CYK/2023/PH201 Mathematical Physics

Tutorial 9: Group Theory

Group Theory: Continuous Groups.

1. Verify that the following sets of $n \times n$ matrices for a real Lie algebra and find corresponding Lie groups (obtained by exponentiating them):

   (a) all real matrices;

   (b) all real upper triangular matrices;

   (c) all real upper trinagular traceless matrices;

   (d) all real upper triangular matrices with zero diagonal elements;

   (e) all real traceless matrices.

(a) Closed under matrix addition, additive inverse and commutator bracket.

Gives a Lie group $GL(n)$. Since e^A is a nonsingular matrix.

(b) all real upper triangular matrices: Again closed under addition, additive inverse and commutator bracket.

First Product and sum of upper triangular matrices is upper triangular.

Now $e^A = 1 + A + \underset{\substack{\uparrow \\ \text{upper}}}{\tfrac{1}{2}A^2} + \cdots \Rightarrow e^A$ is upper triangular.

$\Rightarrow$ Lie Group = {nonsingular upper triangular matrices}

(c) Traceless $\Rightarrow \det e^A = \exp(\operatorname{Tr} A) = 1.$

$\Rightarrow$ Lie Group = {non singular upper triangular matrices with det = +1}

(d) Zero diagonal elements $\Rightarrow$ Traceless matrices +

$$e^A = 1 + \underbrace{A + A^2}_{\text{Zero diagonal elements.}}$$

$\Rightarrow$ Lie Group = {non singular, upper triangular with 1 on diagonal, det = +1}

(e) Lie Group = $SL(n)$ = {nonsingular matrices with det = +1}.

2. Let E_{ij} $(i, j = 1, \ldots, n)$ be $n \times n$ matrices such that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Verify that the following sets constitute bases of the Lie algebras of the indicated groups

   (a) E_{ij} for $GL(n, \mathbb{R})$

   (b) E_{ij} and iE_{ij} for $GL(n, \mathbb{C})$

   (c) $E_{|ij|} = \frac{1}{2}(E_{ij} - E_{ji})$ and $iE_{(ij)} = \frac{i}{2}(E_{ij} + E_{ji})$ for $U(n)$ and

   (d) $E_{|ij|}$ and $\tilde{E}_{(ij)} = E_{(ij)} - \frac{1}{n} I \operatorname{tr}(E_{(ij)})$ for $SU(n)$.

(a) The Lie algebra of $GL(n)$ is $gl(n)$ = Vector space of all $n\times n$ matrices

Clearly $\quad A = \sum_{ij} A_{ij} E_{ij}$

For example, when $n=2$, say $A = \begin{pmatrix} 3 & 7 \\ 9 & 2 \end{pmatrix}$ then

$$A = 3 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}} + 7 \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}} + 9 \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{13}} + 2 \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{14}}$$

$$= 3E_{11} + 7E_{12} + 9E_{13} + 2E_{14} \quad \text{etc.}$$

$\Rightarrow$ Thus, $B = \{ E_{ij} \mid i,j = 1,\dots,n \}$ is the basis of $gl(n)$.

(b) The Lie algebra of $GL(n,\mathbb{C})$ is $gl(n,\mathbb{C})$ = Vector space of all $n\times n$ complex matrices.

These are complex now

Now $\quad A = \sum_{ij} A_{ij} E_{ij}$

$$= \sum_{ij} Re(A_{ij}) E_{ij} + \sum_{ij} Im(A_{ij}) (i E_{ij})$$

Thus $gl(n,\mathbb{C})$ is $2n^2$ dimensional REAL Lie algebra.

(c) For $U(n)$, the Lie algebra is a vector space of traceless antihermitian matrices, since, if $A \in U(N)$ and $A = e^L$ then

$\det(A^\dagger A) = 1 \Rightarrow |\det A|^2 = 1 \Rightarrow \det A = e^{i\alpha}$ for some α.

$A^\dagger A = I \Rightarrow A^\dagger = A^{-1} \Rightarrow e^{L^\dagger} = e^{-L} \Rightarrow L^\dagger = -L$.

$\Rightarrow$ diagonals are purely imaginary. Thus structure of L is

$$L = \begin{pmatrix} iL_{11} & L_{12} & & \\ -L_{12}^* & iL_{22} & & \\ & & \ddots & \end{pmatrix} \qquad L_{ii} \text{ are real nos.}$$

$$= L_{11}(iE_{11}) + Re(L_{12})\cdot(E_{12} - E_{21}) + Im(L_{12})\cdot i(E_{12}+E_{21}) + \cdots$$

$$= L_{11}\tfrac{i}{2}(E_{11}+E_{11}) + 2Re(L_{12})\tfrac{1}{2}(E_{12}-E_{21}) + 2Im\, L_{12}\cdot\tfrac{i}{2}(E_{12}+E_{21}) + \cdots$$

Then, the set $\quad B = \{ \tfrac{i}{2}(E_{ij}+E_{ji}),\ \tfrac{1}{2}(E_{ij}-E_{ji}) \mid i \geq j \}$

there $\quad n(n+1)/2 + n(n-1)/2 = n^2$. generators.

(d) Same as above but in addition L must be traceless.

This can be done by subtracting the $\tfrac{1}{n}tr(E_{ii})$ from the diagonal elements.

For example $\quad iE_{11} = i\begin{pmatrix} 1-\tfrac{1}{n} & & & \\ & -\tfrac{1}{n} & & \\ & & -\tfrac{1}{n} & \\ & & & \ddots \end{pmatrix}$

3. Show that the set of all $(n+1)\times(n+1)$ real matrices of the form

$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$$

where A is a $n\times n$ non-singular matrix, a is column matrix with n rows, for a Lie group G that is isomorphic to the affine group $A(n,R)$. What is the Lie algebra of group G? Obtain the commutation relations of the suitable basis. (Note that the affine $A(n,R)$ is group of transformations on $\mathbb{R}^n$ which map $x \mapsto Ax+a$. This group contains translations in addition to transformations in $GL(n)$.)

Answer:

The group multiplication in affine group is given by

$$T_{A,b}\circ T_{c,d} = T_{AC,\,Ad+b}$$

Consider Product of two matrice from the given group

$$\underset{X}{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}}\underset{Y}{\begin{pmatrix} C & d \\ 0 & 1 \end{pmatrix}} = \underset{Z}{\begin{pmatrix} AC & Ad+b \\ 0 & 1 \end{pmatrix}}$$

For $i,j\le n$, $Z_{ij} = \sum_k X_{ik}Y_{kj}$

$$= \sum_{k=1 \text{ to } n} X_{ik}Y_{kj} + X_{i,n+1}Y_{n+1,j}$$

$$= \sum_{k=1,\text{to } n} A_{ik}C_{kj} + b_i\cdot 0 = (AC)_{ij}$$

For $i\le n$ $\quad Z_{i\,n+1} = \sum_{k=1,..,n} X_{ik}Y_{k\,n+1} + X_{i\,n+1}Y_{n+1;n+1}$

$$= \sum_k A_{ik}d_k + b_i\cdot 1 = (Ad+b)_i$$

Thus the two groups are isomorphic.

$T_{A,b}: x \longmapsto Ax+b$

$T_{c,d}: x \longmapsto Cx+d$

Then $T_{A,b}\circ T_{c,d}(x)$

$= T_{A,b}(Cx+d)$

$= A(Cx+d)+b$

$= ACx + (Ad+b).$

4. Find the axis and angle of rotation for the following rotation matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \quad \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{6} & 1 \\ \sqrt{6} & 2 & -\sqrt{6} \\ 1 & \sqrt{6} & 3 \end{pmatrix}$$

Answer:

(a) $R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \cos\theta = \frac{1}{2}(\mathrm{tr}(R)-1) = -1/2 \Rightarrow \theta = 120^\circ.$ $\quad \sin\theta = \sqrt{3}/2$

$a_1 = (R_{32}-R_{23})/2\sin\theta = 1/\sqrt{3}$

$a_2 = (R_{13}-R_{31})/2\sin\theta = 1/\sqrt{3}$

$a_3 = (R_{21}-R_{12})/2\sin\theta = 1/\sqrt{3}.$

(b) $R = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$ $\quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $\quad \sin\theta = 1.$

$a_1 = \frac{1}{\sqrt{2}}, \; a_2 = \frac{1}{\sqrt{2}}, \; a_3 = 0.$

(c) $R = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{6} & 1 \\ \sqrt{6} & 2 & -\sqrt{6} \\ 1 & \sqrt{6} & 3 \end{pmatrix}$ $\quad \cos\theta = 1/2 \Rightarrow \theta = 60^\circ$ $\quad \sin\theta = \sqrt{3}/2$

$a_1 = 2\sqrt{6}/4.2.\sqrt{3}/2 = 1/\sqrt{2}, \; a_2 = 0, \; a_3 = 1/\sqrt{2}.$

5. Show that the two elements of $SO(3)$ belong to the same conjugacy class if and only if they correspond to the same angle of rotation.

Answer:

The matrix for rotation by θ about the unit vector $u = (u_x, u_y, u_z)$ is given by

$$R = \begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \mathrm{co} \end{bmatrix}$$

Note that the Trace of the matrix is given by

$$\operatorname{Tr} R = 1 + 2\cos\theta.$$

Now, let P be any other rotation matrix. Then PRP^{-1} is a similarity transformation and keeps the trace invariant. Which means PRP^{-1} is a rotation with the same angle of rotation.
Also,

$$Re^{\theta S_u}R^{-1} = e^{\theta S_{Ru}},$$

thus, we can always find a way to map one axis of rotation to another.

6. Show that the axis of rotation is an eigenvector of rotation matrix with eigenvalue $+1$ and the other two eigenvalues are complex for angle of rotation $0 < \theta < \pi$.

Answer:

(a) $R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \cos\theta = \frac{1}{2}\left(\operatorname{tr}(R) - 1\right) = -\frac{1}{2} \Rightarrow \theta = 120^\circ.$ $\quad \sin\theta = \sqrt{3}/2$

$a_1 = (R_{32} - R_{23})/2\sin\theta = 1/\sqrt{3}$

$a_2 = (R_{13} - R_{31})/2\sin\theta = 1/\sqrt{3}$

$a_3 = (R_{21} - R_{12})/2\sin\theta = 1/\sqrt{3}.$

(b) $R = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$ $\quad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $\quad \sin\theta = 1.$

$a_1 = \frac{1}{\sqrt{2}}, \quad a_2 = \frac{1}{\sqrt{2}}, \quad a_3 = 0.$

(c) $R = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{6} & 1 \\ \sqrt{6} & 2 & -\sqrt{6} \\ 1 & \sqrt{6} & 3 \end{pmatrix}$ $\quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$ $\quad \sin\theta = \sqrt{3}/2$

$a_1 = 2\sqrt{6}/4 \cdot 2 \cdot \sqrt{3}/2 = 1/\sqrt{2}, \quad a_2 = 0, \quad a_3 = 1/\sqrt{2}.$

7. An infinitesimal Lorentz transformation and its inverse can be written as

$$\begin{aligned} x'^{\alpha} &= \left(g^{\alpha\beta} + \epsilon^{\alpha\beta}\right) x_\beta \\ x^{\alpha} &= \left(g^{\alpha\beta} + \epsilon'^{\alpha\beta}\right) x'_\beta \end{aligned}$$

where $\epsilon^{\alpha\beta}$ and $\epsilon'^{\alpha\beta}$ are infinitesimal.

(a) Show from the definition of the inverse that $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta}$.

(b) Show from the preservation of the norm that $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.

(c) By writing the transformation in terms of contravariant components on both sides of the equation, show that $\epsilon^{\alpha\beta}$ is equivalent to the matrix $-\xi \cdot K - \omega \cdot S$ where K and S are the six generators of the Lorentz group.

Answer:

(a) Since $x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) g_{\beta\gamma} x^{\gamma}$

$= (g^{\alpha\beta} + \epsilon^{\alpha\beta}) g_{\beta\gamma} (g^{\gamma\delta} + \epsilon'^{\gamma\delta}) g_{\delta\lambda} x'^{\lambda} = \delta^{\alpha}{}_{\lambda} x^{\lambda}$

$\Rightarrow (g^{\alpha\beta} + \epsilon^{\alpha\beta}) g_{\beta\gamma} (g^{\gamma\delta} + \epsilon'^{\gamma\delta}) g_{\delta\lambda} = \delta^{\alpha}{}_{\lambda}$

$\Rightarrow g^{\alpha\beta} g_{\beta\gamma} g^{\gamma\delta} g_{\delta\lambda} + \epsilon^{\alpha\beta} g_{\beta\gamma} g^{\gamma\delta} g_{\delta\lambda} + g^{\alpha\beta} g_{\beta\gamma} \epsilon'^{\gamma\delta} g_{\delta\lambda} = \delta^{\alpha}{}_{\lambda}$

$\Rightarrow \delta^{\alpha}{}_{\lambda} + \epsilon^{\alpha\beta} g_{\beta\lambda} + \epsilon'^{\alpha\delta} g_{\delta\lambda} = \delta^{\alpha}{}_{\lambda}$

$\Rightarrow \epsilon^{\alpha\beta} = -\epsilon'^{\alpha\beta}$

(b) Now $x'^{\alpha} x'_{\alpha} = x'^{\alpha} g_{\alpha\beta} x'^{\beta}$

$= (g^{\alpha\delta} + \epsilon^{\alpha\delta}) x_{\delta} \cdot g_{\alpha\beta} \cdot (g^{\beta\gamma} + \epsilon^{\beta\gamma}) x_{\gamma}$

$= (g^{\alpha\delta} + \epsilon^{\alpha\delta}) g_{\alpha\beta} (g^{\beta\gamma} + \epsilon^{\beta\gamma}) g_{\gamma\lambda} x_{\delta} \cdot x^{\lambda}$

$\Rightarrow (g^{\alpha\delta} + \epsilon^{\alpha\delta}) g_{\alpha\beta} (g^{\beta\gamma} + \epsilon^{\beta\gamma}) g_{\gamma\lambda} = \delta^{\delta}{}_{\lambda}$

$\Rightarrow \epsilon^{\alpha\delta} g_{\alpha\beta} g^{\beta\gamma} g_{\gamma\lambda} + g^{\alpha\delta} g_{\alpha\beta} \epsilon^{\beta\gamma} g_{\gamma\lambda} = 0$

$\Rightarrow \epsilon^{\alpha\delta} g_{\alpha\lambda} + \underbrace{\epsilon^{\delta\gamma} g_{\gamma\lambda}}_{\text{change dummy variable } \gamma \to \alpha} = 0$

$\Rightarrow \epsilon^{\alpha\delta} = -\epsilon^{\delta\alpha}$

(c) Clearly $\epsilon^{\alpha\beta}$ is antisymmetric then clearly

$\delta^{\alpha}{}_{\delta} + \epsilon^{\alpha\beta} \cdot g_{\beta\delta} = e^{+\epsilon^{\alpha\beta} \cdot g_{\beta\delta}}$

$\Rightarrow \epsilon^{\alpha}{}_{\delta} = -\omega \cdot S - \xi \cdot K$ (by comparison)

8. For the Lorentz boost and rotation matrices $\mathbf{K}$ and $\mathbf{S}$ show that

$$
\begin{aligned}
(\epsilon' \cdot \mathbf{K})^3 &= \epsilon' \cdot \mathbf{K} \\
(\epsilon \cdot \mathbf{S})^3 &= -\epsilon \cdot \mathbf{S}
\end{aligned}
$$

where ϵ and ϵ' are any real unit 3-vectors.
Use the results of part a to show that

$$
\exp\left(-\xi \hat{\beta} \cdot \mathbf{K}\right) = I - \hat{\beta} \cdot \mathbf{K} \sinh \xi + \left(\hat{\beta} \cdot \mathbf{K}\right)^2 (\cosh \xi - 1).
$$

Answer:

9.