Inverse Calculation

Let the vector $(y_0, y_1, \ldots y_{n-1})$ represent the values of polynomial A of degree n−1 where the points $x = w_n^k$ be given. We want to restore the coefficients $(a_0, a_1, \ldots, a_{n-1})$ of the polynomial.
So the whole FFT process can be shown like the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w_n & w_n^2 & \ldots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \ldots & w_n^{2(n-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & w_n^{n-1} & w_n^{2(n-1)} & \ldots & w_n^{(n-1)*(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ | \\ | \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ | \\ | \\ y_{n-1} \end{pmatrix}$$

The square matrix on the left is the **Vandermonde Matrix (V_n)**, where the **(k, j)** entry of V_n is $\mathbf{w_n^{kj}}$
Now for finding the inverse, we can write the above equation as

$$\begin{pmatrix} a_0 \\ a_1 \\ | \\ | \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w_n & w_n^2 & \ldots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \ldots & w_n^{2(n-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & w_n^{n-1} & w_n^{2(n-1)} & \ldots & w_n^{(n-1)*(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_1 \\ | \\ | \\ y_{n-1} \end{pmatrix}$$

A quick check can verify that the inverse of the matrix has the following form:

$$1/n \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w_n^{-1} & w_n^{-2} & \ldots & w_n^{-(n-1)} \\ 1 & w_n^{-2} & w_n^{-4} & \ldots & w_n^{-2(n-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & w_n^{-(n-1)} & w_n^{-2(n-1)} & \ldots & w_n^{-(n-1)*(n-1)} \end{pmatrix}$$

Thus we obtain the formula:

$$\mathbf{a_k=1/n\sum y_j w_n^{-kj} \quad for\ j = 0\ to\ n\text{-}1}$$

Comparing this to the formula for y_k

$$\mathbf{y_k=1/n\sum a_j w_n^{kj} \quad for\ j = 0\ to\ n\text{-}1}$$

We notice that these problems are almost the same, so the coefficients a_k can be found by the same divide and conquer algorithm, only instead of $\mathbf{w_n^k}$ we have to use $\mathbf{w_n^{-k}}$, and at the end, we need to divide the resulting coefficients by n.