PH504 Mathematical Physics - II, IITG

L	Outline of the Course
10	• Groups + Continuous Groups
12	• Vector Spaces, Inner Product Spaces, Hilbert Spaces
6	• Group Representations
8	• Tensor Analysis

Groups:
- Definition, Examples, Properties, ~~Iso~~
- Multiplication Table, Rearrangement Theorem
- Conjugate Classes, Multiplication of classes
- Subgroups, Cosets, Normal Subgroup, Factor Groups
- Product of Groups
- Isomorphism, Homomorphism
- Permutation Group
- Rotation Group O(3), SO(3)
- Unitary Groups SU(1), SU(2), SU(N)
- Point Groups and Space groups in Crystals
- Lorentz & Poincare' Groups

With 10 Lectures. It is important to acheive some depth and not just definition level.

- 3-Tutorial Sheets -

Books
1. Elements of Group Theory for Physicists, A W Joshi, New Age
2. Group Theory in Physics, Wu-Ki Tung, World Scientific
3. Topics in Algebra, I N Herstein, Wiley Eastern
4. Linear Algebra, Hoffman - Kunze, PHI
5. Schaum Series, Linear Algebra
6. Linear Algebra and Group Theory for physicists, K.N. Srinivasa Rao, New Age

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$(a\cdot b)\cdot c = b$

$a\cdot(b\cdot c) = b$

Problem 14/pp 36 Herstein

~~Suppose there is no identity~~ $\cancel{\Rightarrow \text{ for every } a \in G \ \exists\ y(a) \text{ s.t. } a\cdot y(a) \neq y(a)}$

The left cancellation law $(a\cdot w = a\cdot u \Rightarrow w = u)$ implies that the multiplication by a is a one to one function on G. Since G is finite, this function is also a bijection $\Rightarrow \exists\ e \in G$ s.t. $a\cdot e = a \Rightarrow e\cdot e = e$

~~To show that e is unique let~~ $\cancel{a, b \in G \Rightarrow \exists\ e_a, e_b \text{ s.t. } e_a\cdot a = a,\ e_b\cdot b = b}$

$$\cancel{(e_a\cdot a\cdot b) = a\cdot b}$$

- $a\cdot e_R = a \qquad e_L\cdot a = a$
- $e_L\cdot a\, a\, e_R = a^2$

Groups GROUPS

2|4 **Definition:** A nonempty set G is said to form a group if in G there is defined a binary operation, called multiplication and denoted by $\cdot$ s.t.

1. $a, b \in G$ implies that $a \cdot b \in G$ (closure)
2. $a, b, c \in G$ implies that $(a\cdot b)\cdot c = a \cdot (b \cdot c)$
3. $\exists\ e \in G$ s.t. $a \cdot e = e \cdot a = a \quad \forall\ a \in G$
4. For every $a \in G$ $\exists\ a^{-1} \in G$ s.t. $a a^{-1} = a^{-1} a = e$

Examples
1. $G = \{e\}$ $\quad e \cdot e = e$
2. $G = \{e, a\}$ $\quad a\cdot e = e \cdot a = a$, $\ e \cdot e = e$, $\ a \cdot a = e$
3. $G = \{1, -1\}$ under usual multiplication
4. $G = \{e, a, b\}$ $\quad e$ is identity, $a \cdot b = e = b \cdot a$, notation $a^2 = b$

   Multiplication Table.
5. $G = I$ = Set of integers $\quad m \cdot n = m + n$
6. $G = \{e, a, a^2, a^3, \cdots a^{n-1}\}$ $\quad a^m \cdot a^p = a^{(m+p) \bmod n}$, $\quad \omega^n = 1, \omega = e^{2\pi i/n}$
7. $G = \cancel{Q}\ I$ under usual multiplication is not a group
8. $G = Q^*$ Set of Rationals under multiplication.

3|1
9. $G = S_3 = A(S)$ where $S = \{a, b, c\}$ ~~set of~~

   $A(S)$ is set of all bijections on S: Set of permutations.
10. $G = C_3$ set of rotation about z axis by $120°$
11. $G = SO(2)$ set of all rotations about z axis
12. $G = O(2)$ Set of all 2×2 orthogonal matrices
13. $G = O(N)$ Set of all $N \times N$ orthogonal matrices $\quad R^T R = 1$
14. $G = U(N)$ Set of all $N \times N$ Unitary matrices $\quad \cancel{R}\ U^\dagger U = 1$

Definition: Abelian Groups: A group G is called abelian if $\forall\ a, b \in G$ $a \cdot b = b \cdot a$: (commutative)

3|1 **Properties:**
1. identity is unique : e, f are identities $e = e\cdot f = e \cdot f = f$
2. Inverse is unique : $ab = e = ac \Rightarrow (ca)b = c \Rightarrow b = c$
3. $(a^{-1})^{-1} = a$: $(a^{-1})(a^{-1})^{-1} = e = a^{-1}a \Rightarrow (a^{-1})^{-1} = a$ use 2
4. $(ab)^{-1} = b^{-1}a^{-1}$
5. $a \cdot u = a \cdot w \Rightarrow u = w$
6. $u \cdot a = w \cdot a \Rightarrow u = w$

$\left.\right\}$ Cancellation laws

Elaboration Let $G = A(S)$ where $S = \{a, b, c\}$

Let $A(S) = \{e, \psi, \psi^2, \sigma, \sigma\circ\psi, \psi\circ\sigma\}$ where

$e(\{a,b,c\}) = \{a,b,c\}$

$\psi(\{a,b,c\}) = \{b,c,a\}$

$\sigma(\{a,b,c\}) = \{a,c,b\}$

$\psi^2(\{a,b,c\}) = \{c,a,b\}$

$\sigma\circ\psi(\{a,b,c\}) = \{c,b,a\}$

$\psi\circ\sigma(\{a,b,c\}) = \{b,a,c\}$

Multiplication table

	e	ψ	ψ^2	σ	$\sigma\psi$	$\psi\sigma$
e	e	ψ	ψ^2	σ	$\sigma\psi$	$\psi\sigma$
ψ	ψ	ψ^2	e	$\psi\sigma$	σ	$\sigma\psi$
ψ^2	ψ^2	e	ψ	$\sigma\psi$	$\psi\sigma$	σ
σ	σ	$\sigma\psi$	$\psi\sigma$	e	ψ	ψ^2
$\sigma\psi$	$\sigma\psi$	$\psi\sigma$	σ	ψ^2	e	ψ
$\psi\sigma$	$\psi\sigma$	σ	$\sigma\psi$	ψ	ψ^2	e

defining Equations

$$\boxed{\begin{gathered}\psi^3 = e = \sigma^2 = e \\ \psi\sigma = \sigma\psi^2\end{gathered}}$$

$\psi\sigma\psi = \sigma\psi^2\psi = \sigma$

$\psi^2\sigma = \psi\sigma\psi^2 = \sigma\psi$

$\sigma\psi^2\sigma = \psi\sigma\sigma = \psi$

$\sigma\psi\psi\sigma\psi = \psi^2$

~~$\psi\sigma\psi = \sigma$~~

$\psi^2\sigma\psi = \psi\sigma$

$\psi\sigma\psi^2 = \sigma\psi$

Conjugacy Classes $C(e) = \{e\}$

$C(\psi) = \{\psi, \psi^2\}$

$C(\sigma) = \{\sigma, \psi\sigma, \sigma\psi\}$

Rearrangement theorm: By Cancellation Laws it is clear that no row or column of multiplication can have an element appearing twice.

Conjugacy Classes CONJUGACY CLASSES

Definition: $a, b \in G$, b is conjugate to a if $\exists\, c \in G$ s.t.

$$b = c^{-1} a c.$$

Theorem: Conjugacy is an equivalence relation $\quad a \sim a,\ a \sim b \Rightarrow b \sim a,\ a \sim b,\ b \sim c \Rightarrow a \sim c$

Equivalence classes for conjugacy are called conjugate classes.

$C(a) = \{x \in G \,/\, x \text{ is conjugate to } a\}$

Conjugate classes are disjoint (Property of equivalence Relation)

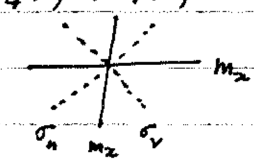
Examples. :1. $G = \{e, a\}$, $a^2 = e$

$C(e) = \{e\}$ $\quad C(a) = \{a\}$

2. $G = \{e, a, a^2\}$ $\quad C(e) = \{e\}$, $C(a) = \{a\}$, $C(a^2) = a^2$

3. ~~$G \cong I$~~ $A(S)$ See Above.

4. C_{4v} : $\quad (E), (C_4, C_4^3), (C_4^2), (m_x, m_y), (\sigma_u, \sigma_v)$



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Multiplication of classes:

Definition : $C_1 C_2 = (A_1 B_1, A_1 B_2, \ldots, A_m B_n)$

$\Rightarrow C_1 C_2$ contains whole classes. $= \sum a_{ijk} C_k$

Example : 1. $G = A(S)$, $S = \{a, b, c\}$

$C(\psi) \cdot C(\sigma) = (\psi\sigma, \psi^2\sigma, \psi^2\sigma, \psi^3\sigma, \psi\sigma\psi, \psi^2\sigma\psi)$

$= (\psi\sigma, \sigma\psi, \sigma\psi, \sigma, \sigma, \psi\sigma)$

$= 2 \cdot C(\sigma)$

9/1/03

Subgroups: SUBGROUPS

Definition: Let G be a group. A subset H is called a subgroup if H is group by itself.

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Theorem: A subset H of a group G is a subgroup if and only if
(i) Multiplication is closed in H
(ii) $a \in H$ implies $a^{-1} \in H$

Proof: ~~It is only required~~ if H is a subgroup (i) and (ii) are obvious
if (i) and (ii) are true then It is only required to show that ~~if~~ $e \in H$.
Let $a \in H \Rightarrow a^{-1} \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$

Corollary: If H is a nonempty finite subset of a group G and H is closed under multiplication then H is a subgroup of G

Examples: 1. Trivial Subgroups
2. $G = I$ set of integers, $m \cdot n = m + n$
$H_5 = \{ \ldots, -10, -5, 0, 5, 10, \ldots \}$
3. S_3: Permutations: $A(S) = \{ e, \sigma, \psi, \psi^2, \sigma\psi, \psi\sigma \}$
$$\left. \begin{array}{l} \{e,\sigma\}, \{e, \sigma\psi\}, \{e, \psi\sigma\} \\ \{e, \psi, \psi^2\} \end{array} \right\} \text{subgroups}$$
$\{e, \psi\}$ is not a subgroup.
4. ~~O(2): Set of all orthog~~ G: Set of all 2×2 nonsingular matrices
$$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \Big/ ad \neq 0 \right\}$$
H is subset, multiplication is closed, inverses are included.
$$K = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \Big/ \cancel{\text{H is not}}\ b \text{ is real} \right\}$$
$K \subset H \subset G$, K is subgroup of H and G.
~~4.~~ other known subgroups $\emptyset$ $G \supset O(2) \supset SO(2) \supset C_6 \supset C_3$ etc.
5. Let G be a group, $a \in G$ $H = \{e, a, a^2, \cdots\}$
6. Let G be C^*: nonzero complex nos. $H = \{ a + bi \,/\, a^2 + b^2 = 1 \}$

16/1/03 **Cosets:**

Definition: If H is a subgroup of a group G then $Ha = \{ ha \,/\, h \in H \}$.
Ha is called a right coset of H in G

Lemma: H is a subgroup of G and $a, b \in G$, Ha & Hb are either disjoint or equal.

Proof: $x \in Ha \Rightarrow$ for some h, $x = ha$ & $x \in Hb \Rightarrow$ for some h', $x = h'b \Rightarrow h'^{-1}ha = b$
$y \in Ha \Rightarrow$ for some h_3 $y = h_3 a = (h_3 h^{-1} h')b \Rightarrow y \in Hb$

Multiplication of Cosets $Ha \cdot Hb = \{ xy \mid x \in Ha, y \in Hb \}$

in Example of S_3. $\psi H \cdot \psi H = \{ \psi\psi, \psi\psi\sigma, \psi\sigma\psi, \psi\sigma\psi\sigma \}$

$= \{ \psi^2, \sigma\psi, \sigma, e \}$

Lemma: There is one-one correspondance between two right cosets of H in G

$f : Ha \to Hb \qquad f(x) = xa^{-1}b$

$x, y \in Ha, \ x \neq y, \quad x = h_1 a \quad y = h_2 a \qquad h_1 \neq h_2$

$f(x) = h_1 b \qquad f(y) = h_2 b \Rightarrow f(x) \neq f(y)$

Theorem: If G is a finite group ~~then~~ and H is a subgroup $\Rightarrow$ $o(H)$ divides $o(G)$

Example: 1. $G = I$ under addition

$H = \{5a \mid a \in I\}$ is a subgroup $\qquad \{\ldots, -10, -5, 0, 5, 10, \ldots\}$

$H_1 = \{5a+1 \mid a \in I\}$ $\qquad \{\ldots, -9, -4, 1, 6, 11\}$

$H_2 = \{5a+2 \mid a \in I\}$ $\qquad \{\ldots, -8, -3, 2, \cancel{7}, 12\}$

$\vdots \qquad\qquad \vdots$

Index of H is 5

2. $G = S_3$ of $A(S)$ $\qquad o(G) = 6$

$\Rightarrow H = \{e, \sigma\}$ $\qquad o(H) = 2$

$H\psi = \{\psi, \sigma\psi\} \neq \psi H = \{\psi, \psi\sigma\}$

$H\psi^2 = \{\psi^2, \psi\sigma\} \neq \psi^2 H = \{\psi^2, \sigma\psi\}$

No subgroup of order 4 is possible

$\Rightarrow H = \{e, \psi, \psi^2\}$

$H\sigma = \{\sigma, \sigma\psi, \psi\sigma\} \stackrel{\vee}{=} \sigma H = \{\sigma, \sigma\psi, \psi\sigma\}$

$\Big\}$ Normal Subgroup.

NORMAL SUBGROUPS

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Definition: A subgroup N of G is said to be normal or invariant subgroup if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Lemma: N is a normal subgroup of G if and only if $gNg^{-1} = N$

Proof: if $gNg^{-1} = N \Rightarrow N$ is normal.

if N is normal $\Rightarrow gNg^{-1} \subset N$

Let $n \in N \Rightarrow g^{-1}ng \in N \Rightarrow g(g^{-1}ng)g^{-1} \in gNg^{-1}$

$\Rightarrow n \in gNg^{-1}$

Lemma: N is normal subgroup of G if every right coset of N in G is also a left coset of N in G

Proof: Let N be normal, $g \in G$. $x \in Ng \Rightarrow x = ng = g(g^{-1}ng)$

$= gn_1$ where $n_1 \in N$

$\Rightarrow Ng \subset gN$, similarly $gN \subset Ng \Rightarrow Ng = gN$

if $Ng = gN$ then $ng = gn' \Rightarrow g^{-1}ng \in N \Rightarrow N$ is normal

$det(AB) = det(A) \cdot det(B)$

group

group

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$Na \quad (Na)(Nc) = N(ac)$

$= Nb \quad (Nb)(Nc) = N(bc)$

since $bc \in N(ac)$

Factor GROUPS

Def: A & B are subsets of G. Define product $AB = \{x \in G \mid x = ab$ for some $a \in A$ & $b \in B\}$

Lemma: A subgroup N of G is normal if and only if Product is two right cosets of N in G is also a right coset of G.

Proof: if N is normal $\Rightarrow$ $Na = aN$ $\Rightarrow$ $(Na)(Nb) \Rightarrow N(Na)b = Nab$

~~Suppose $(Na)(Nb) = Nc$ $\Rightarrow$ $n_1 a n_2 b = c$~~

Let $n \in N$, $a \in G$ $\quad (Na)(Na^{-1}) = N \Rightarrow nana^{-1} = n'$

$\Rightarrow nan = n'a$

$\Rightarrow an = (nn')a \Rightarrow aN \subset Na$

$\Rightarrow aN = Na.$

Let G/N be a collection of cosets of a normal subgroup H of G. And the product of cosets is defined above then, G/N is a group under this multiplication. (check all properties)

$o(G/N) = o(G)/o(N)$ = index of H.

Example $G = S_3 = \{e, \psi, \psi^2, \sigma, \sigma\psi, \psi\sigma\}$

$N = \{e, \psi, \psi^2\} \qquad N\sigma = \{\sigma, \sigma\psi, \psi\sigma\}$

$G/N = \{Ne, N\sigma\}$

Homomorphisms

Definition: A function f from a group G to $\bar{G}$ is called a homomorphism if for all $a, b \in G$, $f(ab) = f(a)f(b)$

Examples: 1 Trivial homomorphism $f(a) = \bar{e} \quad \forall \ a \in G$

2. $G = \{1, -1\}$ under multiplication

$\bar{G} = \{0, 1\}$ under $a \cdot b = (a+b) \bmod 2$

3. $G = I$ under addition

$\bar{G} = \{2^a \mid a \in I\}$ under multiplication

4. $G = S_3 = \{e, \psi, \psi^2, \sigma, \sigma\psi, \sigma\psi^2\}$

$\bar{G} = \{e, \phi\}$ let $f(\sigma^i \psi^j) = \sigma^i$

5. Let G = { 2x2 matrices (nonsingular real) }

$\bar{G} = R^*$

$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

6. $G = SO(2)$, $\bar{G} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$

$f\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \to e^{i\theta}$

Theorem: If f is a homomorphism of G into $\bar{G}$ then

ⓐ $f(e) = \bar{e}$ $\qquad f(x\cancel{e}) = f(x)\cdot f(e) = f(x)\cdot\bar{e} \Rightarrow f(e) = \bar{e}$

ⓑ $f(x^{-1}) = [f(x)]^{-1}$ $\qquad f(xx^{-1}) = f(e) = \bar{e}$

$f(x)\cdot f(x^{-1}) = \bar{e}$

~~Theorem~~

Definition: Kernel. If f is a homomorphism ~~from~~ of G into $\bar{G}$, the Kernel K of homomorphism is defined as $K = \{ x \in G \,/\, f(x) = \bar{e} \}$

Theorem: Kernel of a homomorphism of G in $\bar{G}$ is a subgroup (Normal)

Definition: ISomorphism: A homomorphism f of G into $\bar{G}$ is said to be an isomorphism if f is one-to-one

Definition: Two groups G and $\bar{G}$ are said to be isomorphic if there exists an isomorphism of G onto $\bar{G}$.

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⑥

Continuous Groups

Example:

a. $C_4 = \{e, \sigma, \sigma^2, \sigma^3\}$ Finite

b. I = Set of integers Infinite, discrete

c. $G = \{2^i \,/\, i \in I\}$ under multiplication Infinite, discrete

d. $G = \{ \tau_{ab} : R \to R \,/\, \tau_{ab}(x) = ax + b, a \neq 0 \}$ under function composition Infinite, Continuous 2 parameter

e. $G = \{ e^{i\theta} \,/\, \theta \in [0, 2\pi) \}$ Infinite, Continuous, 1 parameter.

f. $G = O(2)$, 2x2, Real orthogonal matrices (?)

$a^2+c^2=1$, $b^2+d^2=1$, $\cancel{ad+bc=0}$ $ab+cd=0$

① $\cancel{a^2=1-c^2 \Rightarrow a^2=1-\left(-\frac{ad}{b}\right)^2}$

$ab=-cd \Rightarrow a^2b^2=c^2d^2 \Rightarrow a^2(1-d^2)=c^2d^2 \Rightarrow a^2=(a^2+c^2)d^2 \Rightarrow a^2=d^2 \Rightarrow \boxed{a=\pm d}$

② $a^2b^2=c^2d^2 \Rightarrow (1-c^2)b^2=c^2d^2 \Rightarrow \boxed{b=\pm c}$

③ $(ad-bc)^2=a^2+b^2=1 \Rightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\pm 1$

O(2) and SO(2) Groups

Definition: O(2) is set of all 2×2 real, orthogonal matrices. $O^TO = \mathbb{I}$

SO(2) is set of all 2×2 real, orthogonal matrices with det = +1

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} a^2 + c^2 = 1, & ab + cd = 0 \\ b^2 + d^2 = 1, & ab + cd = 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} = -\frac{d}{b} \text{ or } \frac{a}{d} = -\frac{b}{c}$$

$$\Rightarrow (ad - bc) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$$

$$\Rightarrow a = \pm d \text{ and } c = \pm b \text{ and } ad - bc = 1$$

$$\Rightarrow \begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix} \text{ and } a^2 + b^2 = 1$$

$$\Rightarrow a = \cos\theta \quad b = \sin\theta$$

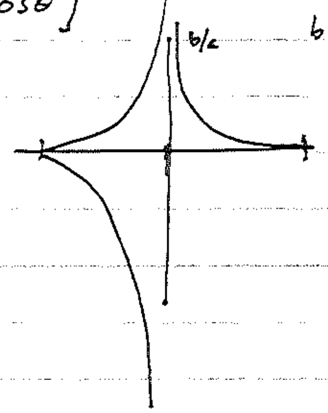
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ +\sin\theta & -\cos\theta \end{bmatrix}$$

$\theta = 0$ to 2π

$-1 \leq a \leq 1, \quad b = \pm\sqrt{1 - a^2}$

$\theta = \tan^{-1}(b/a) \qquad -\pi/2 \leq \theta \leq \pi/2 \quad b > 0$

$\theta = \tan^{-1}(b/a) \qquad \pi/2 \leq \theta \leq 3\pi/2, \quad b < 0$



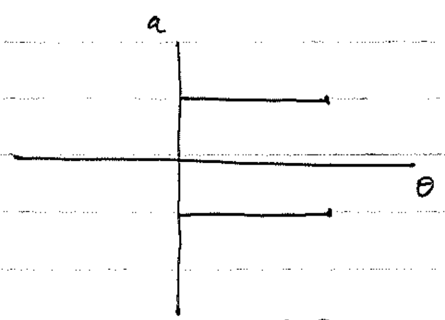
$\Rightarrow a = \cos\theta$

$\Rightarrow b = \sin\theta$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ (-1)^a \sin\theta & (-1)^a \cos\theta \end{bmatrix} \qquad \theta \in [0, 2\pi), \quad a \in \{0, 1\}$$

SO(2) is a single parameter group.

Parameter Space: Domain in $R^n \supset D$



GENERATORS AND EXPONENTIAL MAPS

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(7)

$$F(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad I = (\theta)$$

$$\frac{\partial F}{\partial \theta} = \lim_{h \to 0} \frac{F(\theta + h) - F(\theta)}{h}$$

$$= \lim_{h \to 0} \begin{bmatrix} \frac{\cos(\theta + h) - \cos\theta}{h} & \frac{\sin(\theta + h) - \sin\theta}{h} \\ \frac{-\sin(\theta + h) - \sin\theta}{h} & \frac{\cos(\theta + h) - \cos\theta}{h} \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix}$$

$$\text{Generator } I = \frac{1}{i} \frac{dF}{d\theta}\bigg|_{\theta = 0} = \frac{1}{i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_y$$

Suppose that $\theta = n\epsilon$ where ϵ is small (infinitesimal)

$$f(\theta) = f(\epsilon)\, f(\epsilon) \cdots = [f(\epsilon)]^n$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} -i^2 = 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since. $f(\epsilon) \approx \mathbb{1} + i\sigma_y \epsilon$

$$\Rightarrow \quad f(\theta) \approx (\mathbb{1} + i\sigma_y \theta/n)^n$$

$$= \lim_{n\to\infty} (\mathbb{1} + i\sigma_y \theta/n)^n = \exp(i\sigma_y \theta)$$

$$\Rightarrow \quad f(\theta) = \exp(i\sigma_y\theta)$$

$$\Rightarrow \quad = \mathbb{1} + i\sigma_y\theta + \frac{i^2\sigma_y^2\theta^2}{2} + \cdots$$

$$= \mathbb{1} + i\sigma_y\theta - \tfrac{1}{2}\theta^2 \mathbb{1} \cancel{\theta} - \tfrac{1}{6} i\sigma_y\theta^3 + \cdots$$

$$= \mathbb{1}\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + i\sigma_y\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \mathbb{1}\cos\theta + i\sigma_y \sin\theta$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Rotation Group on Functions

Let $\mathcal{L} = \{$ set of all analytic real fn on $\mathbb{R}^2 \}$

When ~~~~ there is a rotation of coordinates

Then $f \to f' : f'(P) = f(P')$

$P \equiv (x, y)$

$P' \equiv (x', y')$

$$\Rightarrow \quad f'(x,y) = f(x', y') = f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

$$= (\cancel{\mathbb{1}}\, i\theta L_z)\, f(x,y)$$

$$\Rightarrow \quad f' \approx (\mathbb{1} + i\theta L_z) f \quad \text{if } \theta \text{ is small}$$

$$= \exp(i\theta L_z) f \qquad L_z = i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Ex $\quad f(x,y) = x^2 + y^2$

$$L_z f(x,y) = i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) f(x,y)$$

$$= i(2xy - 2yx) \cancel{f(x,y)} = 0$$

$$\Rightarrow \quad \exp(i\theta L_z) f = f$$

Ex $\quad f(x,y) = x \qquad L_z x = i(-y) = -iy \qquad L_z^2 x = -i \cdot i \cdot x = x$

$\exp(i\theta L_z)\, x$

$= \left(1 + i\theta L_z - \frac{1}{2}\theta^2 L_z^2 - \frac{1}{3!}\theta^3 i L_z^3 \cdots\cdots \right) x$

$= x(1 - \frac{1}{2}\theta^2 + \theta^4/4! \cdots) + y(\theta - \theta^3/3! + \cdots)$

$= x\cos\theta + y\sin\theta$

8|2 SO(3) Group and its Generators

⑧ Def: SO(3) all 3x3 real orthogonal matrices ~~3x3~~ with det = 1

Requires $A^T A = I$ $\quad A \in SO(3)$

$\Rightarrow$ Since $A^T A$ is symmetric there are six ind. equations

$\Rightarrow$ SO(3) is a three parameter group

Definitiona ~~A rotation R about an axis (passing thro' origin) $\hat{a}$~~

A co-ordinate transformation ~~is called~~ $T: R^3 \to R^3$ is called a rotation if it preserves distances and ~~fixes the origin~~. is linear.

$T: R^3 \to R^3 \qquad |T(x) - T(y)| = |x - y| \qquad x, y \in R^3$

$T(x)^T \cdot T(y) = x^T \cdot y$

$\therefore (Tx)^T Ty = x^T y$

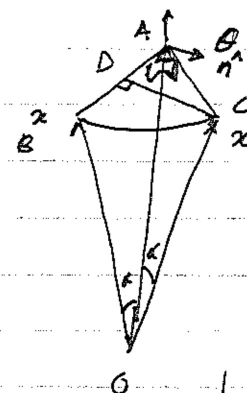
$\therefore x^T T^t T y = x^T y$

$\therefore T^t T = I$

SO(3) is set of rotations on R^3.

A. Axis-Angle Parametrization.

$\hat{a}$ axis of rot: θ angle of rotation



$\vec{x}' = \overline{OA} + \overline{AD} + \overline{DC}$

$\hat{a} \times \hat{x} = \hat{n}\sin\alpha$

$\hat{a} \cdot \hat{x} = \cos\alpha$

$\hat{n} \times \hat{a} = \dfrac{(\hat{a} \times \hat{x}) \times \hat{a}}{\sin\alpha}$

$\vec{x}' = \hat{a}\, x\cos\alpha + \hat{n}\times\hat{a}\, AC\cos\theta + \hat{n}\, AC\sin\theta$

$= x\,(\hat{a}\cos\alpha + \hat{n}\times\hat{a}\,\sin\alpha\cos\theta + \hat{n}\sin\alpha\sin\theta)$

$= x(\hat{x} - \hat{n}\times\hat{a}\sin\alpha(1 - \cos\theta) + \hat{a}\times\hat{x}\sin\theta)$

$= \vec{x} + \hat{a}\times\vec{x}\sin\theta + (1 - \cos\theta)\,(\hat{a}\times(\hat{a}\times\vec{x}))$

$\hat{n} \times \hat{a}$

$= (\hat{a}\times\hat{x})\times\hat{a} / \sin\alpha$

$= + \dfrac{(\hat{a}\cdot\hat{a})\hat{x} - \hat{a}\,(\hat{a}\cdot\hat{x})}{\sin\alpha}$

$=$

$\hat{n}\times\hat{a}\sin\alpha = \hat{x} - \hat{a}\cos\alpha$

$$\hat{a} \times \vec{x} = (a_2 x_3 - x_2 a_3)\, i + (a_3 x_1 - a_1 x_3)\, j + \hat{k}\, (a_1 x_2 - x_1 a_2)$$

$$= \begin{bmatrix} a_2 x_3 - a_3 x_2 \\ -a_1 x_3 + a_3 x_1 \\ a_1 x_2 - a_2 x_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ +a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S_a$$

$$\hat{a} \times (\hat{a} \times \hat{x}) = \begin{bmatrix} -a_2^2 - a_3^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & -a_3^2 - a_1^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & -a_2^2 - a_1^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = S_a^2$$

$$\cancel{[x''] = I[x] + (1 - \cos\theta)\, S_a\, [x] +}$$

$$[x'] = I[x] + \sin\theta\, S_a [x] + (1-\cos\theta) S_a^2 [x]$$

$$= R(\hat{a}, \theta)\, [x]$$

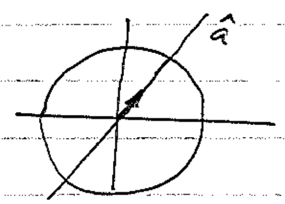
$$R(\hat{a}, \theta) = I + \sin\theta\, S_a + (1 - \cos\theta)\, S_a^2$$

$$= \begin{bmatrix} a_1^2 (1-\cos\theta) + \cos\theta & a_1 a_2 (1-\cos\theta) - a_3 \sin\theta & a_1 a_3 (1-\cos\theta) + a_2 \sin\theta \\ a_2 a_1 (1-\cos\theta) + a_3 \sin\theta & \cdots & \cdots \\ & & \end{bmatrix}$$

~~Each Rotation is an~~ $0 \le \theta \le \pi$ at π $R(\hat{a}, \pi) = R(-\hat{a}, \pi)$

Parametrization.

Inverse map



$$a_1^2 (1 - \cos\theta) + \cos\theta = a_{11}$$

$$a_2^2 (1 - \cos\theta) + \cos\theta = a_{22}$$

$$a_3^2 (1 - \cos\theta) + \cos\theta = a_{33}$$

$$1 + 2\cos\theta = Tr(A)$$

$$\cos\theta = \frac{1}{2} (Tr(A) - 1)$$

$$\cancel{2 a_1 a_2 (1-\cos\theta)} = 2 a_3 \sin\theta = a_{21} - a_{12}$$

$$a_3 = (a_{21} - a_{12}) / 2\sin\theta$$

$$a_2 = (a_{13} - a_{31}) / 2\sin\theta$$

$$a_1 = (a_{32} - a_{23}) / 2\sin\theta$$

$\theta \neq 0$ - no axis

$\theta = \pi$ - Symmetric

in Case of $\pi = \theta$,

$$a_1 = (1 + a_{11}) / \sqrt{2} (1 + a_{11})^{1/2}$$

$$a_2 = a_{12} / \sqrt{2} (1 + a_{11})^{1/2}$$

$$a_3 = a_{13} / \sqrt{2} (1 + a_{11})^{1/2}$$

10/2

9

Thm: $R(\hat{a},\theta) = \exp(\cancel{i}S_a\theta)$ $\qquad \vec{a} = \theta\hat{a}_p$

$= \cancel{\exp}$

Notice $S_a^3 = \begin{bmatrix} 0 & +a_3 & -a_2 \\ -a_3 & 0 & +a_1 \\ +a_2 & -a_1 & 0 \end{bmatrix} = -S_a$

$$= 1 + \cancel{i}S_a\theta + \frac{-i^2}{2}S_a^2\theta^2 + iS_a\frac{1}{3}\theta^3 + \cdots$$

$$= 1 + \cancel{i}S_a\left(\theta - \frac{\theta^3}{3}\cdots\right) + S_a^2\left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \cdots\right)$$

$$= 1 + \sin\theta\, S_a + (1-\cos\theta)\, S_a^2$$

Generator of rotation about a axis as

$$J_a = \frac{1}{i}S_a$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{bmatrix}$$

$[L_x, L_y] = iL_z$

$[L_y, L_z] = iL_x$

$\vdots$

$$J_y = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

a

$$J_z = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

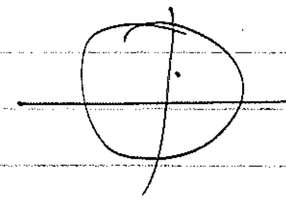
$$J_a = a_1 J_x + a_2 J_y + a_3 J_z$$

$$R(\hat{a},\theta) = \exp[i(a_1 L_x + a_2 L_y + a_3 L_z)\theta)$$

(a_x, a_y, a_z)
$= (a_1\theta, a_2\theta, a_3\theta)$

Not Multiplication.

Parametrization



SU(2) Group

Definition $SU(2) =$ ~~set~~ Group of all 2×2 unitary matrices.

Require, $A \in SU(2) \Rightarrow A^{\dagger}A = I$

$$\Rightarrow \begin{bmatrix} a^* & b^* \\ b^* & d^* \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = I$$

Four equations reduce $U(2)$ to a 4 parameter Group.

General Form

$$A = \begin{bmatrix} \cos\theta\, e^{i\alpha} & \sin\theta\, e^{i\gamma} \\ -\sin\theta\, e^{i(\beta-\gamma)} & \cos\theta\, e^{+i(\beta-\alpha)} \end{bmatrix}$$

det of the matrix $e^{i\beta}$. $|\det A| = 1$

$SU(2)$ elements $\beta = 0$ or $2\pi \Rightarrow \det A = 1$

Identity element $\alpha = \gamma = \theta = 0$

Generators

$$I_\theta = \frac{1}{i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \sigma_y$$

$$I_\alpha = \frac{1}{i}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z$$

$$I_\gamma = \frac{1}{i}\begin{bmatrix} 0 & \cancel{\ } i \\ \cancel{1}-i & \cancel{X} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \sigma_x$$

Algebra of Generators

$$[\sigma_x, \sigma_y] = i\sigma_z \;,\; [\sigma_y, \sigma_z] = i\sigma_x \;,\; [\sigma_z, \sigma_x] = i\sigma_y \text{ etc.}$$

There is some connection bet $SO(3)$ and $SU(2)$

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10 13/2 1. Definition of Fields $\mathbb{R}$ and $\mathbb{C}$ as examples

~~1~~ Definition of Vector Spaces, Examples

11 14/2 2. More Examples, Some basic Properties

3. Linear Combinations, Linear Independece of a set, Geometric interpretation of Linear independence with examples.

12 24/2 4. Span of a set, examples, Basis and Dimension

13 26/2 5. Examples of Bases. Polynomials, Finite and infinite, Legendre basis

14 27/2 · Real fns on lattices

6. Co-ordinates, Change of Basis, matrix manipulations.

7. Subspaces,

15 28/2 8. Homomorphism, Kernel, Range, Isomorphism, Isomorphic spaces.

16 3/3 9. Inner product, Inner product spaces, <u>Cauchy Schwarz Inequality</u>, Gram-Schmidt Orthogonalization.

17 5/3 (GS)

18 6/3 10. Metric, Metric spaces, Sequences, Cauchy Sequences, Completeness.

19 7/3 ~~11~~ Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^N$, $C[a,b]$, l^∞_∞, $L^2[a,b]$,

20 12/3 11. Hilbert Space: formal Definition

statement: For every IPS, there exists a Hilbert Space.

Cauchy-Schwarz Inequality (Should have done earlier)

Orthonormal Sets: Are linearly independent.

If ~~X/~~ B, ~~an orth~~ is an orthonormal set then if $x \in \mathrm{span}(B)$ then

$$x = \sum_k \langle x, e_k \rangle e_k$$

and if $x \notin \mathrm{Span}(B)$ then $x = y + z$ where $y \in \mathrm{Span}(B)$ and $z \perp y$

Infinite Sum: $B = \{e_1, e_2, \ldots, e_n, \ldots\}$

let $S_n = \sum_1^n x_k e_k$

if Seq. $S_n \to S$

then S_n is Cauchy seq. $S \in H$.

denote $S = \sum_1^\infty x_k e_k$

$$\boxed{\begin{array}{l} \|S_n - S\| \to 0 \\ \big|\|S_n\| - \|S\|\big| \to 0 \\ \|S_n\| \to \|S\| \end{array}}$$

Basis (Orthonormal): B is an orthonormal set in H then B is called a basis if $\overline{\mathrm{Span}(B)} = H$

Examples: Fourier Series, Legendre, Hermite, Laguerre

Homomorphism

Ex $T: R^3 \to R^3$, $B = \{i, j, k\}$

$T(x,y,z) = (x,0,0)$. Projection.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Range (T) = x-axis, Kernel $(T) = \{(0,y,z)\}$ Yz plane

Not an isomorphism.

Ex $T: R \to R$ $T(x) = 5x$ $[T] = [5]$

Range $(T) = R$, Kernel $(T) = \{0\}$, Isomorphism

Ex $T: R \to R^2$ $T(x) = (3x, 5x)$ $y' = 5x$ $x' = 3x$ $\Rightarrow 5x' - 3y' = 0$

Range (T) = St-line, Ker $(T) = \{0\}$, Isomorphism, R and R^2 not isomorphic

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

- Homorphism maps $0 \to 0$. $T(x) = x'$ $T(0) = \xi'$

$x' = T(x) = T(x+0) = T(x) + T(0) = x' + \xi' \Rightarrow \xi' = 0$

- ~~Homomorphism~~ isomorphism maps basis of $V \to$ Basis of $T(V)$

$B = \{e_1, e_2, \dots, e_n\}$ $B' = \{e_1', e_2', \dots e_n'\} = T(B)$

$T(e_i) = e_i'$

$\alpha_i e_i' = \alpha_i T(e_i) = T(\alpha_i e_i) = 0$ implicit sum

$\Rightarrow \alpha_i e_i = 0 \Rightarrow \alpha_i = 0 \ \forall i.$

$x' = x_i' e_i'$ has a preimage $x = x_i e_i$

$T(x) = x' = T(x_i e_i) = x_i T(e_i) = x_i e_i'$

- $F^{(n)}$ is isomorphic $F^{(m)}$ only if $n = m$
- $\dim(V) = n$ V is isomorphic to $F^{(n)}$

Let B be basis of V $T(e_i) = (0,0,\dots,\underset{i\text{th place}}{1},0,0)$

- Matrix of Homomorphism $B \subset V_n$ $B' \subset V'_m$ $T: V \to V'$

$T(e_i) = \sum T_{ji} e_i'$

$T(x_i e_i) = \sum_{ji} x_i T_{ji} e_i' = \sum_{j}(\sum_i T_{ji} x_i)\, e_i'$ $T_{m \times n}$

$= x_i' e_i'$

$$\Rightarrow \begin{bmatrix} \vdots \\ x_i' \\ \vdots \end{bmatrix} = T \begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix}$$

GS orthogonalization

1. $\{1, x, x^2, \dots\}$

$P_0 = 1 \qquad P_0 \cdot P_0 = \int_{-1}^{1} dx = 2$

$P_1 = x \qquad P_1 \cdot P_1 = \int_{-1}^{1} x^2 dx = 2/3$

$P_2' = x^2 - \frac{P_2' \cdot P_0}{P_0 \cdot P_0} P_0 + \frac{P_2' \cdot P_1}{P_1 \cdot P_1} P_1 \qquad P_2 \cdot P_0 = \int x^2 dx = 2/3$

$\qquad P_2 \cdot P_1 = 0$

$= x^2 - \frac{2/3}{2} P_0$

$= x^2 - 1/3 \qquad \text{since } P_2'(1) = 2/3$

$P_2 = 1/2\,(3x^2 - 1)$

$P_3' = x^3 - \frac{P_3' \cdot P_1}{P_1 \cdot P_1} P_1 \qquad P_3 \cdot P_1 = \int x^4 dx = 2/5$

$= x^3 - \frac{2}{5} x \qquad P_3'(1) = \frac{3}{5}$

$P_3 = \frac{1}{3}(5x^3 - 2x)$

$P_4' = x^4 - \frac{P_4' \cdot P_2}{P_2 \cdot P_2} P_2 - \frac{P_4' \cdot P_0}{P_0 \cdot P_0} P_0 \qquad P_4' \cdot P_0 = 2/5$

$\qquad P_4' \cdot P_2 = \frac{1}{2}\left(3\,2/7 - 2/5\right) = \frac{1}{2}\left(\frac{30-14}{35}\right) = \cancel{32}\,16/70$

$= x^4 - \frac{{}^{8}16/70}{2/5} P_2 - \frac{2/5}{2} P_0 \qquad P_2 \cdot P_2 = \frac{1}{4}\left[9 \cdot \frac{2}{5} + 2 - 2 \cdot 3 \cdot 2/3\right]$

$= x^4 - \frac{4}{7} \cdot \frac{1}{2}(3x^2 - 1) - 1/5 \qquad = \frac{1}{4}\left[\frac{18}{5} - 2\right] = \frac{2}{5}$

$= x^4 - 6/7\,x^2 + 2/7 - 1/5$

$= x^4 - 6/7\,x^2 + 3/35 \qquad P_4'(1) = 1 - 6/7 + 3/35 = \frac{35 - 30 + 3}{35} = 8/35$

$P_4 = 35/8\,x^4 - \frac{5 \cdot 6}{8} x^2 + 3/8$

$= \frac{1}{8}(35x^4 - 30x^2 + 3)$

Inner Product Spaces

- Introduce IP as generalization of inner product of 3-d vectors
- Definition: V vector space over F. $\phi: V \times V \to F$ s.t.

  ⓐ $u \cdot v = \overline{v \cdot u}$ $\qquad u, v, w \in V$

  ⓑ $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$ $\qquad \alpha, \beta \in F$

  ⓒ $(\alpha u + \beta v) \cdot w = \bar{\alpha}\, u \cdot w + \bar{\beta}\, v \cdot w$

- Ex/ Cartesian product. in $R^{(n)}$

  Ex/ R^2: $\quad x \cdot y = 2x_1 y_1 + x_1 y_2 + y_1 x_2 + x_2 y_2$

  Ex/ V: Set of all complex valued fns on $[0,1]$

  $$f \cdot g = \int_0^1 \bar{f}(x) g(x) dx$$

- length of a vector
- Orthogonality of two vectors, Gram-schmidt process
- Ex/ V: Set of all Polynomials ≤ 2
- Ex/ V: Set of all polynomials ≤ 2 in x on R

  $$p \cdot q = \int_{-\infty}^{\infty} p(x) q(x) e^{-x^2} dx$$

  $\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = 0$ $\quad$ n odd

  $= \frac{1}{a^{\frac{n+1}{2}}} \left(\frac{n-1}{2}\right)!$

- If a set B is orthogonal then B is L.I.
- Definition: length : $\|u\|$

  distance $d(u,v) = \|u - v\|$

- Cor: $\quad \|\alpha u\| = |\alpha| \|u\|$
- Thm: If $u, v \in V$ then $u \cdot v \leq \|u\| \|v\|$

  $$(\lambda u + v) \cdot (\lambda u + v) = \lambda^2 \|u\|^2 + 2\lambda\, u \cdot v + \|v\|^2 \geq 0 \qquad \lambda \text{ arbitrary}$$

  $$\Rightarrow \lambda^2 a^2 + 2\lambda b + c \geq 0$$

  $$\Rightarrow \tfrac{1}{a}(\lambda a + b)^2 + (c - b^2/a) \geq 0$$

  $$\Rightarrow b^2 \leq ac \qquad \text{choose } \lambda = -b/a$$

  $$\Rightarrow u \cdot v \leq \|u\| \|v\| \qquad \text{Cauchy-Schwarz Inequality}$$

- Ex in R^3 $\quad \cos\theta = \frac{u \cdot v}{\|u\| \|v\|} \leq 1$
- Ex in Set of Polynomials

  $$\left| \int_{-1}^{1} p(x) q(x) dx \right|^2 \leq \left[\int_{-1}^{1} p^2(x) dx \right] \left[\int_{-1}^{1} q^2(x) dx \right]$$

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20/3/03 L21 Operators on IPS

- Definition, Examples; Polynomials space, $\hat{x}$, $\hat{D}$ on $C[a,b]$, Simple transformations in $\mathbb{R}^2$
- Hermitian adjoint, Existence $\boxed{\text{Finite D}}$, Matrix Rep w.r.t. orthonormal basis, Hermitian or Self Adjoint
- Unitary Transformations, Isomorphism, Matrix Rep w.r.t. orthonormal basis

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* If A an operator on V. and B is another operator s.t.

$$(A\alpha, \beta) = (\alpha, B\beta)$$ then we say B is hermitian adjoint of A

and denote $B \equiv A^\dagger$ (and vice versa)

- If V is finite dimensional then the adjoint always exists, otherwise it may not.
- $\{e_1, e_2, \dots, e_n\}$ be the basis of V. $[A]$ be the matrix of A and $[B]$ of B

then $$A e_i = \sum_j A_{ji} e_j \Rightarrow A_{ji} = \cancel{(e_j \cdot A e_i)} (A e_i \cdot e_j)$$

$$B_{ji} = (B e_i \cdot e_j) = \overline{(e_j \cdot B e_i)}$$

$$\overline{B_{ij}} = (e_i \cdot B e_j) = A_{ji}$$

$$[B] = [A^*]^T$$

- If $A = A^\dagger$ then we say operator is hermitian

Operators on ~~C/R/C/R~~. IPS.

- Definition of an operator: A LT from $V \to V$ is called an operator on V
- Examples ⓐ Rotation on $\mathbb{R}^2$, matrix rep.
  - ⓑ Set of Polynomials of deg. ≤ 2, $\hat{D}$ operator
  - ⓒ Set of all polynomials, $\hat{D}, \hat{x}$.
  - ⓓ $C[0,1]$, $\hat{x}$, $\hat{D}(?)$.
- Isomorphism from $V \to W$: Is an isomorphism of vector spaces V onto W that preserves the inner product.
  - V & W must have same dimension
  - T carries every orth. Basis of V to orthnormal basis of W.
  - Preserves distances.
- Examples · ⓐ Rotations above
  - ⓑ $\hat{D}$ is not an isomorphism in set of polynomials of deg ≤ 2
- A unitary operator is an isomorphism of V onto itself.
  - If U is unitary operator then $UU^{\dagger} = U^{\dagger}U = \mathbb{1}$

$$(U\alpha, \beta) = (U\alpha, UU^{-1}\beta) = (\alpha, U^{-1}\beta) \Rightarrow U^{-1} = U^{\dagger} \Rightarrow \text{required!}$$

  - Corresponding matrices also are unitary.
  - Set of all $n \times n$ invertible matrices is a group $GL(n)$
  - Set of all $n \times n$ unitary operators is also a group $U(n) \supset GL(n)$
- Normal Operators and Spectral theorem.

Group Representations: Let G be a group. V be ~~a vector s~~ an IPS. $GL(V)$ be the group of invertible operators on V. Then a homomorphism $\phi : G \to GL(V)$ is called a representation of group G. V is called representation of V.
If the dimension of V is n, then $GL(n)$ is isomorphic to $GL(V)$
If $\phi : \cancel{GL(V)}\ G \to GL(n)$ then matrix representation of G.
If $\phi : G \to U(n)$ then unitary matrix representation of G

Examples: 1. G be any group and V be abstract. Identity Representation.
2. $G = \{1, -1\}$. $V = R^2$
ⓐ Identity Representation $\phi_1(1) = I_2$ $\phi_1(-1) = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Unfaithful
ⓑ $\phi_2(1) = I_2$, $\phi(-1) = m_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Faith Full Representations
ⓒ $\phi_3(1) = I_2$, $\phi(-1) = R_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
ⓓ $\phi_4(1) = I_2$, $\phi_4(-1) = m_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

③ $G = \{1, -1\}$, V = Set of all polynomials of deg ≤ 2 in x on $[-1, 1]$
Let $\mathbb{P}$ be parity operator $\mathbb{P}p(x) = p(-x)$. $\phi(1) = I$, $\phi(-1) = \mathbb{P}$, matrices

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- Equivalence of Matrix Representations: $\phi_1 : G \to GL(n)$ & $\phi_2 : G \to GL(n)$ are two distinct representations (matrix), the the two representations are called equivalent if $\exists$ matrix of order n, nonsingular S, such that

$$\phi_1(g) = S^{-1} \phi_2(g) S \qquad \forall g \in G$$

Example given above.

- Invariant Subspaces under a group

Let W be a subspace of V. Let T be an operator on V. If $Tx \in W$ for all $x \in W$ then we say that W is an invariant subspace under T.

Examples (a) $T(x,y) = (2x, x+y)$ $\qquad$ x axis is invariant

(b) $T(x,y) = (y,x)$ $\qquad$ $y=x$ line is invariant

(c) Set of all polynomials of deg ≤ 2, $\mathbb{P}$ parity operator

Then $W :: p(x) = a + bx^2$ ~~is~~ is invariant under T.

- In a suitable basis the matrix of T can be transformed to

$$\left[\begin{array}{c|c} D^{(1)} & 0 \\ \hline \chi & D^{(2)} \end{array}\right]$$

- The subspace is called invariant subspace under the group G if all the matrices can be cast in the same form

i.e.
$$\phi(g) = \left[\begin{array}{c|c} D^{(1)}(g) & 0 \\ \hline \chi(g) & D^{2}(g) \end{array}\right] \qquad \forall g \in G$$

In this case the representation is said to be reducible.

- Theorem 1. Each finite dimensional matrix representation is equivalent to a unitary matrix representation.

Proof: Let $G = \{g_1, g_2, \dots, g_m\}$ $\qquad \phi : G \to GL(n)$

$$H = \sum_{g \in G} \phi(g) \phi^{\dagger}(g)$$ Hermitian $\Rightarrow$ diagonalizable $\Rightarrow \exists$ U unitary

s.t.

$$H_d = U^{-1} H U$$: diagonal matrix with real entries.

$$= \sum_g U^{-1} \phi(g) U U^{-1} \phi^{\dagger}(g) U$$

$$= \sum_g \phi'(g) \phi'^{\dagger}(g)$$

$$[H_d]_{kk} = \sum_g \sum_l [\phi'(g)]_{kl} [\phi'(g)]^*_{kl} = \sum_g \sum_l \left|[\phi'(g)]_{kl}\right|^2$$

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right]^{+} = \left[\begin{array}{c|c} A^{+} & X^{+} \\ \hline 0 & B^{+} \end{array}\right]$$

This is also the matrix of represent.
In other words, if $\phi(g)$ has the form $\left[\begin{smallmatrix} A & 0 \\ X & B \end{smallmatrix}\right]$ then $\phi(g^{-1}) = [\phi(g)]^{-1} = [\phi(g)]^{+} = \left[\begin{smallmatrix} A^{+} & X^{+} \\ 0 & B^{+} \end{smallmatrix}\right]$ ~~also~~

must have the same form $\Rightarrow X = X^{+} = 0$

$X = 0 \Rightarrow$ if unitary.

$V = \mathcal{L}_2(\cancel{-\infty, \infty}\ R)$

$\hat{p} = i\hbar \hat{D}$ $\quad \hat{x}$ $\quad (\hat{R} f)(x) = + f(-x)$

$\cancel{R_\pi} : R \to R \Rightarrow R_\pi x = -x \quad G = \{I, R_\pi\}$

$$V = V_e \oplus V_o$$

V_e and V_o are ~~$\hat{R}$~~ invariant under $\hat{R}$ and I

$$H = \hat{p}^2/2m + k\hat{x}^2$$

$[\hat{R}, H] = 0 \Rightarrow$

$T : V_1 \to V_2$

(*) If T is an isomorphism $\Rightarrow m \le n$ $\quad \dim(V_1) = m$

if $m < n$ then let $W' = \mathrm{Range}(T)$ $\quad \dim(V_2) = n$

for every $x \in W'$, $g \in G$

$\phi_2(g) x$ $\quad$ Let $Ty = x$

$= \phi_2(g) T y$

$= T(\phi_1(g) y) \in \mathrm{Range}(T)$

$\Rightarrow W'$ is ~~$\Rightarrow$~~ inv. ~~Subspace~~ V_2 subpace under G which is not possible hence.

$\Rightarrow H_d > 0$ Positive definite Matrix

$$V = U H_d^{1/2}$$

$$\phi''(g) = V^{-1} \phi V = H_d^{-1/2} U^{-1} \phi U H_d^{1/2}$$

$$\Rightarrow \phi''(g) \phi''^{\dagger}(g) = \left[H_d^{-1/2} U^{-1} \phi'(g) H_d^{1/2}\right]\left[H_d^{1/2} \phi'^{\dagger}(g) H_d^{-1/2}\right]$$

$$= H_d^{-1/2} \phi'(g) H_d \phi'^{\dagger}(g) H^{-1/2}$$

$$= \sum_{g'} H_d^{-1/2} \phi'(g) \phi'(g') \phi'^{\dagger}(g') \phi'^{\dagger}(g) H^{-1/2}$$

$$= \cancel{\sum_{g'}} H_d^{-1/2} H_d H_d^{-1/2} = I$$

— Let ϕ be a unitary representation and reducible

$$\left[\begin{array}{c|c} \phi_1(g) & 0 \\ \hline x & \phi_2(g) \end{array}\right] \rightarrow \left[\begin{array}{c|c} \phi_1(g) & 0 \\ \hline 0 & \phi_2(g) \end{array}\right] \text{ — must have this form —}$$

$$\phi(g) = \left[\begin{array}{c|c|c} \phi_1 & 0 & 0 \\ \hline 0 & \phi_2 & 0 \\ \hline 0 & 0 & \phi_3 \end{array}\right]$$

Reduction of a representation in irreducible representations

— Schur lemma

If $\phi_1 : G \rightarrow GL(V_1)$ and $\phi_2 : G \rightarrow GL(V_2)$ are two m and n dimensional IR representations, let T be a ~~operator~~ LT $T: V_1 \rightarrow V_2$ s.t.

$$T\phi_1(g) = \phi_2(g) T \quad \forall g \in G$$

then (i) if $m \neq n$ $T = 0$

(ii) if $m = n$ then $T = cI$

Proof: $W = \operatorname{Ker} T \subset V_1$. $\operatorname{Ker} T = \{x_1 \mid Tx_1 = 0\}$

- if $x_1 \in W \Rightarrow T(\phi_1(g)(x_1)) = \phi_2(g) T x_1 = \phi_2(g)(0) = 0$

$\Rightarrow \phi_1(g)(x_1) \in W \Rightarrow W$ is inv. under ϕ_1 G.

- But ϕ_1 doesnot have inv. Subspaces $\Rightarrow W = \{0\}$ or V_1 itself.

— $\operatorname{Ker} T = \{0\} \Rightarrow T$ is an isomorphism* but $m \neq n$ hence this is not possible

— $\operatorname{Ker} T = V_1 \Rightarrow T = 0$.

if $m = n$ then T is an isomorphism and [illegible] $\neq 0$ let c be any eigenvalue of T then $(T - cI) = 0 \Rightarrow T = cI$.

C_{3v} : $\{e, \psi, \psi^2, \sigma\psi, \sigma\psi^2, \sigma\}$

(1.2)

$$
\begin{array}{ccccccc}
 & e & \psi & \psi^2 & \sigma & \sigma\psi & \sigma\psi^2 \\
D^{(1)} & 1 & 1 & 1 & 1 & 1 & 1 \\
D^{(2)} & 1 & 1 & 1 & -1 & -1 & -1 \\
D^{(3)} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} & \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} & \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ +\sqrt{3}/2 & 1/2 \end{bmatrix}
\end{array}
$$

$\sigma\psi = \psi^2\sigma$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$
\begin{array}{ccccccc}
 & e & \psi & \psi^2 & \sigma & \sigma\psi & \sigma\psi^2 \\
D^{4} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

ψ 120

$\cos(120) = -\sin(30)$
$= -1/2$

$\sin(120) = +\cos(30)$
$= \sqrt{3}/2$

$\sin(240) = 2\sin(120)\cos(120)$
$= -2 \cdot \sqrt{3}/4 = -\frac{\sqrt{3}}{2}$

$\cos(240) = -1/2$

Change the basis by $S^T = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix}$

Show that

$$S^T D^4(\psi) S = \left[\begin{array}{c|c} D^1(\psi) & \\ \hline & D_2^{(2)}(\psi) \end{array}\right]$$

Character Rep:

$$
\begin{array}{cccc}
 & \{e\} & \{\psi, \psi^2\} & \{\sigma, \sigma\psi, \sigma\psi^2\} \\
D^1 & 1 & 1 & 1 \\
D^2 & 1 & 1 & -1 \\
D^3 & 2 & -1 & 0
\end{array}
$$

$\rightarrow 1\cdot1 + 2\cdot1 + 3\cdot1 = 6$

orth. $1\cdot1\cdot2 + 2\cdot1\cdot(-1) + 3\cdot1\cdot0 = 0$

$\rightarrow 4 + 2 = 6$

Let $T = \sum_{g'} \phi_2(g') U \phi_1(g'^{-1})$ For Finite groups

Then $\phi_2(g') T = T \phi_1(g')$

Let matrix of $\phi_1(g)$ as $D_1(g)$ and $\phi_2(g)$ as $D_2(g)$

Choose matrix U s.t. $U_{pq} = 1$ only on p^{th} row and r^{th} col.

$$[T]_{rs} = \sum_{g'} \sum_{t,u} [D_2(g)]_{rt} U_{tu} [D_1(g'^{-1})]_{us}$$

$$= \sum_{g'} [D_2(g)]_{rp} [D_1(g')]_{qs} \qquad p, q \text{ arbitrary}$$

But if $\phi_1 \neq \phi_2$ then $T = 0$

$$\Rightarrow \sum_{g'} [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} = 0$$

$$\Rightarrow \sum_{g} [D_2(g)]_{rp} = 0 \qquad \text{by setting } D_1 = 1$$

Case two when $\phi_1 = \phi_2$ $T = \lambda_{pq} I$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1(g)]_{qs} = \lambda_{pq} \delta_{rs}$$

Sum over, put $r = s$ and sum over r.

$$\Rightarrow \sum_g \sum_r [D_1(g)]_{rp} [D_1(g^{-1})]_{qr} = \lambda_{pq} \cdot n$$

$$\Rightarrow \sum_g [D_1(gg^{-1})]_{qp} = \lambda_{pq}\, n$$

$$\Rightarrow \sum_g I_{qp} = \lambda_{pq}\, n$$

$$\Rightarrow \delta_{pq} \cdot O_g = \lambda_{pq}\, n$$

$$\Rightarrow \lambda_{pq} = \delta_{pq} \frac{O_g}{n}$$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1(g^{-1})]_{qs} = \frac{O_g}{n} \delta_{pq} \delta_{rs}$$

$$\Rightarrow \sum_g [D_2(g)]_{rp} [D_1^*(g)]_{sq} = \frac{O_g}{n} \delta_{pq} \delta_{rs}$$

→ Organize

→ Argue that if there are c IRReps then

$$\sum n_i^2 \leq g.$$

→ Character Representations

$$\chi(g) = \sum_k D_{kk}(g) \qquad \text{equivalent}$$

if ~~g~~ $h = f^{-1} g f$

then $\chi(h) = \chi(g)$ All characters in equivalence class are equal.

→ Orthogonality of Characters

$$\sum_g [D^i_{pq}(g)]_{pq} [D^j(g)]^*_{rs} = \frac{O_g}{n_i} \delta_{ij} \delta_{pr} \delta_{qs}$$

put $p = q$ and $r = s$

$$\sum_g \chi^i(g) \chi^j_i(g)^* = O_g \delta_{ij}$$

$$\sum_k \sqrt{\frac{n_k}{O_g}} \chi^i_g(k) \chi^{j*}(k) \frac{n_k}{g} = \delta_{ij}$$

$\#$ IRReps $\leq$ $\#$ Classes in G

$\mathbb{D}_k$

→ Reducing the representation

$$\chi(g) = \sum_i a_i \chi^i(g) \qquad \text{block diagonal/}$$

$$\sum_g \chi^{(i)*}(g) \chi(g) = \sum_g \sum_i a_i \, \chi^j(g)^* \chi^i(g) = a_i \cdot g$$

$$\Rightarrow a_i = \frac{1}{g} \sum_g \chi^{(i)*}_g \chi(g).$$

→ Condition for reducibility

$$\sum_g \chi^*(g) \chi(g) = O_g \Leftrightarrow \text{IRReps.}$$

$\phi : G \to V$

$\phi_s(t) = st \qquad (\phi_s \circ \phi_{s'})(t)$

$\cancel{\phi_s(t)} \qquad \phi_s(\phi_{s'}(t)) = \phi_s(s't) = ss't$

$= \phi_{ss'}(t)$

Matrix $\quad [D(s)](\cancel{t}) = u$

$D(s)_{t,u} = \delta_{\cancel{s}t, su}$

$tu^{-1} = s$

$t = \cancel{\psi} su$

		e	ψ^2	ψ	σ (u)	$\sigma\psi$	$\sigma\psi^2$
	e	e	ψ^2	ψ	σ	$\sigma\psi$	$\sigma\psi^2$
	ψ	ψ					
t	ψ^2	ψ^2			s		
	σ	σ					
s	$\sigma\psi$	$\sigma\psi$					
	$\sigma\psi^2$	$\sigma\psi^2$					

$[D(s)D(s)](t)$

$\sum_{r'} [D(s)]_{pr'} [D(s')]_{r'q} = \sum_{r} \delta_{sr',p}\, \delta_{s'q,r'}$

$= \sum_{r} \delta_{ss'q,p} \qquad\qquad sr' = p \quad s'q = r'$

$= [D_s(ss')]_{p,q}$

$[D(s)]_{t,u} = 1$ if $tu = s$

$= 0$ otherwise

Regular Repre., contains all

$\chi(e) = O_g \quad \chi(g) = 0$

$a_i = \frac{1}{O_g} \sum_g \chi(g)\, \chi^i_g \cancel{O_g} = \frac{1}{O_g} \cdot O_g \cdot l_i = l_i$

$= \sum_g \chi(g)\chi^*(g^{-1}) = O_g \sum_i l_i^2$

$O_g^2 = O_g \sum l_i^2$

$\Rightarrow \sum l_i^2 = O_g$ ✓

$\sum_u [D(s)]_{tu} [D(s')]_{u,w}$

$= \sum_u \delta_{s,tu} \cdot \delta_{s',uw}$

$tu = s$

$uw = s'$

$ts'w^{-1} = s$

$\rightarrow$

$$\begin{array}{ccc} e & a & b \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$$a^2 = e, \ \forall a \Rightarrow$$
$$(ab)^2 = (ab)(ab) = e$$
$$\boxed{ab = b^{-1}a^{-1} = ba}$$

C_{2v}

e	$R_{\pi/2}$	σ_x	σ_y
$R_{\pi/2}$	e	σ_y	σ_x
σ_x	σ_y	e	$R_{\pi/2}$
σ_y	σ_x	$R_{\pi/2}$	e

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e \quad \begin{array}{c} e \\ R_{\pi/2} \\ \sigma_x \\ \sigma_y \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\pi/2} \quad \begin{array}{c} e \\ R_{\pi/2} \\ \sigma_x \\ \sigma_y \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_x \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

• Regular Representation: G: group. $g = O(G)$. Consider a Matrix ($g\times g$) Repr.

s.t. $[D(s)]_{t,u} = \delta_{t,su}$

<u>Example</u> C_{2v} : above

⇒ To show that this is a representation

$$\sum_u [D(s)]_{t,u}\,[D(s')]_{u,w} = \sum_u \delta_{t,su}\cdot\delta_{u,s'w} \quad\Rightarrow\quad t=su,\ u=s'w$$
$$= \delta_{t,(ss')w}$$
$$= [D(ss')]_{t,w} \quad\Rightarrow\quad t=(ss')w$$

• The characters of Reg. Rep.

$\chi_e = g$, $\chi_s = 0$ $s\neq e$.

• Suppose χ_g^α $\alpha=1,\cdots,k$ (= # of classes ?)

$$\chi_g = \sum a_\alpha \chi_g^\alpha$$

$$a_\alpha = \frac{1}{g}\sum_g \chi_g^{\alpha *}\chi_g = \frac{1}{g}\cdot\chi_e^\alpha\cdot\chi_e = \chi_e^\alpha = l_\alpha = \text{dimension of } \alpha^{th}\ \text{IR}.$$

Each l_α dimensional IR, appears in Reg. Rep., l_α times

• since $\chi_g = \sum_\alpha a_\alpha \chi_g^\alpha$

$$\Rightarrow\quad g = \sum l_\alpha\cdot l_\alpha = \sum_\alpha l_\alpha^2$$

$$\Rightarrow\quad \boxed{\sum l_\alpha^2 = g}$$

Key Results

• Every Finite dimensional matrix Repr. is equivalent to a unitary matrix Rep.
• Reduction of Matrix Repr. and inv. subspaces
• Grand orth. Theorem $\displaystyle\sum_g [D^\alpha(g)]_{rp}\,[D^\beta(g)]^*_{sq} = \frac{O_g}{l_\alpha}\delta_{\alpha,\beta}\,\delta_{rs}\,\delta_{pq}$
• $\sum l_\alpha^2 \leq O_g$. # of IRReps
• # of IRs $\leq$ # Classes.
• $\sum_g \chi_g^\alpha (\chi_g^\beta)^* = O_G\,\delta_{\alpha\beta}$ IRs

4. $G = C_4$ $\quad C_1 = \{e\}$, $C_2 = \{R_{\pi/2}\}$, $C_3 = \{R_\pi\}$ $\quad C_4 = \{R_{3\pi/2}\}$

$n_1 = n_2 = n_3 = n_4 = 1$

	C_1	C_2	C_3	C_4
D^1	1	1	1	1
D^2	1	i	-1	$-i$
D^3	1	-1	1	-1
D^4	1	$-i$	-1	i

$R_{\pi/2}^4 = 1$ $\quad \chi_{R_{\pi/2}} = \pm 1, \pm i$

- Reducible Representation

$$\chi_g = \sum_\alpha a_\alpha \chi_g^\alpha \qquad \therefore\ a_\alpha = \frac{1}{O_g}\sum_g (\chi_g^\alpha)^* \chi_g$$

- Criterion of Reducibility

$$\sum_g \chi_g^* \chi_g = O_G \iff IR$$

<u>Examples</u> 1. $G = \{1, -1\}$

$C_1 = \{1\}$ $\quad C_2 = \{-1\}$, $\quad n_1^2 + n_2^2 = 2 \Rightarrow n_1 = 1,\ n_2 = 1$

	C_1	C_2
D^1	1	1
D^2	1	-1

- Find Classes (k)
- $\sum n_\alpha^2 = g \Rightarrow n_\alpha$
- Construct k×k table
- Identity Rep.
- For 1d rep, <u>$A^n = E$</u>

2. $G = C_3$, $\quad C_1 = \{1\}$, $C_2 = \{R_{2\pi/3}\}$, $C_3 = \{R_{4\pi/3}\}$

$n_1^2 + n_2^2 + n_3^2 = 3 \Rightarrow n_1 = n_2 = n_3 = 1$

	C_1	C_2	C_3
D^1	1	1	1
D^2	1	$e^{i2\pi/3}$	$e^{i4\pi/3}$
D^3	1	$e^{i4\pi/3}$	$e^{2\pi/3}$

$R_{2\pi/3}^3 = e$

$\chi_{R_{2\pi/3}}^3 = 1 \Rightarrow \chi_{2\pi/3} = e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$

3. $G = C_{4v} = \{e, R_{\pi/2}, R_\pi, R_{3\pi/2}, m_x, m_y, \sigma_x, \sigma_y\}$

$C_1 = \{e\}$, $C_2 = \{R_{\pi/2}, R_{3\pi/2}\}$ $\quad C_3 = R_\pi$ $\quad C_4 = \{m_x, m_y\}$ $\quad C_5 = \{\sigma_x, \sigma_y\}$

$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8 \qquad 1, 1, 1, 1, 2$

	$C_1(1)$	$C_2(2)$	$C_3(1)$	$C_4(2)$	$C_5(2)$
D^1	1	1	1	1	1
D^2	1	-1	1	-1	1
D^3	1	-1	$+1$	$+1$	-1
D^4	1	$+1$	$+1$	-1	-1
D^5	2	$a=0$	$b=(-2)$	$c=0$	$d=0$

$R_{\pi/2}^4 = e$

$\chi_{R_{\pi/4}} = 1, -1, i, -i$

$2a + b + 2c + 2d + 2 = 0$

$-2a + b - 2c + 2d + 2 = 0$

$-2a + b + 2c - 2d + 2 = 0$

$2a + b - 2c - 2d + 2 = 0$

$-4a + b = -4$

$4a + b = -4$

$a = 0$

$b = -2$

$4c + 4d = 0$

$-2c + 2d = 0$

$c + d = 0$

$c - d = 0$

① $P_3[x] = \{1, x, x^2\}$ on $\{-1, 1\}$

$G = \{I, T\}$ $\quad (Tf)(x) = f(-x) \quad (If)(x) = f(x)$

	I	T	
D^1	1	+1	χ^1
D^2	1	−1	χ^2

$D \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\chi_I = 3 \qquad \chi_T = 1$

$\chi_g = \sum a_\alpha \chi_g^\alpha \qquad a_1 = \frac{1}{2}(1 \cdot 3 + 1 \cdot 1) = 2$

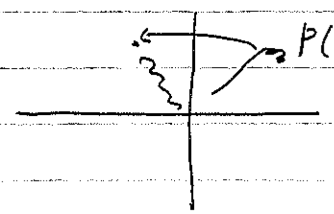
$a_2 = \frac{1}{2}(1 \cdot 3 - 1 \cdot 1) = 1$

② $P_3[x,y] = \{1, x, y, x^2, xy, y^2\}$ 6 dimensional vector space.

$G = \{I, R_\pi\}$ Group of co-ordinate transformation

$GL(P_3[x,y]) \ni U_\pi$

$$(U_\pi f)(x,y) = f(R_\pi^{-1}(x,y))$$



	before	after
Point	P	P
Co-ordinates	(x,y)	$(x',y') = R_\pi(x,y) = (-x,-y)$
fn.	$f(x,y)$	$f'(x',y') = f(x,y) = f(R_\pi^{-1}(x',y'))$

$$(U_\pi f)(x,y) = f(R_\pi^{-1}(x,y)) = f(-x,-y)$$

$U_\pi(1) = 1$

$U_\pi(x) = -x$

$U_\pi(y) = -y$

$U_\pi(x^2) = x^2$

$U_\pi(xy) = xy$

$U_\pi(y^2) = y^2$

$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{6\times 6} \qquad \begin{bmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix}$

$\chi_I = 6 \quad \chi_{U_\pi} = 2 \Rightarrow a_1 = 4$ and $a_2 = 2$

Plugpoint

- PC
- Pagemaker, office.
- Scanner
- Printer
- Sketch Pens -
- Robring Pen (0.3 & 0.5) Twosel
- Papers
- Plastic Scales
- HB, B,