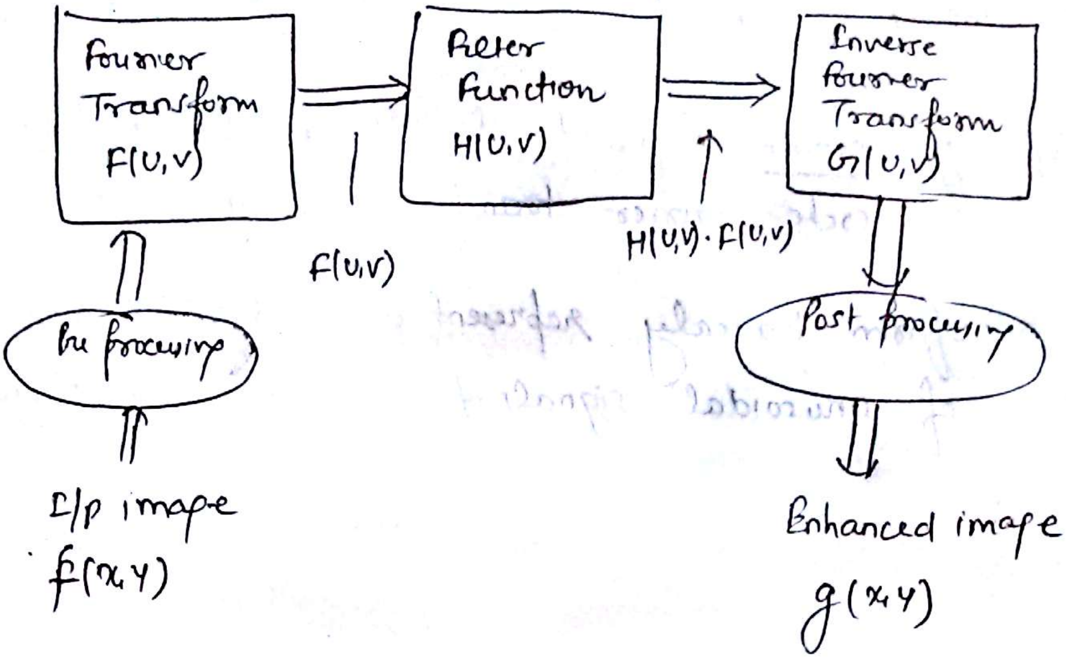
Image Enhancement in Frequency Domain



Frequency Domain filtering operation

$$\boxed{G(U,V) = H(U,V) \cdot F(U,V)}$$

Enhanced image — Transfer function — Given image

⇒ In frequency domain, space defined by values of the fourier transform and its frequency variables (u,v).

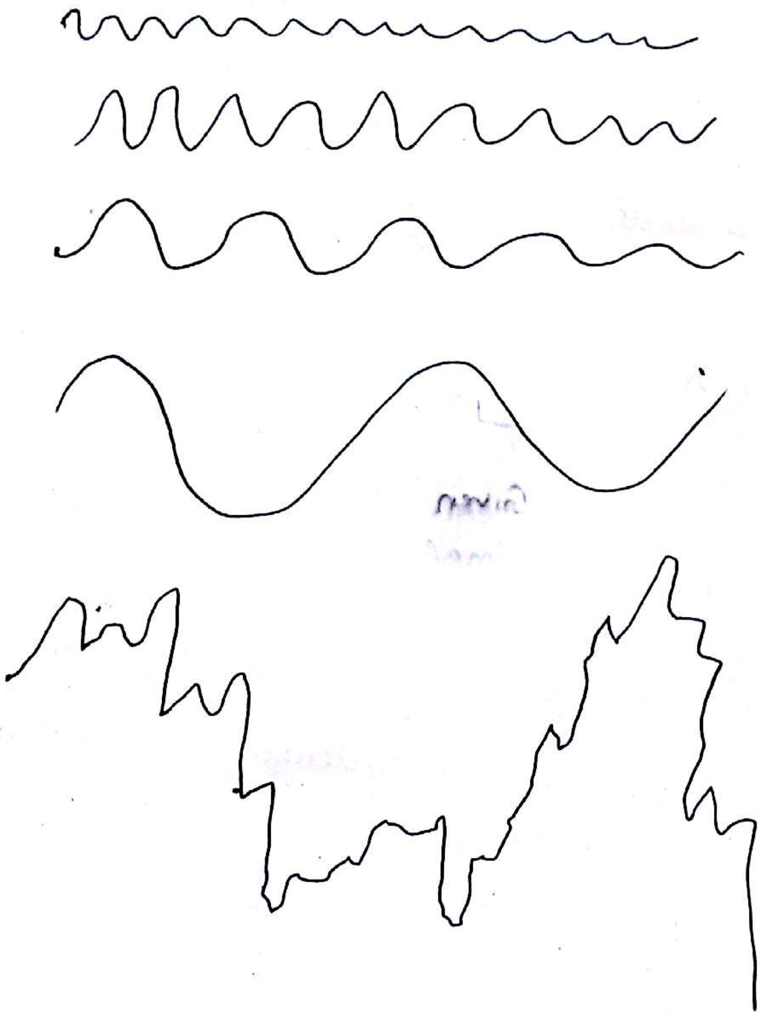
<u>fourier series</u>. Any function that periodically repeat itself can be expressed as the sum of sines /or cosines of different frequency, each multiplied by a different coefficient. (

Fourier transform :

function that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighted function.

The frequency Domain refers to the plane of the two dimensional discrete Fourier transform of an image.

Fourier transform basically represent a signal as a linear combination of sinusoidal signals of various frequencies



Introduction to fourier Transform

Let $f(x)$ be a continuous function of x. The fourier transform of $f(x)$ is

$$F(U) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux}\, dx \qquad \text{where } j = \sqrt{-1}$$

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

$$\boxed{F(U) = \int_{\infty}^{\infty} f(x)\left[\cos 2\pi ux - j\sin 2\pi ux\right] dx}$$

IFT

$$f(x) = \int_{-\infty}^{\infty} F(U)\, e^{j2\pi ux}\, du$$

2D fourier transform and its inverse (In continuous domain)

$$F(U,V) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\,dy$$

IFT

$$f(x,y) = \int_{\infty}^{\infty}\int_{-\infty}^{\infty} F(U,V)\, e^{+j2\pi(ux+vy)}\, du\, \cancel{dy}\, dv$$

1D fourier transform & its inverse (In Discrete case) DFT

$$F(U) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j2\pi ux/M} \qquad \text{for } u = 0, 1, 2 \,\text{—}\, M-1$$

IFT

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(U)\, e^{j2\pi ux/M} \qquad \text{for } x = 0, 1, 2 \,\text{—}\, M-1$$

$F(U) = |F(U)| \, e^{j\phi(U)}$

$F(U) = R(U) + j I(U)$

↓ Real | Imaginary

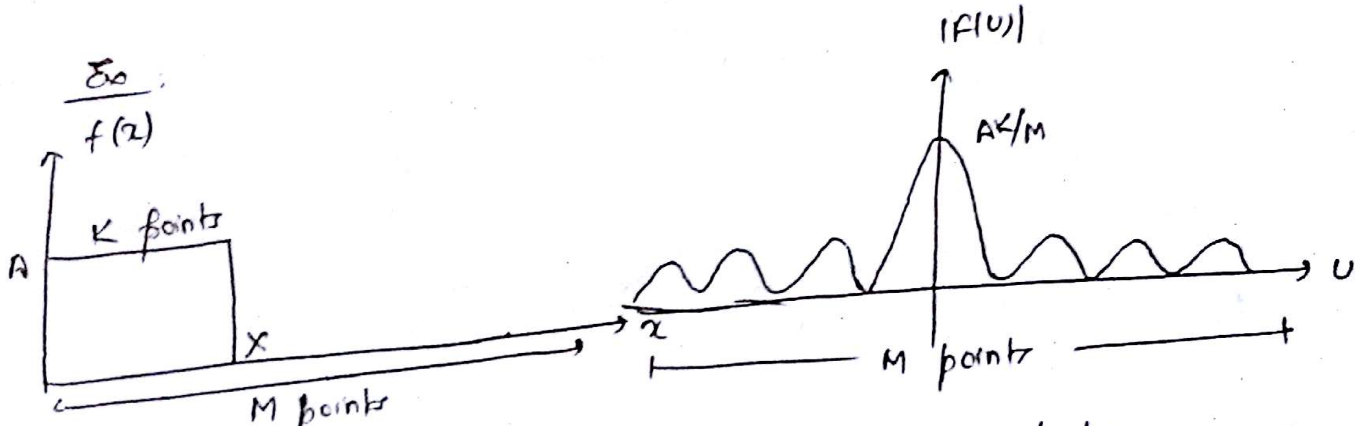
Mag $|F(U)| = [R^2(U) + I^2(U)]^{1/2}$ (Magnitude or spectrum)

$\phi(U) = \tan^{-1}\left[\frac{I(U)}{R(U)}\right]$ (phase angle or phase spectrum)

Power spectrum

$$\boxed{P(U) = |F(U)|^2 = R^2(U) + I^2(U)}$$

Ex



Discrete function of M points

Fourier spectrum

$$F(U) = \int_{-\infty}^{\infty} f(x) \, e^{-j2\pi u x} \, dx = \int_0^{X} A \, e^{-j2\pi u x} \, dx$$

$$F(U) = AX \, \frac{\sin(\pi u x)}{\pi u x} \left[e^{-j\pi u x}\right]$$

$$|F(U)| = AX \left| \frac{\sin(\pi u x)}{\pi u x} \right| \quad 1$$

$= B$

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2D Fourier transform & its inverse in Discrete case

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi (ux/M + vy/N)}$$

for $u = 0,1,2 \ldots M-1$ $\quad v = 0,1,2 \ldots N-1$

I.F.T.

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi (ux/M + vy/N)}$$

for $x = 0,1,2 \ldots M-1$ $\quad y = 0,1,2 \ldots N-1$

u,v = Frequency variable

x,y = Image variable.

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$\boxed{P(u,v) = R^2(u,v) + I^2(u,v)}$$

Properties of D.F.T. :-

① separability Property

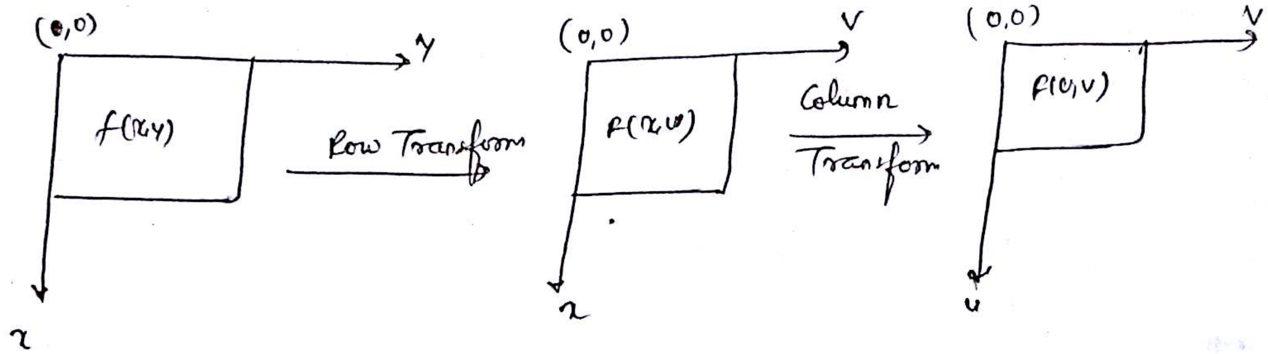
2D DFT can be seperated into two 1D D.F.T

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1} e^{-\frac{j2\pi ux}{M}} \sum_{y=0}^{N-1} e^{-j2\pi vy/N} f(x,y)$$

Let $F(x,v) = \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi vy/N}$ —①

$$\boxed{F(u,v) = \sum_{x=0}^{M-1} f(x,v)\, e^{-j2\pi ux/M}}$$



② Translation Property (shifting property)

$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$

$f(x-M/2, y-N/2) = F(u,v)(-1)^{u+v}$

$$f(x,y)\, e^{j2\pi \frac{(u_0x+v_0y)}{N}} \xrightarrow{FT} F(u-u_0, v-v_0)$$

if $f(x,y)$ is multiplied by an exponential, the original Fourier transform $F(u,v)$ gets shifted in frequency by $F(u-u_0, v-v_0)$.

$$F\{f(x,y)\} \longrightarrow F(u,v)$$

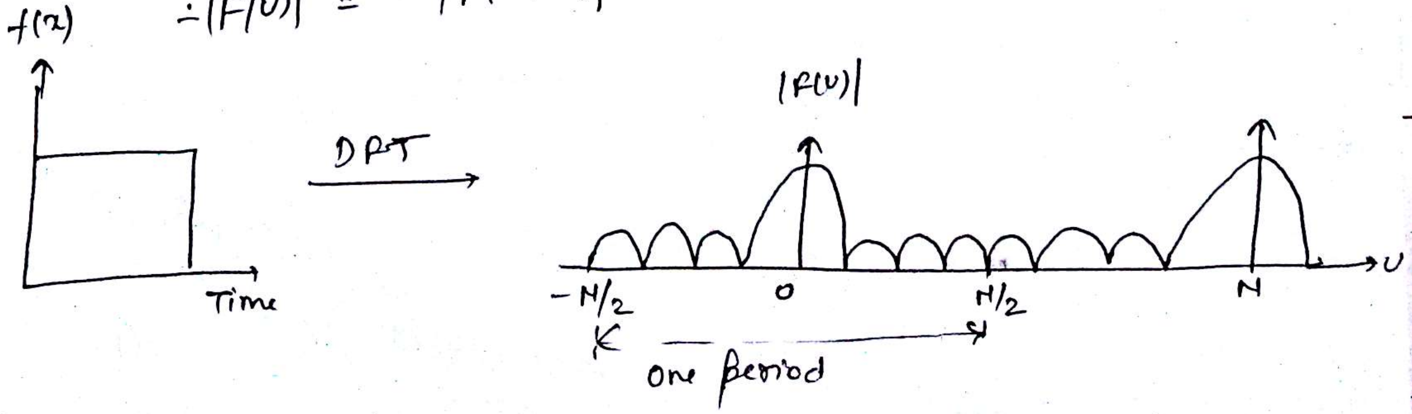
$$F\left\{f(x,y)\, e^{j2\pi \frac{(u_0x+v_0y)}{N}}\right\} \longrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\, e^{-j2\pi (ux_0/M + vy_0/N)}$$

③ Periodicity & conjugate symmetry property.

$$F(u) = F(u+N)$$

$$\therefore |F(u)| = |F(u+N)|$$



$$\left.\begin{aligned} F(U) &= F^{*}(-u) \\ |F(U)| &= |F(-u)| \end{aligned}\right\}$$

④ <u>Rotation property</u> :-

$x = r\cos\theta \qquad y = r\sin\theta$

$u = \omega\cos\phi \qquad v = \omega\sin\phi$

then $f(x,y)$ & $F(u,v)$ will become $f(r,\theta)$ &

$F(\omega,\phi)$

$f(r, \theta+\theta_0) \Rightarrow F(\omega, \phi+\phi_0)$

⑤ <u>Distributivity & scaling property</u> :-

$F\{f_1(x,y) + f_2(x,y)\} = F\{f_1(x,y)\} + F\{f_2(x,y)\}$

$F\{f_1(x,y) \cdot f_2(x,y)\} \neq F\{f_1(x,y)\} \cdot F\{f_2(x,y)\}$

Distributive over addition but not over multiplication

$af(x,y) \Leftrightarrow aF(u,v)$

& $f(ax, by) \Rightarrow \frac{1}{|ab|} F(u/a, v/b)$

⑥ <u>Average value property</u> :-

$$\bar{f}(x,y) = \frac{1}{M.N} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \quad \text{--- (1)}$$

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

when $u=0$ & $v=0$

$$F(u,v) = F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \quad \text{--- (2)}$$

by (1) & (2)

$$\bar{f}(x,y) = \frac{1}{M.N} F(0,0)$$

Laplacian property :

The laplacian of 2D function is defined as

$$\nabla^2 f(x,y) = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2}$$

From 2D D.F.T.

$$\boxed{F\{\nabla^2 f(x,y)\} = -(2\pi)^2 (u^2+v^2)\, F(u,v)}$$

we know that

$$f(x,y) = \frac{1}{MN^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\frac{(ux+vy)}{N}} \quad \text{— IDFT}$$

we ignore $1/N^2$ term

$$= \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\, e^{+j\frac{2\pi ux}{N} + j\frac{2\pi vy}{N}}$$

$$\frac{\delta}{\delta x}\{f(x,y)\} = \frac{\delta}{\delta x} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \frac{\delta}{\delta x} e^{\left(\frac{j2\pi ux}{N} + \frac{j2\pi vy}{N}\right)}$$

$$= \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} j2\pi u\, F(u,v)\, e^{\frac{j2\pi ux}{N} + \frac{j2\pi vy}{N}}$$

$$\frac{\delta}{\delta x} f(x,y) = F^{-1}\{j2\pi u\, F(u,v)\}$$

$$F\left\{\frac{\delta}{\delta x} f(x,y)\right\} = (j2\pi u)\, F(u,v)$$

$$\therefore F\, \frac{\delta^2}{\delta x^2} f(x,y) = (j2\pi u)^2\, F(u,v)$$

Similarly $F\, \frac{\delta^2}{\delta y^2} f(x,y) = (j2\pi v)^2\, F(u,v)$

we know that $\nabla^2 f(x,y) = \frac{\delta^2 f(x,y)}{\delta x^2} + \frac{\delta^2 f(x,y)}{\delta y^2}$

$$F\{\nabla^2 f(x,y)\} = (j2\pi u)^2 F(u,v) + (j2\pi v)^2 F(u,v)$$

$$= -(2\pi)^2 [u^2+v^2]\, F(u,v)$$

Convolution property :-

$g(x,y) = f(x,y) * h(x,y)$ in spatial Domain

$G(u,v) = F(u,v) \times H(u,v)$ in the frequency Domain

Hence to find $G(u,v)$ we take the 2D DFT of the I/p image, find the transfer function of the filter, $H(u,v)$ and simply multiply two.

Finally we take the inverse fourier transform of $G(u,v)$ to get the modified image ie $g(x,y)$.

Frequency domain

$$
\begin{tikzcd}
f(x,y) \arrow[r] & \boxed{\text{Multiplying } (-1)^{x+y}} \arrow[r] & \boxed{\text{2D DFT}} \arrow[r] & \boxed{F(u,v)} \arrow[d] \\
 & & \boxed{H(u,v)} \arrow[r] & \boxed{\times} \arrow[d] \\
 & & & \boxed{G(u,v)} \arrow[d, "\text{2D IDFT}"] \\
 & & & \boxed{g(x,y)}
\end{tikzcd}
$$

Centre → Low frequency

Peripherals → High frequencies

Low pass filters (In frequency Domain)

For filtering we use the formula

$G(u,v) = F(u,v) \times H(u,v)$

There are 3 basic types of low pass filters

a) Ideal b) Butterworth c) Gaussian

Ideal low pass filter :-

It cuts off all high frequencies components of the Fourier transform that are at a distance greater than a specified distance D_0

$$H(U,V) = \begin{cases} 1; & \text{if } D(U,V) \le D_0 \\ 0 & \text{if } D(U,V) > D_0 \end{cases}$$

D_0 = specified non negative distance

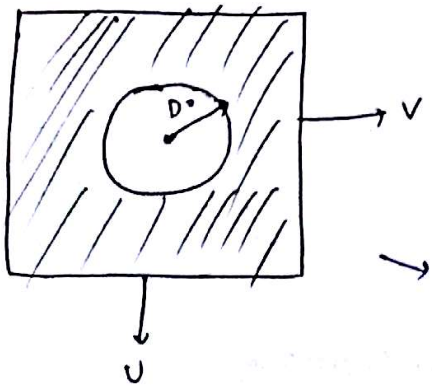
$D(U,V)$ is the distance from the pt (U,V) to the origin of the frequency reactangle for an $M \times N$ image.

$$D(U,V) = \left[(U - M/2)^2 + (V - N/2)^2 \right]^{1/2} \quad (\text{It centres our } H(U,V))$$

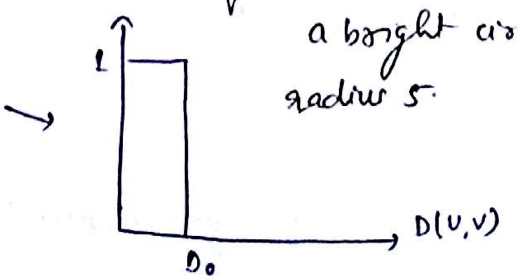
when $U = M/2$, $V = N/2$ $\quad D(U,V) = 0$

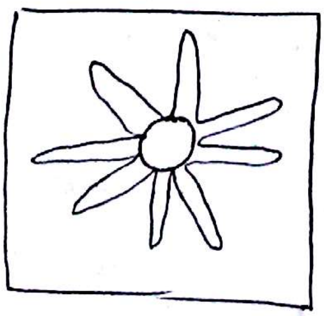
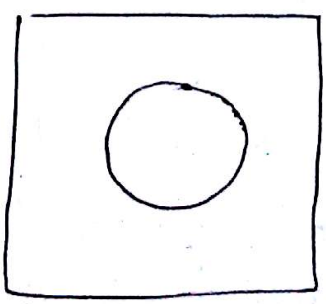
Fourier transform that we plot is centred using the formula $f(x,y) \cdot (-1)^{x+y}$.

D_0 = Cut off frequencies



$H(U,V)$ if $D_0 = 5$; then $H(U,V)$ will be a bright circle having radius 5.



$\therefore G(U,V) =$  $\times$ 

F(U,V) $\quad$ H(U,V)

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Drawbacks

1. Ringing effect which occurs along the edges of filtered real domain image

What should be the ideal value of D_0

H(U,V) is a circular structured based on the value of D_0.

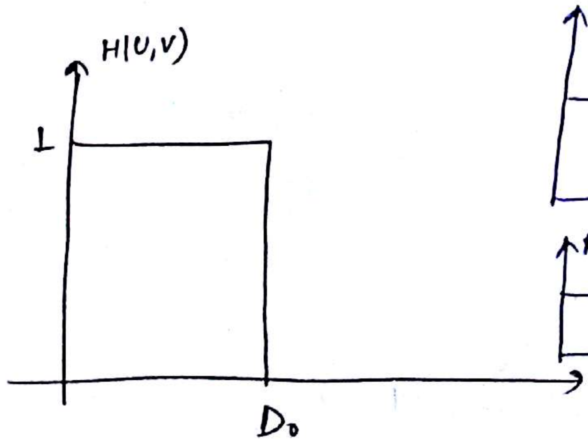
One way to establish a standard cut-off frequency is to compute circles that enclose specified amount of total image power P_{total}

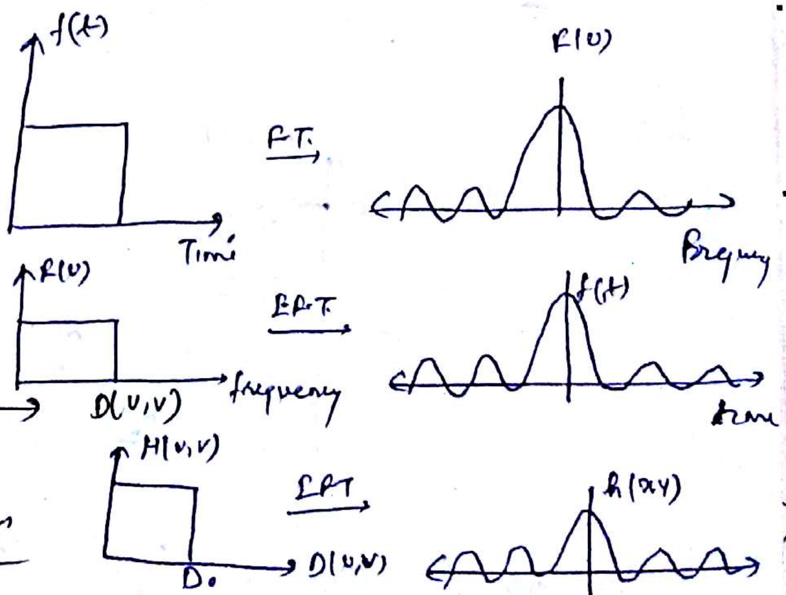
we know that $P(U,V) = (F(U,V))^2 = [R^2(U,V) + I^2(U,V)]$

$$P_{total} = \sum_{U=0}^{M-1} \sum_{V=0}^{N-1} P(U,V)$$

A circular radius of value r with the origin at the centre of the frequency rectangle encloses a percent of power where

$$\alpha = 100 \times \left[\sum_{V=0}^{M-1} \sum_{V=0}^{N-1} P(U,V) / P_{total} \right]$$





Filter radial cross section

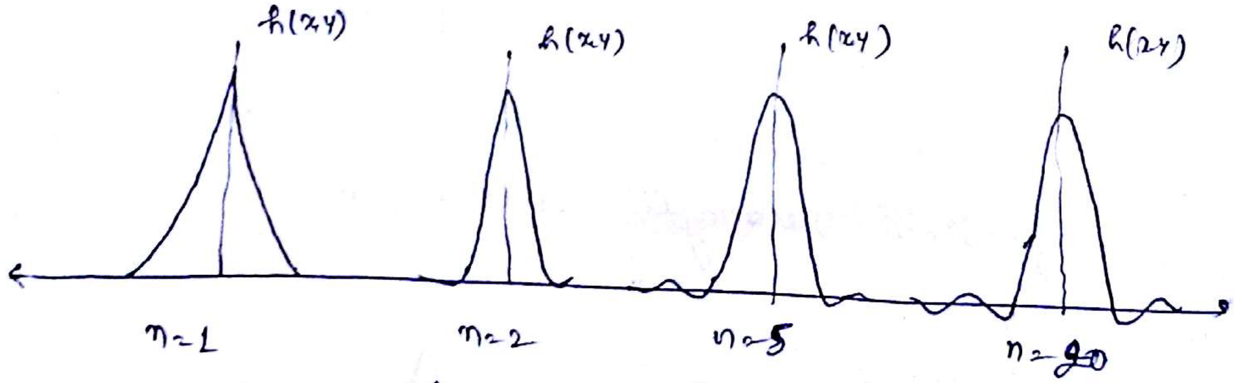
LPF is a type of non physical filters and can not be realized with electronic component and is not very practical

Butterworth low pass filter:

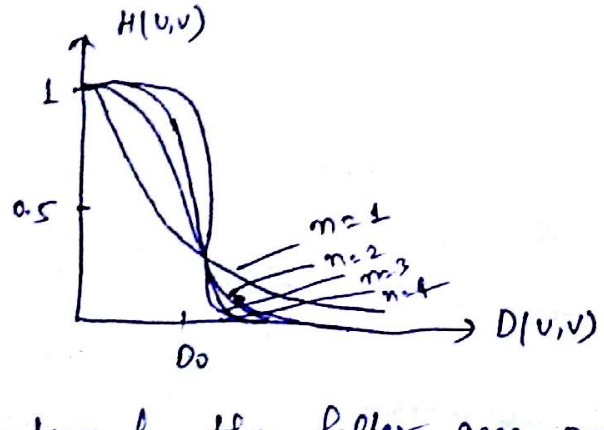
The ringing effects are due to the sharp cutoffs in the ideal filter. To get rid of ringing effects BLPF are generally used.

The transfer function of the Butterworth low pass filter of order n and cutoff frequency at a distance D_0 from the origin is defined as

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$



spatial representation of B.L.P.F



⇒ As a order of the filter goes on increasing, a small amount of ringing effects does creep in because the Butterworth low pass filter tends to be an ideal filter.

⇒ Hence for good result use an order ≤5.

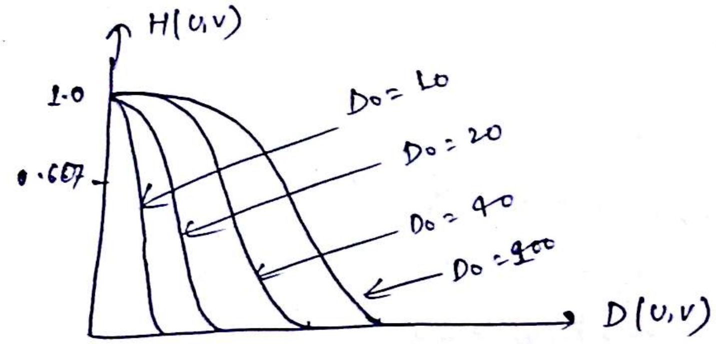
Gaussian low pass filter :

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

σ is the standard deviation and is a measure of spread of the Gaussian curve.

if $\sigma = D_0$

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$



filter radial cross section for various values of Do

when $D(u,v) = D_0$, the GLPF is down to 0.607 of its maximum value.

In gaussian low pass filter we are assured that there will be no ringing effects no matter what filter order we chose to work with.

Image sharpening using frequency Domain filter:

A high pass filter is obtained from a given lowpass filter using the following eqn

$$\boxed{H_{HP}(u,v) = 1 - H_{LP}(u,v)}$$

There are 3 basic types of High pass filters.

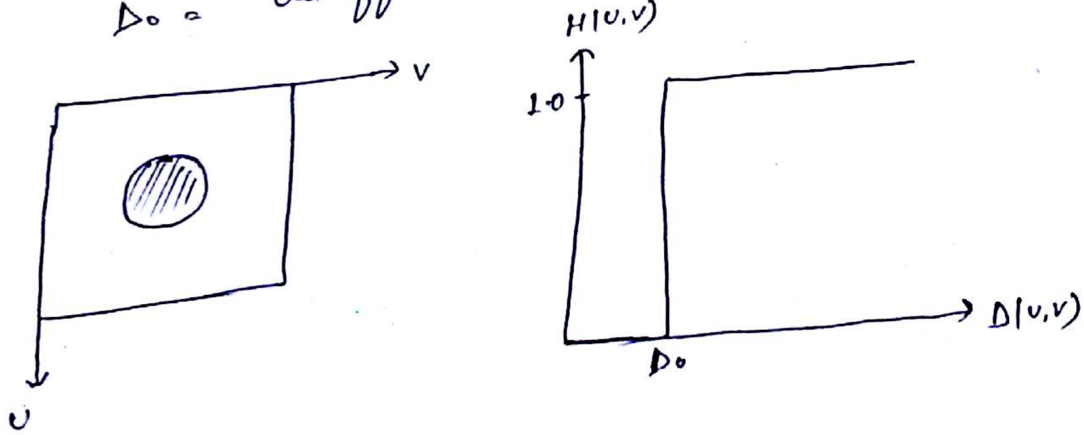
1) Ideal High pass
2) Butterworth high pass
3) Gaussian high pass

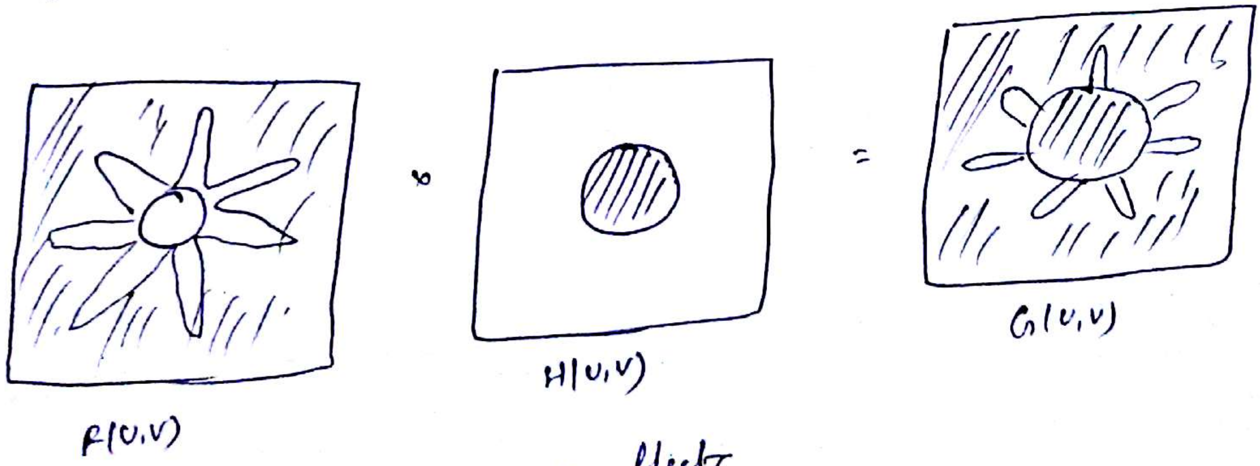
① Ideal High pass filter :-

2D ideal High pass filter defined as

$$H(u,v) = \begin{cases} 0 ; & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

D_0 = cutoff





Disadvantage   ringing effects

Butterworth high pass filters:

$$\boxed{H_{HBWF}(U,V) = \frac{1}{1 + \left[\frac{D_0}{D(U,V)}\right]^{2n}}}$$

$$H_{LBWF}(U,V) = \frac{1}{1 + \left[\frac{D(U,V)}{D_0}\right]^{2n}}$$

$$H_{HBWF}(U,V) = 1 - \frac{1}{1 + \left[\frac{D(U,V)}{D_0}\right]^{2n}}$$

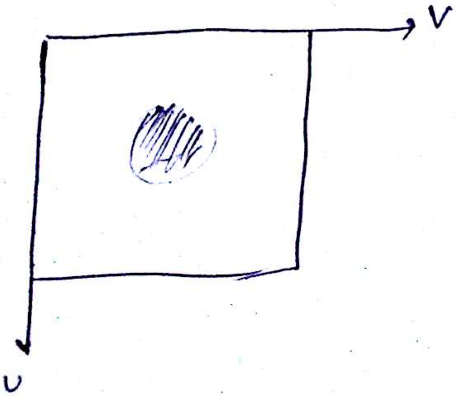
Let $\left[\frac{D(U,V)}{D_0}\right]^{2n} = x$

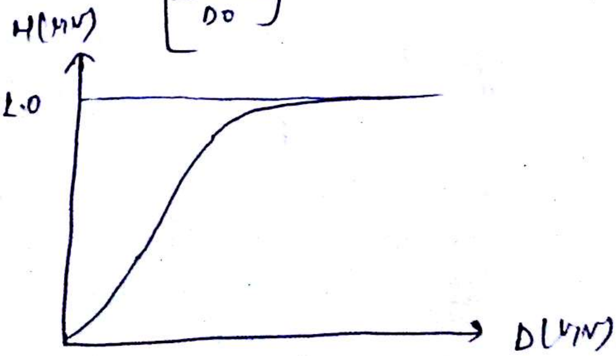
$$1 - \frac{1}{1+x}$$

$$\frac{1+x-1}{1+x} = \frac{x}{1+x}$$

$$= \frac{1}{1+\frac{1}{x}}$$

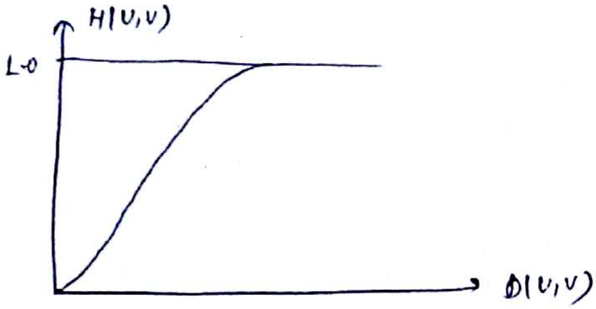
$$= \frac{1}{1+\frac{1}{\left[\frac{D(U,V)}{D_0}\right]^{2n}}} = \frac{1}{1+\left[\frac{D_0}{D(U,V)}\right]^{2n}}$$





Gaussian High pass filter:

$$H(U,V) = 1 - e^{-D^2(U,V)/2D_0^2}$$



Unsharp masking, High boost filtering:

$$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

with $f_{LP}(x,y) = F^{-1}\{H_{LP}(U,V)\, F(U,V)\}$

Here $f_{LP}(x,y)$ is a smoothed image analogous to $f(x,y)$

$$g(x,y) = f(x,y) + K * g_{mask}(x,y)$$

this expression define unsharp masking when $K=1$

and high boost filtering when $K>1$

$$g(x,y) = f(x,y) + K * [f(x,y) - f_{LP}(x,y)]$$

$$g(x,y) = F^{-1}[F(U,V) + K * [F(U,V) - H_{LP}(U,V)\, F(U,V)]]$$

$$= F^{-1}[F(U,V)\,(1 + K * [1 - H_{LP}(U,V)])]$$

$$\boxed{g(x,y) = F^{-1}\{[1 + K * H_{HP}(U,V)]\, F(U,V)\}}$$

Homomorphic filtering :-

$f(x,y)$ can be expressed as the product of illumination $i(x,y)$ & reflectance, $r(x,y)$ components.

$$f(x,y) = i(x,y)\ r(x,y)$$

It is not used directly to operate on frequency components of illumination & reflectance because

$$F[f(x,y)] \neq F[i(x,y)] \cdot F[r(x,y)]$$

Now suppose

$$z(x,y) = \ln f(x,y)$$
$$= \ln (i(x,y) \cdot r(x,y))$$
$$= \ln i(x,y) + \ln r(x,y)$$

$$F[z(x,y)] = F[\ln i(x,y)] + F[\ln r(x,y)]$$

$$Z(u,v) = F_i(u,v) + F_r(u,v)$$

we can filter $Z(u,v)$ using filter $H(u,v)$ so that

$$S(u,v) = H(u,v)\ Z(u,v)$$
$$= H(u,v)\ [F_i(u,v) + F_r(u,v)]$$
$$= H(u,v)\ F_i(u,v) + H(u,v) \cdot F_r(u,v)$$

in spatial domain

$$s(x,y) = F^{-1}(S(u,v))$$
$$= \underbrace{F^{-1}[H(u,v)\ F_i(u,v)]}_{i'(x,y)} + \underbrace{F^{-1}[H(u,v) \cdot F_r(u,v)]}_{r'(x,y)}$$

$$s(x,y) = i'(x,y) + r'(x,y)$$

Because $z(x,y)$ was formed by taking the natural logarithm of the input image, we reverse the process by taking the exponential of the filtered result to form the o/p image.

$$g(x,y) = e^{s(x,y)}$$
$$= e^{i'(x,y)}\ e^{r'(x,y)}$$
$$= i_0(x,y)\ r_0(x,y)$$

$$i_0(x,y) = e^{i'(x,y)}$$

$$r_0(x,y) = e^{r'(x,y)}$$

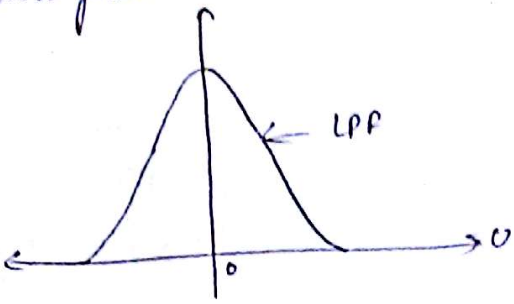
$$f(x,y) \Rightarrow \boxed{\ln} \Rightarrow \boxed{DFT} \Rightarrow \boxed{H(u,v)} \Rightarrow \boxed{(DFT)^{-1}} \Rightarrow \boxed{\exp} \Rightarrow g(x,y)$$

summary of steps in Homomorphic filter

Correspondance b/w Filtering in the spatial & frequency domain

Generating $h(x,y)$ from $H(U,V)$:

Let us start this with a gaussial low pass filter



$$H(U) = A e^{-u^2/2\sigma^2}$$

$$h(x) = \int_{-\infty}^{\infty} H(U)\, e^{j2\pi ux}\, du$$

$$\Rightarrow \int_{-\infty}^{\infty} A e^{-u^2/2\sigma^2} e^{j2\pi ux}\, du$$

$$= A \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[u^2 - j4\pi\sigma^2 ux\right]}\, du$$

Now

$$\frac{e^{(2\pi)^2 x^2 \sigma^2}}{2} \cdot \frac{e^{-(2\pi)^2 x^2 \sigma^2}}{2} = 1$$

$$= A \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[u^2 - j4\pi\sigma^2 ux + (2\pi)^2 x^2 \sigma^4\right]} \cdot e^{-(2\pi)^2 x^2 \sigma^2/2}\, du$$

$$\Rightarrow A\left[e^{(2\pi)^2 x^2 \sigma^2} \int_{-\infty}^{\infty} e^{-1/2\sigma^2\left[u - j2\pi x\sigma^2\right]^2} du\right]$$

$\Rightarrow$ let $r = u - j2\pi x\sigma^2$

$dr = du$

$$h(x) = A\left[e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}}\, dr\right]$$

Multiplying & dividing by $\sqrt{2\pi}\,\sigma$

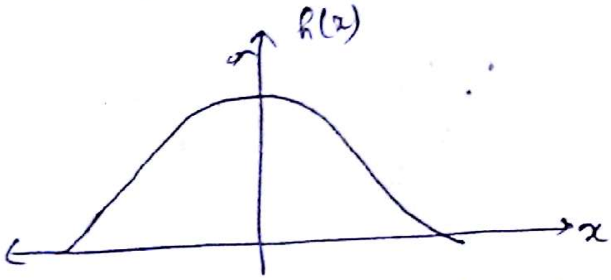
$$h(x) = A\sqrt{2\pi}\,\sigma\, e^{-2\pi^2 x^2 \sigma^2} \underbrace{\left[\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}}\, dr\right]}_{\text{Gaussian function}}$$

Gaussian function is 1 in the range $-\infty$ to ∞.

$\therefore\ h(x) = \sqrt{2\pi}\,\sigma A\, e^{-2\pi^2\sigma^2 x^2}$

similarly a 2D gaussian function is

$h(x,y) = \sqrt{2\pi}\,\sigma A\, e^{-2\pi^2\sigma^2(x^2+y^2)}$

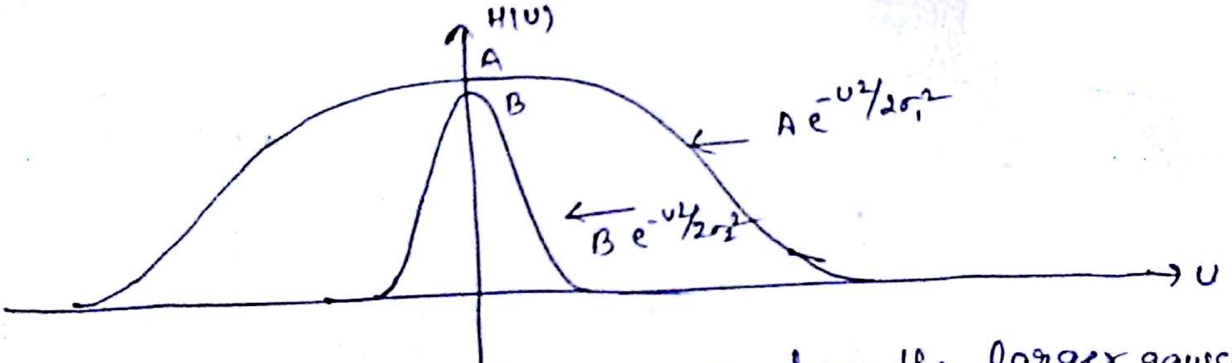


Here h(x) is broader than H(U). Narrower the H(U), broader will be h(x). All the values of h(x) are (+)ve

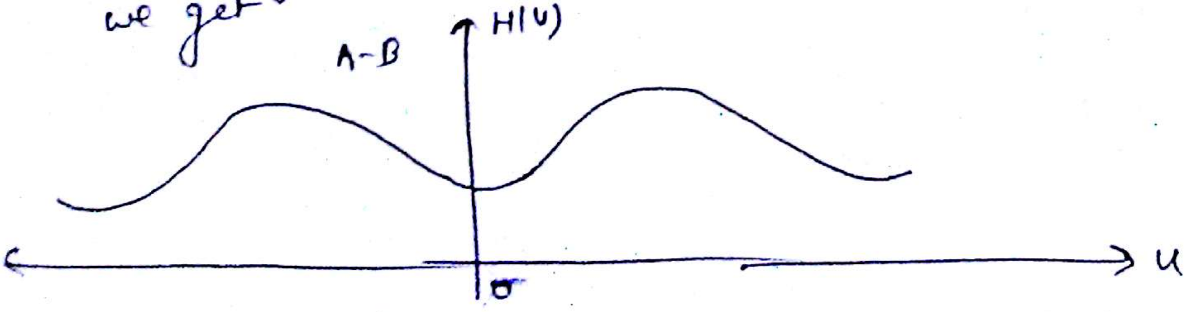
Here we used a GLPF because more complex filters can be constructed from the basic gaussian function

A high pass filter in frequency domain can be constructed using a difference of two gaussian low pass filters.

$H(U) = A e^{-u^2/2\sigma_1^2} - B e^{-u^2/2\sigma_2^2}$



Subtracting the smaller gaussian from the larger gaussian we get



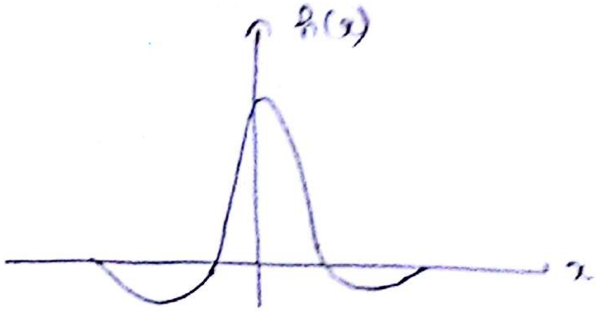
Inverse fourier transform of $H(u)$, we get

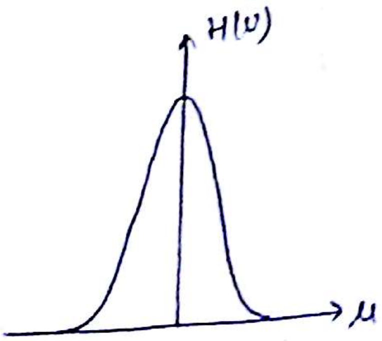
$$\boxed{h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}}$$

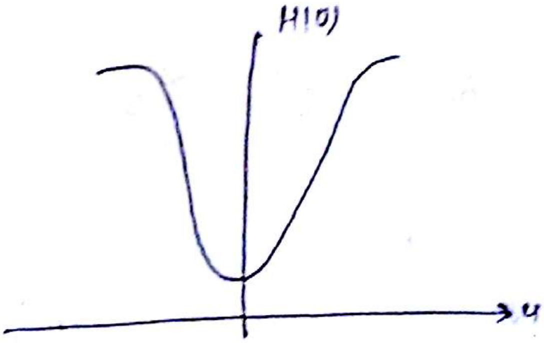
-1	-1	-1
-1	8	-1
-1	-1	-1

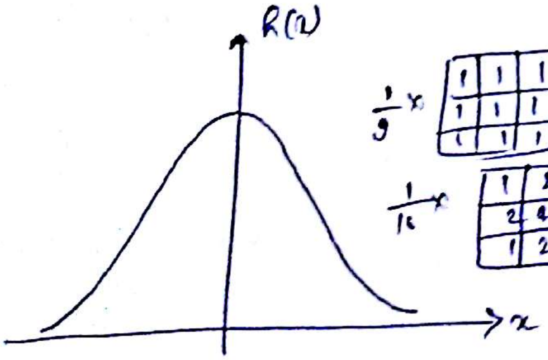
or

0	-1	0
-1	4	-1
0	-1	0









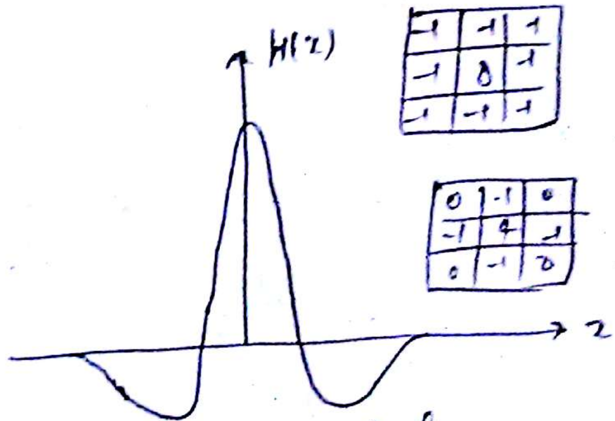
$\frac{1}{9} \times$

1	1	1
1	1	1
1	1	1

$\frac{1}{16} \times$

1	2	1
2	4	2
1	2	1

Low pass



-1	-1	-1
-1	8	-1
-1	-1	-1

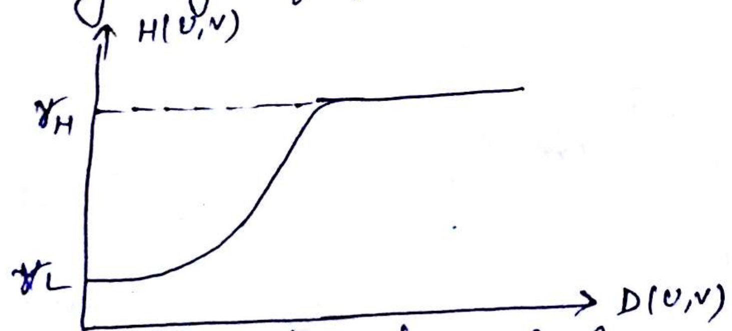
0	-1	0
-1	4	-1
0	-1	0

High pass

Remaining part of homomorphic filtering

The key approach in homomorphic filtering is to seperate the illumination and reflectance components. Between them $i(x,y)$ contributes to the ~~f~~ low frequency because illumination is more or less uniform and $r(x,y)$ is the high frequency component since $r(x,y)$ tends to vary abruptly.

$H(u,v)$ tends to decrease the contribution made by low frequencies (illumination) and amplify the contribution made by high frequencies (reflectance).



Radial crossection of circularly symmetric homomorphic filter function

$\gamma_L < 1$ & $\gamma_H > 1$

$H(U,V)$ can be implemented by a simple formula.

$$H(U,V) = (\gamma_H - \gamma_L)\left[1 - e^{-c[D^2(U,V)/D_0^2]}\right] + \gamma_L$$

c is a constant used to controls the ~~f~~ sharpness of the slope of the function as it transitions b/w γ_L and γ_H.

Generating $H(u,v)$ from $h(x,y)$:-

① Multiply $h(x,y)$ by $(-1)^{x+y}$ to centre the frequency domain filter

② Compute the forward DFT of the result in ①

③ Set the real part of the result DFT to 0 to account for parasitic real parts ($H(u,v)$ has to be purely imaginary)

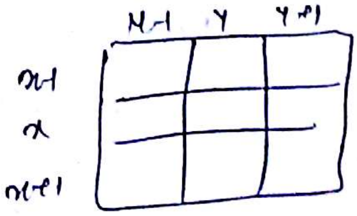
④ Multiply the result by $(-1)^{u+v}$

⇒ we perform the conversion from $h(x,y)$ to $H(u,v)$ to understand what different masks would do and what kind of filtering would they achieve

⇒ Once we plot $H(u,v)$, we can analyse the filter by its shape

⇒ One common property that we use for this conversion is:- (translation)

$$F\left[f(x-x_0, y-y_0)\right] \overset{F}{\Leftrightarrow} F(u,v)\, e^{-j2\pi\frac{(ux_0+vy_0)}{N}}$$



$F[f(x,y)] \Rightarrow F(u,v)$

$F[f(x+1,y)] \Rightarrow e^{\frac{j2\pi u}{N}} F(u,v)$

$F[f(x-1,y)] \Rightarrow e^{-\frac{j2\pi u}{N}} F(u,v)$

$F[f(x,y-1)] \Rightarrow e^{-j2\pi v/N} F(u,v)$

$F[f(x,y+1)] \Rightarrow e^{j2\pi v/N} F(u,v)$

$F[f(x-1,y-1)] = F(u,v)\, e^{-j2\pi\left(\frac{u+v}{N}\right)}$

$F[f(x+1,y+1)] \Rightarrow F(u,v)\, e^{j2\pi\frac{(u+v)}{N}}$

Ex Given a spatial mask $h(x,y)$, find the equivalent filter $H(U,V)$ in the frequency domain.
Comment on what kind of filter $h(x,y)$ is

0	1/6	0
1/6	1/3	1/6
0	1/6	0

	y-1	y	y+1
x-1	0	$f(x,y-1)$	0
x	$f(x,y-1)$	$f(x,y)$	$f(x,y+1)$
x+1	0	$f(x+1,y)$	0

we know that

$$F[f(x-x_0, y-y_0)] \Rightarrow F(U,V)\, e^{-j2\pi\frac{(Ux_0+Vy_0)}{N}} \quad \text{—}\ (A)$$

$$\frac{1}{6} f(x, y-1) \Rightarrow \frac{1}{6} F(U,V)\, e^{-\frac{j2\pi V}{N}} \quad \text{—}\ (1)$$

$$\frac{1}{6} f(x-1, y-1) = \frac{1}{6} F(U,V)\, e^{-j2\pi\frac{(U+V)}{N}} \quad \text{—}\ (2)$$

$$\text{—}\ (3)$$

$$\frac{1}{3} f(x,y) = \frac{1}{3} F(U,V)$$

$$\frac{1}{6} f(x, y+1) = \frac{1}{6} F(U,V)\, e^{j2\pi V/N} \quad \text{—}\ (4)$$

$$\frac{1}{6} f(x+1, y) = \frac{1}{6} F(U,V)\, e^{j2\pi U/N} \quad \text{—}\ (5)$$

$$G(U,V) = (1) + (2) + (3) + (4) + (5)$$

$$= \frac{1}{3} F(U,V) + \frac{1}{6}\left[F(U,V)\, e^{\frac{j2\pi V}{N}} + e^{-\frac{j2\pi V}{N}}\right] + \frac{1}{6}\left[e^{\frac{j2\pi U}{N}} + e^{-\frac{j2\pi U}{N}}\right]$$

$$= F(U,V)\left[\frac{1}{3} + \frac{1}{6}\left(e^{\frac{j2\pi V}{N}} + e^{-\frac{j2\pi V}{N}}\right) + \frac{1}{6}\left(e^{\frac{j2\pi U}{N}} + e^{-\frac{j2\pi U}{N}}\right)\right]$$

$$= F(U,V)\left[1/3 + \frac{1}{3}\cos\left(\frac{2\pi U}{N}\right) + \frac{1}{3}\left(\cos\frac{2\pi V}{N}\right)\right]$$

$$G(U,V) = H(U,V) \cdot F(U,V)$$

$$H(U,V) = \left[\frac{1}{3} + \frac{1}{3}\cos\left(\frac{2\pi U}{N}\right) + \frac{1}{3}\cos\left(\frac{2\pi V}{N}\right)\right]$$

$$H(U,V)_{U=0,\,V=0} \quad \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$U = V = N/4 \quad \frac{1}{3} + \frac{1}{3}\cos \pi/2 + \frac{1}{3}\cos \pi/2$$

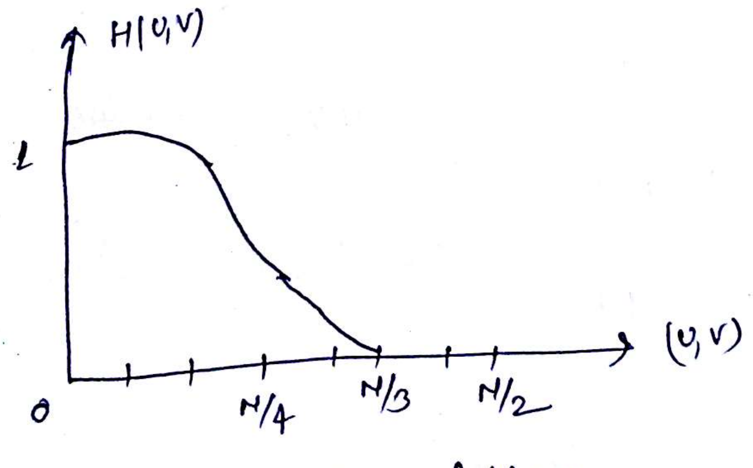
$$= \frac{1}{3} = 0.33$$

$$U = V = N/3 \quad \frac{1}{3} + \frac{1}{3}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{3}\cos\left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3}\cos(90+30) + \frac{1}{3}\cos(90+30)$$

$$= \frac{1}{3} + \frac{1}{3}(-0.5) + \frac{1}{3}(-0.5)$$

$$= \frac{1}{3} - \frac{1}{3} = 0$$



This is low pass filter.

Que Given $h(x,y)$

1	-3	1
-3	9	-3
1	-3	1

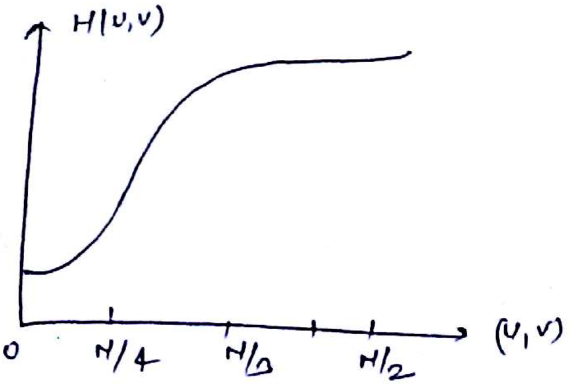
Find frequency domain filter $H(u,v)$.

$$G(u,v) = \left[9 - 6\cos\left(\frac{2\pi u}{N}\right) - 6\cos\left(\frac{2\pi v}{N}\right) + 2\cos\left(\frac{2\pi(u+v)}{N}\right) + 2\cos\left(\frac{2\pi(u-v)}{N}\right) \right] F(u,v)$$

$$H(u,v)\Big|_{u=0,v=0} = [9-6-6+2+2] = 1$$

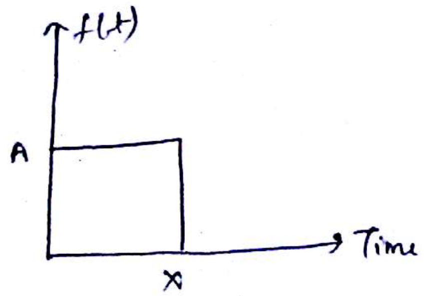
$$u = v = N/4 \quad = 16$$

$$u = v = N/2 \quad = 23$$



High pass filter

Que: Find the Fourier Transform of the signal shown



$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi u t}\, dt$$

$$= \int_{0}^{x} A\, e^{-j2\pi u t}\, dt$$

$$\Rightarrow \frac{A.}{-j2\pi u} \left[e^{-j2\pi u t} \right]_0^x$$

$$= \frac{-A}{j2\pi u} \left[e^{-j2\pi u x} - 1 \right]$$

$$= \frac{A}{j2\pi u} \left[1 - e^{-j2\pi u x} \right]$$

$$\Rightarrow \frac{A}{j2\pi u} \left[e^{+j\pi u x} - e^{-j\pi u x} \right] e^{-j\pi u x}$$

$$\Rightarrow \frac{A}{\pi u} \left[\frac{e^{j\pi u x} - e^{-j\pi u x}}{2j} \right] e^{-j\pi u x}$$

$$F(u) = \frac{A}{\pi u} \left[\sin(\pi u x)\, e^{-j\pi u x} \right]$$

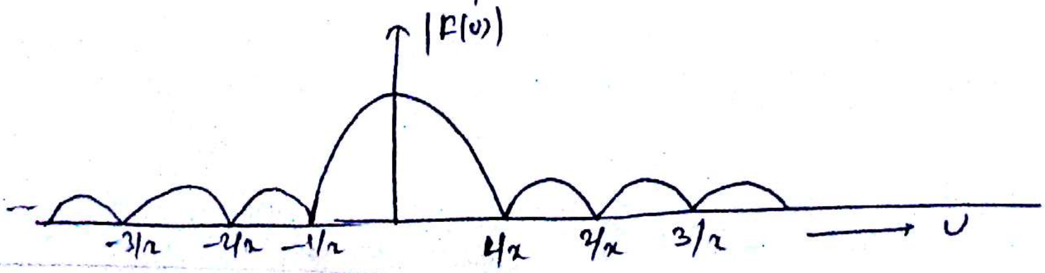
$$= A\, \frac{\sin(\pi u x)}{\pi u} \left[e^{-j\pi u x} \right]$$

$$F(u) = Ax\, \frac{\sin(\pi u x)}{\pi u x} \left[e^{-j\pi u x} \right]$$

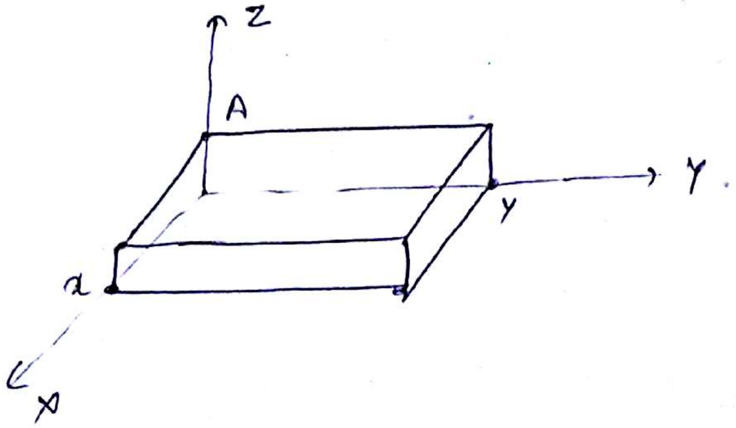
$$|F(u)| = |Ax| \left| \frac{\sin(\pi u x)}{\pi u x} \right| \underbrace{\left| e^{-j\pi u x} \right|}_{1}$$

$$= Ax \left| \frac{\sin(\pi u x)}{\pi u x} \right| \times 1$$

$$u = 0 \qquad |F(u)| = Ax \left| \frac{\sin 0}{0} \right| = Ax$$



Find the fourier transform of the given function.



$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \int_0^{x}\int_0^{Y} A\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

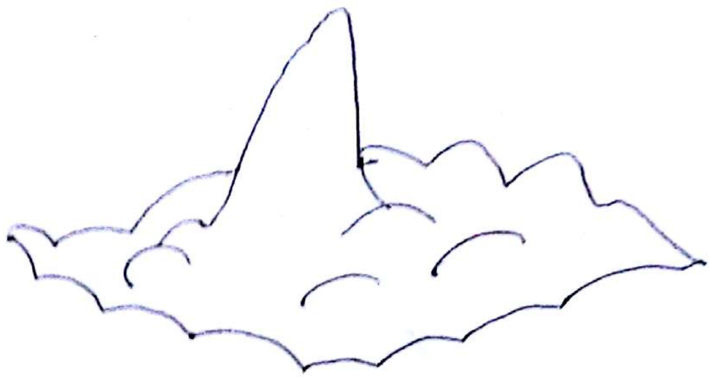
$$= A\left[\int_0^{x} e^{-j2\pi ux}\, dx \int_0^{Y} e^{-j2\pi vy}\, dy\right]$$

$$\Rightarrow A\left\{\left[\frac{1-e^{-j2\pi ux}}{j2\pi u}\right]\left[\frac{1-e^{-j2\pi vY}}{j2\pi v}\right]\right\}$$

$$\Rightarrow A\left\{\left[\frac{e^{j\pi ux}-e^{-j\pi ux}}{j2\pi u}\right] e^{-j\pi ux}\left[\frac{e^{+j2\pi vy}-e^{-j2\pi vy}}{j2\pi v}\right] e^{-j\pi vy}\right\}$$

$$F(u,v) = Axy\left[\frac{\sin(\pi ux)}{\pi ux}\right]\left[\frac{\sin \pi vy}{\pi vy}\right] e^{-j\pi ux}\, e^{-j\pi vy}$$

$$|F(u,v)| = |Axy|\left|\frac{\sin \pi ux}{\pi ux}\right|\left|\frac{\sin \pi vy}{\pi vy}\right| \times 1$$



<u>Que</u> Find the D.F.T. of $f(x) = \{0, 1, 2, 1\}$

$$F(u) = \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi\left(\frac{ux}{N}\right)}$$ $N = 4$ for $u = 0, 1, \ldots N-1$

$$F(u) = \sum_{x=0}^{3} f(x)\, e^{-j2\pi\left(\frac{ux}{N}\right)}$$ $u = 0, 1, 2, 3 \ldots$

$$F(0) = \sum_{x=0}^{3} \left[f(0)\, e^{-j2\pi\frac{0\times 0}{4}} + f(1)\, e^{-j2\pi\frac{0\times 1}{4}} + f(2)\, e^{-j2\pi\frac{0\times 2}{4}} + f(3)\, e^{-j2\pi\frac{0\times 3}{4}} \right]$$

$$\Rightarrow \left[f(0) + f(1) + f(2) + f(3) \right] = 0 + 1 + 2 + 1$$

$F(0) = 4$

$F(1) = -2$

$F(2) = 0$

$F(3) = -2$

$F(u) = \{4, -2, 0, -2\}$

D.F.T. is obtained by taking the magnitude

$|F(0)| = 4$

$|F(1)| = 2$

$|F(2)| = 0$

$|F(3)| = 2$

consider two image subsets S_1 & S_2

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

for $V = \{2\}$ determine whether S_1 & S_2 are

a) 4 connected

b) 8 connected

c) m connected

$$F(u) = \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N}$$

$$W_N = e^{-j2\pi/N}$$

$$\boxed{F(u) = \sum_{x=0}^{N-1} f(x)\, e^{W_N ux}}$$

To solve this we form a square matrix WN of $N \times N$

$f(x) = \{0, 1, 2, 1\}$ $\quad N = 4$

$$W_4 = \begin{array}{c} x\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \rightarrow \text{DFT Matrix}$$

W = Twiddle factor

$F(u) = W_4 \times f(x)$

Scanned by CamScanner

Que Find the D.F.T. of the image

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ -2 \end{bmatrix} \rightarrow \text{DFT of } 1^{st} \text{ row}$$

intermediate stage

$$\begin{bmatrix} 4 & -2 & 0 & -2 \\ 8 & -2 & 0 & -2 \\ 12 & -2 & 0 & -2 \\ 8 & -2 & 0 & 2 \end{bmatrix}$$