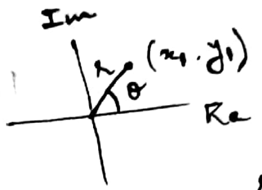
Complex Analysis

$x^2 = -1 \Rightarrow i = \sqrt{-1}$



$z_1 = x_1 + iy_1 \qquad x_1, y_1 \in \mathbb{R}$

$z_2 = x_2 + iy_2 \equiv (x_2, y_2)$

$r = |z| = \sqrt{x_1^2 + y_1^2}$

$z_1 \oplus z_2 = (x_1 + x_2) + i(y_1 + y_2) \equiv (x_1 + x_2, y_1 + y_2)$

$\theta = Arg(z_1)$

$= \tan^{-1}(y_1/x_1)$

$z_1 = r_1 \cos\theta_1 + i r_1 \sin\theta_1$

$z_1 = r_1 e^{i\theta_1}$

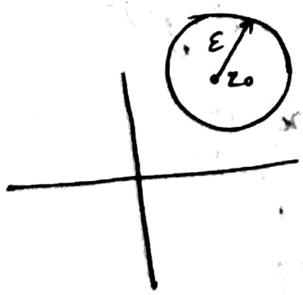
$z_1 . z_2 = (x_1 x_2 - y_1 y_2, \; x_1 y_2 + x_2 y_1)$

$wz_1 = wx_1 + iwy_1$

$z = (x, y) \Rightarrow \frac{1}{z} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$

$z = (x, y) \Rightarrow \bar{z} = (x, -y)$

$\frac{1}{z} = \frac{\bar{z}}{r^2}$



$|z - z_0| < \varepsilon$ (neithourhood of z_0)

S

Interior point $\Rightarrow$ Any Neighbourhood of z_0 should be completely inside S

Exterior point $\Rightarrow$ Any neighbourhood of z_0 should be completely outside S

Boundary point $\Rightarrow$ Every neighbourhood should contain points both inside & outside S

Open sets $\Rightarrow$ Contains interior point only

Closed Sets $\Rightarrow$ Contains all boundary points also

Connected Sets $\Rightarrow$ $\exists$ Any two points should be able to connect by a line that doesn't go outside the set

Singly Connected Set

Multiply Connected Set

Functions of Complex Variables

$f : D \subset \mathbb{C} \rightarrow \mathbb{C}$, $z \in D$

$$w = f(z) = \underbrace{u(x,y)}_{D \rightarrow \mathbb{R}} + i\underbrace{v(x,y)}_{D \rightarrow \mathbb{R}}$$

Eg. $f(z) = z^2$

$= (x+iy)^2$ $\qquad u(x,y) = x^2 - y^2$

$= x^2 - y^2 + 2xyi$ $\qquad v(x,y) = 2xy$

$$f(z) = \frac{1}{z^2} = \left(\frac{\bar{z}}{r^2}\right)^2 = \frac{(x-iy)^2}{r^4} = \frac{x^2 - y^2 - 2xyi}{(x^2+y^2)^2}$$

$$u = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$v = \frac{-2xy}{(x^2+y^2)^2}$$

$z = re^{i\theta}$

$$f(z) = \frac{1}{z^2} = \frac{1}{r^2 e^{2i\theta}} = \frac{1}{r^2}(+\cos 2\theta - i\sin 2\theta)$$

$$u(r,\theta) = \frac{1}{r^2}\cos 2\theta$$

$$v(r,\theta) = \frac{-1}{r^2}\sin 2\theta$$

Polynomials

$f(z) = a_0 + a_1 z + \dots + a_n z^n \qquad a_i \in \mathbb{C}$

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \qquad$ Defined over $\mathbb{C}$

$e^{i\theta} = \cos\theta + i\sin\theta$

Trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{Zeroes} \Rightarrow z = n\pi$$

Hyperbolic Functions

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

Logarithmic Functions

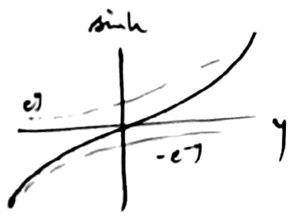
$$\log z = \log r + i(\theta + 2\pi n) \qquad z = re^{i(\theta + 2n\pi)}$$

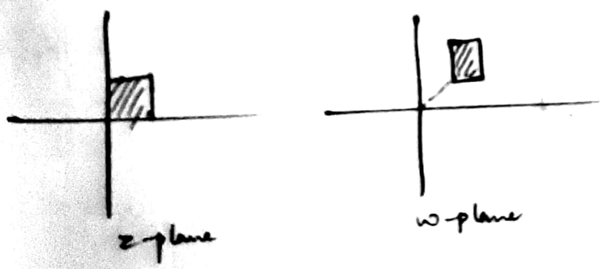
Principle branch $\Rightarrow 0 \le \theta + 2n\pi \le 2\pi$

$\sin iy = i\sinh y$

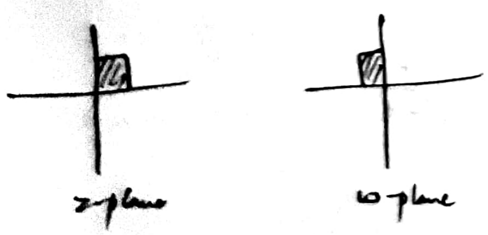
$\cos iy = \cosh y$

$$\sin iy = \frac{e^{-y} - e^{y}}{2i} = i\left(\frac{e^{y} - e^{-y}}{2}\right) = i\sinh y$$





$f(z) = z + (1+i)$



$f(z) = e^{i\pi/2} z$

Mappings : $f(z) = z^2$ $\qquad z = (x+iy)$

$w = f(z) = z^2$ $\qquad C: \; z(t) = (x(t), y(t))$

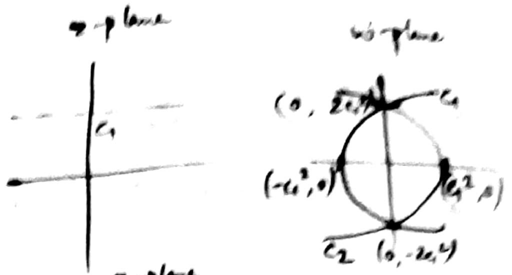
$= (u, v)$ $\qquad \Gamma: \; w(t) = (u(t), v(t))$

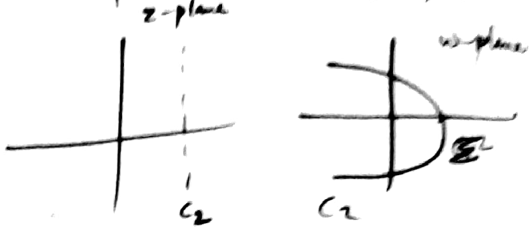
$= (x^2 - y^2, 2xy)$

$C_1: \; z(t) = (t, c_1) \Rightarrow x(t) = t$
$\qquad y(t) = c_1$

$\Gamma_1: \; w(t) = (t^2 - c_1^2, \; 2tc_1)$
$\qquad u(t) \qquad v(t)$

$u = \frac{v^2}{4c_1^2} - c_1^2$





$C_2: \; z(t) = (c_1, t)$

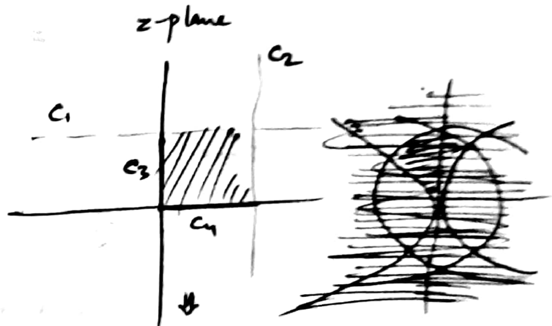
$\Gamma_2: \; w(t) = (c_1^2 - t^2, \; 2tc_1)$

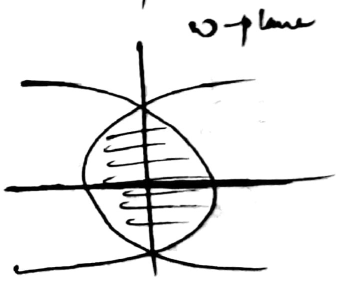
$v = 2tc_1$ $\qquad u = c_1^2 - \frac{v^2}{4c_1^2}$

$t = \frac{v}{2c_1}$

$C_3 \Rightarrow v = 0^-$

$C_4 \Rightarrow v = 0^+$





$w = \frac{1}{z}$ $\qquad$ [redacted], $x = c, \; y = t$

$w = \frac{1}{c + it} = \frac{c - it}{c^2 + t^2}$

$u = \frac{c}{c^2 + t^2}$ $\qquad v = \frac{-t}{c^2 + t^2}$

$\frac{1}{c^2 + t^2} = \frac{u}{c}$

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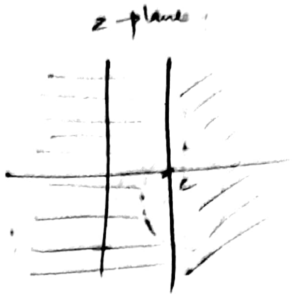
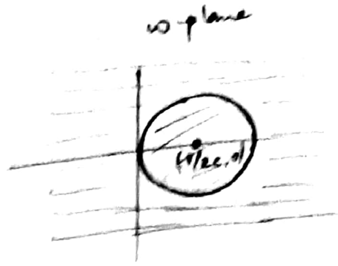
$$u^2 + v^2 = \frac{1}{c^2+t^2} = \frac{u}{c}$$

$$u^2 + v^2 - \frac{1}{c} u = 0$$





Limit of Complex No.

$$\lim_{z \to z_0} f(z) = w_0$$

if $\forall\ \delta > 0\ \exists\ \varepsilon > 0$ s.t. $|z - z_0| < \varepsilon \Rightarrow |f(z) - w_0| < \delta$

$f(z) = 5z \to 5z_0$ as $z \to z_0$

$\delta > 0$, $\varepsilon = \delta/5$

Given $|z - z_0| < \varepsilon$ $\quad |f(z) - 5z_0| = 5|z - z_0| < 5\varepsilon = \delta$

$f(z) = z^2 \to z_0^2$ as $z \to z_0$

$\delta > 0$, $|z - z_0| < \varepsilon$

$$|f(z) - z_0^2| = |z^2 - z_0^2| = |z - z_0||z + z_0|$$

$$\leq |z - z_0|(|z - z_0| + 2|z_0|)$$

$$\leq \underbrace{|z - z_0|^2}_{<\delta/2} + \underbrace{2|z_0||z - z_0|}_{<\delta/2} < \delta$$

$\downarrow \qquad \downarrow$

$\varepsilon < \sqrt{\delta/2} \qquad \varepsilon < (\delta/4)|z_0|$

$$\varepsilon = \text{Min}\left\{\sqrt{\delta/2}, \frac{\delta|z_0|}{4}\right\}$$

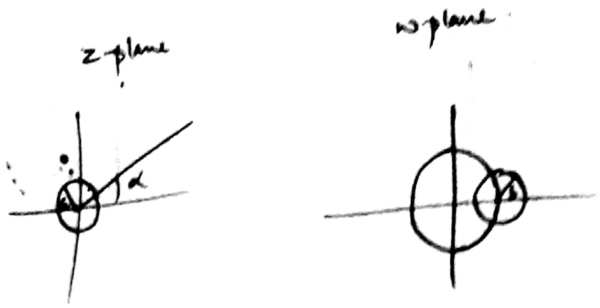
$f(z) = \frac{z}{\bar{z}}$ limit does not exist as $z \to z_0 = 0$

$z - z_0 = x + iy$

$$f(z) = \frac{x+iy}{x-iy}$$

$x \to 0 \quad \frac{x}{x} = 1$

$y \to 0 \quad -1$



$f(z) = e^{2i\theta}$

limit does not exist

Theorem ⇒ $f(z) = u(x,y) + iv(x,y)$

$w_0 = u_0 + iv_0$

$\lim_{z \to z_0} f(z) = w_0$ iff $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$ & $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$

As $z \to z_0$ $\quad f(z) \to w_0$, $\quad g(z) \to s_0$ $\quad \Rightarrow \quad f+g = w_0 + s_0$

$f \cdot g = w_0 \cdot s_0$

$f/g = w_0/s_0$

$\lim_{z \to z_0} \sin z$

$$\begin{aligned}
\sin z &= \sin(x+iy) \\
&= \sin x \cos iy + \cos x \sin iy \\
&= \sin x \cosh y + i \cos x \sinh y \\
\lim_{z \to z_0} &= \sin x_0 \cosh y_0 + i \cos x_0 \sinh y_0 \\
&= \sin(x_0 + iy_0) \\
&= \sin(z_0)
\end{aligned}$$

Continuity

Continuous if $\lim_{z \to z_0} f(z) = f(z_0)$

Derivative

If $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists

$\frac{df}{dz}$, $f'(z_0)$

$f(z) = z^2$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{z^2 - z_0^2}{z - z_0} = z + z_0$$

$= 2z_0$ as $z \to z_0$

$$\frac{d}{dz}(z^2) = 2z$$

$f(z) = |z|^2$ at $z \to 0$

$$\lim_{z \to 0^+} \frac{f(z) - f(z_0)}{z - z_0} = \frac{|z|^2}{z} = \cancel{\bar{z} \cdot z \to 0} \quad \frac{z\bar{z}}{z} = \bar{z} = 0$$

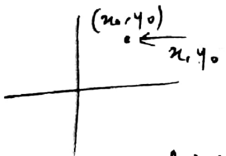
$$\lim_{z \to 0^-} \frac{|z|^2}{z} = \frac{z^2}{z} = \bar{z} = 0$$

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \cancel{\frac{(|z| - |z_0|)(|z| + |z_0|)}{z - z_0}}$$

$$= \frac{z\bar{z} - z_0\bar{z_0}}{z - z_0} = \frac{x^2 + y^2 - x_0^2 - y_0^2}{x - x_0 + i(y - y_0)}$$

Horizontal limit $\Rightarrow$ $x + x_0 = 2x_0$

(x, y_0)



Vertical limit $\Rightarrow$ $\frac{y^2 - y_0^2}{i(y - y_0)} = \frac{y + y_0}{i} = -2iy_0$

Not differentiable anywhere except $z = 0$

$z = (x, y)$ $\quad f(z) = u(x, y) + iv(x, y)$

If $f'(z_0)$ exists then f is continuous at z_0

& u_x, u_y, v_x, v_y exist

$$\frac{df}{dz}(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \left(\frac{f(z)-f(z_0)}{z-z_0} (z-z_0) \right)$$

$$= \lim_{z \to z_0} \underbrace{\left(\frac{f(z)-f(z_0)}{z-z_0} \right)}_{\text{exists}} \underbrace{\lim (z-z_0)}_{0}$$

$$= 0$$

If derivative exists then function is continuous

Rules of Differentiation

$$\frac{d}{dz}(f \pm g)(z) = \frac{df(z)}{dz} \pm \frac{dg(z)}{dz}$$

$$\frac{d}{dz}(f \circ g)(z) = \frac{df}{dg}\frac{dg}{dz}$$

$$\frac{d}{dz}(f \cdot g)(z) = \frac{df(z)}{dz} g(z) + \frac{dg(z)}{dz} \cdot f(z)$$

$$f(z) = z^n, \quad \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \frac{\cancel{z^n} + nz^{n-1}\cancel{\Delta z} + \cancel{\Delta z^2(\ldots)} \ldots - \cancel{z^n}}{\cancel{\Delta z}}$$

$$= nz^{n-1}$$

$$f(z) = e^{az} \qquad \frac{d}{dz}(e^{az}) = \lim_{\Delta z \to 0} \frac{(e^{a(z+\Delta z)}) - e^{az}}{\Delta z}$$

$$= e^{a} \cancel{1 + e^{a(z+\Delta z)}} + e^{a}$$

$$= \cancel{1 + a(z+\Delta z)}$$

$$= \frac{e^{az}(e^{a\Delta z} - 1)}{\Delta z}$$

$$= \frac{e^{az}\left(\cancel{1} + a\cancel{\Delta z} + \cancel{\frac{(a\Delta z)^2}{2}} \ldots - \cancel{1}\right)}{\cancel{\Delta z}}$$

$$= ae^{az}$$

$f(z) = \cos z \quad \frac{d}{dz}\cos z = \cancel{\lim_{\Delta z \to}} \frac{d}{dz}\left(\frac{1}{2}\left(e^{iz}+e^{-iz}\right)\right)$

$$= \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)$$

$$= \frac{i}{2}\left(e^{iz} - e^{-iz}\right)$$

$$= \frac{-(e^{iz}-e^{-iz})}{2i} = -\sin z$$

$f(z) = (2z^2+1)^5$

$$\frac{d}{dz}(2z^2+1)^5 = 5(2z^2+1)^4 \times 4z$$

$$= 20z(2z^2+1)^4$$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = |z|^2 \Rightarrow u = x^2+y^2$

$v = 0$

$u_x = 2x, \; u_y = 2y, \; v_x = v_y = 0$

$f(z) = z^2 \Rightarrow u = x^2-y^2$

$v = 2xy$

$f'(z) = 2z = 2x + 2yi$

$u_x = 2x, \; u_y = 2y, \; v_x = 2y, \; v_y = 2x$

$f(z) = u + iv$ has derivative then all partial derivatives of u, v exist & they satisfy ~~Cau~~ Cauchy - Reimann condition (C.R.),

(Necessary but not sufficient)

$u_x = v_y$

$u_y = -v_x$

$f(z) = \cancel{f}\, u(x,y) + iv(x,y)$

$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$ (Approaching limit from horizontal direction $\Delta y = 0$)

$= u_x + iv_x$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x, y+\Delta y) - u(x,y) + i(v(x,y+\Delta y) - v(x,y))}{i\Delta y}$$

(Approaching limit from vertical direction, $\Delta x = 0$)

$$= -iu_y + v_y$$

$$f(z) = e^z$$
$$= e^{x+iy}$$
$$= e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$$

$u = e^x \cos y$
$v = e^x \sin y$

$u_x = e^x \cos y$ $\qquad u_y = -e^x \sin y$
$v_x = e^x \sin y$ $\qquad v_y = e^x \cos y$

$u_x = v_y$
$u_y = -v_x$

Converse Theorem: $f(z) = u + iv$, u_x, u_y, v_x, v_y exist and are continuous then $f'(z)$ exists (C.R. condition also satisfy)

$$f'(z) = u_x + iv_x$$
$$= v_y - iu_y$$

$$f(z) = \sin z$$
$$= \sin(x+iy)$$
$$= \sin x \cosh y + i \cos x \sinh y$$

$$f'(z) = u_x + iv_x$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$= \cos x \cos iy - \sin x \sin iy$$
$$= \cos(x+iy)$$

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \Rightarrow \quad \begin{aligned} u_{xx} &= v_{xy} \\ u_{yy} &= -v_{yx} \end{aligned}$$

$$u_{xx} + u_{yy} = 0 \qquad (\text{Laplace Eq}^n)$$

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \end{aligned}\right\} \text{Harmonic Functions}$$

Harmonic Functions

If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ then function is harmonic.

$$f(z) = u + iv$$

$$u_x = v_y \qquad u_y = -v_x$$

$$u_{xx} = v_{xy} \qquad u_{yy} = -v_{yx}$$

$$u_{xx} + u_{yy} = 0$$

$$\left.\begin{aligned} &u, v \Rightarrow \text{harmonic} \\ &u_x = v_y \\ &u_y = -v_x \end{aligned}\right\} \; v \text{ is harmonic conjugate of } u \text{ (not vice-versa)}$$

$u = y \Rightarrow$ harmonic

$$u_x = v_y = 0 \qquad u_y = -v_x = +1$$

$$v_x = -1$$

$$v = -x + c$$

$$u(x,y) = \log(x^2+y^2)$$

$$u_x = \frac{2x}{x^2+y^2} = v_y \qquad u_y = \frac{2y}{x^2+y^2}$$

$$v_x = \frac{-2y}{x^2+y^2}$$

$$\cancel{2x \cdot \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)} \qquad \cancel{-2y \cdot \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right)}$$

$$\cancel{-2 \tan^{-1}\left(\frac{x}{y}\right)}$$

$$v = 2\tan^{-1}\left(\frac{y}{x}\right)$$

$$f(z) = \log(x^2+y^2) + \cancel{2\tan}\; i\, 2\tan^{-1}\left(\frac{y}{x}\right) = 2\log|z| \text{ [illegible] } 2\tan^{-1}(y/x) =$$

$$f(z) = 2\left(\log|z| + i\theta\right)$$
$$= 2\left(\log r + i\theta\right)$$
$$= 2\log\left(r_0 e^{i\theta}\right)$$

Analytic Functions

f on open set S is analytic if f is diff. at all points of S

f analytic at z

f is entire if analytic at $\mathbb{C}$

If f is analytic in open set S, then u & v are harmonic.

Singular Point

$f = 1/z$ diff. except $z=0$ (singular)

$f = |z|^2$ diff at $z=0$ (rest points not singular)

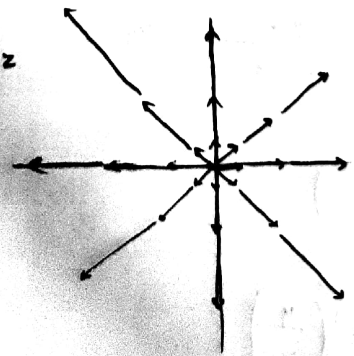
A point z_0 is singular if there exists ~~any~~ neighbourhood which contains a point not analytic.

Vector fields

$$f(z) = u(x,y) + iv(x,y)$$

$\vec{f} = (u, v)$ $\qquad f(z) = z$

$\vec{z} = (x, y)$



$f = (u, v, w)$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \qquad (\vec{\nabla} \times \vec{f})_z = v_x - u_y$$

Integration

$f(t) = u(t) + iv(t)$

$f'(t) = u'(t) + iv'(t)$

$$\int_a^b f(t)\, dt = \int u(t)\, dt + i \int v(t)\, dt$$

$\int f(z)$ w.r.t. z over curves (contours)

$z(t) = x(t) + iy(t) \qquad t : a \to b$

$x(t) = t \qquad y(t) = t^2 \qquad t : 0 \to 1$

$x(t) = \cos t \qquad y(t) = \sin t \qquad t : 0 \to \pi/2$

$x(t) = 1 - t \qquad y(t) = \sqrt{1-(1-t)^2} \qquad t : 0 \to 1$

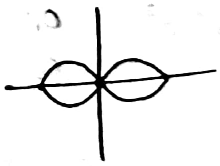
} same path, different trajectory

$z(t) = \cos t + i \sin 2t \qquad t : 0 \to 2\pi$

$y = 2 \sin t \cos t \qquad x = \cos t$

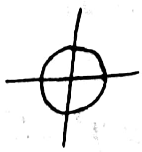
$= 2 \cos t \sqrt{1-\cos^2 t}$

$= \pm 2x \sqrt{1-x^2}$



$z(t) = \cos t + i \sin t \qquad t : 0 \to 2\pi$

$x^2 + y^2 = 1$



closed path $\Rightarrow z(a) = z(b)$

Simple curve $\Rightarrow t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2) \quad \forall\, t_1, t_2$

Differentiable $\Rightarrow z'(t)$ exists

Smooth curve $\Rightarrow$ differentiable & $z'(t) \neq 0$

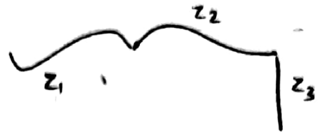
Length of curve $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$L = \int_a^b |z'(t)|\, dt$$

Contour $\Rightarrow$ smooth curves joined end to end

$z'(t)$ exists

piecewise continuous



Contour Integral $\Rightarrow$ $c: z(t)$ $t = a \to b$

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$f = u + iv$

$z(t) = x(t) + iy(t)$

$$= \int_a^b (u + iv)\,(x'(t) + iy'(t))\, dt$$

$$= \int_a^b [(ux' - vy') + i(vx' + uy')]\, dt$$

$$= \int_a^b (ux' - vy')\, dt + i\int_a^b (vx' + uy')\, dt$$

$$= \int_a^b (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

$f(z) = z^2$ C_1: straight line $(0,0)$ to $(2,1)$

C_2: straight line $(0,0)$ to $(2,0)$ & $C_3: (2,0)$ to $(2,1)$

$f(z) = z^2 = (x+iy)^2$

$= x^2 - y^2 + 2xyi$

$u = x^2 - y^2$

$v = 2xy$

$$= \int (x^2 - y^2)\,dx - \int 2xy\,dy + i\left[\int 2xy\,dx + \int (x^2 - y^2)\,dy\right]$$

$= \cancel{\frac{x^3}{3} \quad xy^2}$

$$= \left[\frac{x^3}{3} - xy^2\right]_{(0,0)}^{(2,1)} - \left[xy^2\right]_{(0,0)}^{(2,0)} + i\left[x^2y\right]_{(0,0)}^{(2,0)} + i\left[x^2y - \frac{y^3}{3}\right]_{(0,0)}^{(2,1)}$$

$$= \frac{8}{3} - 2 - 2 + i\left[4 + 4 - \frac{1}{3}\right]$$

$$= \frac{8}{3} - 4 + i\left[8 - 1/3\right] = -4/3 + 23/3\, i$$

Aliter $C_1 : z(t) = (2t + it) \qquad t : 0 \to 1$

$$\int f(z)\,dz = \int (2t+it)^2 (2+i)\,dt$$
$$= \int_0^1 (2+i)^3\, t^2$$
$$= (2+i)^3 \left[\frac{t^3}{3}\right]_0^1 = \frac{(2+i)^3}{3} = \frac{8 - i + 3\cdot 2\cdot i(2+i)}{3}$$
$$= \frac{8 - i + 6i(2+i)}{3}$$
$$= \frac{8 - i + 12i - 6}{3}$$
$$= 2/3 + \frac{11}{3} i$$

$C_2 : z(t) = 2t \qquad t : 0 \to 1$

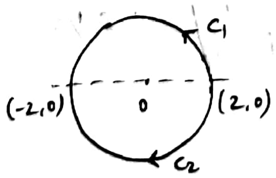
$$\int f(z)\,dz = \int f(z(t)) \cdot z'(t)\,dt$$
$$= \int (2t)^2 \cdot 2\,dt$$
$$= \frac{8t^3}{3} = 8/3$$

$C_3 : z(t) = 2 + it \qquad t : 0 \to 1$

$$\int f(z(t)) \cdot z'(t)\,dt = \int (2+it)^2 \cdot i\,dt$$
$$= \int (4 - t^2 + 4ti)\, i\,dt$$
$$= \int (4i - it^2 - 4t)\,dt$$
$$= \left(4it - \frac{it^3}{3} - 2t^2\right)_0^1$$
$$= 4i - \frac{i}{3} - 2$$
$$= -2 + \frac{11}{3} i$$

$f(z) = 1/z$

$\int f(z)\,dz$



$C_1 : z(t) = 2e^{it} \qquad t : 0 \to \pi$

$C_2 : z(t) = 2e^{-it} \qquad t : 0 \to \pi$

$$C_1 \Rightarrow \int_0^\pi \frac{1}{2e^{it}} \cdot 2e^{it} \cdot i\,dt = i\pi$$

$$C_2 \Rightarrow \int_0^\pi \frac{1}{2e^{-it}} \cdot 2e^{-it} (-i)\,dt = -i\pi$$

$$\int w\, f(z)\, dz = w \int f(z)\, dz \qquad w \Rightarrow \text{complex const.}$$

$$\int (f(z) + g(z))\, dz = \int f(z)\, dz + \int g(z)\, dz$$

$$\int_{C_1 + C_2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

$$\left| \int f(z)\, dz \right| \leq \int \left| f(z)\, z'(t)\, dt \right|$$

$$\left| \int_C f(z)\, dz \right| \leq m \cdot L$$

$m \Rightarrow$ max. of $|f(z)|$, $z \in C$ $\qquad L \Rightarrow$ length of curve

Show $\left| \int \frac{dz}{z^2 - 1} \right| \leq \pi/3$ $\qquad C : (2, 0) \to (0, 2)$ along circular path

$$z(t) = 2e^{it}$$

$$z(t) = 2(\cos t + i \sin t)$$

$$\frac{1}{z^2 - 1} = \frac{1}{4e^{2it} - 1}$$

$$= \frac{1}{4(\cos 2t + i \sin 2t) - 1}$$

$$= \frac{1}{4(\cos 2t + i \sin 2t) - 1}$$

$$= \cancel{\frac{1}{4}}$$

$$\max.\left(\frac{1}{z^2 - 1} \right) = 1/3$$

$$\left| \int \frac{dz}{z^2 - 1} \right| \leq m \cdot L$$

$$\leq \frac{1}{3} \cdot \pi$$

$$\leq \pi/3$$

Alter

$$|z^2 - 1| \geq \left| |z|^2 - 1 \right|$$

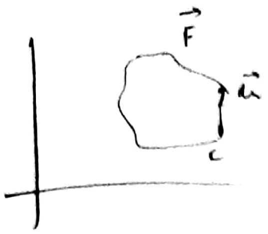
$$\geq |4 - 1|$$

$$\geq 3$$

$$\left| \frac{1}{z^2 - 1} \right| \leq 1/3$$

$$\text{Volume} = \int_{\Omega} f(x, y)\, ds$$

$$\text{Line Integral} = \int_C f(x, y)\, dl$$



$\vec{F} = (f_x, f_y)$

$$\oint_c \vec{F}\cdot d\vec{r} = 0 \iff \nabla \times F = 0$$

$$\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} = 0$$

$$\oint_c (f_x\,dx + f_y\,dy) = 0$$

$$\int(u\,dx - v\,dy) + i\int(v\,dx - u\,dy)$$

$$\int F\cdot dr$$

$\vec{F} = (u, -v)$

$d\vec{r} = (dx, dy)$

$$\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} = 0$$

$$\frac{\partial(-v)}{\partial x} - \frac{\partial(u)}{\partial y} = 0$$

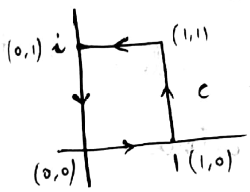
$$u_y = -v_x$$

$\bar{z}^2$

$z\bar{z} = |z|$

$\bar{z} = \frac{|z|^2}{z^2}$

Compute $\int_C \bar{z}^2\,dz$



$z_1(t) = t \quad 0\to 1$

$z_2(t) = 1 + it \quad 0 \to 1$

$\cancel{z_3(t) = t + i \quad 1\to 0}$

$\cancel{z_4(t) = it \quad 1 \to 0}$

$z_3(t) = (1-t) + i \quad 0\to 1$

$z_4(t) = (1-t)i \quad 0 \to 1$

$$\cancel{\int_0^1 \frac{1}{t^2}\,t\,dt + \int_0^1 \frac{1}{(1+it)^2}\,i\,dt + \int_1^0 \frac{1}{(i+t)^2}\cdot 1\,dt + \int_1^0 \frac{1}{i^2t^2}\cdot i\,dt}$$

$$\cancel{= [\ln t]_0^1 + \int_0^1 \frac{i}{1-t^2+2it}\,dt + \int_1^0 \frac{1}{i^2+t^2+2ti} + \int}$$

$$= \int_0^1 t^2\,dt + \int_0^1 (1 + i^2t^2 - 2it)\,i\,dt + \int_1^0 (t^2 + i^2 - 2it)\cdot 1\,dt + \int_1^0 i^2t^2\cdot i\,dt$$

$$= \frac{1}{3} + i\left[t - \frac{t^3}{3} - it^2\right]_0^1 + \left[\frac{t^3}{3} - t - it^2\right]_1^0 - \left[\frac{it^3}{3}\right]_1^0$$

$\cancel{\frac{1}{3}} + \cancel{\frac{i}{3}} - \cancel{\frac{1}{3}} - \lambda \quad - \frac{1}{3} + 1 + 1 + \frac{1}{3}$

$\cancel{\frac{1}{3}} + i \quad \cancel{\frac{i}{3}} + 1 \qquad \cancel{\frac{1}{3}} + 1 + i + \cancel{\frac{i}{3}}$

$\cancel{z+i} = \underline{2 + 2i}$

~~Alite~~ $\cancel{\bar{z}^2 = (x^2+y^2) - 2ixy}$

Alite $\bar{z}^2 = (x^2+y^2) - 2ixy$

Jordan Curve Lemma

Every simple & closed contour in complex plane splits the entire plane into 2 domains one of which is bounded. The bounded domain is called interior of the contour & the other one is called exterior of the contour.

Green's Theorem

Let C be a ~~...~~ simple closed contour with positive orientation & let D be the interior of C. If P & Q are continuous & have continuous partial derivatives P_x, P_y, Q_x & Q_y at all points on C & D, then

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint [Q_x(x,y) - P_y(x,y)]\,dx\,dy$$

Cauchy - Goursat Theorem

Let f be analytic in a simply connected domain D. If C is any simple closed contour in D, then

$$\int_C f(z)\, dz = 0$$

$\oint_C f(z)\, dz = 0$

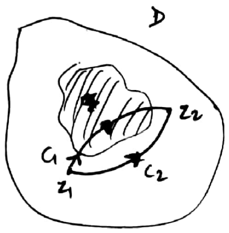
$$\int f(z)\, dz = \int (u\,dx - v\,dy) + i\int (u\,dy + v\,dx)$$

$$\text{Re} \oint f(z)\, dz = \oint_C u\,dx - v\,dy = \iint_S dx\,dy\,(-v_x - u_y) = 0$$

$(u_y = -v_x)$

$$\text{Im} \oint f(z)\, dz = \oint_C u\,dy + v\,dx = \iint_S (u_x - v_y)\, dx\,dy = 0$$

$(u_x = v_y)$



↓

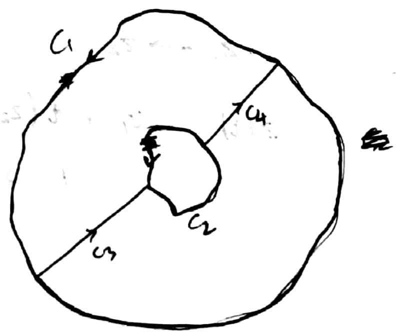
Path independence

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz = \int_{z_1}^{z_2} f(z)\, dz$$

$$\int_{C_1 + (-C_2)} f(z)\, dz = 0$$

Multiply connected regions

Analytic b/w C_1 & C_2 & not inside C_2.



$\Gamma_1 \Rightarrow$ ~~$C_3 \to C_4$~~

$C_3 \to -\tfrac{1}{2}C_2 \to C_4 \to +\tfrac{1}{2}C_1$

~~$\Gamma_2 \Rightarrow C_3 \to \tfrac{1}{2}C_2 \to C_4 \to +\tfrac{1}{2}C_1$~~

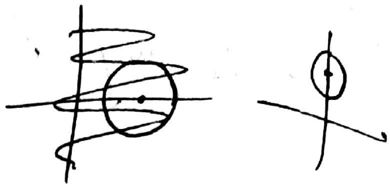
$\Gamma_2 \Rightarrow +\tfrac{1}{2}C_1 \to -C_4 \to -\tfrac{1}{2}C_2 \to -C_3$

$$\int_{\Gamma_1} f(z)\, dz = 0$$

$$\int_{\Gamma_2} f(z)\, dz = 0$$

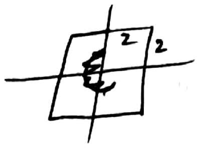
$$\int_{\Gamma_1 + \Gamma_2} f(z)\, dz = \int_{C_1 + (-C_2)} f(z)\, dz = 0$$

$$\oint \frac{1}{z}\,dz \qquad C_1 : \cancel{|z-2i|} \quad |z-2i| = 1$$

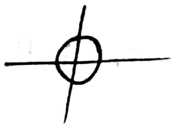


$$\int_{C_1} dz/z = 0$$

$$\oint_{C_2} \frac{1}{z}\,dz = \oint_{C_2} \bar{z}\,dz \qquad C_2 :$$



$$= \oint_{C_3} \bar{z}\,dz \qquad C_3 :$$



$$= \int_0^{2\pi} e^{-i\theta} \cdot i \cdot e^{i\theta}\,d\theta$$

$$= 2\pi i$$

$$\oint \frac{1}{z-z_0}\,dz = 2\pi i \text{, if } c \text{ contains } z_0$$

$$= 0 \text{, otherwise}$$

$$\int_C \frac{2z\,dz}{z^2+2} \qquad c : |z| = 2$$

$$z \neq \pm\sqrt{2}i$$

$$= \int \frac{2z\,dz}{(z-\sqrt{2}i)(z+\sqrt{2}i)}$$

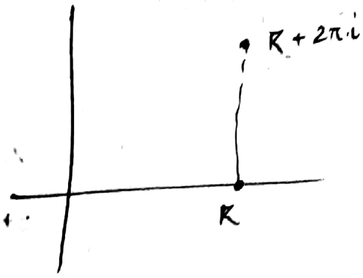
$$= \int \left[\frac{+1}{z+\sqrt{2}i} + \frac{1}{z-\sqrt{2}i}\right] dz = 4\pi i$$

Parameterization for straight line $\Rightarrow$ $z_1 \rightarrow z_2$

$$z(t) = z_1 + t(z_2 - z_1)$$

$$t : 0 \rightarrow 1$$

$$\left|\int_C \frac{e^{3z}}{1+e^{z}}\,dz\right| \le \frac{2\pi\, e^{3R}}{e^{R}-1}$$



$z = R$ to $R + 2\pi i$

$$\left|\frac{e^{3z}}{1+e^{z}}\right| = \frac{|e^{3z}|}{|1+e^{z}|}$$

$$|e^{3z}| = |e^{3(R+iy)}|$$
$$= |e^{3R}|\,|e^{3yi}|$$
$$= e^{3R}$$

$$|1+e^{z}| \ge \big|\,|e^{z}| - |1|\,\big| = |e^{R}-1| = e^{R}-1$$

$$\therefore \left|\frac{e^{3z}}{e^{z}+1}\right| \le \frac{e^{3R}}{e^{R}-1}$$

$L = 2\pi$

$$\left|\int_C \frac{e^{3z}}{1+e^{z}}\,dz\right| \le M\cdot L = 2\pi\,\frac{e^{3R}}{e^{R}-1}$$

Antiderivative $\Rightarrow$ If F is analytic in D, & $F'(z) = f(z)$ $\forall z \in D$
F is antiderivative of f

Theorem $\Rightarrow$ Simply connected domain D, f analytic
then there is F such that $F'(z) = f(z)$

Fundamental Theorem of Integration $\Rightarrow$

$$F(z) = \int_{z_0}^{z} f(s)\,ds$$

F is antiderivative of f.

$$F(z) = \int_{z_0}^{z} f(s)\,ds$$

$$\lim_{\Delta \to 0} \frac{F(z+\Delta) - F(z)}{\Delta} = f(z)$$

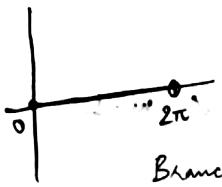
$$\lim_{\Delta \to 0} \frac{1}{\Delta}\left[\int_{z_0}^{z+\Delta} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds\right] = f(z)$$

$$\int_0^1 \sin z\,dz = -[\cos z]_0^1 = -\cos 1 + 1$$

$$\oint_C \frac{1}{z}\,dz \qquad C: |z| = 1$$

$$\log z = \log r + i\theta \qquad 0 \le \theta < 2\pi$$

$$\operatorname{Log} z = \log r + i\theta$$
$$-\pi \le \theta \le \pi$$

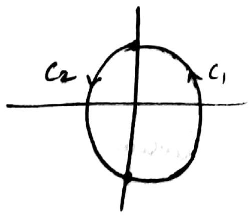


Not continuous

Branch cut

$\oint \frac{1}{z}\,dz = \log z$ (if there is no branch cut in the domain)

$$\oint_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz$$



$$= \operatorname{Log}(i) - \operatorname{Log}(-i) + \log(-i) - \log(i)$$

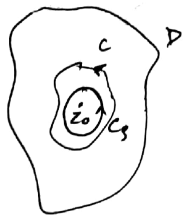
$$= \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) + \frac{i3\pi}{2} - \frac{i\pi}{2}$$

$$= 2\pi i$$

Cauchy Integral Formula

If f is analytic in D & C is simply ~~connected~~ closed contour in D & z_0 is in interior of C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}$$



$$\Rightarrow \oint \frac{f(z)\,dz}{z - z_0} = 2\pi i \, f(z_0)$$

$$\oint \frac{\sin z}{z}\,dz = 2\pi i \sin 0 = 0$$

$$\oint \frac{dz}{z - z_0} = 2\pi i$$

$$I = \oint \frac{f(z)\,dz}{z - z_0} - 2\pi i\, f(z_0)$$

$$= \oint_C \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

$$|I| = \left| \oint \frac{f(z) - f(z_0)}{z - z_0}\,dz \right|$$

$\varepsilon > 0$ Find δ s.t.
$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

$\rho = \delta/2$

$$|I| = \oint_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz \le \frac{\varepsilon}{\rho}$$

$|I| = 0 \Rightarrow I = 0$

~~$f'(z) = 1$~~

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(s)\,ds}{s - z}$$

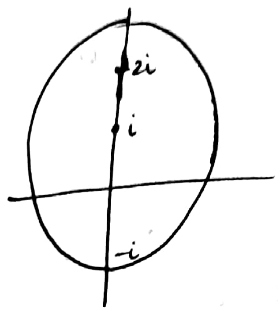
$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s - z)^2}$$

$$f''(z) = \frac{2!}{2\pi i} \oint \frac{f(s)\, ds}{(s-z)^3}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(s)\, ds}{(s-z)^{n+1}}$$

If f is analytic in D, then all its derivatives exist & are analytic.

Q $\oint \frac{dz}{z^2+4}$ $\quad C: |z-i| = 2$



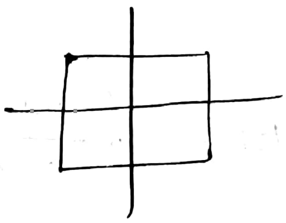
$$\oint \frac{dz}{(z-2i)(z+2i)}$$

$$\oint \frac{1/(z-(-2i))}{z-2i}\, dz \qquad f(z) = \frac{1}{z+2i} \text{ (Analytic)}$$

$$= f(2i) \cdot 2\pi i$$

$$= \frac{2\pi i}{4i} = \pi/2$$

Q $\oint \frac{z\, dz}{2z+1}$ $\quad C:$ square with $(\pm 2, \pm 2)$



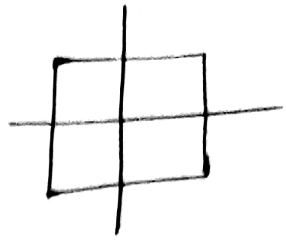
$$= \frac{1}{2} \int \frac{z\, dz}{z-(-1/2)}$$

$$= \frac{1}{2} f(z_0) \cdot 2\pi i$$

$$= \frac{1}{2} \times \frac{-1}{2} \times 2\pi i$$

$$= \cancel{-\frac{\pi i}{2}} \; \frac{-\pi i}{2}$$

Q $\oint_C \frac{\cos z}{z(z^2+8)}\, dz$ C : square $\pm 2 \pm 2i$



$\oint \frac{\cos z / z^2+8}{z-0}\, dz$

$= f(0) \cdot 2\pi i$

$= \frac{2\pi i}{8} = \frac{\pi i}{4}$

Q $\oint_C \frac{e^{-z}\, dz}{z-(\pi i/2)} = 2\pi i \; e^{+\pi i/2}$

$= 2\pi i^2$

$= -2\pi$

Q $\oint \frac{\tan(z/2)}{(z-x_0)^2}$ $(-2 < x_0 < 2)$

$= \cancel{\int \frac{\tan(z/2)/(z-x_0)}{(z-x_0)}\, dz}$

$= \cancel{2\pi i f'(z)}$

$z/2 = s$

$dz = 2ds$

$= \oint \frac{\tan(z/2)}{4(\frac{z}{2}-\frac{x_0}{2})^2}\, dz$

$= \oint \frac{\tan s \cdot 2ds}{4(s-x_0/2)^2}$

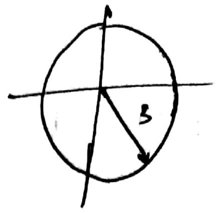
$= \frac{2\pi i}{2} \cdot \frac{1}{2} f'(z)$

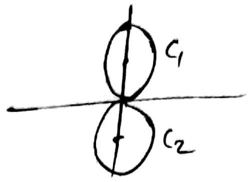
$= 2\pi i \cdot \frac{1}{2} \sec^2 s$ $s = x_0/2$

$= \cancel{\frac{\pi i \cdot 1/4}{x_0^2}}$ $\pi i \sec^2(x_0/2)$

$= \cancel{\frac{4\pi i}{x_0^2}}$

Q $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}\,dz$ $\qquad C : |z| = 3$, $t > 0$





$$= \frac{1}{2\pi i}\left[\oint_{C_1} \frac{e^{zt}/z+i}{z-i}\,dz + \oint_{C_2} \frac{e^{zt}/z-i}{z+i}\,dz\right]$$

$$= \frac{1}{\cancel{2\pi i}}\left[\cancel{2\pi i}\cdot\frac{e^{it}}{2i} - \cancel{2\pi i}\,\frac{e^{-it}}{2i}\right]$$

$$= \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$

$$= \sin t$$

$f(z) = \sqrt{z}$

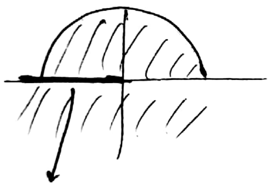
$= \sqrt{r}\, e^{i\theta/2}$

$z = re^{i\theta}$ ...

$0 \le \theta < 2\pi$

or

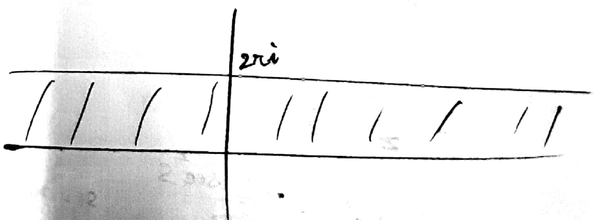
$0 < \theta \le 2\pi$



Branch cut

$e^{z+2\pi i} = e^{z}\cdot e^{2\pi i}$

$= e^{z}$



Sequences

$f : \mathbb{N} \to \mathbb{C}$

$z_n = 1/n + i\,(-1)^n/n^2$ $\qquad\qquad z_n = x_n + i y_n$

$z_n = -2 + i \cos(5n)$

If $\epsilon > 0 \quad \exists \; N$ s.t. $|z_n - l| < \epsilon$ (For real nos.)

$n > N$

$z_n \to l$ iff $x_n \to \alpha$, & $y_n \to \beta$ (For complex nos.)

$z_n = z^n = r^n e^{in\theta}$ (Convergent if $r < 1$)

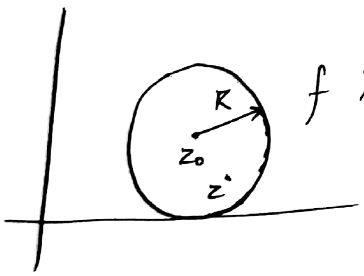
Series

$$\sum_{n=0}^{\infty} z_n \to l$$

$$S_N = \sum_{n=0}^{N} z_n \to l \Rightarrow \sum_{n=0}^{\infty} z_n = l$$

$$\sum_{n=0}^{\infty} z^n , \quad S_N = \frac{1 - z^{N+1}}{1 - z} \to \frac{1}{1-z} \text{ as } N \to \infty \quad (\text{For } |z| < 1)$$

Taylor Series



f is analytic

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(s)}{s - z} ds$$

$$= \frac{1}{2\pi i} \oint \frac{f(s)\, ds}{(s - z_0) - (z - z_0)}$$

$$= \frac{1}{2\pi i} \oint \frac{1}{s - z_0} \frac{f(s)\, ds}{1 - \frac{(z - z_0)}{s - z_0}}$$

$$= \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{f(s)\, ds}{(s - z_0)^{n+1}} (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Q $\sin z$ ~~about~~ about $z_0 = 0$

$$\sum_{n=0}^{\infty} \frac{f(0)(z-0)^1}{1!} + \frac{f'(0)(z-0)^2}{2!} + \frac{f''(0)}{3!}(z)^2 + f$$

$f(z) = \sin z$
$f'(z) = \cos z$
$f''(z) = -\sin z$
$f'''(z) = -\cos z$
$f^{iv}(z) = \sin z$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots$$

Radius of convergence $= R = \infty$

Q $\frac{1}{1+z}$ about $z_0 = 0$

$f(z) = \frac{1}{1+z}$
$f'(z) = -1/(1+z)^2$
$f''(z) = 2/(1+z)^3$
$f'''(z) =$

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \frac{1}{1-w}$$

$(w = -z)$

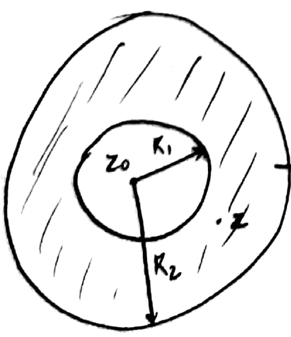
$$= \sum_{n=0}^{\infty} (-z)^n \qquad R = 1.$$

Q $1/z$ about $z_0 = 0$

$$= \frac{1}{1-(1-z)} = \frac{1}{1-w} \qquad w = 1-z$$

$$\sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty} (1-z)^n$$

Laurent Series

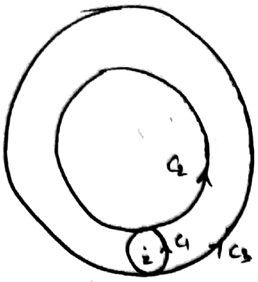


— f analytic

$$R_1 < |z - z_0| < R_2$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z_0)^{n+1}} \qquad b_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z_0)^{-n+1}}$$



$$\oint_{C_1} \frac{f(s)\,ds}{s-z} + \oint_{C_2} \frac{f(s)\,ds}{s-z} - \oint_{C_3} \frac{f(s)\,ds}{s-z} = 0$$

$$2\pi i\, f(z) = -\underbrace{\oint_{C_2} \frac{f(s)\,ds}{s-z_0-(z-z_0)}}_{|s-z_0| \le |z-z_0|} + \underbrace{\oint_{C_3} \frac{f(s)\,ds}{s-z_0-(z-z_0)}}_{|s-z_0| > |z-z_0|}$$

$$= \oint_{C_2} \frac{f(s)}{z-z_0} \frac{ds}{1-\frac{s-z_0}{z-z_0}} + \oint_{C_3} \frac{f(s)}{s-z_0} \cdot \frac{1}{1-\frac{z-z_0}{s-z_0}}$$

$$= \sum_{n=0}^{\infty} \int_{C_2} \frac{1}{(z-z_0)^{n+1}} f(s)\,ds\,(s-z_0)^n + \sum_{n=0}^{\infty} \frac{f^{n}(z_0)(z-z_0)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \int_C \frac{1}{(z-z_0)^n} f(s)\,ds\,(s-z_0)^{n-1} + \cancel{\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int \frac{f(s)\,ds}{(s-z_0)^{n+1}} (z-z_0)^n$$

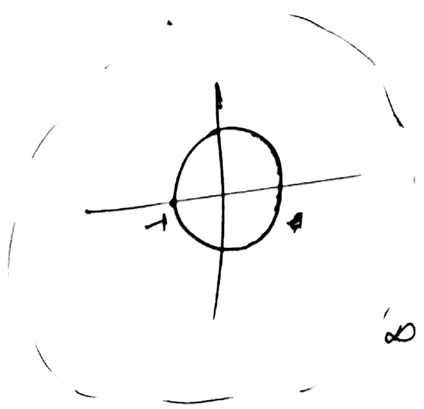
Q $f(z) = \dfrac{1}{1+z}$ series about $z_0 = 0$

$w = 1/z$

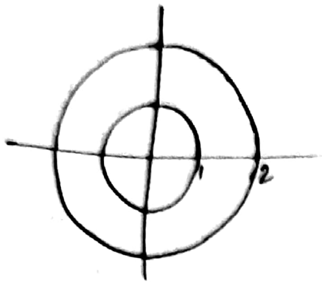
$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z} = \frac{1}{z} \cdot \frac{1}{1+w}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$



Q $\quad f(z) = \dfrac{-1}{(z-1)(z-2)} = \dfrac{1}{z-1} - \dfrac{1}{z-2}$



$|z| < 1$

$|z/2| < 1 \qquad \dfrac{-1}{1-z} + \dfrac{1}{1-z/2}\cdot\dfrac{1}{2}$

$1 < |z| < 2 \qquad \dfrac{1}{z}\,\dfrac{1}{1-1/z} + \dfrac{1}{2}\,\dfrac{1}{1-z/2}$

$|z/2| < 1$

$|z| > 2 \qquad \dfrac{1}{z}\cdot\dfrac{1}{1-1/z} - \dfrac{1}{1-2/z}\cdot\dfrac{1}{2}$

Singularity of f at z_0 if

Every neighbourhood of z_0, some analytic points exist.

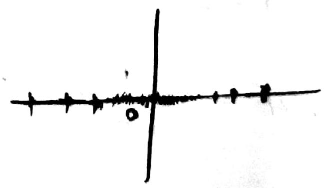
$f(z) = 1/z \qquad z = 0$ singular

$f(z) = |z|^2 \qquad z = 1+i$ non-singular (all other points are non-differentiable)

$f(z) = \dfrac{1}{\sin(1/z)} \qquad z = 1/n\pi$ (isolated singular)

$z = 0$ (singular but not isolated)

Isolated singularity $\Rightarrow$ $0 < |z - z_0| < \varepsilon$ for $\varepsilon > 0$ s.t. all points are analytic



Q $\quad f(z) = \dfrac{1}{z - z_0}$

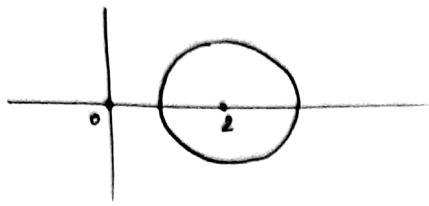
$a_n = 0$

$b_1 = 1$

$b_2 = b_3 = \cdots = 0$

$$\oint f(s)\,ds = 2\pi i b_1 = 2\pi i$$

Q $\displaystyle\oint_C \frac{dz}{z(z-2)^4}$ $\quad C : |z-2| = 1$



$$f(z) = \frac{1}{z(z-2)^4}$$

$$= \frac{1}{(z-2)^4} \cdot \frac{1}{2+(z-2)}$$

$$= \frac{1}{(z-2)^4} \frac{1/2}{1+(z-2)/2}$$

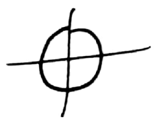
$$= \frac{1}{2(z-2)^4} \sum_{n=0}^{\infty} \left(\frac{z-2}{2}\right)^n (-1)^n$$

$$= \frac{1}{2}\left[\frac{1}{(z-2)^4} - \frac{1}{2(z-2)^3} + \cdots + \frac{-1}{2^3(z-2)} \cdots\right]$$

$$b_1 = \frac{-1}{16}$$

$$\therefore \oint \frac{dz}{z(z-2)^4} = 2\pi i b_1 = 2\pi i\left(\frac{-1}{16}\right) = \frac{-\pi i}{8}$$

Q $f(z) = e^{1/z}$ $\quad C : |z| = 1$



$$f(z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

$$\oint_C f(z)\,dz = 2\pi i b_1 = 2\pi i(1) = 2\pi i$$

Q $f(z) = e^{1/z^2}$ $\quad C : |z| = 1$

$$f(z) = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \cdots$$

$$\oint_C f(z)\,dz = 0 \quad (\because b_1 = 0)$$

Types of Isolated Singularities

f at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}}_{\text{Principle part}}$$

Pole of order m $\quad b_m \neq 0$, $b_i = 0$, $i > m$

Pole of order 1 $\Rightarrow$ simple pole

Essential singularity $\Rightarrow$ principle has infinite ~~te~~ terms

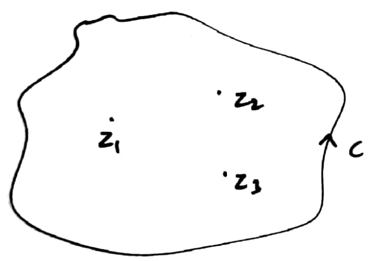
Removable singularity $\Rightarrow$ $b_n = 0 \ \forall n$

$$\text{e.g.} = \frac{\sin z}{z} \quad z \neq 0$$

$$= 1 \quad z = 0$$

Residue of f at $z_0 = b_1 = \text{Res}(f(z_0)) = \underset{z = z_0}{\text{Res}}(f(z))$

Cauchy - Residue Theorem



f has singularities (isolated) at $z_1, z_2, z_3 \dots z_k$

$$\oint_C f(z)\, dz = 2\pi i \sum_{n=1}^{k} \text{Res}\, f(z_n)$$

$$f(z) = P(z)/Q(z)$$

$$Q(z_0) = 0 \qquad Q'(z_0) \neq 0$$

$$Q(z) = Q(z_0) + Q'(z_0)(z - z_0) + Q''(z - z_0)^2/2 + \dots\dots$$

$$f(z) = \frac{P(z)}{(z - z_0)\left[Q'(z_0) + \dots\dots\right]} \Rightarrow \text{simple pole}$$

$$\text{Res}\, f(z_0) = \frac{P(z_0)}{Q'(z_0)}$$

f has a pole of order m at z_0

$$\text{Res } f(z_0) = \lim_{z \to z_0} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(f(z) (z-z_0)^m \right) \right]$$

$\cancel{\text{Res } f(z) =}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \underbrace{b_1}_{c_{m-1}} (z-z_0)^{m-1} + \cdots + \underbrace{b_m}_{c_0} \quad \text{(analytic at } z_0)$$

$$\Downarrow \quad \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$b_1 = c_{m-1}$

$$\frac{1}{z(z-2)^4}$$

Applications

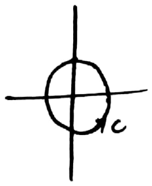
$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

$$F \longrightarrow F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2} \right)$$

$|z| = 1$, $z = e^{i\theta}$

$dz = i.e^{i\theta} d\theta$

$dz = iz\, d\theta$



$$\oint_C F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2} \right) dz/iz$$

$$= \int_0^{2\pi} F(\sin\theta, \cos\theta) \frac{\cancel{i.e^{i\theta}}}{\cancel{i.e^{i\theta}}} d\theta$$

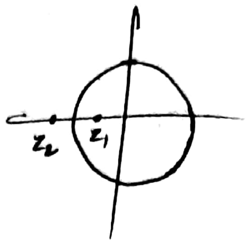
Q $\displaystyle \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$ $|a| < 1$ $\displaystyle f = \frac{1}{1 + a\cos\theta} \to f = \frac{1}{1 + a\left(\frac{z + 1/z}{2}\right)}$

$$= \frac{z}{\frac{az^2}{2} + \frac{a}{2} + z}$$

$$= \frac{2z}{a(z-z_1)(z-z_2)}$$

$$z_1 = \frac{-1}{a} + \frac{1}{a}\sqrt{1-a^2}$$

$$z_2 = \frac{-1}{a} - \frac{1}{a}\sqrt{1-a^2}$$



$$I = \oint_C \frac{2}{a}\,\frac{z}{(z-z_1)(z-z_2)}\,\frac{dz}{iz}$$

$$= \frac{2}{ia}\oint_c \frac{dz}{(z-z_1)(z-z_2)}$$

$$= \frac{2}{ia}\cdot 2\pi i \operatorname{Res} g(z_1) \qquad m=1$$

$$= \frac{2}{ia}\, 2\pi i \lim_{z\to z_1}\Big[\big(g(z)\big).(z-z_1)\Big]$$

$$= \frac{4\pi}{a}\left[\frac{1}{(z-z_1)(z-z_2)}(z-z_1)\right]$$

$$= \frac{4\pi}{a}\,\frac{1}{z_1-z_2}$$

$$= \frac{4\pi}{a}\cdot\left(\frac{2}{a}\sqrt{1-a^2}\right)^{-1}$$

$$= \cancel{\frac{8\pi\sqrt{1-a^2}}{a}}$$

$$= \frac{4\pi}{a}\cdot\frac{a}{2\sqrt{1-a^2}}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Improper Integrals

$$\int_0^\infty dx \qquad \int_{-\infty}^{\infty} dx \quad \Rightarrow \quad \lim_{R\to\infty}\int_0^R f(x)\,dx$$

$$\int_{-\infty}^{\infty} dx = \lim_{R_1\to\infty}\int_0^{R_1} f\,dx + \lim_{R_2\to\infty}\int_{-R_2}^{0} f\,dx$$

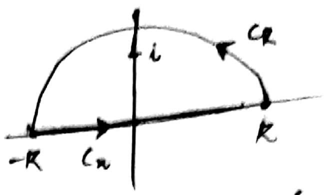
$= \text{Integral of } f$

$$\lim_{R\to\infty}\int_{-R}^{R} f\,dx \Rightarrow \text{Cauchy Principle value}$$

Q $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$f(x) = \dfrac{1}{1+x^2}$

$f(z) = \dfrac{1}{1+z^2} = \dfrac{1}{(z-i)(z+i)}$ $\quad R \gg 1$



$C_x : \; x, y = 0$

$C : C_x + C_R$

$$\int_{C_x} f(z)\,dz + \int_{C_R} f(z)\,dz = \oint_C f(z)\,dz$$

$$\oint f(z)\,dz = 2\pi i \operatorname{Res} f(i)$$

$$= \int_{-R}^{R} \frac{dx}{1+x^2} = 2\pi i \operatorname{Res} f(i) - \int_{C_R} f(z)\,dz$$

$$= \int_{-R}^{R} \frac{dx}{1+x^2} = 2\pi i \operatorname{Res} f(i) - \cancel{\lim_{R\to\infty} \int_{C_R} f(z)\,dz} \to 0$$

$$= 2\pi i \lim_{z\to i} \left[\frac{(z-i)}{(z-i)(z+i)} \right]$$

$$\left| \int_{C_R} f(z)\,dz \right| \le m \cdot L \le \frac{1}{R^2-1} \cdot \pi R = 0$$

$$= 2\pi i \lim_{z\to i} \left(\frac{1}{z+i} \right)$$

$$= \frac{2\pi i}{2i} = \pi$$

$$|1+z^2| \le \big| |1| - |z|^2 \big| = R^2 - 1$$

$$\left| \frac{1}{1+z^2} \right| \le \frac{1}{R^2-1}$$

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

$$f(z) = \frac{P(z)}{Q(z)}$$

~~$\deg(P) < \deg(Q)$~~

$\deg(P) < \deg(Q) - 1$

Q $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

$x^6 = -1$

$x^3 = \pm i$

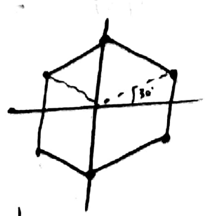
$x = (\pm i)^{1/3}$

$z = e^{i\theta}$

$e^{6i\theta} + 1 = 0$

$e^{6i\theta} = -1$

$e^{6i\theta} = e^{+i(\pi + 2m\pi)}$



$$z = e^{i(\pi/6 + i\, 2m\pi/6)}$$

$$z = e^{i(\pi/6 + i\, m\pi/3)}$$

$$\oint \frac{dz}{z^6+1} = 2\pi i \left(\operatorname{Res} f(\pi/6) + \operatorname{Res} f(\pi/2) + \operatorname{Res} f(5\pi/6) \right)$$

$$= \frac{2\pi i}{6} \left(\frac{1}{z_0^5} + \frac{1}{z_1^5} + \frac{1}{z_2^5} \right)$$

$$= \frac{2\pi i}{6} \left(\frac{1}{e^{5\pi/6 i}} + \frac{1}{e^{5\pi/2 i}} + \frac{1}{e^{25\pi/6 i}} \right)$$

$$= \frac{2\pi i}{6} \left(\frac{1}{-\sqrt{3}/2 + i/2} + \frac{1}{i} + \frac{1}{\sqrt{3}/2 + i/2} \right)$$

$$= \frac{2\pi i}{6} \left(\frac{i}{i^2/2 - 3/4} + \frac{1}{i} \right)$$

$$= \frac{2\pi i}{6} \left(\frac{i}{-5/4} - i \right)$$

$$= \frac{2\pi i}{6} \left(\frac{4}{5} \cancel{-\frac{5}{4}} i - i \right)$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}4} \left(+ \frac{9 \cancel{i}}{4} \right)$$

$$= \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}$$

$$f(z) = \int_{-\infty}^{\infty} \frac{e^{i3z}}{(z^2+1)^2} \qquad z = \pm i$$

$$= \operatorname{Re}\left(2\pi i \left(\operatorname{Res} f(i) \right)\right)$$

$$\operatorname{Re} \int f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx = I$$

$$|f(z)| \le \frac{|e^{i3z}|}{(R^2-1)^2} = \frac{|e^{i3x}|\cdot|e^{-3y}|}{(R^2-1)^2} \le \frac{1}{(R^2-1)^2}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \cdot \frac{1}{(R^2-1)^2} \to 0 \quad \text{as } R\to\infty$$

$$\operatorname{Res} f(i) = \lim_{z\to i} \frac{1}{1!} \frac{d}{dz}\left(\frac{e^{i3z}(z-i)^2}{(z-i)^2(z+i)^2}\right)$$

$$= \frac{d}{dz}\left(\frac{e^{i3z}}{(z+i)^2}\right)$$

$$= \frac{3i\, e^{i3z}(z+i)^2 - 2(z+i)\,e^{i3z}}{(z+i)^4}$$

$$= \frac{3i\,e^{-3} \cdot 4i^2 - 2\cdot 2i\, e^{-3}}{16}$$

$$= \frac{-12e^{-3}i - 4e^{-3}i}{16}$$

$$= \frac{-16e^{-3}\, i}{16} = -i\cdot e^{-3}$$

$$= \operatorname{Re}\left(2\pi i\cdot(-ie^{-3})\right)$$

$$= \operatorname{Re}\left(2\pi e^{-3}\right)$$

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} = \frac{2\pi}{e^3}$$

Applications of Residues

$$I = \oint_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \oint_{|z|=1} F\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2}\right)\frac{dz}{iz}$$

$$I = \int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz$$

$$I = \int g(x)\begin{pmatrix}\cos ax \\ \text{or } \sin ax\end{pmatrix} dx$$

~~f(z) = g(z) =~~

$$f(z) = g(z)\cdot e^{iaz}$$

$$\left| \int_{C_R} f(z)\, e^{iaz}\, dz \right| \le M \left| \int e^{iaz}\, dz \right|$$

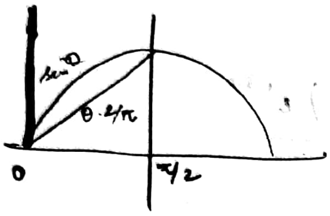
$$M = \text{Max. } f(z)$$

$$z = Re^{i\theta}$$
$$dz = Ri e^{i\theta}$$

$$\left| \int_{C_R} e^{iaz}\, dz \right| = \left| \int_0^{\pi} e^{iax} e^{-ay} R i\, e^{i\theta} d\theta \right|$$

$$\le \left| R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta \right|$$

$$= 2R \left| \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta \right|$$



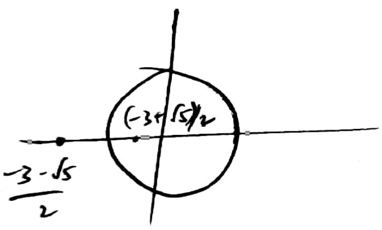
$$\le 2R \left| \int_0^{\pi/2} e^{-Ra(\theta \cdot 2/\pi)}\, d\theta \right|$$

$$\sin\theta > \theta \cdot \frac{2}{\pi}$$

$$e^{-aR\sin\theta} < e^{-Ra(\theta \cdot 2/\pi)}$$

Q $$\int_0^{\pi} \frac{d\theta}{(3 + 2\cos\theta)^2} = \frac{1}{2} \int_0^{2\pi} \frac{dz/iz}{\left[3 + 2\left(\frac{z+z^{-1}}{2}\right)\right]^2}$$

$$= \frac{1}{2i} \oint \frac{z\, dz}{(z^2 + 3z + 1)^2}$$



$$= \frac{2\pi i}{2i} \left[\text{Res } f(z_0) \right]$$

$m = 2$

$$= \pi \left[\cancel{\frac{z}{(z^2+3z+1)}} \right]$$

$z^2 + 3z + 1$

$\Rightarrow z = \frac{-3 \pm \sqrt{5}}{2}$

$$= \pi \left[\frac{1}{1} \frac{d}{dz} \left(\frac{z}{(z^2+3z+1)^2} \cdot \left[z - \left(\frac{-3+\sqrt{5}}{2} \right) \right]^2 \right) \right]$$

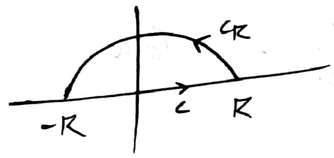
$$\lim_{z \to \frac{-3+\sqrt{5}}{2}} \pi \frac{d}{dz} \left(\frac{z}{\left(z + \frac{3+\sqrt{5}}{2}\right)^2} \right)$$

$$= \pi \cdot \frac{\left(z + \frac{3+\sqrt{5}}{2}\right)^2 - 2\left(z + \frac{3+\sqrt{5}}{2}\right) \cdot z}{\left(z + \frac{3+\sqrt{5}}{2}\right)^4}$$

$$= \frac{\pi \left(5 - 2 \cdot \left(\frac{-3+\sqrt{5}}{2}\right) \sqrt{5}\right)}{25}$$

$$= \pi \left(\frac{5 + 3\sqrt{5} - 5}{25} \right) = \cancel{\frac{6\sqrt{5}\pi}{28}} \quad \frac{3\sqrt{5}\pi}{25}$$

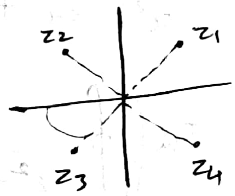
Q $$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx$$



$$\oint \frac{z^2+1}{z^4+1}\, dz$$

$$z_0 = \overset{z_1}{e^{i\pi/4}},\ \overset{z_2}{e^{3i\pi/4}},\ \overset{z_3}{e^{5i\pi/4}},\ \overset{z_4}{e^{7i\pi/4}}$$

$$= 2\pi i \left(\text{Res } f(z_0) \right)$$



$$= \text{Res } f(z_1) = \frac{z_1^2 + 1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}$$

$$= \frac{\cancel{(i+1)}}{\sqrt{2} \cdot \sqrt{2}\cancel{(1+i)}\sqrt{2}i} = \frac{1}{2\sqrt{2}i}$$

$$\cancel{\text{Res } f(z_2) = \frac{z_2^2 + 1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)}}$$

$$= 0$$

$$\cancel{\text{Res } f(z_3) = \frac{z_3^2 + 1}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)}}$$

$$=$$

$$\text{Res } (f(z_2)) = \frac{z_2^2 + 1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)}$$

$$= \frac{\cancel{(-i+1)}}{+\sqrt{2} \cdot \sqrt{2}i \ \sqrt{2}\cancel{(i-1)}}$$

$$= \frac{1}{2\sqrt{2}i}$$

$$\text{Res } f(z_3) = \frac{z_3^2 + 1}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)}$$

$$= \frac{\cancel{i+1}}{-\sqrt{2}\cancel{(i+1)} \cdot +\sqrt{2}i(+\sqrt{2})}$$

$$= \frac{-1}{2\sqrt{2}i}$$

$$\text{Res } f(z_4) = \frac{z_4^2 + 1}{(z_4 - z_1)(z_4 - z_2)(z_4 - z_3)} = \frac{\cancel{(-i+1)}}{(-\sqrt{2}i)(\sqrt{2})\cancel{(1-i)}\sqrt{2}} = \frac{+1}{2\sqrt{2}i}$$

$$\frac{1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} = \sqrt{2}(1+i)$$

$$\frac{1+i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}} \quad \sqrt{2}$$

$$\frac{1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} \quad \sqrt{2}(1+i)$$

$$\frac{-1+i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}} \quad -1+i-1-i$$

$$\frac{-1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}}$$

$$\frac{-1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \quad \sqrt{2}(-1+i)$$

$$\frac{-1-i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \quad \sqrt{2}(-1-i)$$

$$\frac{1-i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}}$$

$$\frac{-1-i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}$$

$$\frac{1-i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}} = -\sqrt{2}i$$

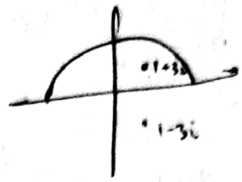
$$\frac{1-i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}} \quad \sqrt{2}(1-2i)$$

$$\frac{1-i}{\sqrt{2}} + \frac{+1+i}{\sqrt{2}}$$

$$2\pi i \left(\text{Res } f(z)\right)$$

$$= 2\pi i \left(\frac{1}{2\sqrt{2}i} + \frac{1}{2\sqrt{2}i} \cancel{\frac{-1}{2\sqrt{2}i}} + \cancel{\frac{1}{2\sqrt{2}i}} \right)$$

$$= 2\pi i \frac{1}{\sqrt{2}i} = \sqrt{2}\pi$$



Q $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10}$$

$$= \text{Im}\left(\oint \frac{z e^{iz}\, dz}{z^2-2z+10}\right)$$

$$z^2-2z+10 \Rightarrow$$

$$z = \frac{2 \pm \sqrt{4-40}}{2}$$

$$= \frac{2\pm 6i}{2} = 1\pm 3i$$

$$= \text{Im}\left(2\pi i \left(\text{Res } f(z_0)\right)\right)$$

$$\text{Im}\left(2\pi i \left(\lim_{z\to 1+3i} \frac{z e^{iz}\left(z-(1+3i)\right)}{z^2-2z+10}\right)\right)$$

$$\text{Im}\left(2\pi i\left(\lim_{z\to 1+3i}\frac{z e^{iz}}{z-(1-3i)}\right)\right)$$

$$\text{Im}\left(2\pi i \frac{(1+3i)\,e^{i(1+3i)}}{1+3i-1+3i}\right)$$

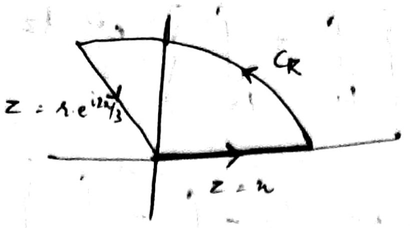
$$\text{Im}\left(2\pi i \frac{(1+3i)\,e^{i-3}}{6i}\right)$$

$$\text{Im}\left(\frac{\pi}{3e^3}(1+3i)\left[\cos 1 + i\sin 1\right]\right)$$

$$\text{Im}\left(\frac{\pi}{3e^3}\left(\cos 1 + i\sin 1 + 3i\cos 1 - 3\sin 1\right)\right)$$

$$= \cancel{\frac{\pi}{3e}}\ \frac{\pi}{3e^3}\left(\sin 1 + 3\cos 1\right)$$

$$\int_0^\infty \frac{dx}{x^3+1}$$



$$\oint \frac{dz}{z^3+1} = I - e^{i2\pi/3}\cdot I + \cancel{I_{C_R}} \to 0$$

$$I = \frac{\oint \frac{dz}{z^3+1}}{1 - e^{i2\pi/3}}$$

$$\int_{-\infty}^{\infty} \sin x/\cos x \; f(x)\, dx \to \left|\int_{C_R} f(z)\, e^{iz}\, dz\right| \le \int_{C_R} |f(z)||e^{iz}|\, dz$$

$$\le M_R \int_0^\pi e^{-R\sin\theta} R\, d\theta$$

$$\le M_R \times 2 \int_0^{\pi/2} e^{-R\sin\theta} R\, d\theta$$

$z = x + iy$

$|e^{iz}| = |e^{i(x+iy)}|$

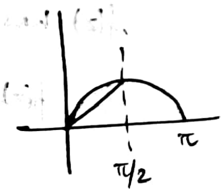
$= |e^{ix-y}|$

$= |e^{ix}\cdot e^{-y}|$

$= |e^{ix}||e^{-y}|$

$= 1\cdot|e^{-y}|$

$= e^{-R\sin\theta}$

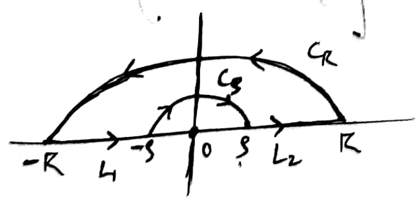


$\frac{2\theta}{\pi} \le \sin\theta$ (if $0 \le \theta \le \pi/2$)

$$\le 2M_R \int_0^{\pi/2} e^{-\frac{2\theta R}{\pi}} R\, d\theta$$

$M_R = \max_{C_R} |f(z)|$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \Rightarrow \int \frac{e^{iz}}{z}\, dz$$



$$\int_{L_1} \frac{e^{iz}}{z}\, dz = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\, dx$$

$$\int_{L_2} \frac{e^{iz}}{z}\, dz = \int_\rho^R \frac{e^{ix}}{x}\, dx$$

$$I = \operatorname{Im} \lim_{\substack{R\to\infty \\ \rho\to 0}} \left\{ \int_{L_1} + \int_{L_2} \right\} \frac{e^{iz}}{z}\, dz$$

$$= \operatorname{Im} \lim_{\substack{R\to\infty \\ \rho\to 0}} \left\{ -\cancel{\int_{C_R}}^{\,0} - \int_{C_\rho} \right\} \frac{e^{iz}}{z}\, dz$$

$$= \operatorname{Im} \left[- \oint_{C_\rho} \frac{e^{iz}}{z}\, dz \right]$$

$$= \cancel{\operatorname{Im} \left[-2\pi i \operatorname{Res} f(z_0) \right]}$$

$$= \cancel{\operatorname{Im} \left[2\pi i \lim_{z\to 0} \frac{e^{iz}}{z} \cdot z \right]}$$

$$\int_{C_\rho} f(z)\, dz$$

$f(z)$ has simple pole at $z=0$

$$f(z) = \underbrace{g(z)}_{\substack{\text{analytic} \\ \text{at } z=0}} + \frac{B_1}{z}$$

$$\left| \int_{C_\rho} g(z)\, dz \right| \le M \cdot \frac{\pi}{\rho} \longrightarrow 0 \quad \text{as } \rho \to 0$$

$$\int_{C_\rho} \frac{B_1}{z}\, dz = i \int_0^{\pi} \frac{B_1}{\rho e^{i\theta}} \rho i\, d\theta \cdot e^{i\theta}$$

$$= B_1 i\pi$$

$$= \pi i B_1$$

$z = \rho e^{i\theta}$

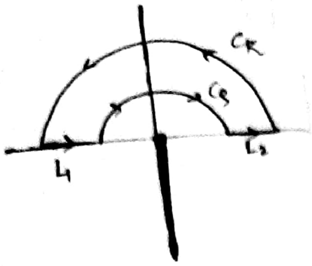
$B_1 = 1$

$$\therefore \operatorname{Im} \left[- \oint_{C_\rho} \frac{e^{iz}}{z}\, dz \right] = \operatorname{Im} \left[\lim_{\rho\to 0} \pi i (1) \right] = \pi$$

Q $\displaystyle\int \frac{\ln(x)}{(x^2+4)^2}\,dx$

$\ln z = \ln r + i\theta$

$-\pi/2 \le \theta \le 3\pi/2$



$$\int_{L_2} \frac{\ln z \cdot dz}{(z^2+4)^2} = \int_\rho^R \frac{\ln x}{(x^2+4)^2}$$

$$\int_{L_1} \frac{\ln z\, dz}{(z^2+4)^2} = \int_{-R}^{-\rho} \frac{\ln x + i\pi}{(x^2+4)^2} \qquad z = re^{i\pi}$$

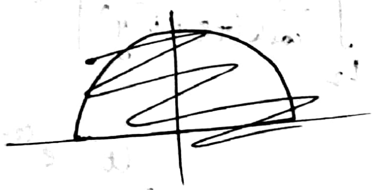
$$= -\int_\rho^R \frac{\ln r + i\pi}{(r^2+4)^2}\,dr$$

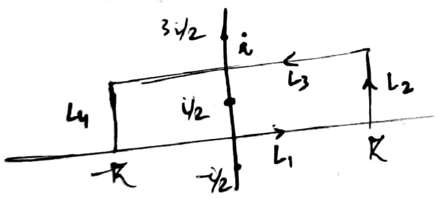
$$\rho \ln \rho = \frac{\ln \rho}{1/\rho} = \frac{1/\rho}{-1/\rho^2} = -\rho \to 0$$

Q $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx = \int_{-\infty}^{\infty} \frac{e^{2x}\,dx}{\cos(i\pi x)}$

$$= \int_{-\infty}^{\infty} \frac{e^{2x}\,2}{e^{-\pi x}+e^{\pi x}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{2z}\,dz}{\cosh(\pi z)}$$





$$\int_{L_1,\,-R}^{R} \frac{e^{2z}}{\cosh(\pi z)}\,dz - \int_{L_3,\,R}^{-R} \frac{e^{2(z-i)}}{\cosh(\pi(z-i))}\,dz$$

$z = x + i$

$x = z - i$

$\cancel{\cos}$ $\cosh(\pi z) = \cosh(\pi R + \pi i y)$

$\geq \frac{1}{2}\left(\left|e^{\pi R} \cdot e^{\pi i y}\right| - \left|e^{-\pi R} \cdot e^{-i z y}\right| \right)$

$\geq \frac{1}{2}\left(e^{\pi R} - \cancel{e^{-\pi R}} \right) \cancel{\left|e^{-iy}\right|}$

$= \cancel{\frac{1}{2}}$

$= \frac{1}{2} e^{\pi R}$

$\lim_{R\to\infty} \left(\int_{L_1} + \int_{L_3} \right) = I \pm e^{2i} I = 2\pi i \operatorname{Res}(i/2)$

$= \int_{L_1} - \int_{-L_3}$

$= \int_{L_1} - \int_{-L_3} \frac{e^{2i} \cdot e^{2x}}{\cosh(\pi z + \pi i)} dx$

$= \int_{L_1} + e^{2i} \int_{-L_3} \frac{e^{2x}}{\cosh(\pi z + \pi i)} dx$

$\left| \int_{L_2} \frac{e^{2(R+iy)}}{\cosh(\pi R + i\pi y)} \right| \leq M_R = e^{(2-\pi)R}$

$= 0$

$\lim_{R\to\infty} e^{(2-\pi)R} \to 0$

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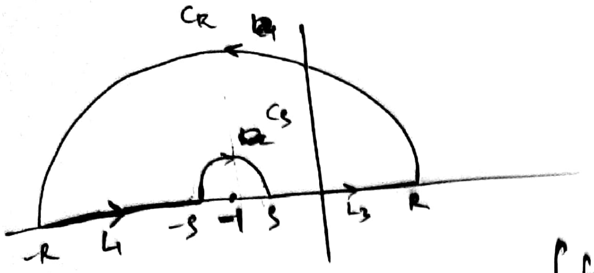
Q $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1}\, dx = \int \frac{e^{2iz}}{z+1}\, dz$



$$\int_{C_s} f(z)\, dz = -\pi i \operatorname{Res} f(z_0)$$

$$\lim_{z \to -1} \frac{e^{2iz}}{z+1}(z+1) = e^{-2i}$$

$$\int_{C_s} f(z)\, dz = -\pi i \cdot e^{-2i}$$

$$I = -\int_{C_s} f(z)\, dz = -(-\pi i e^{-2i}) = \pi i e^{-2i}$$

$\int_{C_s} f(z)dz = -\alpha i \operatorname{Res} f(z_0)$

Q $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)}\, dx = \int \frac{-e^{ix}}{x-1} + \frac{e^{ix}}{x-2}\, dx$

$=$

$$\int_{C_s} = -\pi i\left(-\operatorname{Res} f(1) + \operatorname{Res} f(2)\right)$$

$$= -\pi i \left(-\frac{e^{iz}(z-1)}{z-1} + \frac{e^{iz}(z-2)}{(z-2)} \right)$$

$$\int_{C_s} f(z)dz = -\pi i \left(-e^{i} + e^{2i} \right)$$

$$I = -\int_{C_s} f(z)\, dz = \pi i \left(e^{2i} + e^{-i} \right)$$