Tutorial 5: Laplace Equation

Variable Separation in 3D, Conformal Mapping

1. Find the steady-state temperature distribution in solid cylinder of height h and radius a if the top and the curved surface are held at $0°$ and the base at $100°$.

2. Find the steady-state temperature distribution in a solid semi-infinite cylinder (bounded by $\rho = a$ and $z = 0$) if the boundary temperatures are $T = 0$ at $\rho = a$ and $T = \rho \sin\phi$ at $z = 0$. Hints: This problem is similar to the one we did in the class except the last integral. Look at the recursion relations to integrate integrands with Bessel functions.

3. Find the steady-state, bounded temperature distribution in the interior of a solid cylinder of radius a and height h, given that the temperature of the curved lateral surface is kept at zero, the base is *insulated*, and the top is kept at constant temperature u_0.

4. Discuss the image of the circle $|z - 2| = 1$ and its interior under the following transformations:
   (a) $w = z - 2i$;
   (b) $w = 3iz$;
   (c) $w = \dfrac{z-2}{z-1}$;
   (d) $w = \dfrac{z-4}{z-3}$;
   (e) $w = 1/z$.

5. What is the image of the sector $-\pi/4 < \arg z < \pi/4$ under the mapping $w = z/(z-1)$?

6. Write an equation defining a Möbius transformation that maps the half-plane below the line $y = 2x - 3$ onto the interior of the circle $|w - 4| = 2$. Repeat for the exterior of this circle.

7. Two points z_1 and z_2 are said to be symmetric with respect to a circle (line) C if every straight line or circle passing through z_1 and z_2 intersects C orthogonally.
   (a) Show that if C is a line then it must be a perpendicular bisector of the line segment joining z_1 and z_2.
   (b) Show that if C is a circle then z_1 and z_2 lie on some radius of the circle C and if the radius of the circle is R and if distances of z_1 and z_2 from the center of the circle are a and d then $R^2 = ad$.
   (c) Show that if the center of the circle at z_0 in above question, then the same conditions can be put as

$$\begin{aligned} \arg(z_1 - z_0) &= \arg(z_2 - z_0) \\ |z_1 - z_0| &= \frac{R^2}{|z_2 - z_1|} \end{aligned}$$

   (d) Prove following theorem:
   (Symmetry Principle) Let C_z be a line or a circle in z-plane, and let $w = f(z)$ be any Möbius transformation.. Then two points z_1 and z_2 are symmetric with respect to C_z if and only if their images $w_1 = f(z_1)$ and $w_2 = f(z_2)$ are symmetric with respect to the image C_w of C_z under f.

8. Find a point symmetric to $4 - 3i$ with respect to each of the following circles:
   (a) $|z| = 1$;
   (b) $|z - 1| = 1$;
   (c) $|z - 1| = 2$.

9. By Completing following steps prove that any two non intersecting circles C_1 and C_2 there always exist two distinct points z_1 and z_2 that are symmetric with respect to C_1 and C_2 *simultaneously.*

   (a) Argue that there exists a Möbius transformation that maps C_1 onto the real axis and C_2 onto some circle C of the form $|w - \lambda i| = R$ with λ real and $R < |\lambda|$.

   (b) Show that w_1 and w_2 are symmetric with respect $\mathbf{R}$ and C if and only if

$$w_2 = \overline{w_1} \quad \text{and } w_2 = \frac{R^2}{\overline{w_1} + \lambda i} + \lambda i.$$

   Solve this pair of equations to obtain

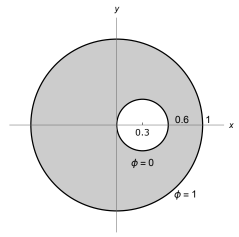
$$w_1 = i\sqrt{\lambda^2 - R^2} \text{ and } w_2 = -i\sqrt{\lambda^2 - R^2}$$

   as simultaneous symmetric points.

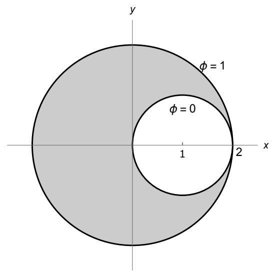
   (c) Use the symmetry principle to conclude that there are points z_1 and z_2 are symmetric with respect to both C_1 and C_2.

10. Use the results of previous problem to show that for any two non-intersecting circles C_1 and C_2 there exists a Möbius transformation that maps C_1 and C_2 onto *concentric circles*. Hint: Map z_1 to origin and z_2 to infinity.

11. Find the function ϕ that is harmonic in the shaded region depicted in the figure and takes values 0 on the inner circle and 1 on the outer circle. This is a cylindrical capacitor with nonconcentric cylinders.



12. Find electrostatic potential in the shaded region in the Figure.



13. Find steady state temp in the shaded region in the Figure.