CSE/AMS 547 Discrete Mathematics October 7th, 2019

Homework Two

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Due by October, 07, 4pm.

Problem 2.22.
If $j < k$:

$$a_j b_k - a_k b_j = -(a_k b_j - a_j b_k)$$
$$A_j B_k - A_k B_j = -(A_k B_j - A_j B_k)$$

so, if $s_{j,k} = (a_j b_k - a_k b_j)(A_j B_k - A_k B_j)$
then $s_{j,k} = s_{k,j}$.

$$\sum_{1 \le j,k \le n} s_{j,k} = \sum_{1 \le j < k \le n} s_{j,k} + \sum_{1 \le j = k \le n} s_{j,k} + \sum_{1 \le k < j \le n} s_{k,j}$$

Clearly, as $s_{j,j} = 0$

$$\sum_{1 \le j,k \le n} s_{j,k} = 2 \sum_{1 \le j < k \le n} s_{j,k}$$

So,expansion of $s_{j,k}$ is $a_j A_j b_k B_k - a_j B_j A_k b_k - A_j b_j a_k B_k + b_j B_j a_k A_k$

$$\sum_{1 \le j,k \le n} a_j A_j b_k B_k = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j A_j b_k B_k$$

$$= \sum_{j=1}^{n} a_j A_j (\sum_{k=1}^{n} b_k B_k)$$

$$= (\sum_{j=1}^{n} a_j A_j)(\sum_{k=1}^{n} b_k B_k)$$

$$= (\sum_{k=1}^{n} a_k A_k)(\sum_{k=1}^{n} b_k B_k)$$

Similarly, Calculating the rest of the terms of expansion we will get,

$$\sum_{1 \le j,k \le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) = (\sum_{k=1}^{n} a_k A_k)(\sum_{k=1}^{n} b_k B_k) - (\sum_{k=1}^{n} a_k B_k)(\sum_{k=1}^{n} A_k b_k)$$

For special case, when $a_k = A_k$and $b_k = B_k$ we can get **Lagrange's identity** without induction as,

$$\sum_{1\le j,k\le n}(a_jb_k - a_kb_j)^2 = (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_kb_k)^2$$

Problem 2.27

$$\Delta c^{\underline{x}} = c^{\underline{x+1}} - c^{\underline{x}}$$
$$= c(c-1)...(c-x+1)(c-x) - c(c-1)...(c-x+1)$$
$$= c^{\underline{x}}(c-x-1)$$

we have,

$$\sum_{k=1}^{n}(-2)^{\underline{k}}/k = \sum_{k=1}^{n}((-2)^{\underline{k-1}}(-2-k+1))/k$$
$$= \sum_{k=1}^{n}((-2)^{\underline{k-2}}(k+1)(k))/k$$
$$= \sum_{k=1}^{n}(-2)^{\underline{k-2}}(k+1)$$

Because,

$$\sum_{k=1}^{n}(-2)^{\underline{k}}/k = \sum_{1}^{n+1}((-2)^{\underline{k}}/k)\delta k$$
$$= (-(-2)^{\underline{k-2}}|_{1}^{n+1}$$
$$= (-2)^{\underline{-1}} - (-2)^{\underline{n-1}}$$
$$= -1 - (-2)(-3)...(-n)$$
$$= (-1)^n n! - 1$$

Problem 4.16.
we can get,

$$/e_1 = 1/2$$
$$1/e_1 + 1/e_2 = 1/2 + 1/3 = 5/6$$
$$1/e_1 + 1/e_2 + 1/e_3 = 1/2 + 1/3 + 1/7 = 41/42$$

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so on.
We can recognize a pattern in the form of $1 - 1/d_n$ where d_n is the product of Euclid numbers like,
$d_1 = 2 = e_1 = e_2 - 1$ for $1/e_1$,
$d_2 = 6 = e_1 e_2 = e_3 - 1$ for $1/e_1 + 1/e_2$, and so on. Let us use induction.
Suppose $1/e_1 + ... + 1/e_{n-1} = 1 - 1/(e_n - 1)$ then,

$$\sum_{k=1}^{n}(1/e_k) = \sum_{k=1}^{n-1}(1/e_k) + 1/e_k$$

$$= 1 - 1/e_n - 1 + 1/e_n$$

$$= 1 - (e_n - (e_n - 1))/(e_n - 1)e_n$$

$$= 1 - 1/(e_1 ... e_{n-1}.e_n)$$

$$= 1 - 1/(e_{n+1} - 1)$$

Problem 4.24.

$$\epsilon_p(n!) = \lfloor(n/p)\rfloor + \lfloor(n/p^2)\rfloor + ... + \lfloor(n/p^n)\rfloor = \sum_{k\geq 1}\lfloor(n/p^k)\rfloor$$

when p = 2,

$$\epsilon_2(n!) = \lfloor(n/2)\rfloor + \lfloor(n/4)\rfloor + \lfloor(n/8)\rfloor + ... = \sum_{k\geq 1}\lfloor(n/2^k)\rfloor$$

when n = 100,

$$\epsilon_2(100!) = 50 + 25 + 12 + 6 + 3 + 1 = 97$$

Therefore, This implies each consecutive number is floor of half previous number.
In binary conversion, 100 = 1100100; 50 = 110010; 25 = 11001; 12 = 1100; 6 = 110; 3 = 11; 1 = 1. We can deduce from that,

$$\epsilon_2(n!) = n - \nu_2(n)$$

where $\nu_2(n)$ = Number of 1's(Non -zero digits).

$$\epsilon_2(100!) = 100 - 3 = 97$$

This is because each 1 that contribute 2^m to n also contributes $1.2^{m-1}+1.2^{m-2}+...+1.2^0 = (2^m - 1)$ to $\epsilon_2(n!)$. Similarly, in radix $3, 4, ...$, the pattern repeats. Therefore, Each digit d that contributes $d.p^m$ to n also contributes as,

$$d.p^{m-1} + d.p^{m-2} + ... + d.p^0 = d(p^m - 1)/(p - 1)$$

to $\epsilon_p(n!)$. Contributions of $d.p^m = n$ and $d = \nu_2(n)$. Therefore,

$$d(p^m - 1)/(p - 1) = (n - \nu_p(n))/(p - 1)$$

Problem 4.26. $m'n - mn' = 1$ is true as G_n defines a subtree of Stern - Brocot tree. Whose represesntation would be LRLLRLLRLLLLRRRL....

Problem 4.30. Proving Chinese Remainder Theorem,

$$m/m_j \perp m_j$$

so, $\exists$b$_j$,

$$(m/m_j)b_j \equiv 1(mod(m_j))$$
$$(m/m_j)b_j \equiv 0(mod(m_i))$$

where i$\neq j$.
$Let x_0 = \sum_{j=1}^{(} rm/m_j)b_j a_j where$x$_0$ is the solution. Then,

$$x_0 = (m/m_i)b_i a_i \equiv a_i(mod(m_i))$$

No other solution exists as c $\rightarrow if$z =a$_i(mod(m_c))$ then 2 is multiple of z.

Problem 4.38. Using Euclid's Algorithm and following equation,

$$a^n - b^n = (a^m - b^m)(a^{n-m}b^0 + a^{n-2m}b^m + ... + a^{nmodm}b^{n-m-nmodm}) + b^{\lfloor n/m \rfloor}(a^{nmodm} - b^{nmodm})$$

so,

$$gcd(a^n - b^n, a^m - b^m) = gcd(a^m - b^m, b^{\lfloor n/m \rfloor}(a^{nmodm} - b^{nmodm}))$$

But $b^{\lfloor n/m \rfloor}$ is realtively prime to $a^n - b^n$, it will relatively prime to gcd. Thus, we can remove that factor from second term.

$$gcd(a^n - b^n, a^m - b^m) = gcd(a^m - b^m, (a^{nmodm} - b^{nmodm}))$$

Solving this further will give us,

$$gcd(a^n - b^n, a^m - b^m) = a^{gcd(m,n) + b^{gcd(m,n)}}$$

Problem 4.42. For given two lowest terms,it is possible if and only if they are relatively prime to each other. which implies, $m \perp n$ and $m' \perp n'$ In other words,

$$gcd(m, n) = 1 = gcd(m', n')$$

But,

$$gcd(m, n) = 1 = gcd(m, n + am)$$

We need to prove,

$$m/n + m'/n' = (mn' + nm')/(nn')$$

is in the lowest terms if and only if nn' is relatively prime.
Therefore, if $m \perp n$ and $m' \perp n$ this implies that mn'$\perp$ n
we know that,
mn'$\perp$ n implies $n \perp (mn' + nm)$

we also know that:

mn' $\perp$ n implies n $\perp$ mn'+nm'

Therefore,

$$(mn' + nm') \perp (nn')$$