CYK/2023/PH201 Mathematical Physics

Mid-Semester Exam

Total Marks: 30; **Duration:** 2 Hours (9AM-11AM); **Date:** 18 Sept 2023, Monday

1. [6×2 **Marks**] Answer the following questions: (Write steps and reasons in **brief**. Writing just final answers will not be awarded marks.)

   (a) Find and sketch the image of the straight line $y = x + 1$ under the transformation $w = \frac{1}{z}$.

   (b) The branch cut of $\ln z$ is chosen along the radial line making an angle of 120 degrees with the positive x axis. If $\ln 1 = 0$, then what are the values of $\ln(i)$, $\ln(1)$ and $\ln(-1)$?

   (c) Show that if a function $f(z) = u + iv$ is analytic at z, level curves of u and v passing through z are orthogonal.

   (d) At which points the function $f(z) = \bar{z}^2$, analytic?

   (e) If $C : |z| = R$ is a positively oriented circular contour, compute

   $$\oint_C \frac{e^z}{\left(z - \frac{\pi i}{4}a\right)^2} dz \qquad (R > a).$$

   write in terms of sin & cos.

   (f) Discuss and classify the singularities of $\frac{1}{\sin(\pi/z)}$.

   essential / isolated case $\left(\frac{\pi}{z}\right) \to \infty$ $m\pi$

2. [2×3 **Marks**] Answer the following questions:

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   (a) [3] The Euler numbers E_n are defined by the power series $\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n$. What is the radius of convergence for this series? Compute E_0 to E_4. 1/2 each

   v. easy (b) [3] Given the function $f(z) = \frac{z}{(z-2)(z+i)}$, expand the function in a series about $z_0 = 0$, in the regions (i) $|z| < 1$ (ii) $1 < |z| < 2$ and (iii) $|z| > 2$.

3. [3×4 **Marks**] Using the method of residues, answer the following questions: (**sketch contours and show contributions from each segment of contours explicitly**)

   Tutorial Question Type-2 (a) Compute

   $$\text{pv} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$$

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   type 3 (b) Compute

   $$\int_{-\infty}^{\infty} \frac{\cos kx}{(x+b)^2 + a^2} dx$$

   where $a > 0$, $k > 0$ and b is a real number.

   v. difficult (c) Compute

   $$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{a^2 - 2a\cos\theta + 1}; \qquad -1 < a < 1.$$