Answer/solutions to problem set I

① a $\vec{G} = \frac{x\cos\varphi}{\rho}\hat{i} + \frac{2yz}{\rho^2}\hat{j} + \left(1 - \frac{x^2}{\rho^2}\right)\hat{k}$

in spherical polar coordinate system

$$\vec{G} = (\sin\theta\cos^3\varphi + 3\cos\theta\sin^2\varphi)\hat{r}$$
$$+ (\cos\theta\cos^3\varphi + 2\tan\theta\cos\theta\sin^2\varphi - \sin\theta\sin^2\varphi)\hat{\theta}$$
$$+ \sin\varphi\cos\varphi(\sin\varphi - \cos\varphi)\hat{\varphi}$$

~~② Flux of vector A → $\int \vec{A}\cdot\vec{da}$~~

1 b $\vec{A} = \rho z \sin\varphi\,\hat{\rho} + 3\rho\cos\varphi\,\hat{\varphi} + \rho\cos\varphi\sin\varphi\,\hat{k}$

in cartesian coordinates

$$\vec{A} = \frac{1}{\sqrt{x^2+y^2}}\left[(xyz - 3xy)\hat{i} + (y^2z + 3x^2)\hat{j} + xy\hat{k}\right]$$

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② $\vec{A} = 5\rho\hat{\varphi}$

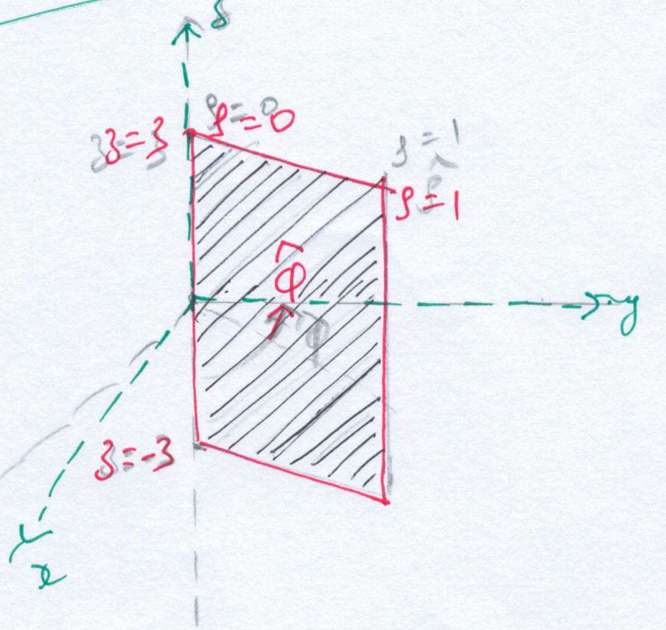
Flux $\int \vec{A}\cdot\vec{da}$,

here $\vec{da} = d\rho\, dz\, \hat{\varphi}$

$$\therefore \int \vec{A}\cdot\vec{da} = \int_{\rho=0,\, z=-3}^{1,\,3} (5\rho\hat{\varphi})\cdot\hat{\varphi}\, d\rho\, dz.$$

$$= 5\left(\int_0^1 \rho\, d\rho\right)\left(\int_{-3}^{3} dz\right)$$

$\boxed{\text{Flux} = 15}$

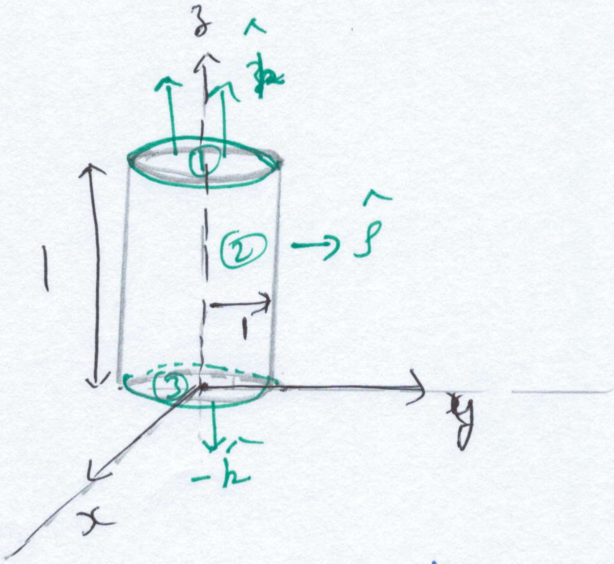


(3)

Divergence Theorem

$$\int (\vec{\nabla} \cdot \vec{G}) \, d\tau = \oint \vec{G} \cdot \vec{da}$$

$$\vec{G} = 10 e^{-2z} (\rho \hat{\rho} + \hat{k})$$



L.H.S.

$$\int (\vec{\nabla} \cdot \vec{G}) \, d\tau = 0$$

$(d\tau = (d\rho)(\rho \, d\phi) \, dz$

limits $\rightarrow$ $\rho \rightarrow 0$ to 1

$\phi \rightarrow 0$ to 2π

$z \rightarrow 0$ to 1

R.H.S $\rightarrow$ Three surfaces assumed

~~i top surface, $\vec{da} = dz \, \rho \, d\phi$~~

i top surface $\vec{da} = (d\rho)(\rho \, d\phi) \hat{k}$, $z = 1$

ii curved surface $\vec{da} = \rho \, d\phi \, dz \, \hat{\rho}$, $\rho = 1$

iii Bottom surface $\vec{da} = (d\rho)(\rho \, d\phi)(-\hat{k})$, $z = 0$

finally $\oint \vec{G} \cdot \vec{da} = 0$

$\Rightarrow$ divergence Theorem is verified.

(4) Stoke's Theorem

$$\int (\vec{\nabla} \times \vec{A}) \cdot \vec{da} = \oint \vec{A} \cdot \vec{dl}$$

~~L.H.S. $\vec{da}$ =~~

$$\vec{A} = \rho \cos\phi \, \hat{\rho} + \sin\phi \, \hat{\phi}$$

Working in cylindrical coordinate system.

of L.H.S $\vec{da} = (d\rho)(\rho \, d\phi) \, \hat{k}$, limits: $\phi \to 30^\circ$ to 60°

$\rho \to 2$ to 5

finally

$$\int (\vec{\nabla} \times \vec{A}) \cdot \vec{da} = 4.941$$

R.H.S. line integration has four parts

i) path ab $\to \vec{dl} = \rho \, d\phi \, \hat{\phi}$, $\rho = 2$, limit $\phi \to 60^\circ$ to 30°

ii) path bc, $\vec{dl} = d\rho \, \hat{\rho}$, $\rho \to 2$ to 5, $\phi = 30^\circ$

iii) path cd $\vec{dl} = \rho \, d\phi \, \hat{\phi}$, $\rho = 5$, $\phi \to 30^\circ$ to 60°

iv) path da, $\vec{dl} = d\rho \, \hat{\rho}$, $\rho = 5$ to 2, $\phi = 60^\circ$

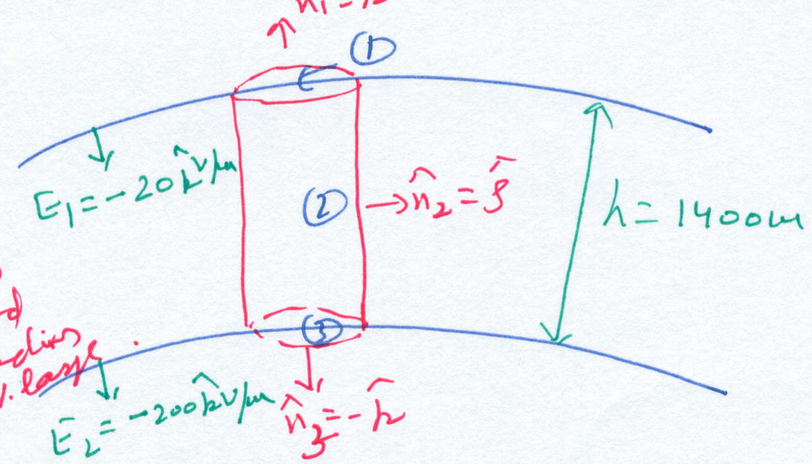
finally

$$\oint \vec{A} \cdot \vec{dl} = \int_{ab} \vec{A} \cdot \vec{dl} + \int_{bc} \vec{A} \cdot \vec{dl} + \int_{cd} \vec{A} \cdot \vec{dl} + \int_{da} \vec{A} \cdot \vec{dl}$$

$$= 4.941$$

Hence Stoke's theorem is verified.

(5)



Note! Here top and bottom surfaces can be considered to be flat as radius of the earth is v. large.

Note: Gaussian surface should be within the layer as shown (in red), From divergence theorem

$$\oint \vec{E}\cdot\vec{da} = \frac{Q_{encl}}{\epsilon_0}, \quad Q_{encl} = \int \rho_v \, \rho \, d\rho \, d\phi \, dz \quad \text{—— (1)}$$

($\rho_v \rightarrow$ average vol. charge density)

L.H.S

$$\oint \vec{E}\cdot\vec{da} = \int_{(1)} \vec{E}\cdot\vec{da} + \int_{(2)} \vec{E}\cdot\vec{da} + \int_{(3)} \vec{E}\cdot\vec{da}$$

2nd term is zero ($\because$ E $\rightarrow$ along $(-\hat{k})$ and da along $\hat{\rho}$ $(\hat{k}\cdot\hat{\rho} = 0)$)

$$\therefore \oint \vec{E}\cdot\vec{da} = \int_{(1)} \vec{E}\cdot\vec{da} + \int_{(2)} \vec{E}\cdot\vec{da}$$

$$= (-20)\int (-\hat{k})\cdot(d\rho)(\rho\, d\phi)\,\hat{k}$$

$$+ (-200)\int (-\hat{k})\cdot d\rho(\rho\, d\phi)\cdot(-\hat{k}) \quad \text{—— (2)}$$

$$= 180\pi\rho^2 = \frac{Q_{encl}}{\epsilon_0}$$

$$Q_{encl} = \int \rho_v \, \rho \, d\rho \, d\phi \, dz = \frac{\rho_v \pi \rho^2 h}{\cancel{\epsilon_0}} \quad \text{—— (3)}$$

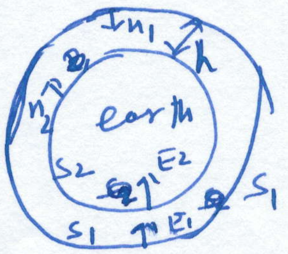
from (1), (2) & (3)

$$\underline{\rho_v} = \frac{180\,\epsilon_0}{h} = \frac{180 \times 8.85 \times 10^{-12}}{1400} = \underline{1.38 \times 10^{-12}\ C/m^3}$$

Alternative to problem 5

Considering the entire vol. in between earth's surface and atmospheric layer 1400m above earth's surface (similar to bicycle tube)

$$\int \vec{\nabla}\cdot\vec{E}\, d\tau = \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho_v\, d\tau \quad \text{--- (1)}$$



$$\oint \vec{E}\cdot d\vec{a} = \int_{S_1} \vec{E}_1\cdot d\vec{a} + \int_{S_2} \vec{E}_2\cdot d\vec{a}$$

$$= -20\times 4\pi (R+h)^2 + 200\, 4\pi R^2 \quad \text{--- (1')}$$

$R \to$ radius of earth

$h << R$

from ~~(1)~~

$$Q_{encl} = \int \rho_v\, d\tau = \rho_v\left[\frac{4}{3}\pi (R+h)^3 - \frac{4}{3}\pi R^3\right] \quad \text{--- (3)}$$

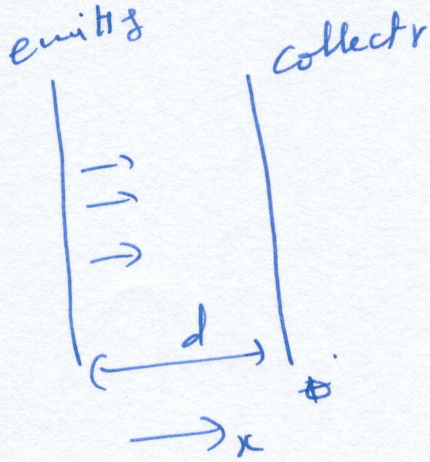
from (1), (2) & 3 and taking $R = 6371$ km (radius of earth)

$\rho = 1.23\times10^{-12}\ C/m^3$, nearly same as previous ~~one~~ case

(6) $V = cx^{4/3}$

$\vec{E} = -\vec{\nabla} V$

$= -\frac{4}{3} c x^{1/3} \hat{i}$



~~surfa~~ at $x = 0$ (emitter)

$E = 0$

at $x = d$ (collector

$\vec{E} = -\frac{4}{3} c d^{1/3} \hat{i}$

As electrodes are ||, Therefore outside the field is zero.

∴ surface charge density

$\sigma = \epsilon_0 E$,

at $x = 0$, $\sigma = 0$

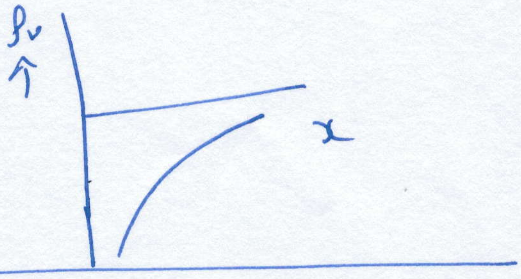
$x = d$ $\sigma = -\frac{4}{3} c d^{1/2} \epsilon_0$ (at the collector.

from the Gauss's law.

$\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$

$\rho_v = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \left(-\frac{4}{3} \cdot \frac{1}{3} c x^{-2/3} \right)$

$$\boxed{\rho_v = -\frac{4}{9} \epsilon_0 c x^{-23}}$$



7. i $\nabla^2 V \neq 0 \Rightarrow$ poisson equation

ii $\nabla^2 V = 0 \rightarrow$ Laplace's equation

iii $\nabla^2 W \neq 0 \rightarrow$ poisson's equation.

(8) $$V = \frac{q e^{-r/\lambda}}{4\pi\epsilon_0 r} \quad \text{—— (1)}$$

$$\vec{E} = -\vec{\nabla} V$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)\hat{r}$$

$$\boxed{\vec{E} = \frac{q e^{-r/\lambda}}{4\pi\epsilon_0}\left[\frac{1}{\lambda r} + \frac{1}{r^2}\right]\hat{r}}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \text{—— (3)}$$

Region (1) $r \to 0$

region (2) $r \neq 0$.

(1) $r \to 0$, $\nabla^2 V = \frac{q}{4\pi\epsilon_0}\nabla^2\left(\frac{1}{r}\right) = \frac{q}{4\pi\epsilon_0}\left(-\delta^3(\vec{r})\right)$

from (3) $\boxed{\rho = \frac{q}{4\pi}\delta^3(\vec{r})}$ —— (4)

(2) $r \neq 0$ $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)$

$$= \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{\lambda^2 r}$$

from (3) $\boxed{\rho(r \neq 0) = -\frac{q}{4\pi\lambda^2}\frac{e^{-r/\lambda}}{r}}$

$\Rightarrow$ electric cloud of $-$ve charge smearing out over the $+$ve charge centred at origin

$\therefore$ Total charge density

$$\rho = \frac{q}{4\pi\epsilon_0}\left[\delta^3(\vec{r}) - \frac{e^{-r/\lambda}}{\lambda^2 r}\right] \quad \text{—— (5)}$$