CSE/AMS 547 Discrete Mathematics October 28th, 2019

Homework Two

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Due by October, 28, 4pm.

Problem 4.33. Function f is multiplicative if gcd(m,n)=1 implies f(m.n)=f(m).f(n).In this case d—mn if and only there exist two integers a,b such that d=ab,a—m,gcd(a,n)=1,b—n,and gcd(b,m)=1. Then gcd(a,b)=1. So,

$$h(mn) = \sum_{d|mn} f(d)g(\frac{mn}{d})$$

$$= \sum_{a|m,b|n} f(ab)g(\frac{mn}{ab})$$

$$= \sum_{a|m,b|n} f(a)f(b)g(\frac{m}{a})g(\frac{n}{b})$$

$$= (\sum_{a|m} f(a)g(\frac{m}{a})).(\sum_{b|n} f(b)g(\frac{n}{b}))$$

$$h(mn) = h(m).h(n)$$

Problem 4.47. As $n^{m-1} \equiv 1(\mod m)$, then $n \perp m$.
Now,

$$n^k \equiv n^j$$

for some $1 \leq j < k < m$. then, $n^{k-j} \equiv 1$ because we can divide by n^j. Therefore, lets assume the numbers

$$n^1 \mod m, ..., n^{m-1} \mod m$$

are not distinct, there is a $k < m-1$ with $n^k \equiv 1$. Using equations from question 4.46 i.e., if $n^j \equiv 1$ and $n^k \equiv 1(\mod m)$ then $n^{gcd(j,k)} \equiv 1$. so, the least such k divides m-1. Then,

$$kq = (m-1)/p$$

for some prime p and some positive integer q. But this is not possible since $n^{kq} \not\equiv 1$. Thus by contradiction,the numbers$n^1 \mod m, ..., n^{m-1} \mod m$ are distinct and relatively prime to m. Therefore the numbers 1, ..., m-1 are relatively prime to m, and m must be prime.

Problem 5.14.

Using symmetry property and $(-1)^k = (-1)^{k-m+m}(-1)^{2l} = (-1)^{m+l}(-1)^{k-m+l}$

$$\sum_{k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{m+l}\sum_{k\le l}\binom{l-k}{l-k-m}\binom{s}{k-n}(-1)^{l-k-m}$$

$$= (-1)^{m+l}\sum_{k\le l}\binom{l-k-m-(-m-1)-1}{l-k-m}\binom{s}{k-n}(-1)^{l-k-m}$$

Using Upper Negation,

$$= (-1)^{m+l}\sum_{k\le l}\binom{-m-1}{l-k-m}\binom{s}{k-n}$$

Lower part need to be an integer. Thus, summation's restriction will change like as follows:

$$= (-1)^{m+l}\sum_{k}\binom{-m-1}{l-k-m}\binom{s}{k-n}$$

Using Vandermonde's Convolution,

$$\sum_{k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{m+l}\binom{s-m-1}{l-m-n}$$

Which is the required identity 5.25.

Now, Using symmetry property,

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} = \sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{q+k-n}$$

Putting k-q=k,

$$= \sum_{0\le k-q\le l}\binom{l-k+q}{m}\binom{q+k-q}{q+k-q-n}$$

$$= \sum_{q\le k\le l+q}\binom{l-k+q}{m}\binom{k}{k-n}$$

Applying Upper Negation again as stated in the question,

$$= \sum_{q\le k\le l+q}\binom{l-k+q}{m}\binom{k-n-k-1}{k-n}(-1)^{k-n}$$

$$= (-1)^{-n}\sum_{q\le k\le l+q}\binom{l+q-k}{m}\binom{-n-1}{k-n}(-1)^k$$

Using equation 5.25,

$$= (-1)^{-n}(-1)^{l+q+m}\binom{-n-1-m-1}{l+q-m-n}$$

$$= (-1)^{l+q-m-n}\binom{l+q-m-n-(l+q+1)-1}{l+q-m-n}$$

Using Upper Negation,

$$= \binom{l+q+1}{l+q-m-n}$$

Using symmetry property,

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}$$

Problem 5.16

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k = \sum_k \frac{(2a)!(2b)!(2c)!}{(a-k)!(a+k)!(b-k)!(b+k)!(c-k)!(c+k)!}(-1)^k$$

$$= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!}\sum_k \frac{(a+b)!(b+c)!(c+a)!}{(a-k)!(a+k)!(b-k)!(b+k)!(c-k)!(c+k)!}(-1)^k$$

Rearranging the terms and removing constant term $\frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(c+a)!}$,

$$= \sum_k \frac{(a+b)!}{(b-k)!(a+k)!}\frac{(b+c)!}{(c-k)!(b+k)!}\frac{(c+a)!}{(a-k)!(c+k)!}(-1)^k$$

$$= \sum_k \binom{(a+b)}{(a+k)}\binom{(b+c)}{(b+k)}\binom{(c+a)}{(c+k)}(-1)^k$$

Using Equation 5.29,

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}$$

Problem 5.37.

We need to prove following,

$$(x+y)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$$

$$RHS = \sum_k \frac{(n)! x^{\underline{k}} y^{\underline{n-k}}}{(n-k)!(k)!}$$

n is not dependent on k. So,

$$= n! \sum_k (\frac{x^{\underline{k}}}{k!})(\frac{y^{\underline{n-k}}}{(n-k)!})$$

$$= n! \sum_k \binom{x}{k}\binom{y}{n-k}$$

Using Vandermonde's convolution,

$$= n! \binom{x+y}{n}$$

$$= n! \frac{(x+y)^{\underline{n}}}{n!}$$

$$= (x+y)^{\underline{n}} = LHS$$

We also need to prove the following,

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}$$

$$RHS = n! \sum_k (\frac{x^{\overline{k}}}{k!})(\frac{y^{\overline{n-k}}}{(n-k)!})$$

$$= n! \sum_k (-1)^k (\frac{(-x)^{\underline{k}}}{k!})(-1)^{n-k}(\frac{(-y)^{\underline{n-k}}}{(n-k)!})$$

$$= (-1)^n n! \sum_k (\frac{(-x)^{\underline{k}}}{k!})(\frac{(-y)^{\underline{n-k}}}{(n-k)!})$$

$$= (-1)^n n! \sum_k \binom{-x}{k}\binom{-y}{n-k}$$

Using Vandermonde's convolution,

$$= (-1)^n n! \binom{-x-y}{n}$$

$$= (-1)^n n! \frac{(-x-y)^{\underline{n}}}{n!}$$

$$= (x+y)^{\overline{n}} = LHS$$

Problem 5.43. We need to prove,

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}$$

By using hint from the question,

$$= \sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\sum_j \binom{r}{m+n-j}\binom{k}{j}$$

$$= \sum_j \sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r}{m+n-j}\binom{k}{j}$$

$$= \sum_j \binom{r}{m+n-j}\sum_k \binom{m-r+s}{k}\binom{k}{j}\binom{n+r-s}{n-k}$$

Using equation 5.21 i.e.

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

$$= \sum_j \binom{r}{m+n-j}\sum_k \binom{m-r+s}{j}\binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}$$

$$= \sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\sum_k \binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}$$

Using Vandermonde's convolution,

$$= \sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\binom{m+n-j}{n-k}$$

$$= \sum_j \binom{r}{m+n-j}\binom{m+n-j}{n-k}\binom{m-r+s}{j}$$

Using equation 5.21 again,

$$= \sum_j \binom{r}{n-j}\binom{r-n+j}{m}\binom{m-r+s}{j}$$

Using symmetry equation,

$$= \sum_j \binom{r}{r-n+j}\binom{r-n+j}{m}\binom{m-r+s}{j}$$

Using equation 5.21 again,

$$= \sum_j \binom{r}{m}\binom{r-m}{r-n+j-m}\binom{m-r+s}{j}$$

Using symmetry equation,

$$= \sum_j \binom{r}{m} \binom{r-m}{n-j} \binom{m-r+s}{j}$$

$$= \binom{r}{m} \sum_j \binom{r-m}{n-j} \binom{m-r+s}{j}$$

Using Vandermonde's convolution,

$$= \binom{r}{m} \binom{r-m+m-r+s}{n-j+j}$$

$$= \binom{r}{m} \binom{s}{n}$$