, $dz=(1+2it)\,dt$. Thus,

$$\begin{aligned}\int_C (x-2xyi)\,dz &= \int_0^1 \left(t-2\cdot t\cdot t^2 i\right)(1+2it)\,dt \\ &= \int_0^1 \left(4t^4+t+2i\left(t^2-t^3\right)\right)dt = \frac{13}{10}+\frac{i}{6}\end{aligned}$$

7. Evaluate $\int_C \overline{z}^2\,dz$ around the circles (a) $|z|=1$, (b) $|z-1|=1$.

Answer:

(a) $|z|=1$ can be parametrized as $z(\theta)=e^{i\theta}$ with $\theta: 0\to 2\pi$. Then, $z'(\theta)=ie^{i\theta}d\theta$.

$$\int_C \overline{z}^2 dz = \int_0^{2\pi}\left(e^{-i\theta}\right)^2 ie^{i\theta}d\theta = i\int_0^{2\pi} e^{-i\theta}d\theta = 0$$

(b) $|z-1|=1$ can be parametrized as $z(\theta)=1+e^{i\theta}$ with $\theta: 0\to 2\pi$. Then, $z'(\theta)=ie^{i\theta}d\theta$.

$$\int_C \overline{z}^2 dz = \int_0^{2\pi}\left(1+e^{-i\theta}\right)^2 ie^{i\theta}d\theta = i\int_0^{2\pi}\left(e^{i\theta}+2+e^{-i\theta}\right)d\theta = 4\pi i$$

8. Verify Green's Theorem in the plane for $\int_C \left(x^2-2xy\right)dx+\left(y^2-x^3y\right)dy$ where C is a square with vertices at $(0,0),(2,0),(2,2)$ and $(0,2)$.

Answer:
We want to verify that

$$\int_C (Pdx+Qdy) = \int_A (Q_x-P_y)\,dxdy.$$

The line integral has four sections corresponding to the 4 sides of the square.

- ▷ Section 1: $y=0,\ dy=0$ and $x:0\to 2$. $I_1=\int_0^2 x^2dx=\frac{8}{3}$
- ▷ Section 2: $x=2,\ dx=0$ and $y:0\to 2$. $I_2=\int_0^2\left(y^2-8y\right)dy=-\frac{40}{3}$
- ▷ Section 3: $y=2,\ dy=0$ and $x:2\to 0$. $I_1=\int_2^0\left(x^2-4x\right)dx=\frac{16}{3}$
- ▷ section 4: $x=0,\ dx=0$ and $y:2\to 0$. $I_1=\int_2^0 y^2dy=-\frac{8}{3}$

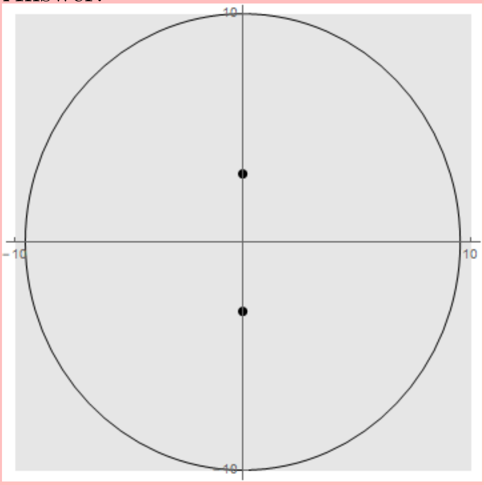
Thus, the required integral is -8. And

$$\int_0^2\int_0^2 dxdy\left(2x-3x^2y\right) = 8-3\cdot\frac{8}{3}\cdot 2 = -8$$

Thus the theorem is verified.

9. Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is the ellipse $|z-3i| + |z+3i| = 20$.

Answer:



Semi-major and semi-minor axes are $a = 10$ and $b = \sqrt{91}$. Then the parametrization for ellipse can be written as

$$z(\theta) = (b\cos\theta, a\sin\theta) \qquad \theta : 0 \to 2\pi$$

Then

$$\int z^3 dz = \int_0^{2\pi} [b\cos\theta + ia\sin\theta]^3 [-b\sin\theta + ia\cos\theta]\, d\theta = 0$$

You can do the remaining terms :) .

10. Show that

(a) $\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{3\pi}{4}$ if $C : |z| = 3$.

(b) $\left|\int_C \frac{e^{3z}}{1+e^z} dz\right| \leq \frac{2\pi e^{3R}}{e^R - 1}$ if C is a vertical line segment from $z = R\,(>0)$ to $z = R + 2\pi i$.

Answer:

(a) Clearly, $\left|z^2 - 1\right| \geq \left||z|^2 - 1\right| \geq 8$ since $|z| = 3$. Thus, $\left|\frac{1}{z^2-1}\right| \leq \frac{1}{8}$. The length of the curve is 6π. By the rules of integration

$$\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{1}{8} \cdot 2\pi \cdot 3 = \frac{3\pi}{4}.$$

(b) Along C, $|e^z| = e^R > 1$. Then $\left|\frac{1}{e^z+1}\right| \leq \frac{1}{||e^z|-1|} = \frac{1}{e^R-1}$ also $\left|e^{3z}\right| = e^{3R}$. The length of the curve is 2π. Thus,

$$\left|\int_C \frac{e^{3z}}{1+e^z} dz\right| \leq \frac{2\pi e^{3R}}{e^R - 1}$$

11. Use antiderivatives to evaluate following integrals:

(a) $\int_i^{i/2} e^{\pi z} dz$;

(b) $\int_0^{\pi+2i} \cos(z/2)\, dz$

(c) $\int_1^3 (z-2)^3\, dz$.

Answer:

(a) Since $\frac{d}{dz}\left(\frac{1}{\pi}e^{\pi z}\right) = e^{\pi z}$, required integral is $\int_i^{i/2} e^{\pi z} dz = \frac{1}{\pi}e^{\pi z}\Big|_i^{i/2} = \frac{1}{\pi}\left(e^{i\pi/2} - e^{i\pi}\right) = \frac{1}{\pi}(i+1)$.

(b) Since $\frac{d}{dz}\left(2\sin\frac{z}{2}\right) = \cos\frac{z}{2}$, we get $\int_0^{\pi+2i} \cos(z/2)\, dz = e^{-1} + e$.

(c) Left for you.

12. Use antiderivatives to show that

$$\int \frac{dz}{z^2 - a^2} = \frac{1}{2a} \log\left(\frac{z-a}{z+a}\right) + c_1 = \frac{1}{a} \coth^{-1}\left(\frac{z}{a}\right) + c_2$$

Answer:
Clearly,

$$\begin{aligned} \frac{1}{2a}\frac{d}{dz} \log\left(\frac{z-a}{z+a}\right) &= \frac{1}{2a}\frac{d}{dz}\left[\log(z-a) - \log(z+a)\right] \\ &= \frac{1}{2a}\left(\frac{1}{z-a} - \frac{1}{z+a}\right) = \frac{1}{z^2-a^2}. \end{aligned}$$

The second part is left for you.

13. Let C be a square with vertices on $z = \pm 2 \pm 2i$. Evaluate following integrals using Cauchy integral formula to evaluate following integrals:

(a) $\int_C \frac{e^{-z}dz}{z-(\pi i/2)}$;

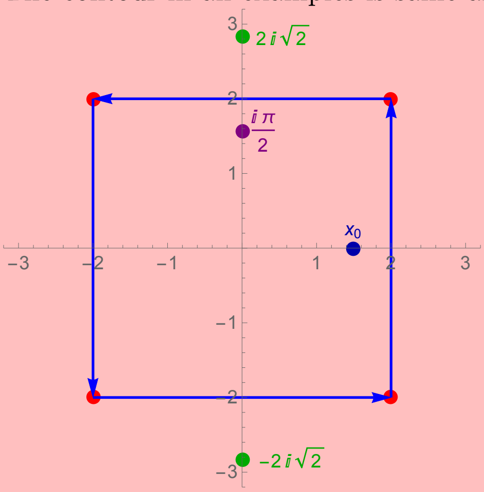
(b) $\int_C \frac{\cos z}{z(z^2+8)} dz$;

(c) $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz \qquad (-2 < x_0 < 2)$.

Answers:
The Cauchy integral formula to evaluate integrals, so we will write it as

$$\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

The contour in all examples is same and is shown in the figure below.



(a) In this integral $\int_C \frac{e^{-z}dz}{z-(\pi i/2)}$, we identify $f(z) = e^{-z}$ and $z_0 = i\pi/2$ (purple point in the figure) and $n = 0$. Since the point z_0 is inside the contour.

$$\int_C \frac{e^{-z}dz}{z-(\pi i/2)} = \frac{2\pi i}{0!} f(i\pi/2) = 2\pi i e^{-i\pi/2} = 2\pi.$$

(b) In this integral $\int_C \frac{\cos z}{z(z^2+8)} dz$, we identify $f(z) = \cos z / \left(z^2+8\right) = \frac{\cos z}{(z+i2\sqrt{2})(z-i2\sqrt{2})}$. Since the points $\pm i2\sqrt{2}$ (green points) are outside the contour, the function $f(z)$ is analytic inside and on the contour. Here, $z_0 = 0$ is inside the contour and hence

$$\int_C \frac{(\cos z)/\left(z^2+8\right)}{(z-0)} dz = 2\pi i \frac{\cos 0}{0+8} = \frac{\pi i}{4}$$

(c) In this integral $\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz$, $f(z) = \tan(z/2)$, $z_0 = x_0$ and $n = 1$.

Question is if f is analytic inside the contour? Since $\tan(z/2) = \sin(z/2)/\cos(z/2)$, the singularities occurs when $\cos(z/2) = 0$, that is at $z/2 = (m + 1/2)\pi$ where m is an integer. That is at points are $(2m+1)\pi$ and notice that all these points are outside the contour. Then

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = \frac{2\pi i}{1!}\frac{d}{dz}\tan\left(\frac{z}{2}\right)\bigg|_{z=x_0} = \pi i \sec^2 x_0$$

14. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$ if $t(>0)$ is a real constant and $C : |z| = 3$.

Answer:
Note that

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz &= \frac{1}{2i}\left(\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z-i}dz - \frac{1}{2\pi i}\int_C \frac{e^{zt}}{z+i}dz\right)\\ &= \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \sin t.\end{aligned}$$

15. Prove Cauchy's Inequality which states that if $f(z)$ is analytic on and inside of a circle of radius R and center a, then

$$\left|f^{(n)}(a)\right| \leq \frac{M \cdot n!}{R^n}$$

where M is a maximum of $|f(z)|$ on the circle.

Answer:
Let the circle be parametrized as $z(\theta) = a + Re^{i\theta}$, $\theta : 0 \to 2\pi$. Using Cauchy integral formula

$$\begin{aligned}\left|f^{(n)}(a)\right| &= \left|\frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-a)^{n+1}}dz\right|\\ &= \frac{n!}{2\pi}\left|\int_0^{2\pi}\frac{f(z)}{R^{n+1}e^{i(n+1)\theta}}iRe^{i\theta}d\theta\right|\\ &= \frac{n!}{2\pi R^n}\left|\int_0^{2\pi} f(z)e^{-in\theta}d\theta\right|\\ &\leq \frac{n!}{2\pi R^n}M \cdot 2\pi = \frac{M \cdot n!}{R^n}\end{aligned}$$