CYK/2023/PH201 Mathematical Physics

Tutorial 1: Analytic Functions

1. Find for each function given below, the domain of definition:

   (a) $f(z) = \frac{1}{z^2+1}$;

   (b) $f(z) = Arg(\frac{1}{z})$;

   (c) $f(z) = \frac{z}{\bar{z}+z}$;

   (d) $f(z) = \frac{1}{1-|z|^2}$.

Answers: (a) $\mathbb{C} - \{i, -i\}$ (b) $\mathbb{C} - \{0\}$ (c) $\mathbb{C}$ except imaginary axis (d) $\mathbb{C}$ except the unit circle about origin.

2. Write each of the following functions in the form $f(z) = u(x, y) + iv(x, y)$:

   (a) $f(z) = z^3 - 1$;

   (b) $f(z) = \sin z$;

   (c) $\mathrm{f}(z) = \log z$.

Answers:

(a) $u(x, y) = x^3 - 3xy^2 - 1$, $v(x, y) = 3x^2y - y^3$

(b) $\sin z = \sin(x + iy)$, $u(x, y) = \sin x \cosh y$, $v(x, y) = \cos x \sinh y$

(c) $u(x, y) = \log\left(\sqrt{x^2 + y^2}\right)$, $v(x, y) = \arctan(x, y) + 2n\pi$.

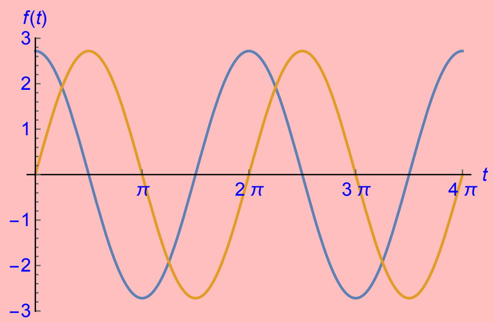
3. A line segment is given by $z_1(t) = (1, t)$ where $0 \le t \le 4\pi$.

   (a) Let $f(t) = \exp(z_1(t)) = u(t) + iv(t)$. Plot $u(t)$ and $v(t)$ as a function of t.

   (b) Do the same for $z_2(t) = (2, t)$ and $z_3 = (t, \pi/6)$.

Answers:

(a) $u(x, y)$: Blue, $v(x, y)$: Orange



4. Show that

   (a) $\sin^2 z + \cos^2 z = 1$;

   (b) $\sin^2(1 + i) = 1.2828 + 1.6489i$ and $\cos^2(1 + i) = -0.2828 - 1.6489i$;

   (c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$;

   (d) $\cosh^2 z - \sinh^2 z = 1$;

   (e) $\cosh^2(1 + i) = -.2828 + 1.6489i$ and $\sinh^2(1 + i) = -1.2828 + 1.6489i$.

(f) $\log(z_1 z_2) = \log z_1 + \log z_2$.

Answers:

(a) $\left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right]^2 + \left[\frac{1}{2}\left(e^{iz} + e^{-iz}\right)\right]^2 = 1$

(c) $\left[\frac{1}{2i}\left(e^{iz_1} - e^{-iz_1}\right)\right]\left[\frac{1}{2}\left(e^{iz_2} + e^{-iz_2}\right)\right] = \frac{1}{4i}\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)}\right)$. Clearly the last two terms will cancel with the second part.

(f) $\log(z_1 z_2) = \log\left(r_1 r_2 e^{i(\theta_1+\theta_2)}\right) = \log(r_1 r_2) + i(\theta_1 + \theta_2) = \log(r_1) + i\theta_1 + \log(r_2) + i\theta_2$

5. Let $w = 1/z$ and $z = x + iy$.

(a) Find u and v if $w = u + iv$.

(b) Show that a curve in z-plane given by

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

$(B^2 + C^2 > 4AD)$ transforms into a curve in w-plane given by

$$D(u^2 + v^2) + Bu - Cv + A = 0$$

(c) Show that a line, not passing through origin in z-plane, maps to a circle passing through origin in w-plane.

(d) Find and sketch a level curve in z-plane for $u(x, y) = 5$.

Answers:

(a) $u(x, y) = x/\left(x^2 + y^2\right)$ and $v(x, y) = -y/\left(x^2 + y^2\right)$

(b) Substitute in u and v in $D(u^2 + v^2) + Bu - Cv + A = 0$.

(c) $A(x^2 + y^2) + Bx + Cy + D = 0$ represents a line, not passing through origin in z-plane when $A = 0$ but $D \neq 0$. This line transforms to $D(u^2 + v^2) + Bu - Cv = 0$ which is equation of a circle passing throuh origin.

(d) $u = 5 \implies x^2 + y^2 - 5x = 0$. This is an equation of a circle with origin at $(5/2, 0)$ and radius $5/2$.

6. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the function $w = \exp z$, where $w = \rho e^{i\phi}$.

Answers:
$w = \exp(x + iy) = e^{ay} e^{iy}$. This gives us $\rho = e^{ay}$ and $\phi = y$. Eliminate y to get $\rho = e^{a\phi}$.

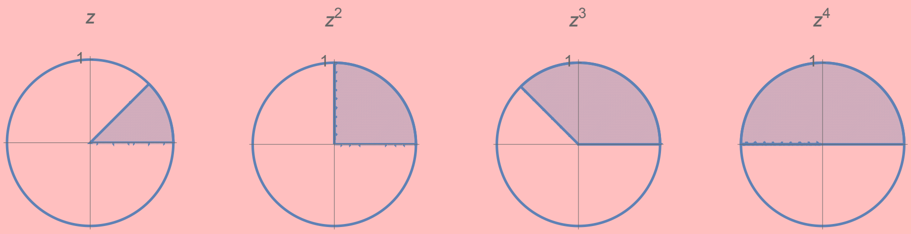
7. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by transformation

(a) $f(z) = z^2$;

(b) $f(z) = z^3$;

(c) $f(z) = z^4$.

Answers:



8. A particle constrained to move in a two dimensional plane, where its coordinates can be given by a complex number z. It is acted upon by a central force $F(z) = f(|z|)z$. Derive the equations of motion

$$\begin{aligned} 2r'\theta' + r\theta'' &= 0 \\ r'' - r\left(\theta'\right)^2 &= \frac{r}{m} f\left(|z|\right) \end{aligned}$$

Answers:
If the trajectory of the particle is given by $z(t) = r(t)e^{i\theta(t)}$, then by Newton's law $m\ddot{z}(t) = F(z)$. Now,

$$\begin{aligned} \dot{z}(t) &= \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta} \\ \ddot{z}(t) &= \ddot{r}e^{i\theta} + 2i\dot{r}\dot{\theta}e^{i\theta} + ir\ddot{\theta}e^{i\theta} + i^2r\dot{\theta}^2e^{i\theta} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)e^{i\theta} + i\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)e^{i\theta}. \end{aligned}$$

Putting this in Newton's law, cancelling $e^{i\theta}$ and equating real and imaginary parts, we get the desired equations of motion.

9. Find each of the following limits.

   (a) $\lim\limits_{z\to 2+3i} (z-5i)^2$
   (b) $\lim\limits_{z\to 2} \frac{z^2+3}{iz}$
   (c) $\lim\limits_{z\to 3i} \frac{z^2+9}{z-3i}$
   (d) $\lim\limits_{z\to 1-i} \left[x + i\left(2x+y\right)\right]$
   (e) $\lim\limits_{z\to \pi i/2} \left(z+1\right)e^z$

Answers: Using the properties of the limits.
(a) $-8i$ (b) $-7i/2$ (c) $6i$ (d) $1+i$ (e) $i\left(1+i\pi/2\right)$

10. Show that the limit of the function $f\left(z\right) = \left(z/\bar{z}\right)^2$ as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin.

Answers:
Along $z = (x, 0)$, $f(z) = 1$ and along $z = (x, x)$, $f(z) = -1$.

11. Find $f'(z)$ when

   (a) $f\left(z\right) = 3z^2 - 2z + 4$;
   (b) $f(z) = (1-4z^2)^3$;
   (c) $f\left(z\right) = \frac{z-1}{2z-1}$;

Answers: Using properties of derivative.

   (a) $f'\left(z\right) = 6z - 2$;
   (b) $f'\left(z\right) = 3\left(1-4z^2\right)^2 \cdot (-8z)$ (chain rule)
   (c) $f'(z) = 1/\left(2z-1\right)^2$

12. Prove that $\frac{d}{dz}z^n = nz^{n-1}$ where n is an integer.

Answers:

$$\begin{aligned} \lim_{z\to z_0} \frac{z^n - z_0^n}{z - z_0} &= \lim_{z\to z_0} \frac{\left(z-z_0\right)\left(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}\right)}{z - z_0} \\ &= nz_0^{n-1} \end{aligned}$$

13. Find the derivative of the given functions using the rules of differentiation:

    (a) $e^z = 1 + \sum_1^\infty z^n/n!$.
    (b) $\sin z = \left(e^{iz} - e^{-iz}\right)/2i$.
    (c) $\cos z = \left(e^{iz} + e^{-iz}\right)/2$.
    (d) $\tan z = \sin z/\cos z$.

Answers:
(a) e^z (b) $\cos z$ (c) $-\sin z$ (d) $\sec^2 z$

14. Show that the derivative of $f(z)$ does not exist for any z for each of the following:

    (a) $f(z) = \bar{z}$.
    (b) $f(z) = \operatorname{Re} z$.
    (c) $f(z) = \operatorname{Im} z$.

Answers:

(a) $u = x$ and $v = -y$. Here, $u_x = 1$ and $v_y = -1$. Thus, CR conditions are not met.

(b) $u = x$ and $v = 0$. Here, $u_x = 1$ and $v_y = 0$. Thus, CR conditions are not met.

(c) $u = 0$ and $v = y$. Here, $u_x = 0$ and $v_y = 1$. Thus, CR conditions are not met.

15. Write the following functions in the form $f(z) = u(x, y) + iv(x, y)$ and find the derivative for each:

    (a) $\cosh z$.
    (b) $\sinh z$.
    (c) $\log z$.

Answers:

(a) $\cosh z = \cosh x \cosh iy + \sinh x \sinh iy = \cosh x \cos y + i \sinh x \sin y$. Thus,

$$\begin{aligned}\frac{d}{dz}\cosh z &= \sinh x \cos y + i \cosh x \sin y \\ &= \sinh(x + iy) = \sinh z\end{aligned}$$

(c) $\log z = \log r + i\theta$, then $u = \frac{1}{2}\log\left(x^2 + y^2\right)$ and $v = \tan^{-1}(y/x)$. Thus,

$$\begin{aligned}\frac{d}{dz}\log z &= u_x + iv_x \\ &= \frac{1}{2}\frac{2x}{x^2 + y^2} + i\frac{1}{1 + (y/x)^2}\cdot\frac{-y}{x^2} \\ &= \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.\end{aligned}$$

16. Prove *L'Hospital rule*: If $f(z)$ and $g(z)$ are analytic at z_0 and $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Find $\lim_{z\to i}\left(1 + z^6\right)/\left(1 + z^{10}\right)$ using L'Hospital rule.

Answers: Since the derivatives of f and g exist at z_0, by definition

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{f(z)}{z - z_0},$$

$$\text{similarly,}\quad g'(z_0) = \lim_{z\to z_0} \frac{g(z)}{z - z_0}.$$

Then,

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} \\ &= \frac{\lim\limits_{z\to z_0} f(z)/(z - z_0)}{\lim\limits_{z\to z_0} g(z)/(z - z_0)} \\ &= \frac{f'(z_0)}{g'(z_0)}.\end{aligned}$$

Applying the rule,

$$\lim_{z\to i} \frac{(1 + z^6)}{(1 + z^{10})} = \left.\frac{6z^5}{10z^9}\right|_{z=i} = \frac{3}{5}$$

17. Let $f(z) = z^3 + 1$, and let $z_1 = \left(-1 + i\sqrt{3}\right)/2$, $z_2 = \left(-1 - i\sqrt{3}\right)/2$. Show that there is no point w on the line segment from z_1 to z_2 such that

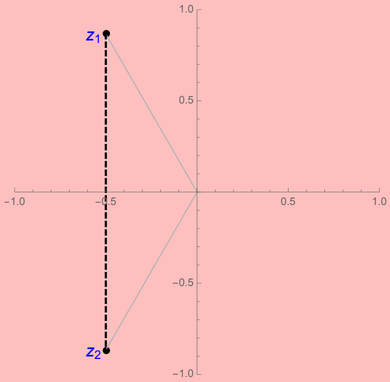
$$f(z_2) - f(z_1) = f'(w)(z_2 - z_1).$$

This shows that the mean-value theorem does not extend to complex functions.

Answers:
Clearly, $z_1 = e^{i2\pi/3}$ and $z_2 = e^{i4\pi/3}$. $f(z_1) = f(z_2) = 2$. Thus,

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = 0.$$

Now, $f'(z) = 3z^2$ and is not zero along the line joining z_1 and z_2.



18. If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic at z, then prove Cauchy-Riemann conditions

$$\begin{aligned}u_r &= \frac{1}{r}v_\theta \\ u_\theta &= -rv_r\end{aligned}$$

and that $f'(z) = e^{-i\theta}(u_r + iv_r)$.

Answers:
Given $z = re^{i\theta}$, we get $\delta z = \delta r\, e^{i\theta} + ir\delta\theta\, e^{i\theta}$. Thus,

$$
\begin{aligned}
f'(z) &= \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta) + iv(r+\delta r, \theta+\delta\theta) - iv(r,\theta)}{(\delta r + ir\delta\theta)\, e^{i\theta}} \\
&= e^{-i\theta}(u_r + iv_r) \qquad \text{along radial direction, } \delta\theta = 0 \\
&= e^{-i\theta}\frac{1}{ir}(u_\theta + iv_\theta) \qquad \text{along tangential direction, } \delta r = 0
\end{aligned}
$$

Comparing these, we get CR conditions in polar coordinates.
Alternatively:

$$
\begin{aligned}
u_r &= \frac{\partial x}{\partial r}u_x + \frac{\partial y}{\partial r}u_y = \cos\theta\, u_x + \sin\theta\, u_y \\
&= \cos\theta\, v_y - \sin\theta\, v_x \qquad \text{CR conditions in cartesian} \\
&= \frac{1}{r}(r\cos\theta\, v_y - r\sin\theta\, v_x) \\
&= \frac{1}{r}\left(\frac{\partial y}{\partial\theta}v_y + \frac{\partial x}{\partial\theta}v_x\right) = \frac{1}{r}u_\theta
\end{aligned}
$$

19. Show that following functions are harmonic and find their harmonic conjugates. Find functions $f(z)$ of which the following are real parts.

(a) y

(b) xy

(c) $\log(x^2+y^2)$

Answers:

(a) Let $u = y$. Clearly, $u_{xx} + u_{yy} = 0 + 0 = 0$. Thus, u is harmonic. Now, since $v_y = u_x$,

$$v_y = 0 \implies v = g(x)$$

and because $v_x = -u_y$, we get

$$v_x = g'(x) = -1 \implies g(x) = -x + c$$

and $v = -x + c$. Thus, $f(z) = y - ix + c = -iz + c$

(b) Answer: $v(x,y) = \frac{1}{2}\left(y^2 - x^2\right) + c$ and $f(z) = \frac{-i}{2}z^2 + c$

(c) Answer: $v(x,y) = 2\tan^{-1}(y/x) + c$ and $f(z) = \log z^2 + c$

20. If $f(z) = u(x,y) + iv(x,y)$, the equations $u(x,y) = c_1, v(x,y) = c_2$ where c_1 and c_2 are constants generate a family of curves in xy plane, namely, *level curves.*

(a) Find the normal vector to these level curves.

(b) Show that the two sets of level curves, one for u function and other for v function are orthogonal to each other if f is analytic.

Answers:

(a) The change in u along a direction $\hat{n}$ is given by $du = (u_x, u_y) \cdot \hat{n}$. Since along the level curve the change must be zero, the vector (u_x, u_y) (also known as the gradient of u, and denoted by ∇u) must be perpendicular to the level curve.

(b) If f is analytic, CR conditions hold. Then,

$$\begin{aligned} \nabla u \cdot \nabla v &= u_x v_x + u_y v_y \\ &= u_x u_y - u_y u_x \qquad \text{CR conditions} \\ &= 0 \end{aligned}$$

hence the level curves of u and v are perpendicular to each other.

21. $f(z) = z + 1/z$. Show that the level curve for $\operatorname{Im} f(z) = 0$ consists of a real axis (excluding $z = 0$) and the circle $|z| = 1$.

Answers:
$u(x,y) = x\left(1 + \frac{1}{x^2+y^2}\right)$ and $v(x,y) = y\left(1 - \frac{1}{x^2+y^2}\right)$. Thus $v = 0$ implies

$$y\left(1 - \frac{1}{x^2+y^2}\right) = 0 \implies y = 0 \text{ or } x^2 + y^2 = 1.$$

22. Consider a wedge bounded by the nonnegative real axis and a line $y = x \ (x \geq 0)$. Find a harmonic function $\phi(x,y)$ which is zero on the sides of the wedge but is not identically zero.

Answers:
Solve a boundary value problem

$$\begin{gathered} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \\ \phi(x,0) = \phi(x,x) = 0. \end{gathered}$$

Write the same equation in polar coordinates

$$\begin{gathered} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \\ \phi(r, \theta = 0) = \phi(r, \theta = \pi/4) = 0. \end{gathered}$$

Start with a guess $\phi(r,\theta) = f(r)\sin 4\theta$. This fits the boundary conditions. Now, let us get $f(r)$ by fitting this to Laplace equation. clearly, $\partial^2\phi/\partial\theta^2 = -16 f(r)\sin 4\theta$

$$\begin{aligned} \sin 4\theta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\left(-16 f \sin 4\theta\right) = 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = \frac{16}{r^2} f \end{aligned}$$

Another guess, $f \sim r^n$. Then $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = n^2 r^{n-2} = \frac{n^2}{r^2} f$. Thus, $n = 4$ and the solution is

$$\phi(r,\theta) = r^4 \sin 4\theta.$$