Tutorial 4: Laplace Equation

1. Let function $\phi(x, y, z)$ be a solution to the Laplace equation $\nabla^2\phi = 0$ in a volume V. Define a vector field $\mathbf{E} = -\nabla\phi$. Let S be any simple closed surface inside V.

   (a) Prove Gauss theorem: $\oint_s \mathbf{E} \cdot d\mathbf{S} = 0$

   (b) Deduce that the function ϕ cannot have a maximum or a minimum value at any point inside S.

   (c) If ϕ is constant over the points of S, it must be constant throughout the interior of S.

Answer:

(a) Note that, in electrostatics, we first define $\mathbf{E}$ and then from Gauss law we deduce the existence of the potential ϕ. Here, it is opposite, the function ϕ is defined first using Laplace equation and then we are trying to prove a theorem concerning $\nabla\phi$. So,

$$\oint_s \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} d\tau = -\int_V \nabla^2\phi \, d\tau = 0$$

since $\nabla^2\phi = 0$ in V.

(b) Intuitively, this theorem is clear. If there is a maximum in ϕ then the function decreases in every direction $\hat{n}$, making $\hat{n} \cdot \nabla\phi < 0$ in some nbd v of the maximum which is contradictory with part (a). Mathematically, the proof depends on the *mean value theorem* for Laplace equation. The thorem says that the value of ϕ at the center of a spherical region is the average of ϕ over the region! This clearly says that there can't be a extremum value inside the region.

(c) Since there are no extrema inside a region, all extremum values must be on the boundary.

2. A simple closed surface S encloses a volume V. Assuming that a solution to the Dirichlet problem of the Lapalce equation in V exists, show that it must be unique.

Answer:
If ϕ_1 and ϕ_2 are two solutions of Laplace equation in V with identical values on boundary, then $\phi_1 - \phi_2$ is the solution of Laplace equation with zero value on the boundary. According to the part (c) of Q1, then $\phi_1 - \phi_2 = 0$ in V.

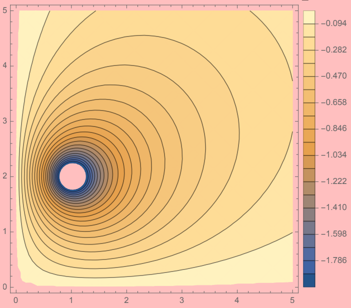
3. Consider a volume $V = \{(x, y, z) \,|x, y \geq 0\}$ bounded by *grounded conducting plates* at surfaces $x = 0$ and $y = 0$. An infinite wire of uniform linear charge density λ is kept parallel to z-axis and passes through point $(1, 2, 0)$. Solve this problem using the method of images and sketch the equipotential lines in XY plane. If you can, use some software to make the sketch (for example, use Mathematica, Maple, Matlab, Gnuplot etc. Or write your own code using Java, Javascript etc.)

Answer:
The images are also wires running parallel to z axis. Two wires with charge density $-\lambda$ pas through $(-1, 2, 0)$ and $(1, -2, 0)$ and one wire with charge density λ passes through $(-1, -2, 0)$. The net potential is given by

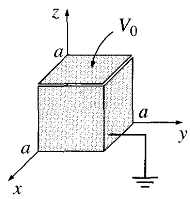
$$\phi(x, y, z) = \frac{\lambda}{2\pi\epsilon_0}\left(\ln\frac{r_1 r_3}{r_2 r_4}\right)$$

where $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ where $(x_i, y_i, 0)$ are the locations of the wires.

The sketch in the first quadrant (our region of interest) is shown below.



4. **Griffiths Problem 3.15** A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (See Fig). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.



Answer:
After variable separation, we get

$$\begin{aligned} V(x, y, z) = &\left(A \cos k_x x + B \sin k_x x\right)\left(C \cos k_y y + D \cos k_y y\right) \\ &\times \left(E \cosh k_z z + F \sinh k_z z\right) \end{aligned}$$

where $k_z^2 = k_x^2 + k_y^2$.

- ▷ BC at $x = 0$ implies that $A = 0$. BC at $x = a$ implies that $k_x = l\pi/a$ for positive integer l.
- ▷ BC at $y = 0$ implies that $C = 0$. BC at $y = a$ implies that $k_y = m\pi/a$ for positive integer m.
- ▷ BC at $z = 0$ implies that $E = 0$.

Thus,

$$V(x, y, z) = \sum_{lm} B_{lm} \sin \frac{l\pi}{a} x \sin \frac{m\pi}{a} y \sinh \frac{\pi\sqrt{l^2+m^2}}{a} z$$

and the final BC says

$$V(x, y, a) = V_0 = \sum_{lm} B_{lm} \sin \frac{l\pi}{a} x \sin \frac{m\pi}{a} y \sinh \pi\sqrt{l^2+m^2}.$$

Using the fourier trick, we get

$$\begin{aligned} B_{lm} \cdot \frac{a}{2} \cdot \frac{a}{2} \cdot \sinh \pi\sqrt{l^2+m^2} &= \int_0^a V_0 \sin \frac{l\pi}{a} x \sin \frac{m\pi}{a} y dx dy \\ &= \frac{V_0 a^2}{\pi^2 lm} \left(1 - (-1)^l\right)\left(1 - (-1)^m\right) \end{aligned}$$

Thus, the final solution is

$$V(x, y, z) = \frac{16 V_0}{\pi^2} \sum_{lm=1,3,\ldots} \left(\frac{1}{lm \sinh \pi\sqrt{l^2+m^2}}\right) \sin \frac{l\pi}{a} x \sin \frac{m\pi}{a} y \sinh \frac{\pi\sqrt{l^2+m^2}}{a} z.$$

5. Consider a semi-infinite plate (bounded by $x = 0$, $x = \pi$ and $y = 0$). The bottom edge is held at temperature $T(x) = \cos x$ and the other sides are at $0°$. Find the steady state temperature in the plate and sketch isotherms.

Answer:
After variable separation, we get

$$T(x, y) = (A\cos kx + B\sin kx)\left(Ce^{-ky} + De^{ky}\right).$$

- ▷ BC at $x = 0$ implies that $A = 0$. BC at $x = \pi$ implies that $k = n$ for positive integer l.
- ▷ BC at $y = \infty$ implies that $D = 0$.

Thus,

$$T(x, y) = \sum_n A_n \sin(n\,x)\, e^{-ny}$$

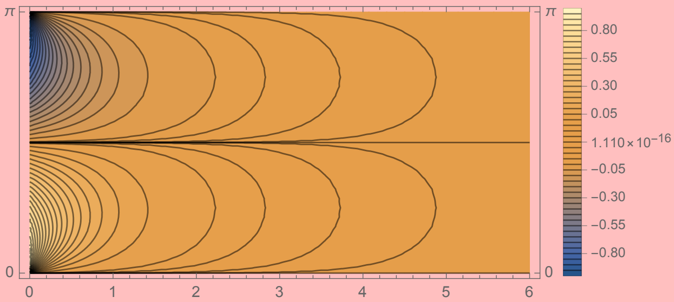
and the final BC says

$$\begin{aligned} T(x, 0) = \cos x &= \sum_n A_n \sin(n\,x) \\ \implies A_n &= \frac{2}{\pi}\int_0^\pi \cos x \sin nx\, dx \\ &= \begin{cases} \frac{2}{\pi}\frac{2n}{n^2-1} & \text{even } n \\ 0 & \text{odd } n \end{cases} \end{aligned}$$

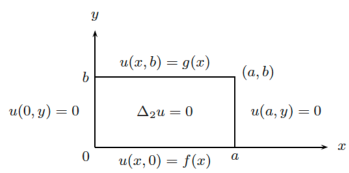
Thus, the final solution is

$$T(x, y) = \frac{4}{\pi}\sum_{n=2,4,\ldots} \frac{n}{n^2-1}\sin(n\,x)\, e^{-ny}.$$

Isotherms are shown in the figure below.



6. Use the method of separation of variables to solve the Dirichlet problem for the Laplace equation, $\nabla^2 u(x, y) = 0$ on the rectangle $R = \{(x, y)\,|0 < x < a, 0 < y < b\}$, satisfying the boundary conditions as shown in the figure. Here, $f(x) = 0$ and $g(x) = x$. (Δ_2 is another less commonly used symbol for the Laplacian operator.) Sketch isocurves of u.



Answer:
After separation, we get

$$u(x, y) = (A\cos kx + B\sin kx)(C\sinh ky + D\cosh ky).$$

- ▷ BC at $x = 0$ implies that $A = 0$.
- ▷ BC at $x = a$ implies that $k = n\pi/a$ for positive integer n.

Then,

$$u(x,y) = \sum_n \sin\frac{n\pi}{a}x\left(C_n\sinh\frac{n\pi}{a}y + D_n\cosh\frac{n\pi}{a}y\right)$$

▷ BC at $y = 0$:

$$f(x) = \sum_n D_n \sin\frac{n\pi}{a}x$$
$$\implies D_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

▷ BC at $y = b$:

$$u(x,y) = \sum_n \sin\frac{n\pi}{a}x\left(C_n\sinh\frac{n\pi b}{a} + D_n\cosh\frac{n\pi b}{a}\right)$$
$$\left(C_n\sinh\frac{n\pi b}{a} + D_n\cosh\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}dx$$

Since D_n is known, we can compute C_n.

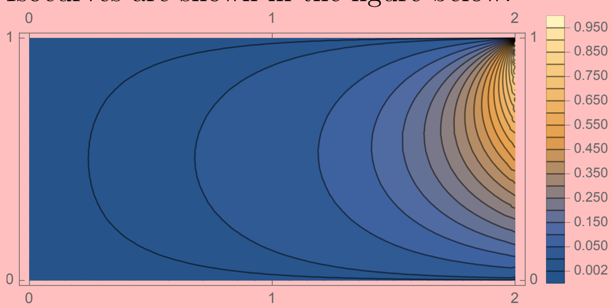
If $f(x) = 0$ then $D_n = 0$ for all n. And

$$C_n = \frac{2}{a\sinh n\pi b/a}\int_0^a x\sin\frac{n\pi x}{a}dx$$
$$= \frac{2}{a\sinh n\pi b/a}\cdot\frac{(-1)^{n+1}a^2}{\pi n}$$

The full solution is then

$$u(x,y) = \frac{2a}{\pi}\sum_n \frac{(-1)^{n+1}}{n\sinh n\pi b/a}\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y.$$

Isocurves are shown in the figure below:



7. **Jackson 2.14** A variant of the preceding two-dimensional problem (we discussed problem the problem 2.13 in the class) is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$. Show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}$$

Answer:
The solution is of the form

$$\Phi(\rho,\phi) = \left(A\rho^n + B\rho^{-n}\right)(C\sin n\phi + D\cos n\phi)$$

Since $\rho = 0$ axis is inside the cylinder, the requirement that Φ is bounded implies that $B = 0$. Thus, we get

$$\Phi(\rho,\phi) = \sum_{n=0}^{\infty}\rho^n(C_n\sin n\phi + D_n\cos n\phi)$$

And

$$
\begin{aligned}
C_n &= \frac{1}{\pi b^n} \int_0^{2\pi} \Phi(b,\phi) \sin n\phi \, d\phi \\
&= \frac{V}{\pi b^n} \left[\int_0^{\pi/2} \sin n\phi \, d\phi - \int_{\pi/2}^{\pi} \sin n\phi \, d\phi + \int_{\pi}^{3\pi/2} \sin n\phi \, d\phi - \int_{3\pi/2}^{2\pi} \sin n\phi \, d\phi \right] \\
&= \frac{V}{\pi b^n n} \left[(1 - \cos n\pi/2) - (\cos n\pi/2 - \cos n\pi) + (\cos n\pi - \cos 3n\pi/2) - (\cos 3n\pi/2 - 1) \right] \\
&= \frac{8V}{\pi b^n n} \qquad \text{when } n = 2, 6, \ldots, 4n+2, \ldots
\end{aligned}
$$

Similarly, show that $D_n = 0$ for all n. This gives the final solution

$$
\Phi(\rho,\phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin\left[(4n+2)\phi\right]}{2n+1}
$$

8. A circular metallic (thermally conducting) disc of radius a is subjected to the following boundary conditions. Find the steady state temperature $T(\rho,\phi)$ in the disc. Sketch isotherms.

   (a) $T(a,\phi) = \frac{1}{2}\left(1 + \cos^3\phi\right), \qquad -\pi < \phi < \pi$

   (b) $T(a,\phi) = |\phi|, \qquad -\pi < \phi < \pi$

Answer:

(a) Since, $T(a,\phi) = \frac{1}{2}\left(1 + \cos^3\phi\right) = \frac{1}{2}\left(1 + \frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi\right)$. By just comparing this expression with the general expression

$$
\Phi(\rho,\phi) = \sum_{n=0}^{\infty} \rho^n \left(C_n \sin n\phi + D_n \cos n\phi\right)
$$

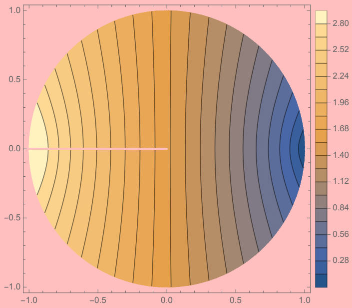
we can confirm, just by inspection, that $C_n = 0$ for all n and $D_0 = \frac{1}{2}$, $D_1 = \frac{3}{8a}$ and $D_3 = \frac{1}{8a^3}$. All other D_n must be zero. Thus

$$
T(\rho,\phi) == \frac{1}{2}\left(1 + \frac{3\rho}{4a}\cos\phi + \frac{1}{4}\left(\frac{\rho}{a}\right)^3 \cos 3\phi\right).
$$

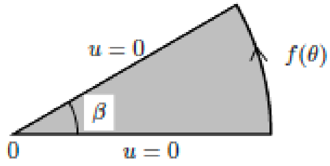
(b) Since this is similar to the Q7, I will give you the answer

$$
T(\rho,\phi) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,\text{odd}}^{\infty} \frac{1}{n^2} \frac{\rho^n}{a^n} \cos n\phi.
$$

Can you guess isotherms?



9. Solve the Laplace equation $\nabla^2 u(\rho,\phi) = 0$ in the wedge with three sides $\phi = 0, \phi = \beta$, and $\rho = a$ (see Figure) and the boundary conditions $u(\rho,0)) = 0 = u(\rho,\beta)$, and $u(a,\phi) = f(\phi)$ for $0 < \phi < \beta$.



Answer:
The solution is of the form

$$u(\rho,\phi) = E + F\ln\rho + \left(A\rho^n + B\rho^{-n}\right)(C\sin n\phi + D\cos n\phi)$$

but now $\phi \in (0,\beta)$. Since the angle ϕ does not span the whole range of $0 \to 2\pi$, we cannot impose singlevaluedness property on u and consenquently n is not necessarily an integer.

- ▷ u must be finite at $\rho = 0$, thus $B = 0$ and $F = 0$.
- ▷ BC at $\phi = 0$ implies that

$$\begin{aligned} u(\rho,0) &= E + A\rho^n (C\sin 0 + D\cos 0) \\ 0 &= E + (AD)\rho^n. \end{aligned}$$

  This implies that $D = 0$ and $E = 0$.

- ▷ BC at $\phi = \beta$ implies that $n\beta = m\pi$ for some integer m. Thus, $n = m\pi/\beta$.

Now, we can write the linear combination of all solutions to obtain general solution,

$$u(\rho,\phi) = \sum_{m=1}^{\infty} A_m \rho^{m\pi/\beta} \sin\left(m\pi\frac{\phi}{\beta}\right)$$

The final boundary condition is

$$f(\phi) = \sum_{m=1}^{\infty} A_m a^{m\pi/\beta} \sin\left(m\pi\frac{\phi}{\beta}\right)$$

which gives us

$$A_m = \frac{2}{\beta} a^{-m\pi/\beta} \int_0^\beta f(\phi)\sin\left(m\pi\frac{\phi}{\beta}\right) d\phi$$