CYK/2023/PH201 Mathematical Physics

Tutorial 6: Wave Equation

Wave Equation in 1D, 2D and 3D

1. Find the solution to the wave equation on an interval $[0, a]$,

$$
\begin{aligned}
c^2 \frac{\partial^2}{\partial x^2} u(x,t) &= \frac{\partial^2}{\partial t^2} u(x,t) \\
u(0,t) = u(a,t) &= 0 \qquad \forall\, t \\
u(x,0) &= f(x) \qquad \forall\, x \\
\frac{\partial}{\partial t} u(x,0) &= g(x) \qquad \forall\, x
\end{aligned}
$$

where $(c = 1)$

(a) $a = \pi$, $f(x) = \sin 3x$ and $g(x) = 4$.

(b) $a = \pi$, $f(x) = x(\pi - x)$ and $g(x) = 0$.

(c) $a = \pi$, $f(x) = \sin^2 x$ and $g(x) = \sin x$.

Answer:
After separation of variables and applying the $u(0,t) = u(a,t) = 0$, the solution is

$$
u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin \omega_n t + b_n \cos \omega_n t\right) \sin(k_n x)
$$

where $k_n = n\pi/a$ and $\omega_n = ck_n$. The coefficients are given by the remaining two conditions.

$$
\begin{aligned}
b_n &= \frac{2}{a} \int_0^a f(x) \sin(k_n x)\, dx \\
a_n &= \frac{2}{a\omega_n} \int_0^a g(x) \sin(k_n x)\, dx
\end{aligned}
$$

(a) Since $a = \pi$, $k_n = n$.

$$
\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} \sin 3x \sin(nx)\, dx = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases} \\
a_n &= \frac{8}{\pi n c} \int_0^{\pi} \sin(nx)\, dx = \begin{cases} \frac{16}{\pi c n^2} & n = 1, 3, \dots \\ 0 & n - 2, 4, \dots \end{cases}
\end{aligned}
$$

The solution is

$$
u(x,t) = \cos(3ct) \sin(3x) + \frac{16}{c\pi} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^2} \sin(nct) \sin(nx)
$$

(b) Since $a = \pi$, $k_n = n$.

$$
\begin{aligned}
b_n &= \frac{2}{a} \int_0^{\pi} x(\pi - x) \sin(nx)\, dx = \begin{cases} \frac{8}{\pi n^3} & \text{odd } n \\ 0 & \text{even } n \end{cases} \\
a_n &= 0 \qquad \forall n
\end{aligned}
$$

The solution is

$$
u(x,t) = \frac{8}{\pi} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^3} \cos nct \sin(nx)
$$

(c) The solutions is

$$u(x,t) = \frac{1}{c}\sin x \sin ct - \frac{8}{\pi}\sum_{n,\text{odd}} \frac{1}{n(n^2-4)} \sin nx \sin nct.$$

2. A tightly stretched string with fixed end points $x = 0$ and $x = a$ is initially in a position given by $u = u_0 \sin^3(\pi x/a)$. If it is released from rest from this position, find the displacement $u(x,t)$.

Answer:
After separation of variables and applying the $u(0,t) = u(a,t) = 0$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \sin \omega_n t + b_n \cos \omega_n t) \sin(k_n x)$$

where $k_n = n\pi/a$ and $\omega_n = ck_n$. Since, the initial velocity of the string is zero, $a_n = 0$ for all n. And

$$\begin{aligned} b_n &= \frac{2}{a}\int_0^a \sin^3(\pi x/a)\sin(n\pi x/a)\,dx \\ &= \frac{2}{a}\int_0^a \frac{1}{4}(3\sin(\pi x/a) - \sin(3\pi x/a))\sin(n\pi x/a)\,dx \\ &= \begin{cases} \frac{3}{4} & n = 1 \\ -\frac{1}{4} & n = 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The solution is

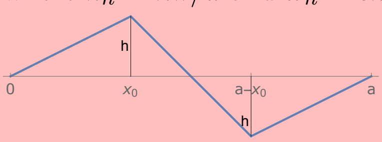
$$u(x,t) = \frac{3}{4}\cos\left(\frac{c\pi t}{a}\right)\sin\left(\frac{\pi x}{a}\right) - \frac{1}{4}\cos\left(\frac{3c\pi t}{a}\right)\sin\left(\frac{3\pi x}{a}\right)$$

3. The points of trisection of a string of length a (with fixed ends) are pulled aside through the same distance h on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string of subsequent time and show that the midpoint of the string remains at rest.

Answer:
After separation of variables and applying the $u(0,t) = u(a,t) = 0$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \sin \omega_n t + b_n \cos \omega_n t) \sin(k_n x)$$

where $k_n = n\pi/a$ and $\omega_n = ck_n$. Since, the initial velocity of the string is zero, $a_n = 0$ for all n.



Initial displacement (see Fig) is given as

$$u(x,0) = \begin{cases} \alpha x & 0 < x < x_0 \\ \alpha x_0 - \alpha\frac{2x_0}{a-2x_0}(x - x_0) & x_0 < x < a - x_0 \\ -\alpha x_0 + \alpha(x - a + x_0) & a - x_0 < x < a. \end{cases}$$

Note that $u(x,0)$ is antisymmetric about $a/2$. Now,

$$b_n = \frac{2}{a}\int_0^a u(x,0)\sin(n\pi x/a)\,dx = 0$$

if n is odd, since $\sin(n\pi x/a)$ is symmetric about $x = a/2$. Thus the solution must contain the modes with even n. But all these modes have zero displacement at $a/2$.

4. Solve the partial DE with initial-boundary conditions

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t}\right) \qquad 0 < x < \pi, \quad t > 0$$
$$u(0,t) = u(a,t) = 0 \qquad \forall\, t$$
$$u(x,0) = x \qquad \forall\, x$$
$$\frac{\partial}{\partial t} u(x,0) = 0 \qquad \forall\, x.$$

Answer:
The separation of variables $u(x,t) = X(x)\,T(t)$ gives us two ODEs

$$\frac{d^2}{dx^2}X(x) + q^2 X(x) = 0,$$
$$\frac{d^2T}{dt^2} + 2k\frac{dT}{dt} + \gamma^2 T(t) = 0, \qquad \gamma = cq.$$

Solving for $X(x)$ and applying BCs $u(0,t) = u(a,t) = 0$, we get

$$X(x) = A\sin(q_n x) \qquad q_n = \frac{n\pi}{a}; n = 1, 2, \ldots.$$

The solution to the second equation is

$$T(t) = e^{-kt}\left(B\sin\omega_n t + b_n\cos\omega_n t\right) \tag{1}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} e^{-kt}\left(a_n\sin\omega_n t + b_n\cos\omega_n t\right)\sin(q_n x)$$

where $\omega_n = \left((n\pi c/a)^2 - k^2\right)^{1/2}$. Since, the initial velocity profile of the string is zero, $a_n = 0$ for all n. Finally, $u(x,t) = x$ becomes

$$x = \sum_{n=1}^{\infty} b_n \sin(q_n x)$$
$$\implies b_n = \frac{a^2(-1)^{n+1}}{\pi n}$$
$$\implies u(x,t) = \frac{2a}{\pi}e^{-kt}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos\omega_n t\sin(q_n x).$$

Here, I have assumed that ω_n^2 is positive for all n. But, suppose $a = \pi$, $c = k = 1$. Now, $\omega_1 = 0$. How will expression for T (Eq 1) change? Now, two linearly independent solutions are e^{-kt} and te^{-kt}. Obtain the solution u considering this.

5. Show that the general solution

$$u(x,t) = \sum (A_n \sin n\pi ct/a + B_n \cos n\pi ct/a) \sin(n\pi x/a)$$

of PDE in problem 1 can be written as

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz.$$

This is known as *d'Alembert's solution*. Thus, to find the solution $u(x,t)$, we need to know only the initial displacement $f(x)$ and the initial velocity $g(x)$. This makes d'Alembert's solution easy to apply as compared to the infinite series. In particular, find the solution on $-\infty < x < \infty, t > 0$ when$f(x) = e^{-|x|}, g(x) = xe^{-x^2}$.

Answer:
After separation of variables and applying the $u(0,t) = u(a,t) = 0$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin(k_n x)$$

where $k_n = n\pi/a$ and $\omega_n = ck_n$. Note that

$$f(x) = \sum_n B_n \sin(k_n x)$$

Then,

$$\begin{aligned} f(x-ct) + f(x+ct) &= \sum_n B_n (\sin(k_n(x-ct)) + \sin(k_n(x+ct))) \\ &= 2\sum_n B_n \sin(k_n x)\cos(k_n ct) \end{aligned}$$

Also,

$$g(x) = \sum_n A_n \omega_n \sin(k_n x)$$

Then,

$$\begin{aligned} \int_{x-ct}^{x+ct} g(z)\,dz &= \sum_n A_n \omega_n \int_{x-ct}^{x+ct} \sin(k_n z)\,dz \\ &= -\sum_n A_n \omega_n \frac{1}{k_n}(\cos(k_n(x+ct)) - \cos(k_n(x-ct))) \\ &= 2c\sum_n a_n \sin(k_n x)\sin(k_n ct) \end{aligned}$$

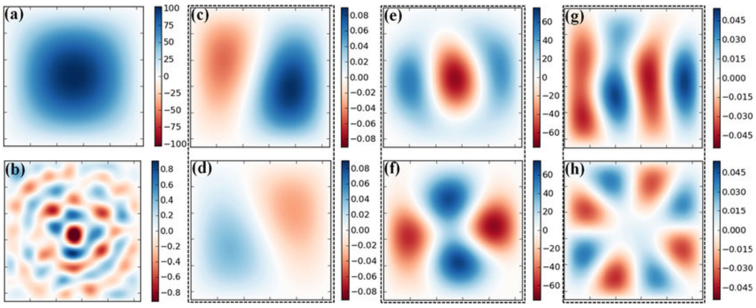
Thus,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz.$$

For given problem,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[e^{-|x+ct|} + e^{-|x-ct|}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} ze^{-z^2}dz \\ &= \frac{1}{2}\left[e^{-|x+ct|} + e^{-|x-ct|}\right] + \frac{1}{4c}\left[e^{-(x-ct)^2} - e^{-(x+ct)^2}\right] \end{aligned}$$

6. The figure[1] shows several representative vibration modes of a ultrathin square (thickness 1nm) carbon nanomembranes with dimensions of $50 \times 50\,\mu$m. The figures (c) and (d) are degenerate vibrational modes at 1.2 MHz. Can you explain these shapes with the calculations we did in the class. Do the same for figures (e) and (f). (Hint: Modes (1,2) and (2,1) are degenerate and any linear combination will have exactly the same mode frequency.)



Answer:

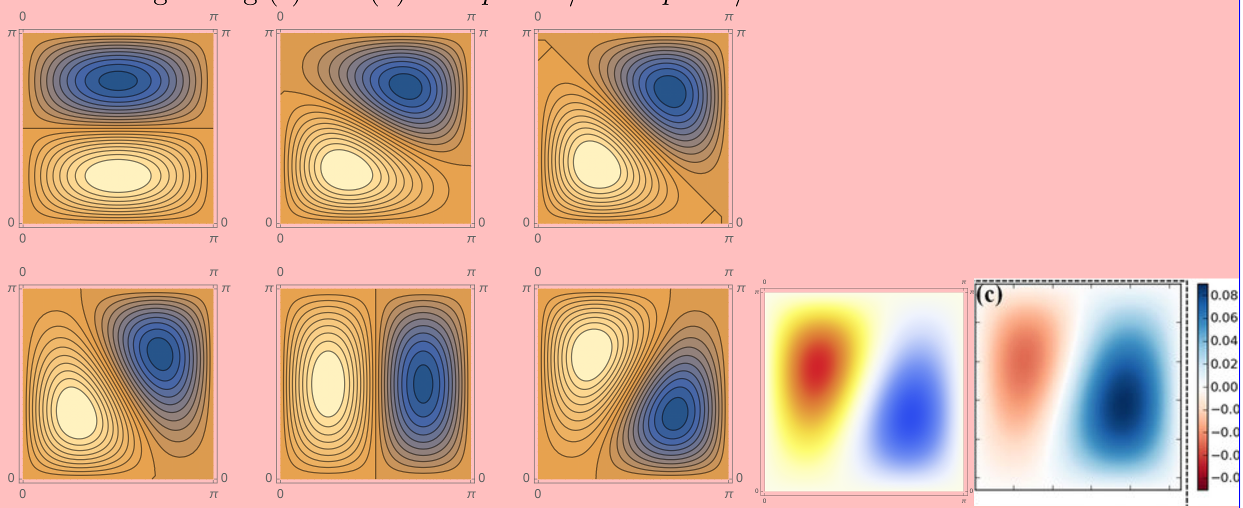
The frequency of the normal modes is given by

$$\omega_{mn} = \frac{\pi c}{a}\sqrt{m^2+n^2} = \omega_{nm}.$$

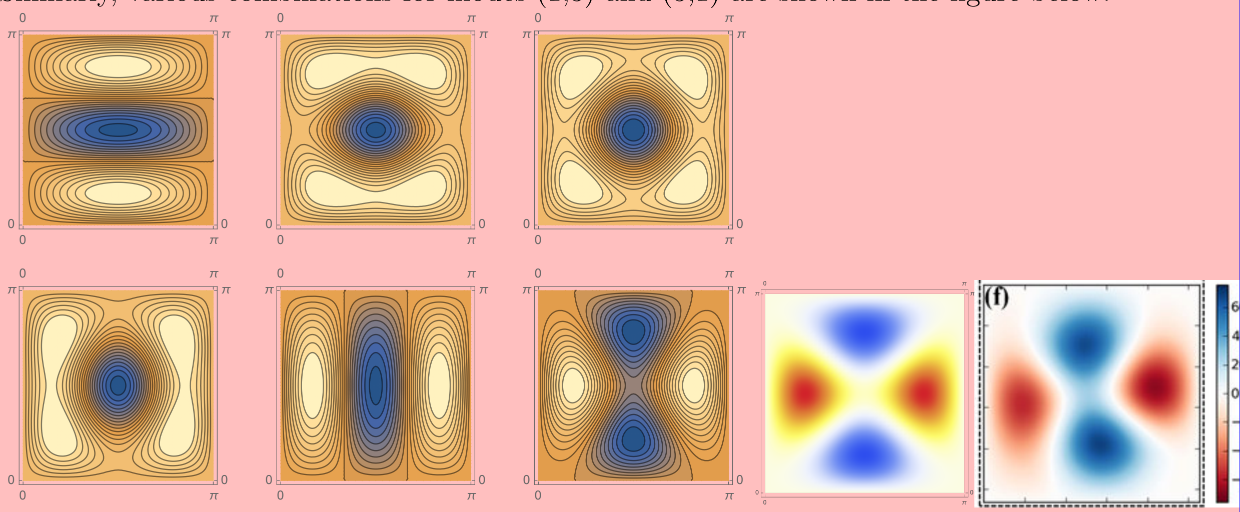
The modes (1,2) and (2,1) are degenerate and any linear combination will have exactly the same mode frequency and will also be a normal mode. Let us construct the linear combinations as

$$\psi_p\left(x,y\right) = \cos p \sin\left(\pi x/a\right)\sin\left(2\pi y/a\right) + \sin p \sin\left(2\pi x/a\right)\sin\left(\pi y/a\right)$$

with parameter p. The figure below shows various modes for $p = 0, \pi/6, \pi/4, \pi/3, \pi/2$ and $2\pi/3$. We can recognize fig (c) and (d) with $p = 2\pi/3$ and $p = \pi/3$.



Similarly, various combinations for modes (1,3) and (3,1) are shown in the figure below:



Some more ineresting patterns are given at the end.

[1]Source: Vibrational modes of ultrathin carbon nanomembrane mechanical resonators, Zhang, Xianghui et al, Applied Physics Letters, Vol 106, pp 063107, 2015.

7. The transverse vibrations of a circular membrane are given by the wave equation (in polar coordinates r and θ)

$$\begin{aligned} c^2\nabla^2 u(\mathbf{r},t) &= \frac{\partial^2}{\partial t^2}u(\mathbf{r},t), \qquad |\mathbf{r}|<a \\ u(a,\theta,t) &= 0 \qquad \forall\, t,\theta \\ u(r,\theta,0) &= f(r) \qquad \forall\, r,\theta \\ \frac{\partial}{\partial t}u(r,0) &= g(r) \qquad \forall\, r,\theta \end{aligned}$$

where

(a) $f(r)=\begin{cases}1 & r<a/2\\ 0 & a/2<r<a\end{cases}$ and $g(r)=0$.

(b) $f(r)=0$ and $g(r)=\begin{cases}\frac{P_0}{\rho\pi\epsilon^2} & r<\epsilon\\ 0 & \epsilon<r<a\end{cases}$. Take a limit $\epsilon\to 0$.

Answer:

The solutions to the 2D wave equation with condition $u(a,\theta,t)=0$ is given by

$$u(r,\theta,t)=\sum_{m=0,n=1}^{\infty,\infty} J_m(k_{mn}r)\left(A_{mn}\cos m\theta+B_{mn}\sin m\theta\right)\left(C_{mn}\cos\omega_{mn}t+D_{mn}\sin\omega_{mn}t\right).$$

Here, $k_{mn}=\chi_{mn}/a$ where x_{mn} is n^{th} zero of J_m. The frequencies of these modes are $\omega_{mn}=k_{mn}$. However, for both parts, the intial conditions are independent of θ and hence the solution u also is independent of θ, reducing it to

$$u(r,\theta,t)=\sum_{k=1}^{\infty} J_0(k_{0n}r)\left(C_{0n}\cos\omega_{0n}t+D_{0n}\sin\omega_{0n}t\right).$$

(a) Since $g(r)=0$, $D_{0n}=0$ for all n. And

$$f(r)=\sum_{k=1}^{\infty} C_{0n}J_0(k_{0n}r)$$

$$C_{0n}\int_0^a \left(J_0(k_{0n}r)\right)^2 r\,dr=\int_0^a f(r)J_0(k_{0n}r)r\,dr$$

Using $\int_0^a \left[J_m(k_{mn}\rho)\right]^2\rho\, d\rho=\frac{a^2}{2}\left[J_{m+1}(\chi_{mn})\right]^2$, we get

$$C_{0n}=\frac{2}{a^2\left[J_1(\chi_{0n})\right]^2}\int_0^{a/2} f(r)J_0(k_{0n}r)r\,dr.$$

Using, $\frac{d}{dx}\left(x^pJ_p(x)\right)=x^pJ_{p-1}(x)$, we finally get

$$C_{0n}=\frac{1}{\chi_{0n}\left[J_1(\chi_{0n})\right]^2}J_1(k_{0n}a/2).$$

(b) Since $f(r)=0$, $C_{0n}=0$ for all n. And

$$g(r)=\sum_{k=1}^{\infty}\omega_{0n}D_{0n}J_0(k_{0n}r).$$

Going through the same steps as part (a), we get

$$\begin{aligned} D_{0n} &= \frac{2}{\omega_{0n}a^2\left[J_1(\chi_{0n})\right]^2}\int_0^a g(r)J_0(k_{0n}r)r\,dr \\ &= \frac{2}{\omega_{0n}a^2\left[J_1(\chi_{0n})\right]^2}\int_0^{\epsilon}\frac{P_0}{\rho\pi\epsilon^2}J_0(k_{0n}r)r\,dr. \end{aligned}$$

(b) (Continued..) Using, $\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x)$, we finally get

$$\begin{aligned} C_{0n} &= \frac{2}{\omega_{0n}\chi_{0n}\left[J_1\left(\chi_{0n}\right)\right]^2}\frac{P_0}{\rho\pi\epsilon^2}k_{0n}\epsilon J_1(k_{0n}\epsilon). \\ &= \frac{2P_0}{\rho\pi c}\frac{1}{\chi_{0n}\left[J_1\left(\chi_{0n}\right)\right]^2}\frac{J_1(k_{0n}\epsilon)}{k_{0n}\epsilon} \\ &= \frac{P_0}{\rho\pi c}\frac{1}{\chi_{0n}\left[J_1\left(\chi_{0n}\right)\right]^2} \end{aligned}$$

This initial condition is like when you strike a drum right in the center with a drumstick to deliver an impulse.

8. (Boas) A sphere initially at 0° has its surface kept at 100° from $t = 0$ on (for example, a frozen potato in boiling water!). Find the *time-dependent* temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer. Can you justify this procedure? The heat equation is

$$\frac{\partial}{\partial t}\Theta = c^2\nabla^2\Theta$$

where Θ is the temperature and c^2 is the diffussivity of the material.)

Answer:
Given boundary condition is $\Theta\left(a,\theta,\phi,t\right) = 100$ and the initial condition is $\Theta\left(r,\theta,\phi,0\right) = 0$. We will solve the problem for $\Theta\left(a,\theta,\phi,t\right) = 0$ and $\Theta\left(r,\theta,\phi,0\right) = -100$. Since the constant value is also a trivial solution to the PDE, we are justified. The solution is similar to the wave equation except the time evolution:

$$\Theta(r,\theta,\phi,t) = \sum_{nlm} A_{nlm}Y_{lm}\left(\theta,\phi\right)j_l\left(k_{ln}r\right)e^{-\omega_{ln}^2 t}$$

The boundary condition at $r = a$ and the initial condition at $t = 0$ suggest that the solution is independent of θ and ϕ. Thus

$$\Theta(r,\theta,\phi,t) = \sum_{n} A_n Y_{00}\left(\theta,\phi\right)j_0\left(k_{0n}r\right)e^{-\omega_{0n}^2 t}.$$

Here $k_{0n}a = \chi_{0n} = n\pi$ which is n^{th} zero of j_0. Now, apply the initial condition to obtain

$$-100 = \frac{1}{\sqrt{4\pi}}\sum_n A_n j_0\left(k_{0n}r\right)$$

Use the following two identities:

$$\int_0^1 j_m\left(\chi_{mn}r\right)j_m\left(\chi_{mp}r\right)r^2dr = \frac{\delta_{np}}{2}\left[J_{m+1}\left(\chi_{mn}\right)\right]^2$$

and

$$\frac{d}{dx}\left(x^2 j_1\left(x\right)\right) = x^2 j_0\left(x\right).$$

The final solution is

$$\Theta = (r,\theta,\phi,t) = 100 + 200\sum_n \left(-1\right)^n j_0\left(k_{0n}r\right)e^{-\omega_{0n}^2 t}$$

This problem has many applications. Most of the time we heat objects from the outside for sufficiently long time to achieve desired temperatures inside. For example, while cooking we put food items inside boiling water. To make sure that all the germs are killed it is necessary that the core temperature reaches at least 60º C. Thus, it is possible to estimate using the solution above to find the duration for which the food item must be immersed in boiling water.