CSE/AMS 547 Discrete Mathematics November 27th, 2019

Homework Four and Five

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Question 1.

Problem 5.18.

Equation 5.35 from the textbook,

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{2k}\binom{2k}{k}/2^{2k}$$

Given,

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$$

$$= \frac{r(r-1)...(r-k+1)}{k!}\frac{(r-1/3)...(r-k+2/3)}{k!}\frac{(r-2/3)...(r-k+1/3)}{k!}$$

$$= \frac{r(r-1/3)(r-2/3)(r-3/3)...(r-k+1)(r-k+2/3)(r-k+1/3)}{k!k!k!}$$

$$= \frac{3r(3r-1)(3r-2)(3r-3)...(3r-3k+3)(3r-3k+2)(3r-3k+1)}{k!k!k!}.\frac{1}{3^{3k}}$$

$$= \frac{(3r)!}{(3r-3k)!k!k!k!3^{3k}}$$

$$= \frac{(3r)!(3k)!(2k)!}{(3r-3k)!(3k)!(2k)!k!k!k!3^{3k}}$$

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k} = \binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k}/3^{3k}$$

Problem 5.19. From equation 5.58 from textbook,

$$B_t(z) = \sum_{k\geq 0}(tk)^{\underline{k-1}}.\frac{z^k}{k!}$$

Using equation 5.60,

$$B_{1-t}(-z)^{-1} = \sum_{k\geq 0}\binom{((1-t)k-1)}{k}(\frac{(-1)}{((1-t)k-1)})(-z)^k$$

$$= \sum_{k \geq 0} \frac{(k-tk-1)(k-tk-2)...(-tk).(-1)^{k+1}.z^k}{k!(k-tk-1)}$$

Taking -1 common from all the terms,

$$= \sum_{k \geq 0} \frac{(tk)(tk-1)...(tk-k+2).(-1)^{k-1}(-1)^{k+1}.z^k}{k!}$$

$$= \sum_{k \geq 0} \frac{(tk)(tk-1)...(tk-k+2).(-1)^{2k}.z^k}{k!}$$

$$= \sum_{k \geq 0} (tk)^{\underline{k-1}}.\frac{z^k}{k!}$$

Thus,

$$B_t(z) = B_{1-t}(-z)^{-1}$$

Problem 5.23.

$$F(-n, 1; ; 1) = F(-n, 1, 1; 1; 1)$$

Using Equation 5.76,

$$= \sum_{k \geq 0} \frac{(-n)^{\overline{k}}(1)^{\overline{k}}(1)^k}{k!}$$

$$= \sum_{k \geq 0} \frac{(-1)^k n^{\underline{k}}(-1)^k(-1)^{\underline{k}}}{k!}$$

$$= \sum_{k \geq 0} \frac{n^{\underline{k}}(-1)^k k!}{k!}$$

$$= \sum_{k \geq 0} \binom{n}{k} k!(-1)^k$$

Using derived equation from 5.49 from book PG195 i.e.

$$n¡ = (-1)^n \sum_{k \geq 0} \binom{n}{k} k!(-1)^k$$

So,

$$F(-n, 1; ; 1) = n¡/(-1)^n$$

Multiplying and dividing by $(-1)^n$,

$$F(-n, 1; ; 1) = (-1)^n n¡$$

Problem 5.40

$$\sum_{j=1}^{m}(-1)^{j+1}\binom{r}{j}\sum_{k=1}^{n}\binom{-j+rk+s}{m-j}$$

switching the summations and using upper negation,

$$=(-1)^{m+1}\sum_{k=1}^{n}\sum_{j=1}^{m}\binom{r}{j}\binom{m-rk-s-1}{m-j}$$

$$=(-1)^{m}\sum_{k=1}^{n}\left(\binom{m-rk-s-1}{m}-\binom{m-r(k-1)-s-1}{m}\right)$$

$$=(-1)^{m}\left(\binom{m-rn-s-1}{m}-\binom{m-s-1}{m}\right)$$

Using Upper negation again,

$$=\binom{rn+s}{m}-\binom{s}{m}$$

Problem 5.41.

$$\sum_{k}\binom{n}{k}\frac{k!}{(n+k+1)!}=n!\sum_{k}\frac{1}{(n-k)!(n+1+k)!}$$

$$=n!\left(\frac{1}{(n+1)!(n)!}+\frac{1}{(n+2)!(n-1)!}+\frac{1}{(n+3)!(n-2)!}+...\right)$$

$$=n!\sum_{k>n}\frac{1}{k!(2n+1-k)!}$$

$$=\frac{n!}{(2n+1)!}\sum_{k>n}\frac{(2n+1)!}{k!(2n+1-k)!}$$

$$=\frac{n!}{(2n+1)!}\sum_{k>n}\binom{2n+1}{k}$$

Using Binomial Expansions,

$$2\sum_{k>n}\binom{2n+1}{k}=\binom{2n+1}{0}+\binom{2n+1}{1}+...+\binom{2n+1}{n}+\binom{2n+1}{n+1}+...+\binom{2n+1}{2n+1}=2^{2n+1}$$

$$\sum_{k>n}\binom{2n+1}{k}=2^{2n}$$

So,

$$\sum_k \binom{n}{k} \frac{k!}{(n+k+1)!} = 2^{2n} \cdot \frac{n!}{(2n+1)!}$$

Problem 5.60.
According to Stirling's equation,

$$n! \approx \sqrt{(2\pi)}(n/e)^n$$

So,

$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$$

Using Stirling's Equation,

$$\approx \frac{\sqrt{2\pi(m+n)}((m+n)/e)^{m+n}}{\sqrt{2\pi(m)}((m)/e)^m \sqrt{2\pi(n)}((n)/e)^n}$$

$$\approx \sqrt{\frac{1}{2\pi}(\frac{m+n}{m.n})} \left(\frac{m+n}{m}\right)^m \left(\frac{m+n}{n}\right)^n$$

$$\approx \sqrt{\frac{1}{2\pi}\left(\frac{1}{m}+\frac{1}{n}\right)} \left(1+\frac{n}{m}\right)^m \left(1+\frac{m}{n}\right)^n$$

For special case, when m=n,

$$\approx \sqrt{\frac{1}{2\pi}\left(\frac{1}{m}+\frac{1}{m}\right)} \left(1+\frac{m}{m}\right)^m \left(1+\frac{m}{m}\right)^m$$

$$= 4^m \sqrt{\frac{1}{\pi m}}$$

Problem 5.73. Given, $X_0 = \alpha$,$X_1 = \beta$,$X_n = (n-1)(X_{n-1} + X_{n-2})$.

lets

$$X_n = \alpha A + \beta B$$

If $X_n = n¡$ then $\alpha = 1$ and $\beta = 0$, Putting in above equation,

$$A = n¡$$

If $X_n = n!$ then $\alpha = 1$ and $\beta = 1$,

$$n! = A + B$$

$$B = n! - n¡$$

Putting the value of A and B in initial equation,

$$X_n = \alpha.n¡ + \beta.(n! - n¡)$$

Problem 5.80. We need to prove,

$$\binom{n}{k} \leq (en/k)^k$$

$$(n^{\underline{k}}/n^k)(k/e)^k \leq k!$$

we know that $(n^{\underline{k}}/n^k) \leq 1$, so

$$(n^{\underline{k}}/n^k)(k/e)^k < (k/e)^k < k!$$

So,we just need to prove $(k/e)^k < k!$ to prove the original inequality.

Using Mathematical Induction,

If $(k/e)^k \leq k!$ is true, we need to prove,

$$(k + 1/e)^{k+1} \leq (k+1)!$$

Using Binomial Expansion on $(k+1)^{k+1}$

$$(k + 1/e)^{k+1} = \frac{1}{e^{k+1}} \sum_{t}^{k} \binom{k+1}{t} k^t$$

Using equation 5.19,

$$= \frac{1}{e^{k+1}} \binom{-1}{k} (-1)^k \sum_{t}^{k} k^t$$

$$= \frac{1}{e^{k+1}} \frac{(-1)^k.k!(-1)^k}{k!} \frac{(k^t - 1)}{(k-1)}$$

$$= \frac{1}{e^{k+1}} \frac{(k^t - 1)}{(k-1)}$$

$$= \frac{1}{e(k-1)} \left(\left(\frac{k}{e} \right)^k - \frac{1}{e^k} \right)$$

As $\dfrac{1}{e(k-1)} < 1$ and $\left(\left(\dfrac{k}{e} \right)^k - \dfrac{1}{e^k} \right) < \left(\dfrac{k}{e} \right)^k$ for $k \geq 0$,

So,

$$\frac{1}{e(k-1)}\left(\left(\frac{k}{e}\right)^k - \frac{1}{e^k}\right) < (k/e)^k$$

$$(k+1/e)^{k+1} < \left(\frac{k}{e}\right)^k$$

But $(k/e)^k \leq k!$ and $k! < (k+1)!$, which gets us,

$$(k+1/e)^{k+1} < (k+1)!$$

Hence by induction it is proved that

$$(k/e)^k < (k)!$$

Which fulfils our original condition for inequality. Thus,

$$(n^{\underline{k}}/n^k)(k/e)^k < (k/e)^k < k!$$

is true. Hence,

$$\binom{n}{k} \leq (en/k)^k$$

Question 2.

Identity 6.20 We have to prove,

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k=0}^{n} \left\{ {k \atop m} \right\} (m+1)^{n-k}$$

Using recurrence from table 264,

$$\left\{ {n+1 \atop m+1} \right\} = (m+1)\left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\}$$

Using Recurrence again from table 264,

$$\left\{ {n+1 \atop m+1} \right\} = (m+1)^2\left\{ {n-2 \atop m+1} \right\} + (m+1)^1\left\{ {n-1 \atop m} \right\} + (m+1)^0\left\{ {n \atop m} \right\}$$

Repeating use of the recurrence will lead us to

$$\left\{ {n+1 \atop m+1} \right\} = (m+1)^{n+1}\left\{ {0 \atop m+1} \right\} + (m+1)^n\left\{ {0 \atop m} \right\} + \cdots + (m+1)^1\left\{ {n-1 \atop m} \right\} + (m+1)^0\left\{ {n \atop m} \right\}$$

but first term is equal to 0. Thus, we get,

$$\left\{ {n+1 \atop m+1} \right\} = (m+1)^n\left\{ {0 \atop m} \right\} + \cdots + (m+1)^1\left\{ {n-1 \atop m} \right\} + (m+1)^0\left\{ {n \atop m} \right\}$$

Which can be further written as,

$$\left\{ n+1 \atop m+1 \right\} = \sum_{k=0}^{n} \left\{ k \atop m \right\} (m+1)^{n-k}$$

Identity 6.22 We have to prove,

$$\left\{ m+n+1 \atop m \right\} = \sum_{k=0}^{m} k \left\{ n+k \atop k \right\}$$

Using the same recurrence from table 264,

$$\left\{ m+n+1 \atop m \right\} = (m)\left\{ m+n \atop m \right\} + \left\{ m+n \atop m-1 \right\}$$

Using the same recurrence from table 264,

$$\left\{ m+n+1 \atop m \right\} = (m)\left\{ m+n \atop m \right\} + (m-1)\left\{ m+n-1 \atop m-1 \right\} + \left\{ m+n-1 \atop m-2 \right\}$$

Repeating use of the recurrence will lead us to

$$\left\{ m+n+1 \atop m \right\} = (m)\left\{ m+n \atop m \right\} + (m-1)\left\{ m+n-1 \atop m-1 \right\} + \cdots + (1)\left\{ n+1 \atop 1 \right\}$$

$$\left\{ m+n+1 \atop m \right\} = (m)\left\{ m+n \atop m \right\} + (m-1)\left\{ m+n-1 \atop m-1 \right\} + \cdots + (1)\left\{ n+1 \atop 1 \right\} + (0)\left\{ n+0 \atop 0 \right\}$$

Which can be further written as,

$$\left\{ m+n+1 \atop m \right\} = \sum_{k=0}^{m} k \left\{ n+k \atop k \right\}$$

Question 3.
Don't know

Question 4.

Problem 6.12
We have to prove

$$g(n) = \sum_{k} \left\{ n \atop k \right\} (-1)^k f(k) \iff f(n) = \sum_{k} \left[n \atop k \right] (-1)^k g(k)$$

We can find,

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k g(k) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k \sum_j \begin{Bmatrix} k \\ j \end{Bmatrix} (-1)^j f(j)$$

$$= \sum_j f(j) \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} (-1)^{k+j}$$

Using inversion formula for Sterling numbers,

$$= \sum_j f(j)(-1)^{j-n} \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} (-1)^{n-k}$$

Again, Using Inversion Formula to replace inner sum by [j=n].
So,

$$= \sum_j f(j)(-1)^{j-n}[j = n]$$

From Unit impulse function, value of sum will be zero everywhere else for j=n where value will be equa to f(n). So,

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k g(k) = f(n)$$

Question 5.

Problem 6.15
We have to prove,

$$m! \begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \left\langle {n \atop k} \right\rangle \binom{k}{n-m}$$

We have,

$$x^n = \sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$$

We know that:

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$

Using $\Delta(\binom{x+k}{n}) = \binom{x+k}{n-1}$ and above equation,

$$\Delta^n x^n = \sum_k \binom{n}{k} (-1)^{n-k} (x+k)^n$$

So,

$$\sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n-m} = \sum_k \binom{m}{k} (-1)^{m-k} (x+k)^n$$

Putting x=0,

$$\sum_k \left\langle {n \atop k} \right\rangle \binom{k}{n-m} = \sum_k \binom{m}{k} (-1)^{m-k} (k)^n$$

Using equation 6.19,

$$m! \left\{ {n \atop m} \right\} = \sum_k \binom{m}{k} k^n (-1)^{m-k}$$

Therefore,

$$m! \left\{ {n \atop m} \right\} = \sum_k \left\langle {n \atop k} \right\rangle \binom{k}{n-m}$$

Question 6.

Problem 6.21
Don't know

Question 7.

Problem 6.26
We need to find,

$$S_n = \sum_{k=1}^{n} \frac{H_k}{k}$$

Using $H_k = H_{k-1} + \frac{1}{k}$ and $H_0 = 0$,

$$S_n = \sum_{k=1}^{n} \frac{H_{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k^2}$$

let

$$S_n = T_n + H_n^{(2)}$$

Using sum by parts on T_n,

$$T_n = \sum_{k=1}^{n+1} u(x) \Delta v(x) \delta x$$

Let, $u(k) = H_{k-1}$ and $\Delta v(k) = \frac{1}{k}$ So,

$$\Delta u(k) = \frac{1}{k} = \Delta v(k)$$

Similarly, we get $v(k) = H_{k-1}$.

$$T_n = u(x)v(x)\Big| x = 1^{x=n+1} - \sum 1^{n+1} v(x+1)\Delta u(x)\delta x$$
$$= (H_{x-1})^2\Big| x = 1^{x=n+1} - \sum k = 1^{n+1}\frac{H_k}{k}$$
$$T_n = H_n^2 - S_n$$

Putting the Value of T_n in original equation of S_n,

$$S_n = H_n^2 - S_n + H_n^{(2)}$$

$$S_n = \frac{H_n^2 + H_2^{(2)}}{2}$$

Question 8.

Problem 6.34
Don't know

Question 9.

Problem 6.39
We have to express $\sum_{k=1}^{n} H_k^2$ in terms of n and H_n.
We are going to use partial sums,

$$\sum_{k=1}^{n} H_k^2 = ((n+1)H_n^2 - n * H_n) - \sum_{k=1}^{n-1} \frac{(k+1)H_k - k}{k+1}$$
$$= ((n+1)H_n^2 - n * H_n) - \sum_{k=1}^{n-1} H_k + \sum_{k=1}^{n-1} \frac{k}{k+1}$$
$$= ((n+1)H_n^2 - n * H_n) - \sum_{k=1}^{n-1} H_k + (n-1) - (H_n - 1)$$
$$= (n+1)H_n^2 - (n+1)H_n + n - \sum_{k=1}^{n-1} H_k$$

Solving for $\sum_k^n H_k$,

$$\sum_k^n H_k = n * H_n - \sum_{k=1}^{n-1} \frac{k}{k+1}$$

$$= n * H_n - (n-1) + \sum_{k=1}^{n-1} \frac{1}{k+1}$$

$$\sum_k^n H_k = n * H_n - n + 1 + (H_n - 1)$$

Putting back in original equation,

$$= (n+1)H_n^2 - (n+1)H_n + n - (n+1)H_n + n + H_n$$

$$\sum_{k=1}^n H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n$$