Tutorial 9: Group Theory

Group Theory: Continuous Groups.

1. Verify that the following sets of $n \times n$ matrices for a real Lie algebra and find corresponding Lie groups (obtained by exponentiating them):
   (a) all real matrices;
   (b) all real upper triangular matrices;
   (c) all real upper trinagular traceless matrices;
   (d) all real upper triangular matrices with zero diagonal elements;
   (e) all real traceless matrices.

2. Let E_{ij} $(i, j = 1, \ldots, n)$ be $n \times n$ matrices such that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Verify that the following sets constitute bases of the Lie algebras of the indicated groups
   (a) E_{ij} for $GL(n, \mathbb{R})$
   (b) E_{ij} and iE_{ij} for $GL(n, \mathbb{C})$
   (c) $E_{|ij|} = \frac{1}{2}(E_{ij} - E_{ji})$ and $iE_{(ij)} = \frac{i}{2}(E_{ij} + E_{ji})$ for $U(n)$ and
   (d) $E_{|ij|}$ and $\tilde{E}_{(ij)} = E_{(ij)} - \frac{1}{n} I \operatorname{tr}\left(E_{(ij)}\right)$ for $SU(n)$.

3. Show that the set of all $(n+1) \times (n+1)$ real matrices of the form
$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$$
   where A is a $n \times n$ non-singular matrix, a is column matrix with n rows, for a Lie group G that is isomorphic to the affine group $A(n, R)$. What is the Lie algebra of group G? Obtain the commutation relations of the suitable basis. (Note that the affine $A(n, R)$ is group of transformations on $\mathbb{R}^n$ which map $x \mapsto Ax + a$. This group contains translations in addition to transformations in $GL(n)$.)

4. Find the axis and angle of rotation for the following rotation matrices:
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \quad \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{6} & 1 \\ \sqrt{6} & 2 & -\sqrt{6} \\ 1 & \sqrt{6} & 3 \end{pmatrix}$$

5. Show that the two elements of $SO(3)$ belong to the same conjugacy class if and only if they correspond to the same angle of rotation.

6. Show that the axis of rotation is an eigenvector of rotation matrix with eigenvalue $+1$ and the other two eigenvalues are complex for angle of rotation $0 < \theta < \pi$.

7. An infinitesimal Lorentz transformation and its inverse can be written as
$$\begin{aligned} x'^{\alpha} &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right) x_{\beta} \\ x^{\alpha} &= \left(g^{\alpha\beta} + \epsilon'^{\alpha\beta}\right) x'_{\beta} \end{aligned}$$
   where $\epsilon^{\alpha\beta}$ and $\epsilon'^{\alpha\beta}$ are infinitesimal.
   (a) Show from the definition of the inverse that $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta}$.
   (b) Show from the preservation of the norm that $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.
   (c) By writing the transformation in terms of contravariant components on both sides of the equation, show that $\epsilon^{\alpha\beta}$ is equivalent to the matrix $-\xi \cdot K - \omega \cdot S$ where K and S are the six generators of the Lorentz group.

8. For the Lorentz boost and rotation matrices $\mathbf{K}$ and $\mathbf{S}$ show that

$$\begin{aligned} \left(\epsilon' \cdot \mathbf{K}\right)^3 &= \epsilon' \cdot \mathbf{K} \\ (\epsilon \cdot \mathbf{S})^3 &= -\epsilon \cdot \mathbf{S} \end{aligned}$$

where ϵ and ϵ' are any real unit 3-vectors.
Use the results of part a to show that

$$\exp\left(-\xi\hat{\beta} \cdot \mathbf{K}\right) = I - \hat{\beta} \cdot \mathbf{K} \sinh\xi + \left(\hat{\beta} \cdot \mathbf{K}\right)^2 (\cosh\xi - 1) .$$

9.