CSCI 665 : Foundations of Algorithms
Homework 1

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1.

$$2^{2^{n+1}} > 2^{2^n} > (n+1)! > n! > e^n >$$

$$n2^n > (3/2)^n > n^{lg\ lg\ n} = (lg\ n)^{lg\ n} > (lg\ n)! >$$

$$n^3 > n^2 = 4^{lgn} > n\ lg\ n = lg(n!) > n = 2^{lgn} > (\sqrt{2})^{lg\ n} >$$

$$2^{\sqrt{2\ lg\ n}} > (lg\ (n))^2 > ln\ n > \sqrt{lg\ n} > ln\ ln\ n >$$

$$2^{lg^*\ n} > lg^*\ (lg\ n) = lg^*\ n > lg\ (lg^*\ n) > 1 = n^{1/lg\ n}$$

Each term represents a different equivalence class.

2.

$$a^x > x^c = \sqrt[k]{x} > log_b\ x$$

The asymptotic growth of the functions f and g using $lim_{x\to\infty} \frac{f(x)}{g(x)}$. If the solution to the limit is ∞, then f dominates. If the limit is 0, then g dominates. If the limit is a constant, then f and g are considered equal (asymptotically).

a^x is an exponential function and rate of growth exceeds the rate of growth for polynomial functions such as x^c and $\sqrt[k]{x}$ which exceeds the rate of growth for the logarithmic function $log_b\ x$. The values of a, c, k, b must be greater than 1 for the order of growth mentioned to hold true.

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3.

(a).

To prove $n \in \mathcal{O}(n^2)$

Consider N = 1, c = 1
Suppose $n \geq N$, therefore, $n \geq 1$

$1 \leq n$ implies $n \leq n^2$
implies $|n| \leq 1.|n^2|$

Hence, proved

(b).

To prove $n^k \in \mathcal{O}(n^{k'})$

Consider N = k, c = 1
Suppose $k' \geq k$

$k \leq k'$ implies $k\ log\ n \leq k'\ log\ n$
implies $n^k \leq n^{k'}$
implies $|n^k| \leq 1.|n^{k'}|$

(c).

To prove $O(f(n)) + O(f(n)) = O(f(n))$

$Consider, f_1 : D \rightarrow R, f : D \rightarrow R,\ where\ D \subseteq R$
$f_1(n) \in \mathcal{O}(f(n))$ if there exists $N \in N$, $c_1 \in R^+$, such that for any $n \in D$, $n \geq N$
$|f_1(n)| \leq c_1 . |f(n)|$ (1)

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$Consider, f_2 : D \to R, f : D \to R, where\ D \subseteq R$
$f_2(n) \in \mathcal{O}(f(n))$ if there exists $N \in N$, $c_2 \in R^+$, such that for any $n \in D$, $n \geq N$
$|f_2(n)| \leq c_2 . |f(n)|$ (2)

(1) + (2)

$|f_1(n)| + |f_2(n)| \leq (c_1 + c_2) . |f(n)|$
$Let\ c = c_1 + c_2, such\ that\ c \in R^+$
$|f_1(n)| + |f_2(n)| \leq c . |f(n)|$

But, $[f_1(n) \in \mathcal{O}(f(n))\ and\ f_2(n) \in \mathcal{O}(f(n))]$
Hence, by definition

$\mathcal{O}(f(n)) + \mathcal{O}(f(n)) = \mathcal{O}(f(n))$

4.

(a).

Given :- $\sum_{k=2}^{n} \frac{1}{k} \leq ln(n) - ln(1)$

To prove:- $H_n \in \mathcal{O}(log\ (n))$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \ln\ (n) - \ln\ (1)$$

$$\sum_{k=2}^{n} \frac{1}{k} + 1 \leq \ln\ (n) - \ln\ (1) + 1$$

$$H_n \leq \ln\ (n) - \ln\ (1) + 1$$

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$H_n \leq \ln(n) + 1$ By ln (1) = 0

$H_n \in \mathcal{O}(log\ (n))$

(b).

Given :- $\sum_{k=2}^{n} \frac{1}{k} \geq ln(n+1) - ln(2)$

To prove:- $H_n \in \Omega(log\ (n))$

$$\sum_{k=2}^{n} \frac{1}{k} \geq \ln(n+1) - \ln(2)$$

$$\sum_{k=2}^{n} \frac{1}{k} + 1 \geq \ln(n+1) - \ln(2) + 1$$

$$H_n \geq \ln(n+1) - \ln(2) + 1$$

$$H_n \geq \ln\left(\frac{n+1}{2}\right) + 1$$ By ln (a/b) = ln(a) - ln(b)

For n = 1

$$H_{n=1} \geq \ln\left(\frac{2}{2}\right) + 1$$
$$H_1 \geq 1$$

$$H_n \in \Omega(log\ (n))$$

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5.

$$
\begin{aligned}
s[4,1,3,2] &= i(4,s([1,3,2])) \\
&= i(4,i(1,s([3,2]))) \\
&= i(4,i(1,i(3,s([2])))) \\
&= i(4,i(1,i(3,i(2,s([]))))) \\
&= i(4,i(1,i(3,i(2,[])))) \\
&= i(4,i(1,i(3,[2]))) \\
&= i(4,i(1,2 :: i(3,[]))) \\
&= i(4,i(1,2 :: [3])) \\
&= i(4,i(1,[2,3])) \\
&= i(4,[1,2,3]) \\
&= 1 :: i(4,[2,3]) \\
&= 1 :: 2 :: i(4,[3]) \\
&= 1 :: 2 :: 3 :: i(4,[]) \\
&= 1 :: 2 :: 3 :: [4] \\
&= [1,2,3,4]
\end{aligned}
$$

6.

The smallest n such that I noticed fib running slowly was $n = 29$.

7.

(a).

Observe that when $n = 2$ we have,

$$
\begin{aligned}
f(2;\ a,\ b) &= f(1;\ b,\ a+b) \\
&= a+b && \text{By } f(1;\ a,\ b) = b \\
&= f(0;\ a,\ b) + f(1;\ a,\ b) && \text{By } f(1;\ a,\ b) = b \\
& && \&\ f(0;\ a,\ b) = a \\
&= f(1;\ a,\ b) + f(0;\ a,\ b)
\end{aligned}
$$

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Assume $f(k;\ a, b) = f(k-1;\ a, b) + f(k-2;\ a, b)$

When $n = k + 1$

$$\begin{aligned} f(k+1;\ a, b) &= f(k;\ b, a+b) \\ &= f(k-1;\ b, a+b) + f(k-2;\ b, a+b) \end{aligned}$$

[By assumption]

$$= f(k;\ a, b) + f(k-1;\ a, b)$$

[By definition]

Hence proved

(b).

By strong form of mathematical induction,

Observe that

when $n = 0$, $F_0 = 0 = f(0;\ 0, 1)$ and
when $n = 1$, $F_1 = 1 = f(1;\ 0, 1)$

Assume $F_k = (f_k;\ 0, 1)$ for every $1 < k < n$

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &= f(n-1;\ 0, 1) + f(n-2;\ 0, 1) \\ &= f(n-2;\ 1, 0+1) + f(n-3;\ 1, 0+1) && \text{[By definition]} \\ &= f(n-1;\ 1, 0+1) \\ &= f(n;\ 0, 1) && \text{[By previous result]} \end{aligned}$$

Hence proved

8.

No, $fibIt$ did not run slowly for the same n.