CSE/AMS 547 Discrete Mathematics September 18, 2019

Homework One

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Due by Wednesday, Sep. 25, 4pm.

Problem 1.10.
Qn is the minimum number of moves to go from peg-A to peg-B in clock-wise direction.
Rn = minimum number of moves to go 2 pegs to the right or 1 peg to the left.
We need to prove the following:

$$Q_n = \begin{cases} 0, & \text{if } n < 0. \\ 2R_n + 1, & \text{if } n > 0. \end{cases} \tag{1}$$

$$R_n = \begin{cases} 0, & \text{if } n < 0. \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0. \end{cases} \tag{2}$$

Proof by induction:

$$Q_0 = 0 => True$$

$$R_0 = 0 => True$$

Assume n = No. of rings to be moved from peg-A to peg-B.
For (n-1) rings from A to intermediate-peg, i.e., 2 pegs across, we need Rn-1 steps.
To get the n^{th} element from peg-A to peg-B. we know that to get (n-1) rings from B to intermediate-peg, we need Rn-1 steps.

So,

$$n = R_{n-1} + R_{n-1} + 1$$

$$Q_n = 2R_{n-1} + 1$$

Assuming, n = No. of rings to be moved from peg-B to peg-A.
For (n-1) rings from B to A, that is 2 pegs across, Rn-1 steps are required.
To get the n^{th} element from B to intermediate-peg,1 step required.
we know that to get (n-1) rings from A to B, Qn-1 steps are required.
To get n^{th} element from intermediate-peg to A-peg, 1 step required.
To get the remaining n-1 rings from peg-B to peg-A, i.e., to the starting-position takes 2

clock-wise step.
So, moving n-1 rings from peg-B to peg-A would take Rn-1 steps. The total steps till now have been can be written as,

$$R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1}$$
$$R_n = 2R_{n-1} + 1 + Q_{n-1} + 1$$
$$R_n = Q_n + Q_{n-1} + 1$$

Problem 1.14
We need to find the maximum number of cheese pieces that can be made with 5 straight slices. Also find the recurrence relation for Pn, the maximum number of three-dimensional regions defined by n different planes.
With multiple planes intersecting the planes at different points, the simplest observation would be that to obtain maximum number of regions, the planes should not be parallel. This would imply that every new plane is passing through (n-1) previous planes.

Given that there would already be P(n-1) regions in the plane... with the addition of a new plane that is not parallel to any of the previous (n-1) planes,

No. of at most regions created = [n(n-1)/2 + 1]
So,

$$P(n) = P(n-1) + n(n-1)/2 + 1$$

We can open this relation as follows:

$$P(1) = 2$$

$$P(2) = 2 + 1 + 1 = 4 = 1 * 2 * 3/6 + 2 + 1$$
$$P(3) = 4 + 3 + 1 = 8 = 2 * 3 * 4/6 + 3 + 1$$
$$P(4) = 8 + 6 + 1 = 15 = 3 * 4 * 5/6 + 4 + 1$$
$$P(5) = 15 + 10 + 1 = 26 = 4 * 5 * 6/6 + 4 + 1$$
$$.....$$
$$.....$$

Thus,
$$P(n) = (n-1)n(n+1)/6 + n + 1$$

Proof by Induction:

$$P(0) = 1$$

Assuming the condition holds for n-1,then,

$$P(n-1) = (n-2)(n-1)(n)/6 + (n-1) + 1$$

So, for n,

$$P(n) = P(n-1) + n(n-1)/2 + 1$$
$$P(n) = (n-2)(n-1)(n)/6 + n + n(n-1)/2 + 1$$
$$P(n) = (n-1)n(n+1)/6 + n + 1$$

So, P(5) = 26.

Problem 1.21.
Suppose 2n people in a circle. first n = 'good' persons and
next n = 'bad' persons.
Every m^{th} person is going to killed. So we need to prove that there exists a number m such that if we kill every m^{th} person, the bad ones are the first to die.
Assuming that n = 3 and m=120, the first execution = 6^{st}th person,
then the sword shifts to 1^{st} person. the second execution= 5^{th} person In the next pass, the 4^{rd} person is going to be executed thus killing all the bad people.

With 2n people and n-good and n-bad guys in the circle, the value of m needs to be:

$$m = 2n * (2n-1)(2n-2)...(n+1)$$

As $2n^{\text{th}}$ person is a multiple of m, he will be the first to go and control shifts to 1.
Similarly, $(2n-1)^{\text{th}}$ person is a multiple of m, and he will be the next one to go and again the control shifts to 1.After n passes, all the bad guys will be dead.
So, for any n, there exists an m such that all the bad guys will be dead first.

Problem 2.14. We know that,

$$k = \sum_{j=1}^{k} 1$$

By substituting,

$$\sum_{k=1}^{n} k2^k = \sum_{k=1}^{n}\sum_{j=1}^{k}(1).2^k$$
$$= \sum_{k=1}^{n}\sum_{k=j}^{n} 2^k$$

By summing first on k the above equation can be rewritten as,

$$\sum_{k=1}^{n} k2^k = \sum_{j=1}^{n} (2^j \sum_{k=0}^{n-j} 2^k)$$

Using Geometry Series formula,

$$= \sum_{j=1}^{n} (2^{n+1} - 2^j)/(2-1)$$

$$= \sum_{j=1}^{n} (2^{n+1} - 2^j)$$

$$= \sum_{j=1}^{n} 2^{n+1} - \sum_{j=1}^{n} 2^j$$

$$= 2^{n+1} \sum_{j=1}^{n} 1 - \sum_{j=1}^{n} 2^j$$

$$= n2^{n+1} - (2^{n+1} - 2)/(2-1)$$

So we get,

$$\sum_{k=1}^{n} k2^k = 2^{n+1}(n-1) + 2$$

Problem 2.21. Given that,

$$S_n = \sum_{k=0}^{n} (-1)^{n-k}$$

$$S_{n+1} = \sum_{0 \leq k \leq n+1} (-1)^{(n+1)-k}$$

According to Perturbation Method and Splitting off the first term,

$$S_{n+1} = a_0 + \sum_{1 \leq k \leq n+1} (-1)^{(n+1)-k}$$

$$S_{n+1} = (-1)^{n+1-0} + \sum_{1 \leq k+1 \leq n+1} (-1)^{(n+1)-(k+1)}$$

$$S_{n+1} = (-1)^{n+1} + \sum_{0 \leq k \leq n} (-1)^{n-k}$$

$$S_{n+1} = (-1)^{n+1} + S_n \qquad ...(1)$$

Again, according to Perturbation Method and Splitting off the last term,

$$S_{n+1} = \sum_{0 \le k \le n+1} (-1)^{(n+1)-k}$$

$$S_{n+1} = \sum_{0 \le k \le n} (-1)^{(n+1)-k} + (-1)^{(n+1)-(n+1)}$$

$$S_{n+1} = -\sum_{0 \le k \le n} (-1)^{(n-k)} + 1$$

$$S_{n+1} = 1 - S_n \qquad ...(2)$$

Equation (1) and Equation (2) are equal, we get,

$$1 - S_n = (-1)^{n+1} + S_n$$

$$S_n = (1 + (-1)^n)/2$$

Given that,

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k$$

$$T_{n+1} = \sum_{0 \le k \le n+1} (-1)^{(n+1)-k} k$$

According to Perturbation Method and Splitting off the first term,

$$T_{n+1} = a_0 + \sum_{1 \le k \le n+1} (-1)^{(n+1)-k}$$

$$T_{n+1} = 0.(-1)^{n+1-0} + \sum_{1 \le k+1 \le n+1} (-1)^{(n+1)-(k+1)}$$

$$T_{n+1} = \sum_{0 \le k \le n} (-1)^{n-k}(k+1)$$

$$T_{n+1} = T_n + S_n \qquad ...(3)$$

Again, according to Perturbation Method and Splitting off the last term,

$$T_{n+1} = \sum_{0 \le k \le n+1} (-1)^{(n+1)-k} k$$

$$T_{n+1} = \sum_{0 \le k \le n} (-1)^{(n+1)-k} k + (-1)^{(n+1)-(n+1)}(n+1)$$

$$T_{n+1} = -\sum_{0 \le k \le n} (-1)^{(n-k)} k + (n+1)$$

$$T_{n+1} = (n+1) - T_n \qquad ...(4)$$

Equation (3) and Equation (4) are equal, we get,

$$(n+1) - T_n = T_n + S_n$$

$$T_n = (n - (-1)^n)/2$$

Given that,

$$U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2$$

$$U_{n+1} = \sum_{0 \le k \le n+1} (-1)^{(n+1)-k} k^2$$

According to Perturbation Method and Splitting off the first term,

$$U_{n+1} = a_0 + \sum_{1 \le k \le n+1} (-1)^{(n+1)-k} k^2$$

$$U_{n+1} = 0.(-1)^{n+1-0} + \sum_{1 \le k+1 \le n+1} (-1)^{(n+1)-(k)} k^2$$

$$U_{n+1} = \sum_{0 \le k \le n} (-1)^{n-k} (k+1)^2$$

$$U_{n+1} = \sum_{0 \le k \le n} (-1)^{n-k} (k^2 + 2k + 1)$$

$$U_{n+1} = U_n + 2T_n + S_n \qquad ...(5)$$

Again, according to Perturbation Method and Splitting off the last term,

$$U_{n+1} = \sum_{0 \le k \le n+1} (-1)^{(n+1)-k} k^2$$

$$U_{n+1} = \sum_{0 \le k \le n} (-1)^{(n+1)-k} k^2 + (-1)^{(n+1)-(n+1)} (n+1)^2$$

$$U_{n+1} = -\sum_{0 \le k \le n} (-1)^{(n-k)} k + (n+1)^2$$

$$U_{n+1} = (n+1)^2 - U_n \qquad ...(6)$$

Equation (5) and Equation (6) are equal, we get,

$$(n+1)^2 - U_n = U_n + 2T_n + S_n$$

Substituting the values,

$$2U_n = (n+1)^2 - (n - (-1)^n) - (1 + (-1)^n$$

$$U_n = (n^2 + n)/2$$

Problem 2.28.

$$Step1 = \sum_{k\geq 1}(k/(k+1) - (k-1)/k)$$

$$Step2 = \sum_{k\geq 1}\sum_{j\geq 1}((k/j)[j = k+1] - (j/k)[j = k-1])$$

$$Step3 = \sum_{j\geq 1}\sum_{k\geq 1}((k/j)[j = k+1] - (j/k)[j = k-1])$$

$$Step4 = \sum_{j\geq 1}\sum_{k\geq 1}((k/j)[k = j-1] - (j/k)[k = j+1])$$

$$Step5 = \sum_{j\geq 1}((j-1)/j - j/(j+1))$$

$$Step6 = \sum_{j\geq 1}((-1)/j(j+1)) = -1$$

Astray Derivation: From $Step2$ and $Step3$ interchange of summation is not valid as the terms of this sum do not absolutely convergent.

Problem 2.31. Given that Riemann's zeta function $\zeta(k)$ is defined as:

$$1 + 1/2^k + 1/3^k + ... = \sum_{j\geq 1} 1/j^k$$

$\zeta(1) = 1 + 1/2 + 1/3 + ...$ is divergent because $\lim_{x\to\infty}\zeta(1)$ is undefined.But the rest of the terms are convergent.
We need to proof the following term: $\sum_{k\geq 0}(\zeta(k) - 1) = 1$,for k$\geq 2$
which will look like as,

$$\zeta(2) - 1 = 1/2^2 + 1/3^2 + ...$$
$$\zeta(3) - 1 = 1/2^3 + 1/3^3 + ...$$
$$\zeta(4) - 1 = 1/2^4 + 1/3^4 + ...$$

....

....

....

$$\sum_{k\geq 2}(\zeta(k)-1) = (1/2^2+1/3^2+...)+(1/2^3+1/3^3+...)+(1/2^4+1/3^4+...)+(...)$$

$$\sum_{k\geq 2}(\zeta(k)-1) = (1/2^2+1/2^3+1/2^4...)+(1/3^2+1/3^3+1/3^4+...)+(...)$$

$$= \sum(\sum(1/n^k))$$

Using sum of convergent Geometric Series, $S_n = a_0/(1-r)$

$$= \sum(1/n^2)/(1-1/n)$$

$$= \sum(1/(n-1)-1/n$$

$$= (1/1-1/2)+(1/2-1/3)+(1/3-1/4)+...$$

$$So, \sum_{k\geq 2}(\zeta(k)-1) = \sum(\sum(1/n^k)) = 1 \qquad ...(1)$$

Similarly, we need to find the value of $\sum_{k\geq 1}(\zeta(2k)-1)$
We can break it down as follows,

$$\zeta(2)-1 = 1/2^2+1/3^2+...$$

$$\zeta(4)-1 = 1/2^4+1/3^4+...$$

$$\zeta(6)-1 = 1/2^6+1/3^6+...$$

....

....

....

$$= (1/2^2+1/3^2+...)+(1/2^4+1/3^4+...)+(1/2^6+1/3^6+...)+(...)$$

$$= (1/2^2+1/2^4+1/2^6...)+(1/3^2+1/3^4+1/3^6+...)+(...)$$

$$= \sum(\sum(1/n^{2k}))$$

Again, using geometric series sum,

$$= \sum(1/n^2)/(1-1/n^2)$$

$$= \sum 1/(n^2-1)$$

$$= (1/2)\sum(1/(n-1)) - (1/(n+1))$$

$$= 1/2(1 - 1/3 + 1/2 - 1/4 + 1/3 - 1/5...)$$

$$= 1/2(1 + 1/2)$$

$$\sum_{k\geq 1}(\zeta(2k) - 1) = 3/4$$