7 Continuous-Time Fourier Series

Recommended Problems

P7.1

(a) Suppose that the signal $e^{j\omega t}$ is applied as the excitation to a linear, time-invariant system that has an impulse response $h(t)$. By using the convolution integral, show that the resulting output is $H(\omega)e^{j\omega t}$, where $H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$.

(b) Assume that the system is characterized by a first-order differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

If $x(t) = e^{j\omega t}$ for all t, then $y(t) = H(\omega)e^{j\omega t}$ for all t. By substituting into the differential equation, determine $H(\omega)$.

P7.2

(a) Suppose that z^n, where z is a complex number, is the input to an LTI system that has an impulse response $h[n]$. Show that the resulting output is given by $H(z)z^n$, where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}.$$

(b) If the system is characterized by a first-order difference equation,

$$y[n] + ay[n-1] = x[n],$$

determine $H(z)$.

P7.3

Find the Fourier series coefficients for each of the following signals:

(a) $x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$

(b) $x(t) = 1 + \cos(2\pi t)$

(c) $x(t) = [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right]$

Hint: You may want to first multiply the terms and then use appropriate trigonometric identities.

P7.4

By evaluating the Fourier series analysis equation, determine the Fourier series for the following signals.

(a)

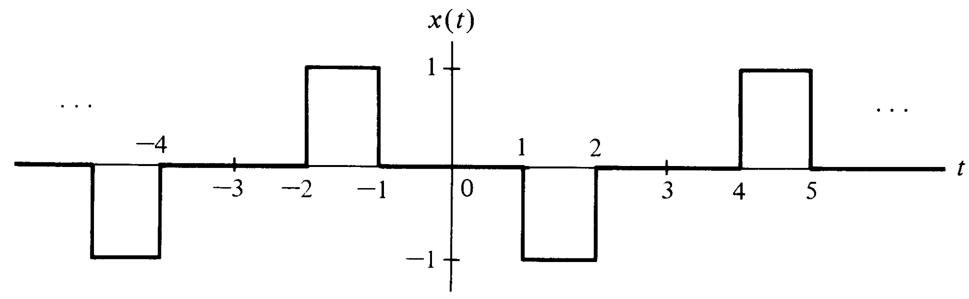

Figure P7.4-1

(b)

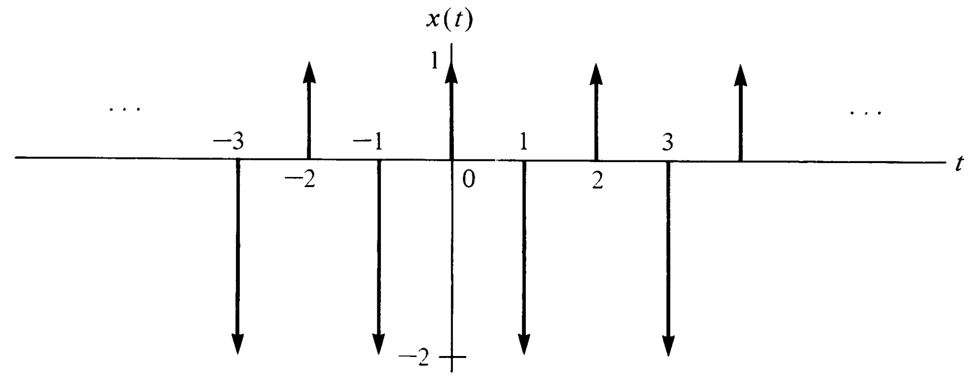

Figure P7.4-2

P7.5

Without explicitly evaluating the Fourier series coefficients, determine which of the periodic waveforms in Figures P7.5-1 to P7.5-3 have Fourier series coefficients with the following properties:

(i) Has only odd harmonics

(ii) Has only purely real coefficients

(iii) Has only purely imaginary coefficients

(a)

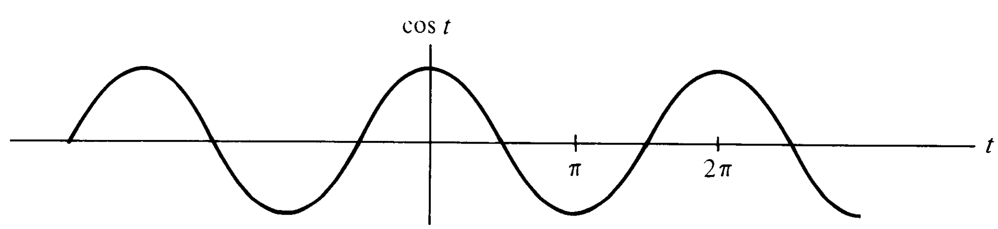

Figure P7.5-1

(b)

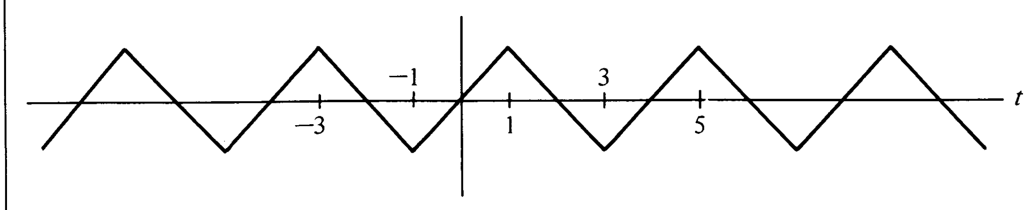

Figure P7.5-2

(c)

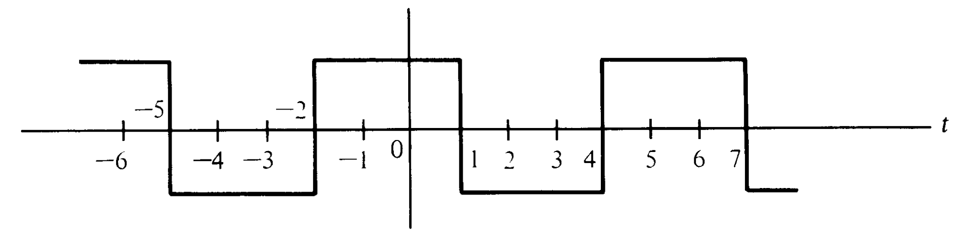

Figure P7.5-3

Optional Problems

P7.6

Suppose $x(t)$ is periodic with period T and is specified in the interval $0 < t < T/4$ as shown in Figure P7.6.

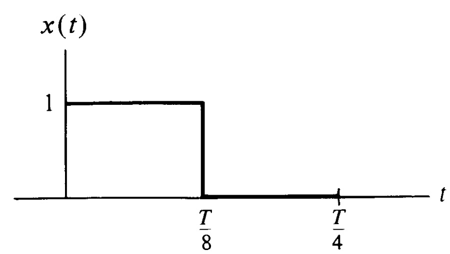

Figure P7.6

Sketch $x(t)$ in the interval $0 < t < T$ if

(a) the Fourier series has only odd harmonics and $x(t)$ is an even function;

(b) the Fourier series has only odd harmonics and $x(t)$ is an odd function.

P7.7

Let $x(t)$ be a periodic signal, with fundamental period T_0 and Fourier series coefficients a_k. Consider the following signals. The Fourier series coefficients for each can be expressed in terms of the a_k as in Table 4.2 (page 224) of the text. Show that the expression in Table 4.2 is correct for each signal.

(a) $x(t - t_0)$

(b) $x(-t)$

(c) $x^*(t)$

(d) $x(\alpha t)$, $\alpha > 0$ (Determine the period of the signal.)

P7.8

As we have seen in this lecture, the concept of an eigenfunction is an extremely important tool in the study of LTI systems. The same can also be said of linear but time-varying systems. Consider such a system with input $x(t)$ and output $y(t)$. We say that a signal $\phi(t)$ is an *eigenfunction* of the system if

$$\phi(t) \rightarrow \lambda\phi(t)$$

That is, if $x(t) = \phi(t)$, then $y(t) = \lambda\phi(t)$, where the complex constant λ is called the *eigenvalue associated with* $\phi(t)$.

(a) Suppose we can represent the input $x(t)$ to the system as a linear combination of eigenfunctions $\phi_k(t)$, each of which has a corresponding eigenvalue λ_k.

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \phi_k(t)$$

Express the output $y(t)$ of the system in terms of $\{c_k\}$, $\{\phi_k(t)\}$, and $\{\lambda_k\}$.

(b) Show that the functions $\phi_k(t) = t^k$ are eigenfunctions of the system characterized by the differential equation

$$y(t) = t^2 \frac{d^2x(t)}{dt^2} + t\,\frac{dx(t)}{dt}$$

For each $\phi_k(t)$, determine the corresponding eigenvalue λ_k.

P7.9

In the text and in Problem P4.10 in this manual, we defined the periodic convolution of two periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ that have the same period T_0. Specifically, the periodic convolution of these signals is defined as

$$\tilde{y}(t) = \tilde{x}_1(t) \circledast \tilde{x}_2(t) = \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau \qquad \text{(P7.9-1)}$$

As shown in Problem P4.10, any interval of length T_0 can be used in the integral in eq. (P7.9-1), and $\tilde{y}(t)$ is also periodic with period T_0.

(a) If $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, and $\tilde{y}(t)$ have Fourier series representations

$$\tilde{x}_1(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T_0)t}, \quad \tilde{x}_2(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk(2\pi/T_0)t}, \quad \tilde{y}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk(2\pi/T_0)t},$$

show that $c_k = T_0 a_k b_k$.

(b) Consider the periodic signal $\tilde{x}(t)$ depicted in Figure P7.9-1. This signal is the result of the periodic convolution of another periodic signal, $\tilde{z}(t)$, with itself.

Find $\tilde{z}(t)$ and then use part (a) to determine the Fourier series representation for $\tilde{x}(t)$.

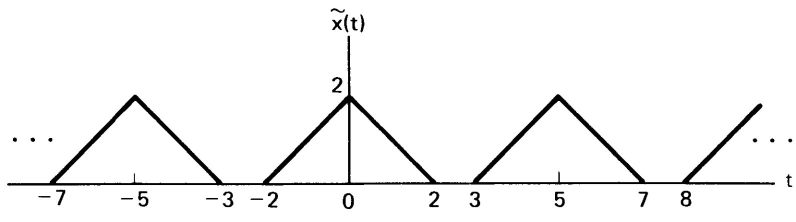

Figure P7.9-1

(c) Suppose now that $x_1(t)$ and $x_2(t)$ are the finite-duration signals illustrated in Figure P7.9-2(a) and (b). Consider forming the periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, which consist of periodically repeated versions of $x_1(t)$ and $x_2(t)$ as illustrated for $\tilde{x}_1(t)$ in Figure P7.9-2(c). Let $y(t)$ be the usual, aperiodic convolution of $x_1(t)$ and $x_2(t)$,

$$y(t) = x_1(t) * x_2(t),$$

and let $\tilde{y}(t)$ be the periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$,

$$\tilde{y}(t) = \tilde{x}_1(t) \circledast \tilde{x}_2(t)$$

Show that if T_0 is large enough, we can recover $y(t)$ completely from one period of $\tilde{y}(t)$, that is,

$$y(t) = \begin{cases} \tilde{y}(t), & |t| \leq T_0/2, \\ 0, & |t| > T_0/2 \end{cases}$$

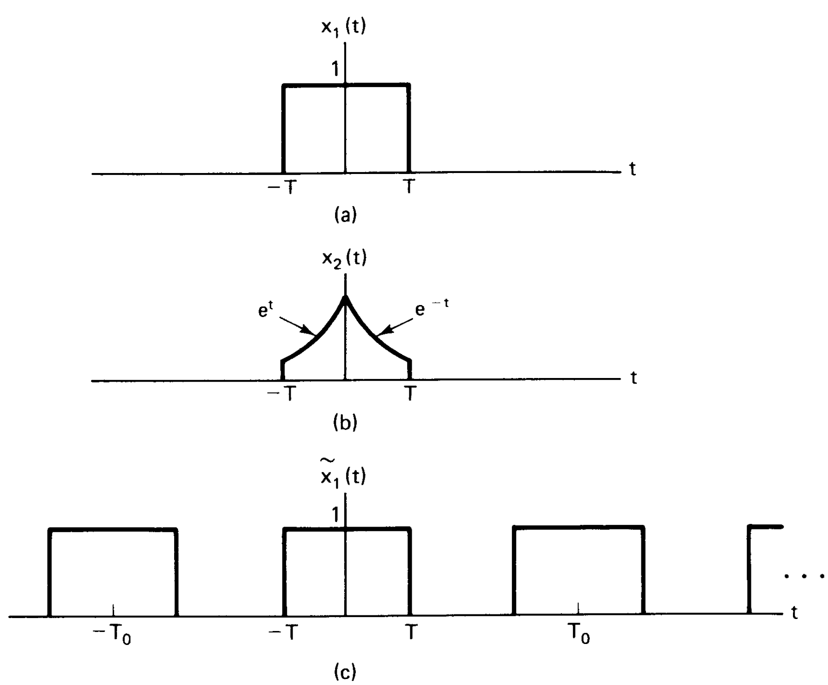

Figure P7.9-2

P7.10

The purpose of this problem is to show that the representation of an arbitrary periodic signal by a Fourier series, or more generally by a linear combination of any set of orthogonal functions, is computationally efficient and in fact is very useful for obtaining good approximations of signals. (See Problem 4.7 [page 254] of the text for the definitions of orthogonal and orthonormal functions.)

Specifically, let $\{\phi_i(t)\}$, $i = 0, \pm 1, \pm 2, \dots$, be a set of orthonormal functions on the interval $a \le t \le b$, and let $x(t)$ be a given signal. Consider the following approximation of $x(t)$ over the interval $a \le t \le b$:

$$\hat{x}_N(t) = \sum_{i=-N}^{+N} a_i \phi_i(t), \qquad \text{(P7.10-1)}$$

where the a_i are constants (in general, complex). To measure the deviation between $x(t)$ and the series approximation $\hat{x}_N(t)$, we consider the error $e_N(t)$ defined as

$$e_N(t) = x(t) - \hat{x}_N(t) \qquad \text{(P7.10-2)}$$

A reasonable and widely used criterion for measuring the quality of the approximation is the energy in the error signal over the interval of interest, that is, the integral of the squared-error magnitude over the interval $a \le t \le b$:

$$E = \int_a^b |e_N(t)|^2 \, dt \qquad \text{(P7.10-3)}$$

(a) Show that E is minimized by choosing

$$a_i = \int_a^b x(t)\phi_i^*(t) \, dt \qquad \text{(P7.10-4)}$$

Hint: Use eqs. (P7.10-1) to (P7.10-3) to express E in terms of a_i, $\phi_i(t)$, and $x(t)$. Then express a_i in rectangular coordinates as $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0, \quad i = 0, \pm 1, \pm 2, \dots, \pm N,$$

are satisfied by the a_i as given in eq. (P7.10-4).

(b) Determine how the result of part (a) changes if the $\{\phi_i(t)\}$ are orthogonal but not orthonormal, with

$$A_i = \int_a^b |\phi_i(t)|^2 \, dt$$

(c) Let $\phi_n(t) = e^{jn\omega_0 t}$ and choose any interval of length $T_0 = 2\pi/\omega_0$. Show that the a_i that minimize E are as given in eq. (4.45) of the text (page 180).

8 Continuous-Time Fourier Transform

Recommended Problems

P8.1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1.

(a) Sketch $x(t)$.

(b) Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.

(c) Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.

(d) Compute a_k, the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm 1, \pm 2, \pm 3$.

(e) Using your answers to (c) and (d), verify that, for this example,

$$a_k = \frac{1}{T_0} X(\omega) \bigg|_{\omega = (2\pi k)/T_0}$$

(f) Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

P8.2

Find the Fourier transform of each of the following signals and sketch the magnitude and phase as a function of frequency, including both positive and negative frequencies.

(a) $\delta(t - 5)$

(b) $e^{-at}u(t)$, $\quad a$ real, positive

(c) $e^{(-1+j2)t}u(t)$

P8.3

In this problem we explore the definition of the Fourier transform of a periodic signal.

(a) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$.

(b) Verify that

$$e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega$$

From this observation, argue that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

(c) Recall the synthesis equation for the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

By taking the Fourier transform of both sides and using the results to parts (a) and (b), show that

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

(d) Sketch $\tilde{X}(\omega)$ for your answer to Problem P8.1(d) for $|\omega| \leq 4\pi/T_0$.

P8.4

(a) Consider the often-used alternative definition of the Fourier transform, which we will call $X_a(f)$. The forward transform is written as

$$X_a(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt,$$

where f is the frequency variable in hertz. Derive the inverse transform formula for this definition. Sketch $X_a(f)$ for the signal discussed in Problem P8.1.

(b) A second, alternative definition is

$$X_b(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-jvt}\, dt$$

Find the inverse transform relation.

P8.5

Consider the periodic signal $\tilde{x}(t)$ in Figure P8.5-1, which is composed solely of impulses.

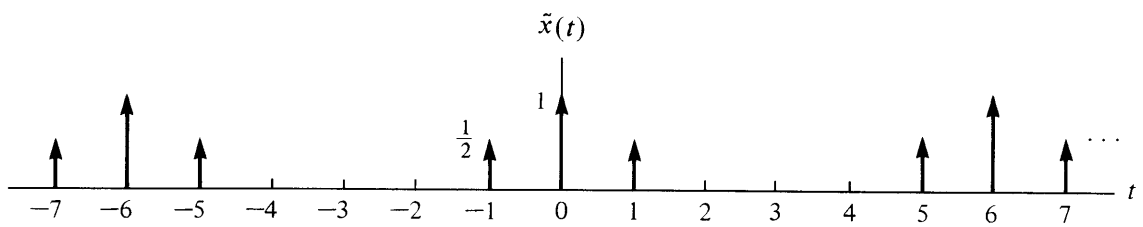

Figure P8.5-1

(a) What is the fundamental period T_0?

(b) Find the Fourier series of $\tilde{x}(t)$.

(c) Find the Fourier transform of the signals in Figures P8.5-2 and P8.5-3.

(i)

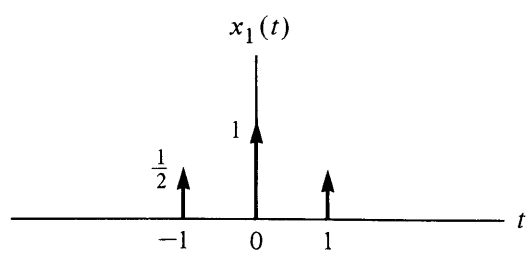

Figure P8.5-2

(ii)

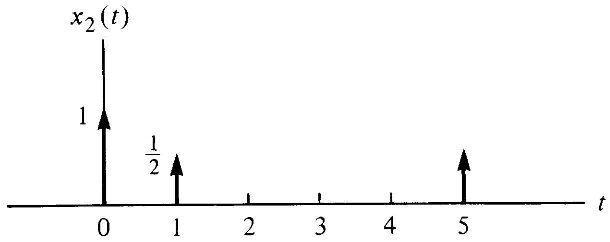

Figure P8.5-3

(d) $\tilde{x}(t)$ can be expressed as either $x_1(t)$ periodically repeated or $x_2(t)$ periodically repeated, i.e.,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_1(t - kT_1), \qquad \text{or} \qquad \text{(P8.5-1)}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_2(t - kT_2) \qquad \text{(P8.5-2)}$$

Determine T_1 and T_2 and demonstrate graphically that eqs. (P8.5-1) and (P8.5-2) are valid.

(e) Verify that the Fourier series of $\tilde{x}(t)$ is composed of scaled samples of either $X_1(\omega)$ or $X_2(\omega)$.

P8.6

Find the signal corresponding to the following Fourier transforms.

(a) $X_a(\omega) = \dfrac{1}{7 + j\omega}$

(b)

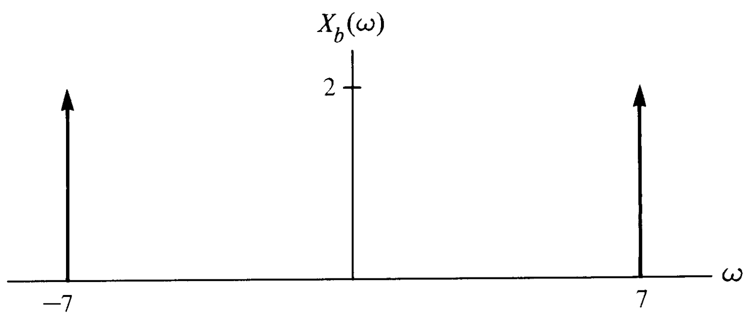

Figure P8.6-1

(c) $X_c(\omega) = \dfrac{1}{9 + \omega^2}$

See Example 4.8 in the text (page 191).

(d) $X_d(\omega) = X_a(\omega)X_b(\omega)$, where $X_a(\omega)$ and $X_b(\omega)$ are given in parts (a) and (b), respectively. Try to simplify as much as possible.

(e)

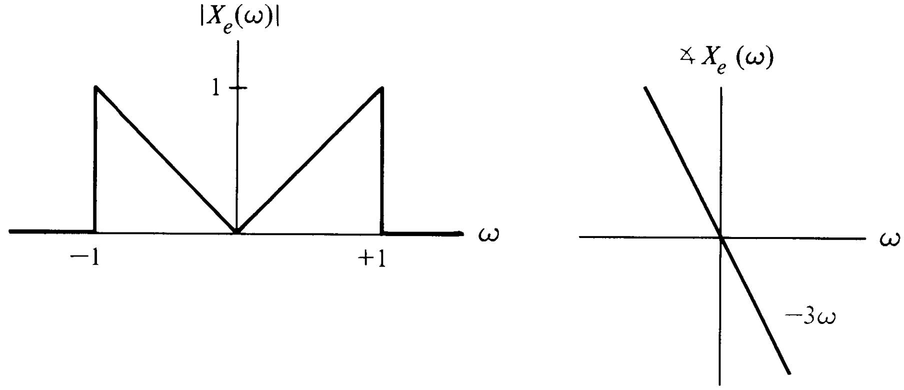

Figure P8.6-2

Optional Problems

P8.7

In earlier lectures, the response of an LTI system to an input $x(t)$ was shown to be $y(t) = x(t) * h(t)$, where $h(t)$ is the system impulse response.

(a) Using the fact that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau,$$

show that

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)e^{-j\omega t}\, d\tau\, dt$$

(b) By appropriate change of variables, show that

$$Y(\omega) = X(\omega)H(\omega),$$

where $X(\omega)$ is the Fourier transform of $x(t)$, and $H(\omega)$ is the Fourier transform of $h(t)$.

P8.8

Consider the impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

shown in Figure P8.8-1.

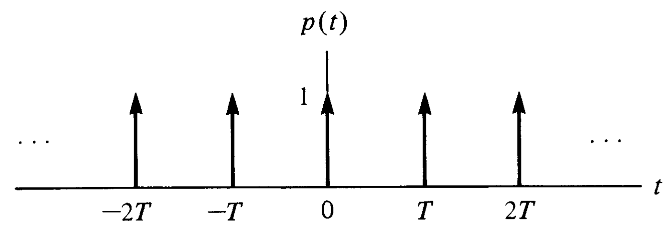

Figure P8.8-1

(a) Find the Fourier series of $p(t)$.

(b) Find the Fourier transform of $p(t)$.

(c) Consider the signal $x(t)$ shown in Figure P8.8-2, where $T_1 < T$.

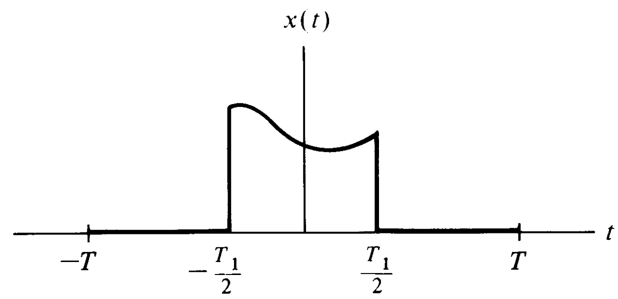

Figure P8.8-2

Show that the periodic signal $\tilde{x}(t)$, formed by periodically repeating $x(t)$, satisfies

$$\tilde{x}(t) = x(t) * p(t)$$

(d) Using the result to Problem P8.7 and parts (b) and (c) of this problem, find the Fourier transform of $\tilde{x}(t)$ in terms of the Fourier transform of $x(t)$.