Tutorial 6: Wave Equation

Wave Equation in 1D, 2D and 3D

1. Find the solution to the wave equation on an interval $[0, a]$,

$$
\begin{aligned}
c^2 \frac{\partial^2}{\partial x^2} u\left(x, t\right) &= \frac{\partial^2}{\partial t^2} u\left(x, t\right) \\
u\left(0, t\right) = u\left(a, t\right) &= 0 \qquad \forall\, t \\
u\left(x, 0\right) &= f(x) \qquad \forall\, x \\
\frac{\partial}{\partial t} u\left(x, 0\right) &= g(x) \qquad \forall\, x
\end{aligned}
$$

where $(c = 1)$

   (a) $a = \pi$, $f(x) = \sin 3x$ and $g(x) = 4$.
   (b) $a = \pi$, $f(x) = x\left(\pi - x\right)$ and $g(x) = 0$.
   (c) $a = \pi$, $f(x) = \sin^2 x$ and $g(x) = \sin x$.

2. A tightly stretched string with fixed end points $x = 0$ and $x = a$ is initially in a position given by $u = u_0 \sin^3\left(\pi x/a\right)$. If it is released from rest from this position, find the displacement $u(x, t)$.

3. The points of trisection of a string of length a (with fixed ends) are pulled aside through the same distance h on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string of subsequent time and show that the midpoint of the string remains at rest.

4. Solve the partial DE with initial-boundary conditions

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} u\left(x, t\right) &= \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t}\right) \qquad 0 < x < \pi, \quad t > 0 \\
u\left(0, t\right) = u\left(a, t\right) &= 0 \qquad \forall\, t \\
u\left(x, 0\right) &= x \qquad \forall\, x \\
\frac{\partial}{\partial t} u\left(x, 0\right) &= 0 \qquad \forall\, x.
\end{aligned}
$$

5. Show that the general solution

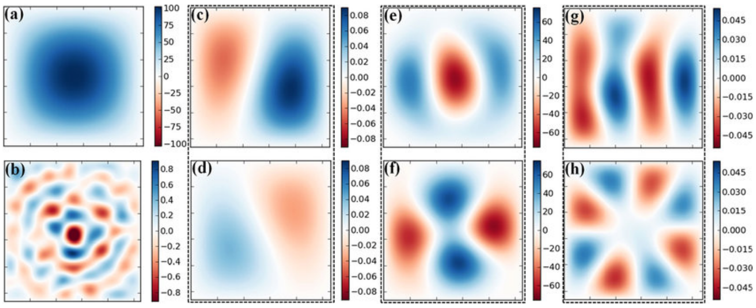
$$
u\left(x, t\right) = \sum \left(A_n \sin n\pi ct/a + B_n \cos n\pi ct/a\right) \sin\left(n\pi x/a\right)
$$

of PDE in problem 1 can be written as

$$
u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz.
$$

This is known as *d'Alembert's solution.* Thus, to find the solution $u(x, t)$, we need to know only the initial displacement $f(x)$ and the initial velocity $g(x)$. This makes d'Alembert's solution easy to apply as compared to the infinite series. In particular, find the solution on $-\infty < x < \infty, t > 0$ when$f(x) = e^{-|x|}, g(x) = xe^{-x^2}$.

6. The figure[1] shows several representative vibration modes of a ultrathin square (thickness 1nm) carbon nanomembranes with dimensions of $50 \times 50\,\mu$m. The figures (c) and (d) are degenerate vibrational modes at 1.2 MHz. Can you explain these shapes with the calculations we did in the class. Do the same for figures (e) and (f). (Hint: Modes (1,2) and (2,1) are degenerate and any linear combination will have exactly the same mode frequency.)



7. The transverse vibrations of a circular membrane are given by the wave equation (in polar coordinates r and θ)

$$
\begin{aligned}
c^2\nabla^2 u(\mathbf{r},t) &= \frac{\partial^2}{\partial t^2}u(\mathbf{r},t), \qquad |\mathbf{r}| < a \\
u(a,\theta,t) &= 0 \qquad \forall\, t,\theta \\
u(r,\theta,0) &= f(r) \qquad \forall\, r,\theta \\
\frac{\partial}{\partial t}u(r,0) &= g(r) \qquad \forall\, r,\theta
\end{aligned}
$$

where

(a) $f(r) = \begin{cases} 1 & r < a/2 \\ 0 & a/2 < r < a \end{cases}$ and $g(r) = 0$.

(b) $f(r) = 0$ and $g(r) = \begin{cases} \frac{P_0}{\rho\pi\epsilon^2} & r < \epsilon \\ 0 & \epsilon < r < a \end{cases}$. Take a limit $\epsilon \to 0$.

8. (Boas) A sphere initially at 0° has its surface kept at 100° from $t = 0$ on (for example, a frozen potato in boiling water!). Find the *time-dependent* temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer. Can you justify this procedure? The heat equation is

$$\frac{\partial}{\partial t}\Theta = c^2\nabla^2\Theta$$

where Θ is the temperature and c^2 is the diffusivity of the material.)

[1]Source: Vibrational modes of ultrathin carbon nanomembrane mechanical resonators, Zhang, Xianghui et al, Applied Physics Letters, Vol 106, pp 063107, 2015.