Solving the 2D Convection-Diffusion Equation with a Source Term Using the ADI Method

1 Introduction

This document presents the **Alternating Direction Implicit (ADI) method** for solving the **two-dimensional convection-diffusion equation** with a **source term**. The source term represents external influences such as heat generation, chemical reactions, or localized perturbations. The ADI method efficiently handles implicit updates in one direction while keeping computations explicit in the other.

2 Mathematical Formulation

2.1 Governing Equation with Source Term

The two-dimensional convection-diffusion equation with a **source term** is given by:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + S(x, y, t), \tag{1}$$

where:

- $u(x, y, t)$ is the transported scalar field (e.g., temperature, concentration).
- a, b are the convection velocities in the x- and y-directions.
- ν is the diffusion coefficient.
- $S(x, y, t)$ is the **source term**, which can be spatially and temporally dependent.
- (x, y) are the spatial coordinates in a 2D domain.
- t represents time.

2.2 Discretized Form

Using a uniform spatial grid with spacing dx and dy, and a time step dt, the finite difference approximation of the equation is:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + a\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2dx} + b\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2dy} = \nu\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{dx^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{dy^2}\right) + S_{i,j}^n. \tag{2}$$

3 Numerical Solution Using ADI

The **Alternating Direction Implicit (ADI) method** solves the equation in **two sequential steps**:

1. **Step 1: Implicit in the x-direction, explicit in the y-direction**. 2. **Step 2: Implicit in the y-direction, explicit in the x-direction**.

3.1 ADI Formulation with Source Term

Step 1: Implicit in x, Explicit in y

$$\frac{u_{i,j}^* - u_{i,j}^n}{\Delta t/2} = \nu\frac{\partial^2 u^*}{\partial x^2} + a\frac{\partial u^n}{\partial x} + \nu\frac{\partial^2 u^n}{\partial y^2} + b\frac{\partial u^n}{\partial y} + S_{i,j}^n. \tag{3}$$

Step 2: Implicit in y, Explicit in x

$$\frac{u_{i,j}^{n+1} - u_{i,j}^*}{\Delta t/2} = \nu\frac{\partial^2 u^{n+1}}{\partial y^2} + b\frac{\partial u^*}{\partial y} + \nu\frac{\partial^2 u^*}{\partial x^2} + a\frac{\partial u^*}{\partial x} + S_{i,j}^*. \tag{4}$$

4 Algorithm

Algorithm 1 ADI Method for 2D Convection-Diffusion with Source Term

```
Initialize domain, grid points, and parameters.
Compute **ADI coefficients** and tridiagonal matrices.
for each time step t = 1 to T/Δt do
    Step 1: Implicit in x, Explicit in y
    for each row j = 1 to N_y − 1 do
        Modify RHS to include source term.
        Solve tridiagonal system using Thomas Algorithm.
    end for
    Step 2: Implicit in y, Explicit in x
    for each column i = 1 to N_x − 1 do
        Modify RHS to include source term.
        Solve tridiagonal system using Thomas Algorithm.
    end for
    Apply boundary conditions.
    Save solution as an image.
end for
Compile saved images into a video.
```

5 Impact of Source Term on the ADI Method

The **source term $S(x, y, t)$ modifies only the right-hand side (RHS)** of the linear system in each ADI step but does not change the structure of the **tridiagonal coefficient matrices**. This allows us to use efficient **Thomas Algorithm-based solvers**.

5.1 Tridiagonal Matrix Formulation

For each implicit step, the discretization leads to a **tridiagonal system**:

$$A_x U^* = \mathbf{b}_x, \quad A_y U^{n+1} = \mathbf{b}_y. \tag{5}$$

where:

$$\begin{aligned}
&\sigma_x = \frac{\nu \Delta t}{dx^2}, \quad \sigma_y = \frac{\nu \Delta t}{dy^2}, \\
&\alpha_x = 0.5\left(\sigma_x + \max(0, a\frac{\Delta t}{dx})\right), \quad \beta_x = 0.5\left(\sigma_x - \min(0, a\frac{\Delta t}{dx})\right), \\
&\alpha_y = 0.5\left(\sigma_y + \max(0, b\frac{\Delta t}{dy})\right), \quad \beta_y = 0.5\left(\sigma_y - \min(0, b\frac{\Delta t}{dy})\right).
\end{aligned}$$

Effect on Right-Hand Side (RHS) The source term modifies only the RHS:

$$\mathbf{b}_x = u^n_{i,j} + \frac{\Delta t}{2}\left[\nu\frac{\partial^2 u^n_{i,j}}{\partial y^2} + b\frac{\partial u^n_{i,j}}{\partial y} + S^n_{i,j}\right]$$

$$\mathbf{b}_y = u^*_{i,j} + \frac{\Delta t}{2}\left[\nu\frac{\partial^2 u^*_{i,j}}{\partial x^2} + a\frac{\partial u^*_{i,j}}{\partial x} + S^*_{i,j}\right]$$

6 Conclusion

The ADI method efficiently solves the **2D convection-diffusion equation with a source term**. The **coefficient matrices remain unchanged**, ensuring computational efficiency, while the **source term only modifies the RHS**.