Optimal Execution Strategy Formulation

Quant Research Task

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1 Objective

The goal of the execution strategy is to purchase a total of S shares over a trading day, which is divided into N one-minute intervals. The strategy must determine the optimal number of shares to buy in each interval, x_i for $i = 1, \ldots, N$, such that the total cost arising from temporary market impact is minimized.

2 Mathematical Formulation

Based on the analysis in our accompanying document, we model the per-share slippage at time i for an order of size x_i as $g_i(x_i) = \beta_i \sqrt{x_i}$, where β_i is the empirically derived impact parameter for that period. The total cost for an order of size x_i is the number of shares multiplied by the per-share cost, i.e., $x_i \cdot g_i(x_i)$. Our optimization problem is therefore to minimize the total cost over all periods, subject to the constraint that we must purchase exactly S shares in total.

- **Objective Function:** Minimize Total Impact Cost

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} x_i \cdot g_i(x_i) = \min_{x_1,\ldots,x_N} \sum_{i=1}^{N} \beta_i x_i^{1.5}$$

- **Constraint:**

$$\sum_{i=1}^{N} x_i = S \quad \text{and} \quad x_i \geq 0 \quad \forall i$$

3 Derivation of the Optimal Solution

This is a constrained optimization problem that can be solved using the method of Lagrange multipliers. We form the Lagrangian $\mathcal{L}$:

$$\mathcal{L}(x_1, \ldots, x_N, \lambda) = \sum_{i=1}^{N} \beta_i x_i^{1.5} - \lambda \left(\left(\sum_{i=1}^{N} x_i \right) - S \right)$$

To find the minimum, we take the partial derivative of $\mathcal{L}$ with respect to each x_i and set it to zero:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 1.5 \cdot \beta_i \sqrt{x_i} - \lambda = 0$$

Solving for x_i, we find its relationship with the Lagrange multiplier λ:

$$\sqrt{x_i} = \frac{\lambda}{1.5\beta_i} \implies x_i = \left(\frac{\lambda}{1.5\beta_i} \right)^2 = \frac{\lambda^2}{2.25\beta_i^2}$$

To find the value of λ, we substitute this expression for x_i back into our constraint equation:

$$\sum_{i=1}^{N} \frac{\lambda^2}{2.25\beta_i^2} = S$$

We can factor out the constant terms involving λ:

$$\frac{\lambda^2}{2.25}\sum_{i=1}^{N}\frac{1}{\beta_i^2} = S \implies \lambda^2 = \frac{2.25S}{\sum_{j=1}^{N}(1/\beta_j^2)}$$

Finally, substituting this expression for λ^2 back into our equation for x_i yields the optimal number of shares to trade in period i, which we denote x_i^*:

$$x_i^* = \frac{1}{\beta_i^2}\cdot\frac{S}{\sum_{j=1}^{N}(1/\beta_j^2)}$$

4 Conclusion and Interpretation

The derived formula provides a closed-form solution for the optimal execution schedule. The quantity x_i^* is inversely proportional to the square of the impact parameter, β_i^2. This has a clear and intuitive interpretation: **we should trade more aggressively (larger x_i) during periods of high liquidity (small β_i) and trade more passively (smaller x_i) when the market is illiquid (large β_i)**. Our execution algorithm is to first use the model to estimate the sequence of β_i values for the day, and then apply this formula to allocate the total order size S across all trading intervals.