Feedback Control System
Unit 1
Introduction

Introduction

Control systems are an integral part of modern society.

Numerous applications are all around us:

The rockets fire, and the space shuttle lifts off to earth orbit; a self-guided vehicle delivering material to workstations in an aerospace assembly plant glides along the floor seeking its destination. These are just a few examples of the automatically controlled systems

The domestic applications are

an air conditioner, a refrigerator, a bathroom toilet tank, an automatic iron, elevators, washing machines and many processes within a car like car braking system, fuel injection system, and son

Introduction

Bio control systems

Within our own bodies we have numerous control systems, such as

the **pancreas**, which regulates our blood sugar.

In time of “fight or flight,” our adrenaline increases along with our heart rate, causing more oxygen to be delivered to our cells.

Our eyes follow a moving object to keep it in view;

our hands grasp the object and place it precisely at a predetermined location.

Definition

A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input.

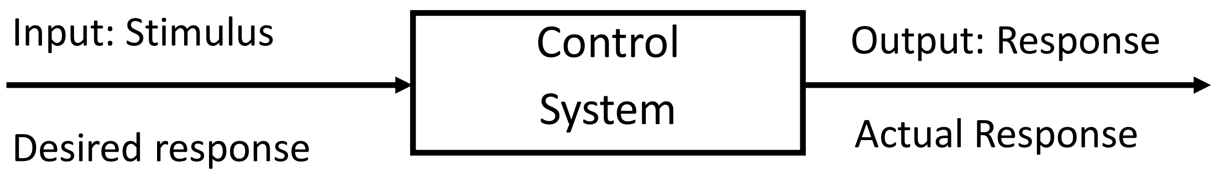
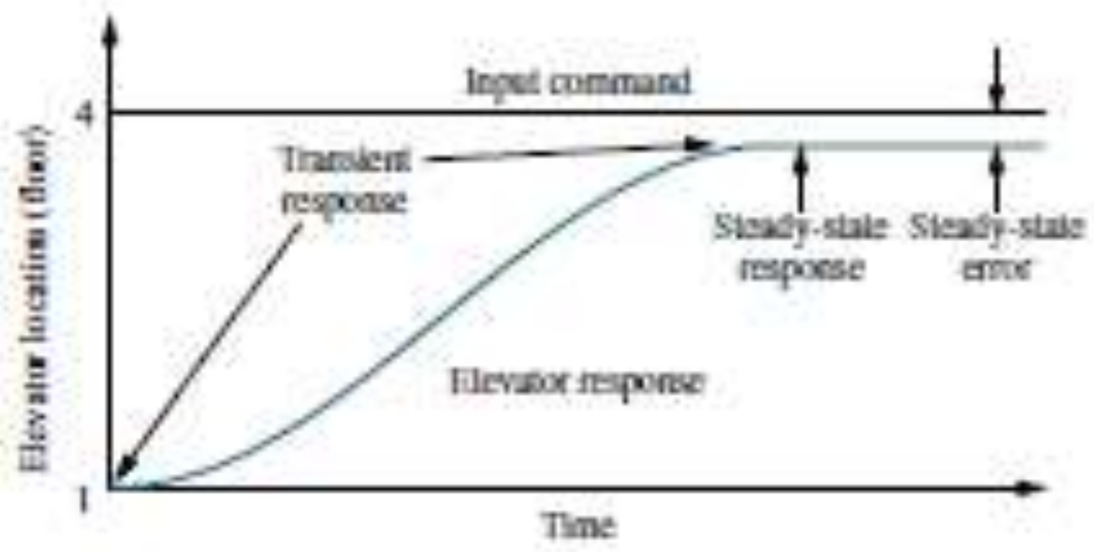


Figure shows a control system in its simplest form, where the input represents a desired output.

Definition

- For example, consider an elevator.
- When the fourth-floor button is pressed on the first floor, the elevator rises to the fourth floor with a speed and floor-leveling accuracy designed for passenger comfort.
- The push of the fourth-floor button is an input that represents our desired output, shown as a step function in Figure



Definition

The performance of the elevator can be seen from the elevator response curve in the figure.

Two major measures of performance are apparent:

(1) the transient response and

(2) the steady-state error.

Definition

In our example, passenger comfort and passenger patience are dependent upon the transient response.

If this response is too fast, passenger comfort is sacrificed;

if too slow, passenger patience is sacrificed.

The steady-state error is another important performance specification since passenger safety and convenience would be sacrificed if the elevator did not properly level.

Advantages of Control Systems

With control systems we can move large equipment with precision that would otherwise be impossible.

We can point huge antennas toward the farthest reaches of the universe to pick up faint radio signals; controlling these antennas by hand would be impossible.

Because of control systems, elevators carry us quickly to our destination, automatically stopping at the right floor.

We alone could not provide the power required for the load and the speed; motors provide the power, and control systems regulate the position and speed.

Advantages of Control Systems

We build control systems for four primary reasons:

1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation for disturbances

For example, a radar antenna, positioned by the low-power rotation of a knob at the input, requires a large amount of power for its output rotation. A control system can produce the needed power amplification, or power gain.

Robots designed by control system principles can compensate for human disabilities.

Control systems are also useful in remote or dangerous locations. For example, a remote-controlled robot arm can be used to pick up material in a radioactive environment.

Advantages of Control Systems

For example, in a temperature control system, the input is a position on a thermostat. The output is heat. Thus, a convenient position input yields a desired thermal output.

Another advantage of a control system is the ability to compensate for disturbances.

When the rope of elevator is cut, or the speed of elevator increases over the rated speed safety brake applied.

The system must be able to yield the correct output even with a disturbance.

For example, consider an antenna system that points in a commanded direction.

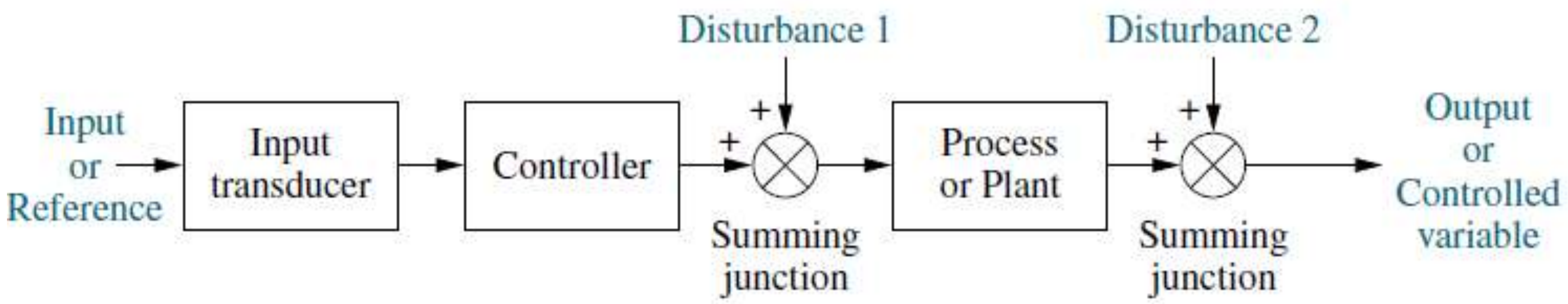
If wind forces the antenna from its commanded position, or if noise enters internally, the system must be able to detect the disturbance and correct the antenna’s position.

Advantages of Control Systems

Obviously, the system's input will not change to make the correction. Consequently, the system itself must measure the amount that the disturbance has repositioned the antenna and then return the antenna to the position commanded by the input.

Open Loop Control Systems

A generic open-loop system is shown in Figure.



(*a*)

It starts with a subsystem called an input transducer, which converts the form of the input to that used by the controller.

The controller drives a process or a plant.

The input is sometimes called the reference, while the output can be called the controlled variable.

Open Loop Control Systems

Other signals, such as disturbances, are shown added to the controller and process outputs via summing junctions, which yield the algebraic sum of their input signals using associated signs.

For example, the plant can be a furnace or air conditioning system, where the output variable is temperature.

The controller in a heating system consists of fuel valves and the electrical system that operates the valves.

The distinguishing characteristic of an open-loop system is that **it cannot compensate for any disturbances** that add to the controller's driving signal (Disturbance 1 in Figure).

Open Loop Control Systems

For example, if the controller is an electronic amplifier and Disturbance 1 is noise, then any additive amplifier noise at the first summing junction will also drive the process, corrupting the output with the effect of the noise. The output of an open-loop system is corrupted not only by signals that add to the controller's commands but also by disturbances at the output (Disturbance 2 in Figure). The system cannot correct for these disturbances, either.

Open Loop Control Systems

Open-loop systems, then, do not correct for disturbances and are simply commanded by the input.

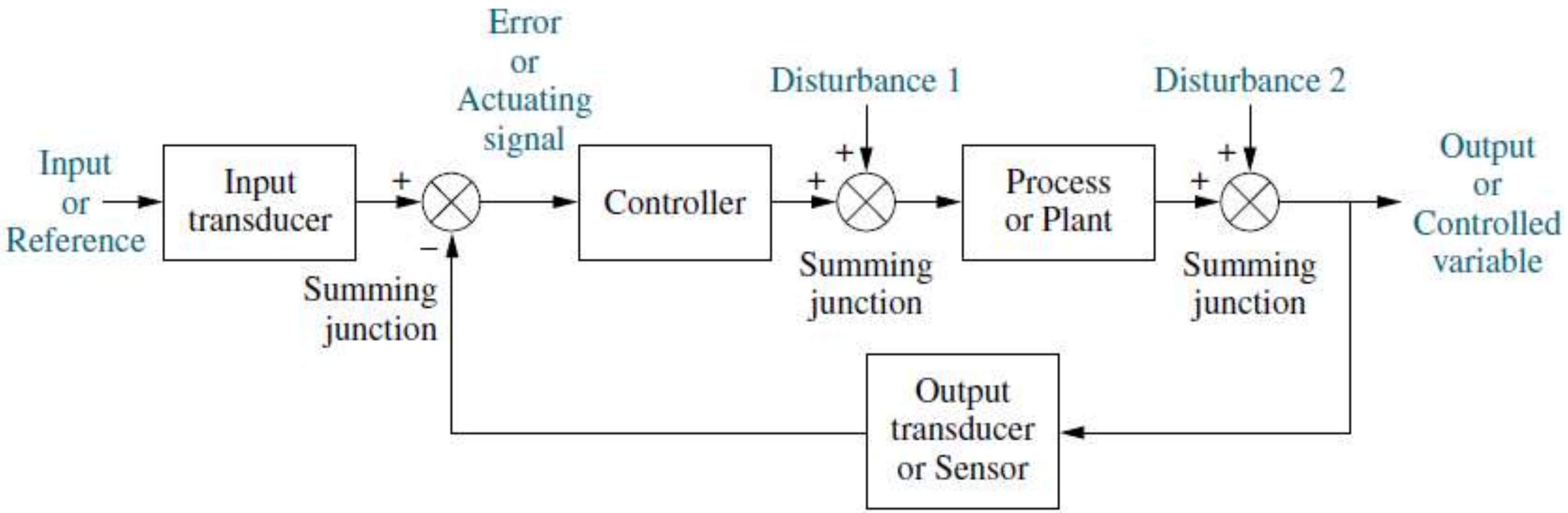
For example, toasters are open-loop systems, as anyone with burnt toast can attest.

The controlled variable (output) of a toaster is the color of the toast. The device is designed with the assumption that the toast will be darker the longer it is subjected to heat.

The toaster does not measure the color of the toast; it does not correct for the fact that the toast is rye, white, or sourdough, nor does it correct for the fact that toast comes in different thicknesses.

Closed Loop Control Systems

The disadvantages of open-loop systems, namely sensitivity to disturbances and inability to correct for these disturbances, may be overcome in closed-loop systems.



The generic architecture of a closed-loop system is shown in Figure.

Closed Loop Control Systems

The input transducer converts the form of the input to the form used by the controller.

An output transducer, or sensor, measures the output response and converts it into the form used by the controller.

For example, if the controller uses electrical signals to operate the valves of a temperature control system, the input position and the output temperature are converted to electrical signals.

The input position can be converted to a voltage by a potentiometer, a variable resistor, and the output temperature can be converted to a voltage by a thermistor, a device whose electrical resistance changes with temperature.

Closed Loop Control Systems

The first summing junction algebraically adds the signal from the input to the signal from the output, which arrives via the feedback path, the return path from the output to the summing junction.

In Figure, the output signal is subtracted from the input signal. The result is generally called the actuating signal.

However, in systems where both the input and output transducers have unity gain (that is, the transducer amplifies its input by 1), the actuating signal's value is equal to the actual difference between the input and the output. Under this condition, the actuating signal is called the error.

Closed Loop Control Systems

The closed-loop system compensates for disturbances by measuring the output response, feeding that measurement back through a feedback path, and comparing that response to the input at the summing junction.

If there is any difference between the two responses, the system drives the plant, via the actuating signal, to make a correction.

If there is no difference, the system does not drive the plant, since the plant's response is already the desired response.

Closed Loop Control Systems

Closed-loop systems have the obvious advantage of greater accuracy than open-loop systems.

They are less sensitive to noise, disturbances, and changes in the environment.

Transient response and steady-state error can be controlled more conveniently and with greater flexibility in closed-loop systems, often by a simple adjustment of gain (amplification) in the loop and sometimes by redesigning the controller.

We refer to the redesign as compensating the system and to the resulting hardware as a compensator.

Closed Loop Control Systems

On the other hand, closed-loop systems are more complex and expensive than open-loop systems.

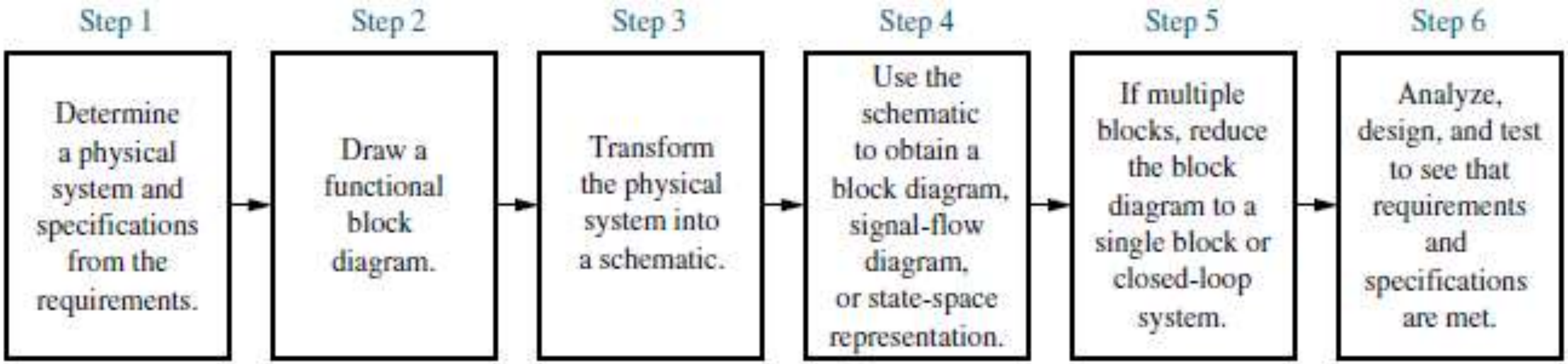
A standard, open-loop toaster serves as an example: It is simple and inexpensive. A closed-loop toaster oven is more complex and more expensive since it has to measure both color (through light reflectivity) and humidity inside the toaster oven.

Thus, the control systems engineer must consider the trade-off between the simplicity and low cost of an open-loop system and the accuracy and higher cost of a closed-loop system.

In summary, systems that perform the previously described measurement and correction are called closed-loop, or feedback control, systems. Systems that do not have this property of measurement and correction are called open-loop systems

Designed Procedure

Orderly sequence for the design of feedback control systems



Designed Procedure: Step 1

Step 1: Transform Requirements Into a Physical System

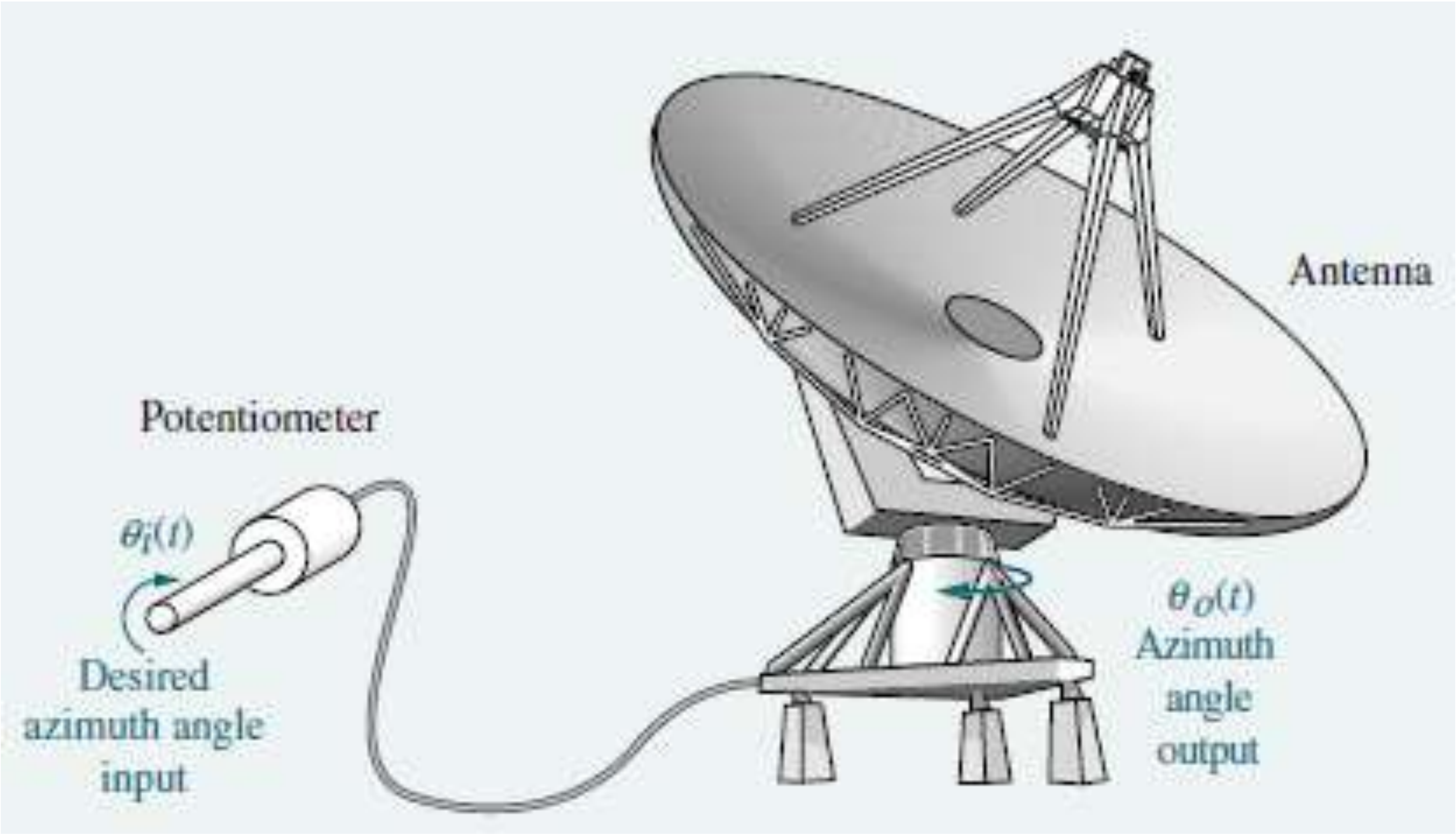
We begin by transforming the requirements into a physical system.

For example, in the antenna azimuth position control system, the requirements would state the desire to position the antenna from a remote location and describe such features as weight and physical dimensions.

Using the requirements, design specifications, such as desired transient response and steady-state accuracy, are determined.

Designed Procedure: Step 1

Step 1



Designed Procedure: Step 2

Step 2: Draw a Functional Block Diagram

The designer now translates a qualitative description of the system into a functional block diagram that describes the component parts of the system (that is, function and/or hardware) and shows their interconnection. Figure 1.9(d) is an example of a functional block diagram for the antenna azimuth position control system. It indicates functions such as input transducer and controller, as well as possible hardware descriptions such as amplifiers and motors. At this point the designer may produce a detailed layout of the system, such as that shown in Figure 1.9(b), from which the next phase of the analysis and design sequence, developing a schematic diagram, can be launched

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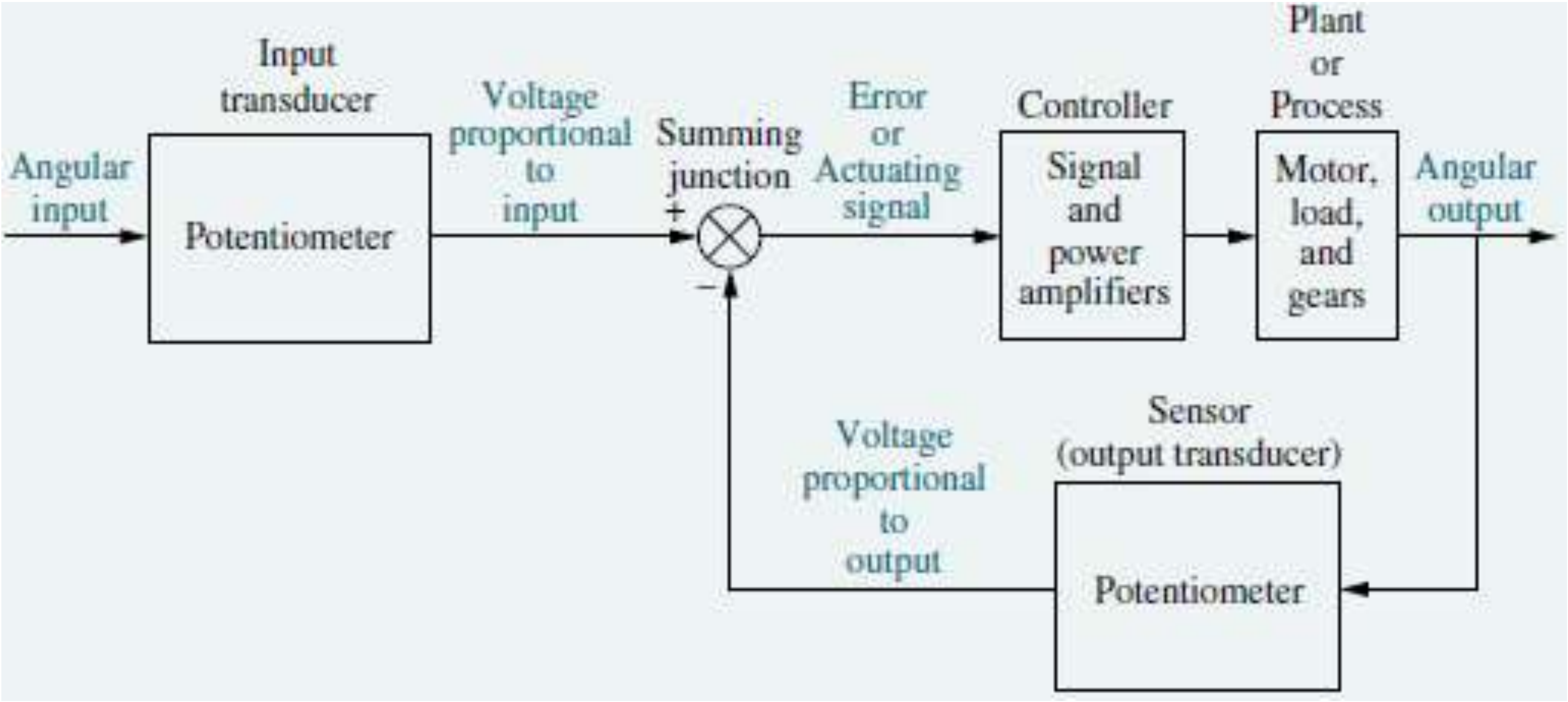

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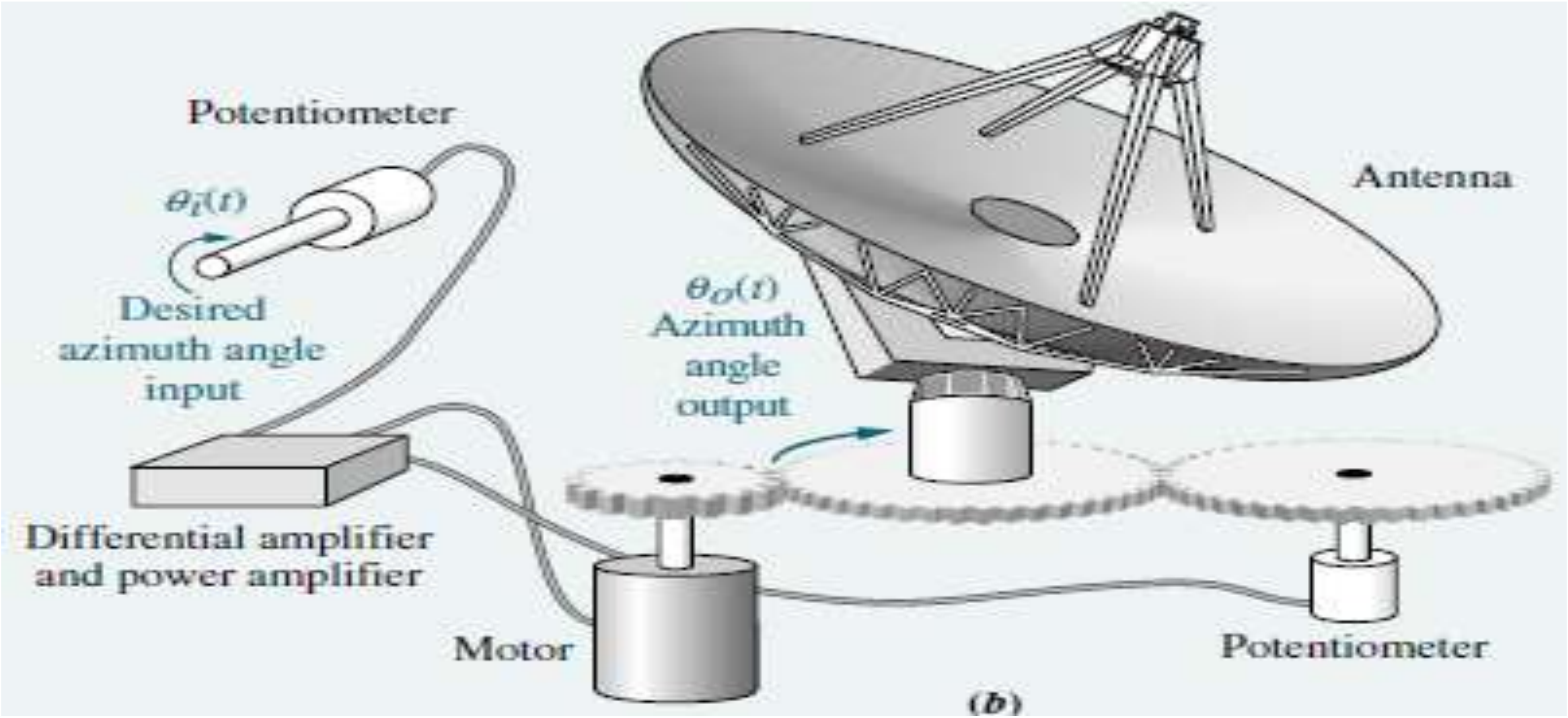


Designed Procedure: Step 3

Step 3: Create a Schematic

As we have seen, position control systems consist of electrical, mechanical, and electromechanical components.

After producing the description of a physical system, the control systems engineer transforms the physical system into a schematic diagram. The control system designer can begin with the physical description, to derive a schematic.

The engineer must make approximations about the system and neglect certain phenomena, or else the schematic will be unwieldy, making it difficult to extract a useful mathematical model during the next phase of the analysis and design sequence.

Designed Procedure: Step 3

Step 3: Create a Schematic

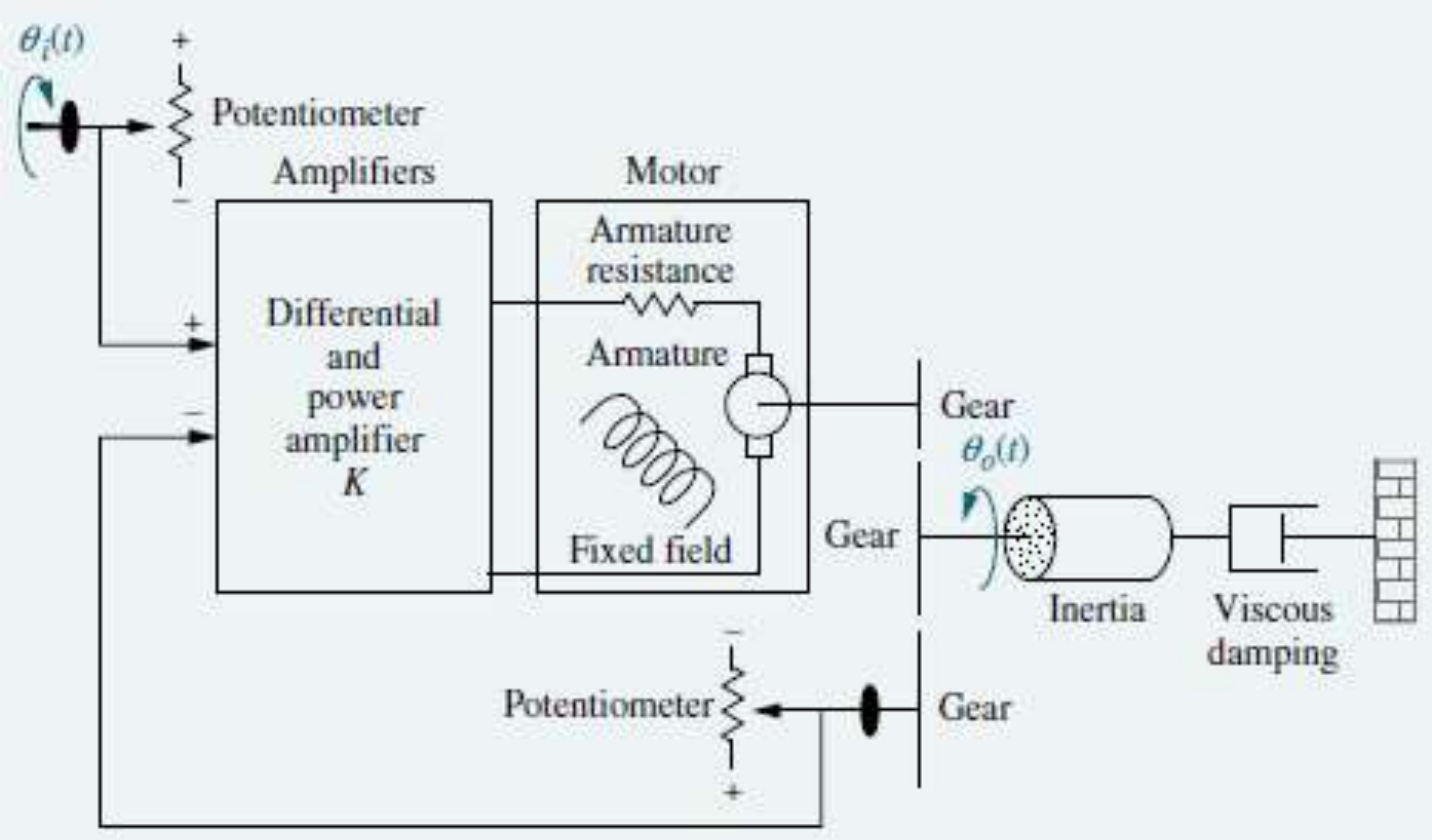
The designer starts with a simple schematic representation and, at subsequent phases of the analysis and design sequence, checks the assumptions made about the physical system through analysis and computer simulation.

If the schematic is too simple and does not adequately account for observed behavior, the control systems engineer adds phenomena to the schematic that were previously assumed negligible.

A schematic diagram for the antenna azimuth position control system is shown in following figure.

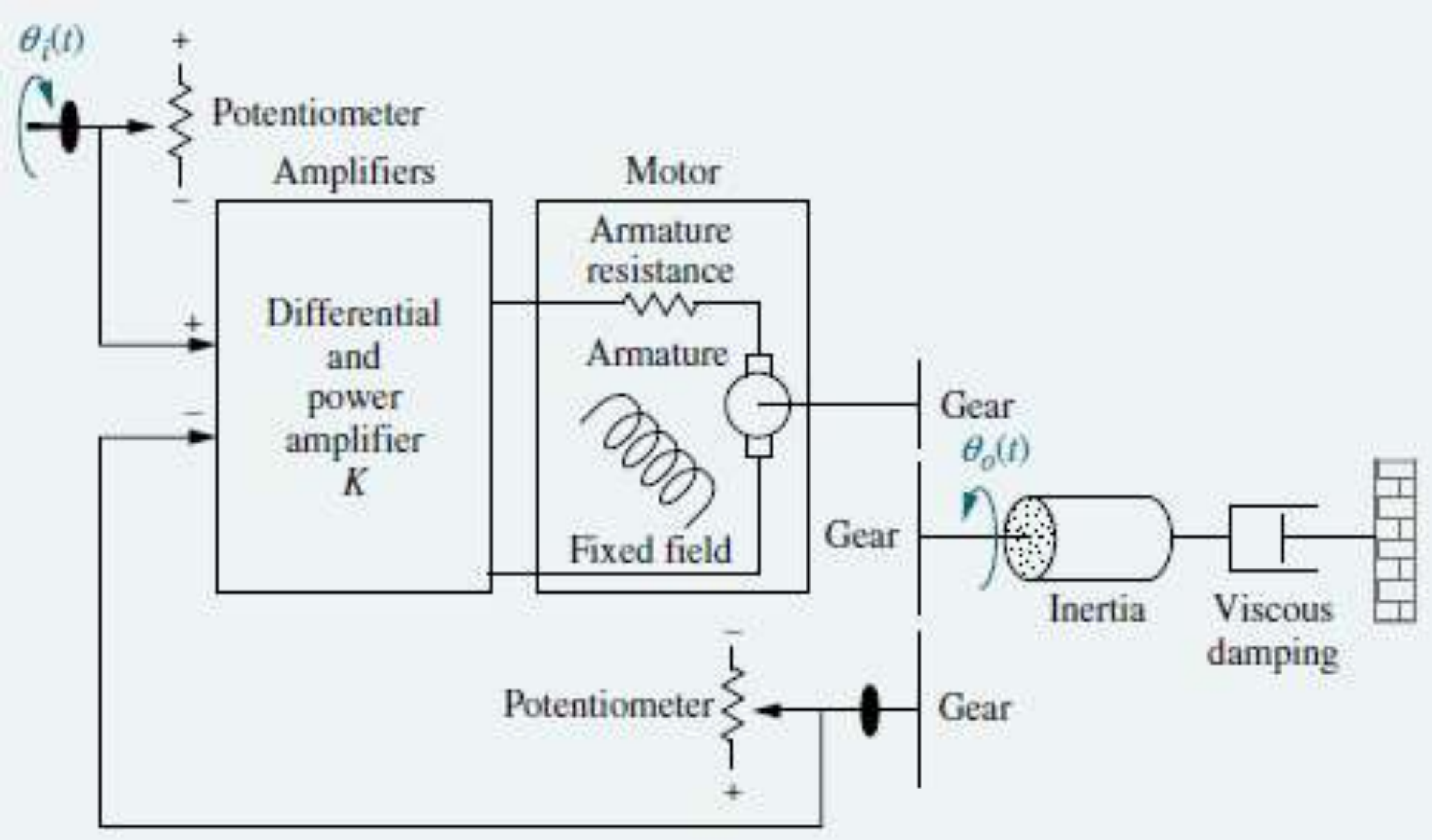
Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic



Designed Procedure: Step 3

Step 3: Create a Schematic

When we draw the potentiometers, we make our first simplifying assumption by neglecting their friction or inertia.

These mechanical characteristics yield a dynamic, rather than an instantaneous, response in the output voltage.

We assume that these mechanical effects are negligible and that the voltage across a potentiometer changes instantaneously as the potentiometer shaft turns.

A differential amplifier and a power amplifier are used as the controller to yield gain and power amplification, respectively, to drive the motor.

Designed Procedure: Step 3

Step 3: Create a Schematic

Again, we assume that the dynamics of the amplifiers are rapid compared to the response time of the motor; thus, we model them as a pure gain, K .

A dc motor and equivalent load produce the output angular displacement. The speed of the motor is proportional to the voltage applied to the motor's armature circuit. Both inductance and resistance are part of the armature circuit.

We assume the effect of the armature inductance is negligible for a dc motor. The designer makes further assumptions about the load.

The load consists of a rotating mass and bearing friction. Thus, the model consists of inertia and viscous damping whose resistive torque increases with speed, as in an automobile's shock absorber or a screen door damper.

Designed Procedure: Step 3

Step 3: Create a Schematic

The decisions made in developing the schematic stem from knowledge of the physical system, the physical laws governing the system's behavior, and practical experience.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Once the schematic is drawn, the designer uses physical laws, such as Kirchhoff's laws for electrical networks and Newton's law for mechanical systems, along with simplifying assumptions, to model the system mathematically. These laws are

Kirchhoff's voltage law: The sum of voltages around a closed path equals zero.

Kirchhoff's current law: The sum of electric currents flowing from a node equals zero.

Newton's laws: The sum of forces on a body equals zero; the sum of moments on a body equals zero

Kirchhoff's and Newton's laws lead to mathematical models that describe the relationship between the input and output of dynamic systems. One such model is the linear, time-invariant differential equation,

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1} c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Many systems can be approximately described by this equation, which relates the output, c(t), to the input, r(t), by way of the system parameters, a_i and b_j.

In addition to the differential equation, the transfer function is another way of mathematically modeling a system.

The model is derived from the linear, time-invariant differential equation using what we call the Laplace transform. Although the transfer function can be used only for linear systems.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

We will be able to change system parameters and rapidly sense the effect of these changes on the system response. The transfer function is also useful in modeling the interconnection of subsystems by forming a block diagram with a mathematical function inside each block.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Another model is the **state-space representation**.

One advantage of state space methods is that they can also be used for systems that cannot be described by linear differential equations.

Further, state-space methods are used to model systems for simulation on the digital computer.

Basically, this representation turns an n^{th} order differential equation into n simultaneous first-order differential equations.

Designed Procedure: Step 4

Step 4: Develop a Mathematical Model (Block Diagram)

Finally, we should mention that to produce the mathematical model for a system, we require knowledge of the parameter values, such as equivalent resistance, inductance, mass, and damping, which is often not easy to obtain. Analysis, measurements, or specifications from vendors are sources that the control systems engineer may use to obtain the parameters.

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Designed Procedure: Step 5

Step 5: Reduce the Block Diagram

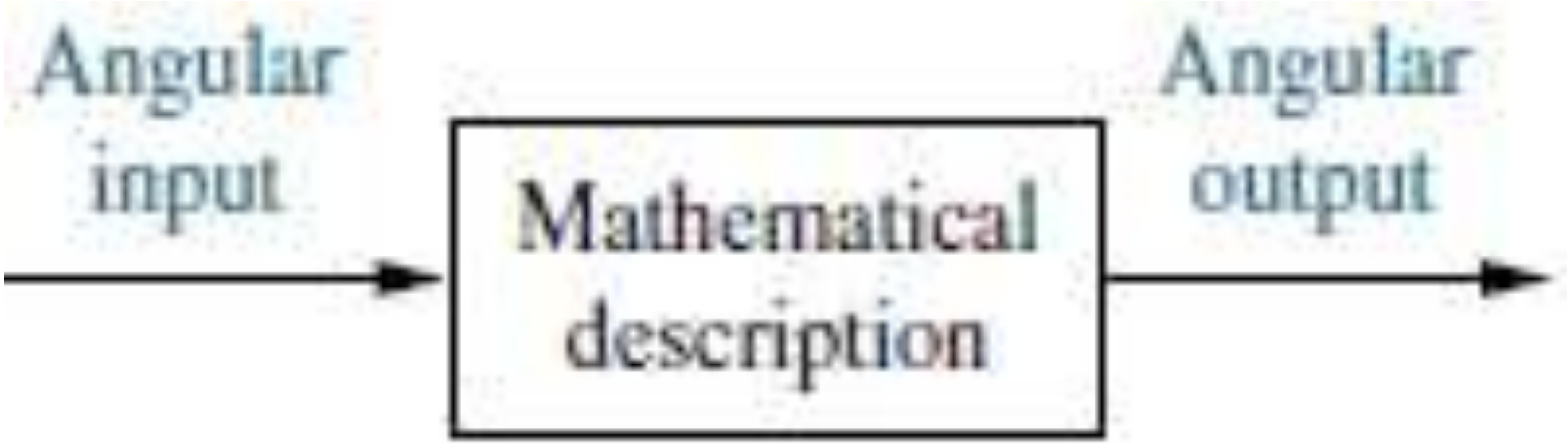
Subsystem models are interconnected to form block diagrams of larger systems where each block has a mathematical description.

Notice that many signals, such as proportional voltages and error, are internal to the system. There are also two signals – angular input and angular output – that are external to the system.

In order to evaluate system response in this example, we need to reduce this large system's block diagram to a single block with a mathematical description that represents the system from its input to its output, as shown in following figure. Once the block diagram is reduced, we are ready to analyze and design the system.

Designed Procedure: Step 5

Step 5: Reduce the Block Diagram



Designed Procedure: Step 6

Step 6: Analyze and Design

The next phase of the process, following block diagram reduction, is analysis and design.

If you are interested only in the performance of an individual subsystem, you can skip the block diagram reduction and move immediately into analysis and design.

In this phase, the engineer analyzes the system to see if the response specifications and performance requirements can be met by simple adjustments of system parameters.

If specifications cannot be met, the designer then designs additional hardware in order to effect a desired performance.

Designed Procedure: Step 6

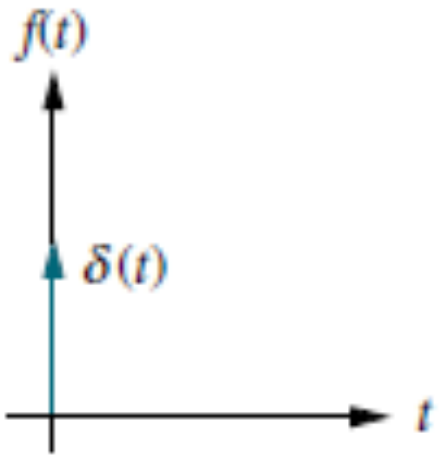
Step 6: Analyze and Design

Test input signals are used, both analytically and during testing, to verify the design.

It is neither necessarily practical nor illuminating to choose complicated input signals to analyze a system's performance.

Thus, the engineer usually selects standard test inputs.

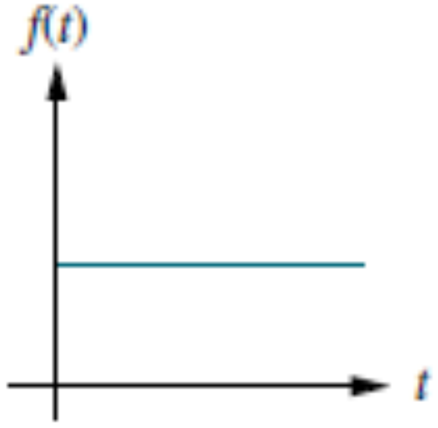
Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Impulse	δ(t)	$\delta(t) = \infty$ for $-0 < t < +0$ $= 0$ elsewhere $$\int_{-0}^{+0} \delta(t)dt = 1$$		Transient response Modelling

An impulse is infinite at t = 0 and zero elsewhere.

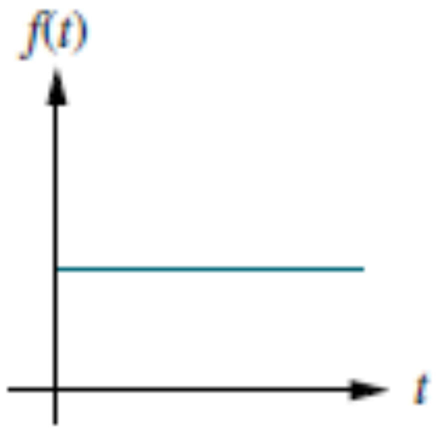
The area under the unit impulse is 1. An approximation of this type of waveform is used to place initial energy into a system so that the response due to that initial energy is only the transient response of a system. From this response the designer can derive a mathematical model of the system

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1 \quad \text{for } t > 0$ $\quad\;\; = 0 \quad \text{for } t < 0$		Transient response Steady-state error

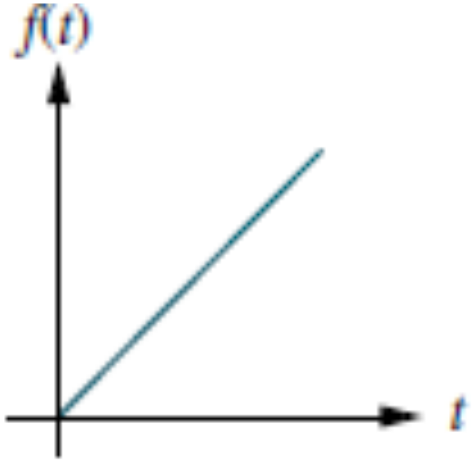
A step input represents a constant command, such as position, velocity, or acceleration. Typically, the step input command is of the same form as the output. For example, if the system's output is position, as it is for the antenna azimuth position control system, the step input represents a desired position, and the output represents the actual position.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Step	u(t)	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error

If the system's output is velocity, as is the spindle speed for a video disc player, the step input represents a constant desired speed, and the output represents the actual speed. The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated

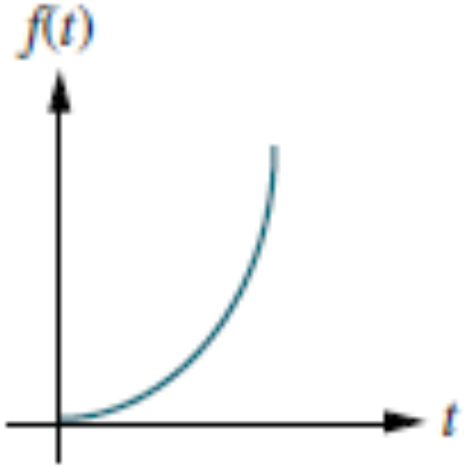
Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Ramp	t u(t)	t u(t) = 1 for t $\geq$ 0 = 0 otherwise		Steady-state error

The ramp input represents a linearly increasing command.

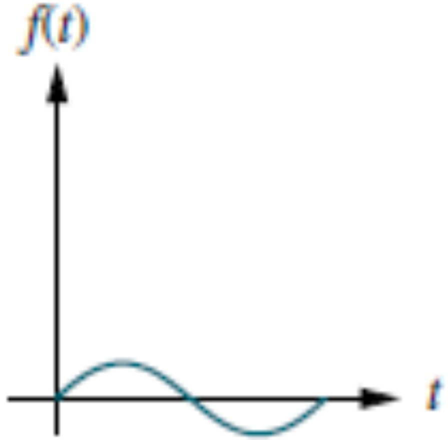
For example, if the system's output is position, the input ramp represents a linearly increasing position, such as that found when tracking a satellite moving across the sky at constant speed.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Parabola	$\frac{1}{2} t^2 u(t)$	$\frac{1}{2} t^2 u(t) = \frac{1}{2} t^2$ for $t \geq 0$ $= 0$ otherwise		Steady-state error

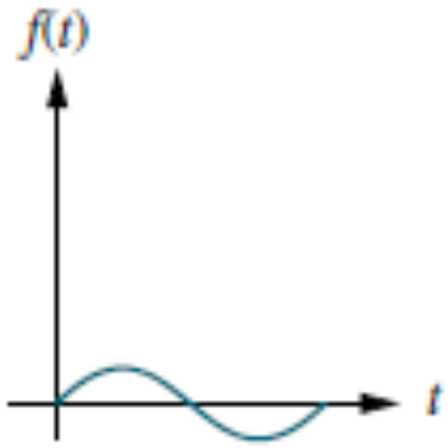
If the system's output is velocity, the input ramp represents a linearly increasing velocity. The response to an input ramp test signal yields additional information about the steady-state error. The previous discussion can be extended to parabolic inputs, which are also used to evaluate a system's steady-state error.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Sinusoid	sin ωt	sin ωt		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Designed Procedure: Step 6

I/P	Function	Description	Sketch	Use
Sinusoid	sin ωt	sin ωt		Transient response Modeling Steady-state error

Sinusoidal inputs can also be used to test a physical system to arrive at a mathematical model.

Laplace Transform

A system represented by a differential equation is difficult to model as a block diagram. By using the Laplace transform, with which we can represent the input, output, and system as separate entities.

Further, their interrelationship will be simply algebraic.

Let us first define the Laplace transform and then show how it simplifies the representation of physical systems

The Laplace transform is defined a

$$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Where $s = \sigma + j\omega$ is a complex variable.

Laplace Transform

The notation for the lower limit means that even if $f(t)$ is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges.

Thus, we can find the Laplace transform of impulse functions.

This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at $t = 0$.

Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity

Laplace Transform

The inverse Laplace transform is defined a

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

Where u(t) = 1; t > 0

= 0; t < 0

is the unit step function. Multiplication of f(t) by u(t) yields a time function that is zero for t < 0.

Laplace Transform Table

SN	f(t)	F(s)
1	$\delta(t)$	1
2	$u(t)$	$1/s$
3	$t\ u(t)$	$1/s^2$
4	$t^n\ u(t)$	$n!\ /(s^n +1)$
5	$e^{-at}\ u(t)$	$1/(s+a)$
6	$\sin \omega t\ u(t)$	$\omega/(s^2 + \omega^2)$
7	$\cos \omega t\ u(t)$	$s/(s^2 + \omega^2)$

Laplace Transform Theorems

SN	Theorem	Name
1	$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2	$L[kf(t)] = kF(s)$	Linearity
3	$L[f_1(t) + f_1(t)] = F_1(s) + F_2(s)$	Linearity
4	$L[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5	$L[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6	$L[f(at)] = \left(\frac{1}{a}\right)F(s/a)$	Scaling theorem
7	$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0-)$	Differentiation theorems

Laplace Transform Theorems

SN	Theorem	Name
8	$L\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorems
9	$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorems
10	$L\left[\int_{0-}^{t} f(\tau)d\tau\right] = F(s)/s$	Integration theorem
11	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem
12	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem

Laplace Transform Theorems

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8	$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0-) - f'(0-)$	Differentiation theorems
9	$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorems
10	$L\left[\int_{0-}^{t} f(\tau)d\tau\right] = F(s)/s$	Integration theorem
11	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem
12	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem

Laplace Transform

Find the Laplace transform of $f(t) = Ae^{-at}\,u(t)$.

Laplace Transform

Find the Laplace transform of f(t) = Ae^{-at} u(t).

Since the time function does not contain an impulse function, we can replace the lower limit of 0- with 0.

Hence,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} Ae^{-at}e^{-st}dt$$

$$= A\int_0^{\infty} e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{A}{s+a}$$

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

We know the frequency shift theorem

$$L[e^{-at}f(t)] = F(s+a)$$

the Laplace transform of $f(t) = tu(t)$ is $1/s^2$,

If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$,

the inverse transform of $F(s + a) = 1/(s + a)^2$ is $e^{-at}\, t\, u(t)$.

Hence, $f_1(t) = e^{-3t}\, t\, u(t)$

Laplace Transform: Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function,

Convert the function to a sum of simpler terms

The result is called a partial-fraction expansion.

If $F1(s) = N(s)/D(s)$,

If the order of $N(s)$ is less than the order of $D(s)$, then a partial-fraction expansion can be made.

If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

Laplace Transform: Partial-Fraction Expansion

For example, if $F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence $F_1(s) = s + 1 + \frac{2}{s^2+s+5}$

Taking the inverse Laplace transform, using the above Tables we obtain $f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + L^{-1}\left[\frac{2}{s^2+s+5}\right]$ Using partial-fraction expansion, we will be able to expand functions like $F(s) = \frac{2}{s^2+s+5}$ into a sum of terms and then find the inverse Laplace transform for each term

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let $F(s) = \frac{2}{(s+1)(s+2)}$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called residues, form the numerators. Hence, $F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(S+1)} + \frac{K_2}{(S+2)}$

To find K_1, we first multiply the equation by (s + 1), which isolates K_1. Thus,

$\frac{2}{(s+1)(s+2)} = K_1 + \frac{K_2(s+1)}{(S+2)}$

Let s → -1 eliminates the last term and gives K1 = 2. Similarly, K2 can be found by multiplying the equation by (s + 2) and then letting s → -2; hence, K2 = -2.

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Let s → -1 eliminates the last term and gives $K_1 = 2$. Similarly,

K_2 can be found by multiplying the equation by (s + 2) and then letting s → -2; hence,

$K_2 = -2$.

Therefore, $F(s) = \frac{2}{S+1} - \frac{2}{s+2}$

By referring the tables above

$f(t) = (2e^{-t} - 2e^{-2t})\, u(t)$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Given the following differential equation, solve for y(t) if all initial conditions are zero. Use the Laplace transform. $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$

Substitute the corresponding Laplace transform for each term in above equation, using Table, and the initial conditions of $y(t)$ and $\frac{dy(t)}{dt}$ given by $y(0-) = 0$ and $\frac{dy(0-)}{dt} = 0$, respectively. Hence, the Laplace transform of equation is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Solving for the response Y(s) we have $Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$

Now y(t) is obtained by taking the inverse Laplace transform, for that we need to partial fraction method. Therefore

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Therefore, $Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

Hence $Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$

This is the simplest form, and we can easily find the LT of each term

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

The u(t) shows that the response is zero until t = 0.

Thus, output responses will also be zero until t = 0. For convenience, we will leave off the u(t) notation. Accordingly, we write the output response as $y(t) = (1 - 2e^{-4t} +$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

Let $F(s) = \frac{2}{(s+1)(s+2)^2}$

The roots of $(s + 2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at 2 is a multiple root of multiplicity 2

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if $F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

For K_1

$$\frac{2}{(s+2)^2} = K_1 + \frac{(s+1)K_2}{(s+2)^2} + \frac{(s+1)K_3}{(s+2)}, \quad \therefore K_1 = 2$$

K_2 can be isolated by multiplying by $(s + 2)^2$, yielding

$\frac{2}{s+1} = (s+2)^2\frac{K_1}{(s+1)} + K_2 + (S+2)K_3$. Letting s → -2 ; K_2 = - 2.

To find K_3 we differentiate above equation with respect to s

$\frac{-2}{(s+1)^2} = \frac{(s+2)K_1}{(s+1)^2} + K_3$, K_3 is isolated and can be found by s → -2 Hence, K3 = -2

$$\therefore F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

$$\therefore f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Let $F(s) = \frac{3}{s(s^2+2s+5)}$ *this*

This function can be expanded in the following form

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2 + 2s + 5)}$$

K_1 is found in the usual way to be 3/5. K_2 and K_3 can be found by first multiplying the above equation by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = 3/5$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right) s^2 + \left(K_3 + \frac{6}{5}\right) s + 3$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Balancing coefficients, $(K_2 + 3/5) = 0$ and $(K_3 + 6/5) = 0$. Hence $K_2 = -3/5$ and $K_3 = -6/5$. Thus,

Let $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using $L[\sin(\omega t)\, u(t)] = \frac{\omega}{(s^2+\omega^2)}$; $L[\cos \omega t\, u(t)] = \frac{s}{(s^2+\omega^2)}$, $L[kf(t)] = kF(s)$; and $L[e^{-at} f(t)] = F(s+a)$

$L[Ae^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$ and $L[Be^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$

If we add these two equations

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

If we add these two equations

$$L[Ae^{-at}\cos \omega t + Be^{-at}\sin \omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

Now rearrange $F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{(s^2+2s+5)}$ by completing the squares in the denominator and adjusting terms in the numerator without changing its value

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{s+1+(1/2)(2)}{(s+1)^2+2^2}$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

The alternative method is

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3}{s(s+1+j2)(s+1-j2)}$$

$$= \frac{k_1}{s} + \frac{k_2}{s+1+j2} + \frac{k_3}{s+1-j2}$$

$$K_2 = \left.\frac{3}{s(s+1-j2)}\right|_{s\to -1-j2} = -\frac{3}{20}(2+j1)$$

We know k1 = 3/5; and K3 is complex conjugate of K2.

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

Hence

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{S+1+j2} + \frac{2-j1}{S+1-j2}\right)$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{20}e^{-t}\left((2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t}\right)$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t}+e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t}+e^{-j2t}}{2j}\right)\right]$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Feedback Control System Unit 1_2
Transfer function of physical systems

The Transfer Function

Consider a n^th^-order, linear, time-invariant continuous time system describe by the differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the a_n, a_{n-1},…, a_0 and b_m, b_{m-1}, …, b_0 are the coefficients. Taking the Laplace transform of both sides,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) =$$
$$b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

If we assume that all initial conditions are zero, then

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

The Transfer Function

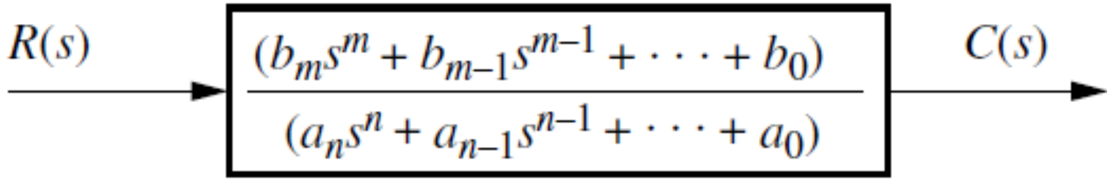
The ratio of the output transform, C(s), divided by the input transform, R(s) is:

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

The transfer function G(s) is defined as the ratio of Laplace Transform of output C(s) to the Laplace Transform of input R(s) with zero initial conditions

The transfer function can be represented as a block diagram, as shown in Figure, with the input on the left, the output on the right, and the system transfer function inside the block.

Also, we can find the output, C(s) by using C(s) = R(s)G(s)



The Transfer Function

Find the transfer function of a system represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s + 2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s + 2}$$

The Transfer Function

Find step response of the linear time invariant system described by the differential equation

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s+2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Now the we know C(s) = R(s) G(s)

The Transfer Function

Now the we know C(s) = R(s) G(s)

r(t) = u(t) therefore R(s) = 1/s

$$C(s) = \frac{1}{s}\frac{1}{s+2}$$

Using partial fraction

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Electrical Network Transfer Functions

- the mathematical modeling of electric circuits
- three passive linear components: resistors, capacitors, inductors, and OPAMP.
- combine electrical components into circuits, decide on the input and output, and find the transfer function.
- Our guiding principles are Kirchhoff's laws.
- Sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the
- result to zero.
- From these relationships write the differential equations for the circuit.
- Then find the Laplace transforms of the differential equations and finally solve for the transfer function

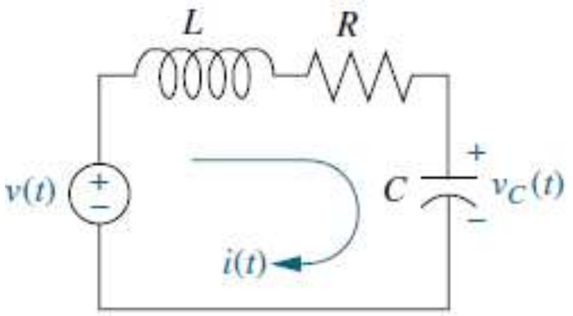
Electrical Network Transfer Functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance Z(s)=V(s)/I(s)	Admittance Y(s)=I(s)/V(s)
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical Network : RLC Circuit (Mesh Analysis)

Transfer functions can be obtained using **Kirchhoff's voltage law** and summing voltages around loops or meshes

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s).



In any problem, the designer must first decide what the input and output should be.

In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current.

In this problem it is stated as statement the capacitor voltage is the output the applied voltage as the input.

Electrical Network : RLC Circuit (Mesh Analysis)

Summing the voltages around the loop, assuming zero initial conditions, we get

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \ldots 1$$

Changing variables from current to charge using $i(t) = dq(t)/dt$

$$\therefore L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \ldots 2$$

From the voltage-charge relationship for a capacitor from the Table $q(t) = Cv_C(t)$

$$\therefore LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \ldots 3$$

Electrical Network : RLC Circuit (Mesh Analysis)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \;\ldots 4$$

Hence the transfer function is

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \;\ldots 5$$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

First, take the Laplace transform of the equations in the voltage-current assuming zero initial conditions.

Component	**Voltage current**	**LT of voltage current**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$V(s) = \frac{1}{Cs}I(s)$
Resistor	$v(t) = Ri(t)$	$V(s) = RI(s)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$V(s) = LsI(s)$

Now define the transfer function $\frac{V(s)}{I(s)} = Z(s)$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current.

But this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function impedance. The impedance for each of the electrical elements is

Component	**Impedance Z(s)=V(s)/I(s)**
Capacitor	$\frac{1}{Cs}$
Resistor	R
Inductor	Ls

Electrical Network : RLC Circuit (Mesh Analysis)

Let us use the concept of impedance for simplified solution

for the transfer function. The Laplace transform of $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$,

assuming zero initial conditions, is

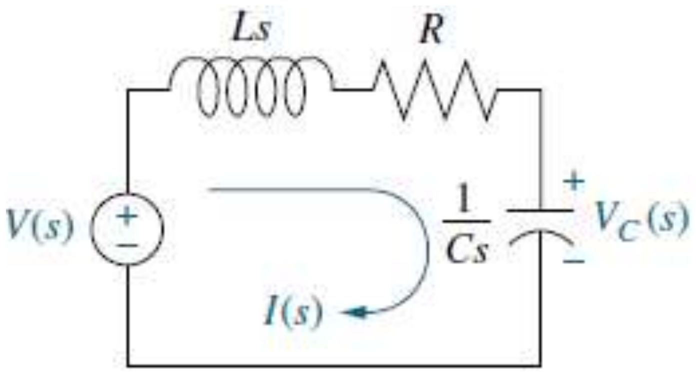
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \ldots 6$$

This equation is in the form

(Sum of impedances) I(s) = (Sum of applied voltages)

Electrical Network : RLC Circuit (Mesh Analysis)

From equation 6 we can have the series circuit as shown. This circuit could have been obtained immediately from the original RLC circuit simply by replacing each element with its impedance. Let us call this altered circuit the transformed circuit.



From the transformed circuit we can write Eq. (6) immediately, if we add impedances in series as we add resistors in series.

Electrical Network : RLC Circuit (Mesh Analysis)

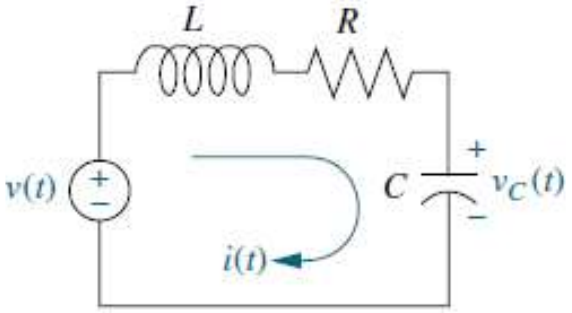
So instead of writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit.

We conclude this discussion as follows

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

Electrical Network : RLC Circuit (Mesh Analysis)

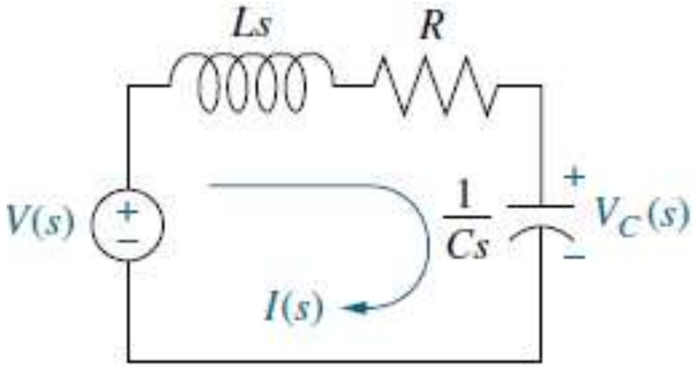
Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s)



by transform method.

Steps

1. Draw the transform network



Electrical Network : RLC Circuit (Mesh Analysis)

2. Write the equation

$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$ solving for I(s)/V(s)

$$\frac{I(s)}{V(s)} = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor.

$$V_{C(s)} = I(s)\frac{1}{Cs} = \frac{1}{Cs}\frac{V(s)}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{V(s)}{(LCs^2 + RCs + 1)} = \frac{(1/LC)V(s)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Electrical Network : RLC Circuit (Mesh Analysis)

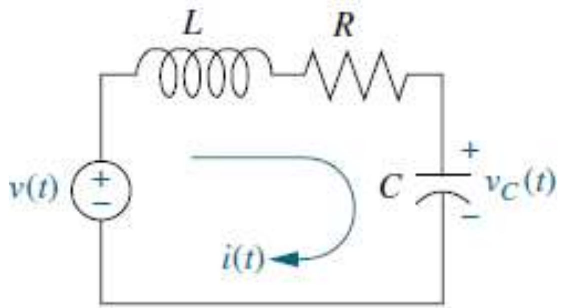
$$\frac{V_{C(s)}}{V(s)} = \frac{(1/LC)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Same as equation 5.

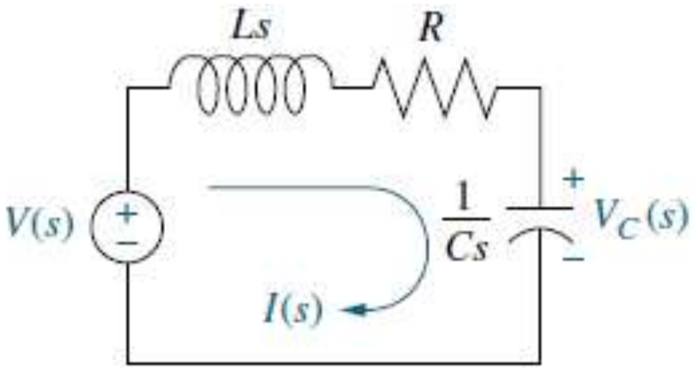
Electrical Network : RLC Circuit (Nodal Analysis)

Transfer functions also can be obtained using **Kirchhoff's current law** and summing currents flowing from nodes. This is nodal analysis method.

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by nodal analysis method and without writing a differential equation i.e. using transform method



transform to



Electrical Network : RLC Circuit (Nodal Analysis)

The transfer function is obtained by summing currents flowing out of the node whose voltage is VC(s) transformed figure with assumption that currents leaving the node are positive and currents entering the node are negative.

The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. Therefore for each I(s) = V(s)/Z(s). Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \ldots 1$$

where $V_C(s)\,/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls$ is the current flowing out of the node through the series resistor and inductor. Solve eq. (1) for the transfer function, $V_C(s)\,/V(s)$

Electrical Network : RLC Circuit (Nodal Analysis)

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = \frac{V_C(s)}{1/Cs} + \frac{V_C(s)}{R + Ls} - \frac{V(s)}{R + Ls} = 0$$

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s)}{R + Ls} = \frac{V(s)}{R + Ls}; \rightarrow \frac{(R + Ls)V_C(s)}{1/Cs} + VC(s) = V(s);$$

$$V_C(s)\left(1 + \frac{(R + Ls)}{1/Cs}\right) = V(s); \rightarrow V_C(s)\left(1 + \frac{(R + Ls)}{1/Cs}\right) = V(s)$$

$$V_C(s)\left(\frac{\frac{1}{Cs} + R + Ls}{1/Cs}\right) = V(s); \rightarrow \frac{V_C(s)}{V(s)} = \frac{1/Cs}{\frac{1}{Cs} + R + Ls}$$

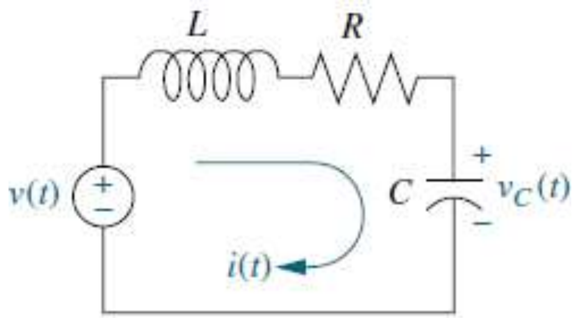
Electrical Network : RLC Circuit (Nodal Analysis)

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}=\frac{1}{LCs^2+RCs+1}$$

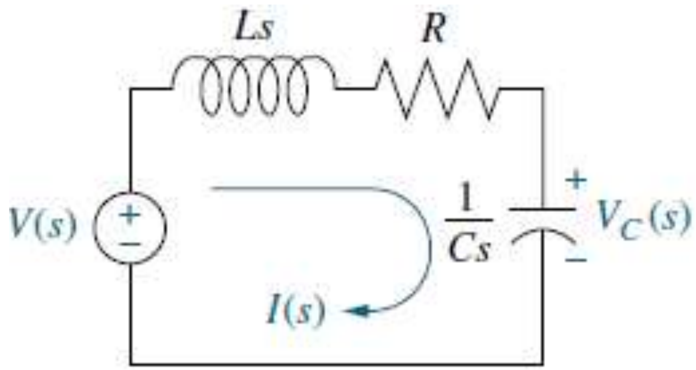
$$=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Electrical Network : RLC Circuit (Voltage Division)

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by votage division method



transformed to



The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus

$$V_C(s) = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} V(s)$$

Solve this equation for the transfer function, $V_C(s)\,/V(s)$

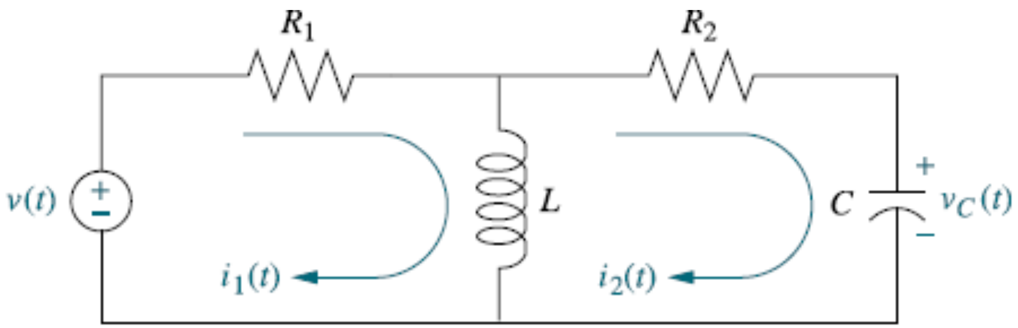
Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

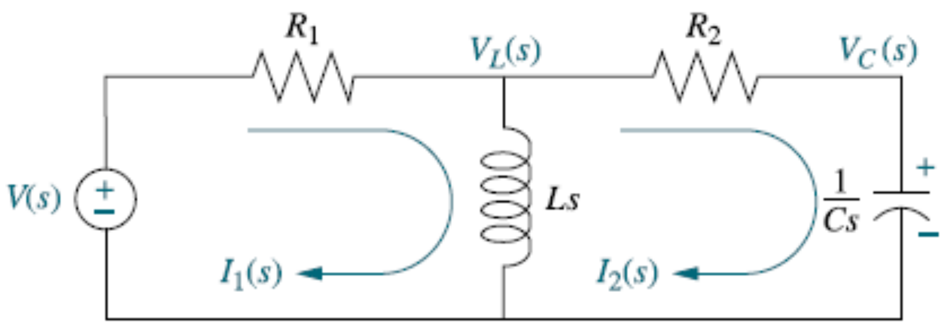
1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function

Complex Circuits via Mesh Analysis

Find the transfer function $I_2(s)/V(s)$ for the circuit given below.



The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.



Complex Circuits via Mesh Analysis

Find the two simultaneous equations for the transfer function by summing voltages around each mesh.

For mesh 1

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \dots 1$$

For mesh 2

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \dots 2$$

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.

Complex Circuits via Mesh Analysis

Combining terms, equation 1 & 2 become simultaneous equat

$$(R_1+Ls)I_1(s) \qquad - LsI_2(s) = V(s) \ldots 3$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2(s) = 0 \ldots 4$$

$$\Delta = \begin{vmatrix} (R_1+Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve equation (3 & 4) for $I_2(s)$. Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

Complex Circuits via Mesh Analysis

Where

$$\Delta = \begin{vmatrix} (R_1+Ls) & -Ls \\ -Ls & \left(Ls+R_2+\frac{1}{Cs}\right) \end{vmatrix}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Therefore, the transfer function $I_2(s)/V(s)$ is

$$G(s) = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Translational Mechanical System: Transfer Function

Mechanical systems are analogous to electrical networks. They mechanical systems also have passive components, energy storage components etc.

The symbols and units are:

f(t): force, N (newtons),

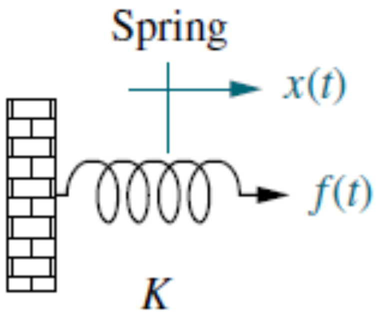
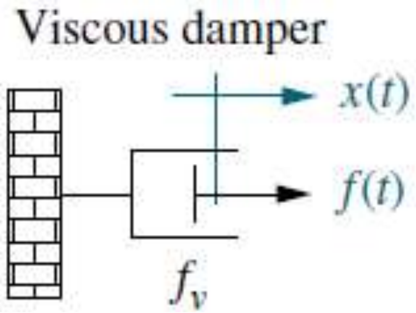
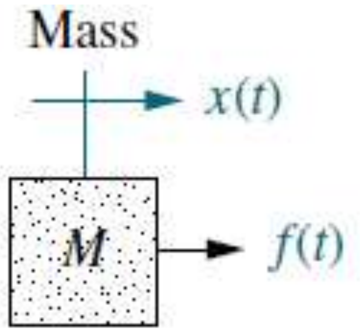
x(t): displacement, m (meters), v(t): velacity m/s (meters/second),

K: spring constant, N/m (newtons/meter),

f_v: coefficient of viscous friction, N-s/m (newton-seconds/meter),

M: Mass, kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System: Transfer Function

Component	Force-velocity	Force-displacement	Impedance ZM(s) =F(s)/X(s)
	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = K\mathrm{x}(\mathrm{t})$	K
	$f(t) = fv\, v(t)$	$f(t) = fv\,\frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Translational Mechanical System: Transfer Function

Component	**Voltage Current**	**Component**	**Force velocity**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	Spring	$f(t) = K\int_0^t v(\tau)d\tau$
Resistor	$v(t) = Ri(t)$	Viscous Damper	$f(t) = f v\, v(t)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	Mass	$f(t) = M\frac{dv(t)}{dt}$

The mechanical **force** is analogous to electrical **voltage** and mechanical **velocity** is analogous to electrical **current**.

The **spring** is analogous to the **capacitor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **inductor**.

Translational Mechanical System: Transfer Function

Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations.

If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations.

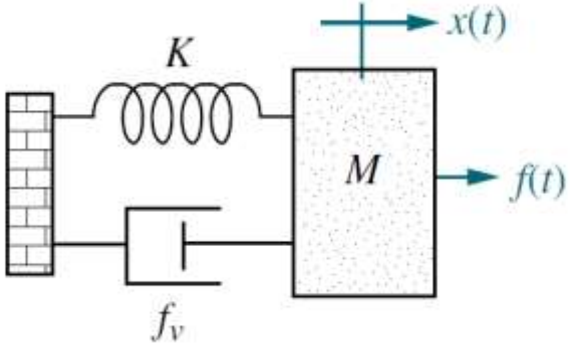
Translational Mechanical System: Transfer Function

Component	Current voltage	Component	Force velocity
Capacitor	$i(t) = C\dfrac{dv(t)}{dt}$	Mass	$f(t) = M\dfrac{dv(t)}{dt}$
Resistor	$i(t) = \dfrac{1}{R}v(t)$	Viscous Damper	$f(t) = fv\, v(t)$
Inductor	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	Spring	$f(t) = K\displaystyle\int_0^t v(\tau)d\tau$

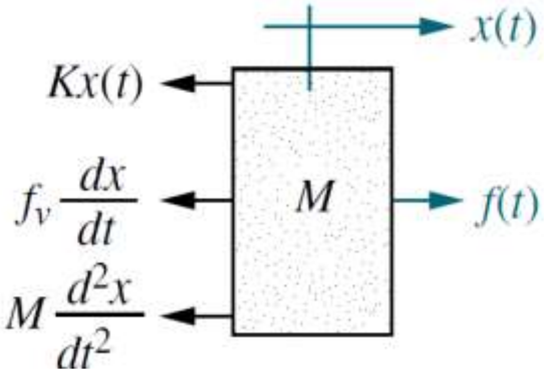
Here the **analogy** is between **force** and **current** and between **velocity** and **voltage**. The **spring** is analogous to the **inductor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **capacitor**. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical **differential equations** are analogous to **nodal** equations.

Translational Mechanical System: Transfer Function

Find the transfer function, X(s)/F(s) for the system as shown



→ draw the free body diagram



Place all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

Write the differential equation of motion using Newton's law to sum of all the forces on the mass is zero

Translational Mechanical System: Transfer Function

$$M\frac{d^2x(t)}{dt^2} + f v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f v\, sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f v\, s + K)X(s) = F(s)$$

Hence the transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + f v\, s + K)}$$

Translational Mechanical System: Transfer Function

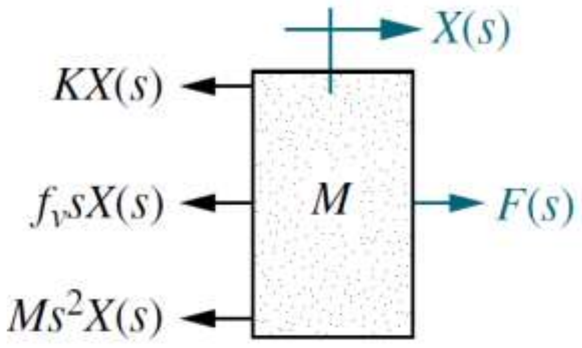
Component	**Force-displacement**	**LT of Force-displacement**	**Impedance $Z_M(s) = F(s)/X(s)$**
Spring	$f(t) = Kx(t)$	$F(s) = KX(s)$	K
Viscous Damper	$f(t) = fv\dfrac{dx(t)}{dt}$	$F(s) = fv\,sX(s)$	$f_v s$
Mass	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	$F(s) = Ms^2X(s)$	Ms^2

Last column is for the impedance of mechanical components.

Replacing each force in free body diagram by its Laplace transform, which is in the format

$F(s) = Z_M(s)X(s)$

Translational Mechanical System: Transfer Function



we obtain the figure as shown, from which we could have obtained the equation immediately without writing the differential equation.

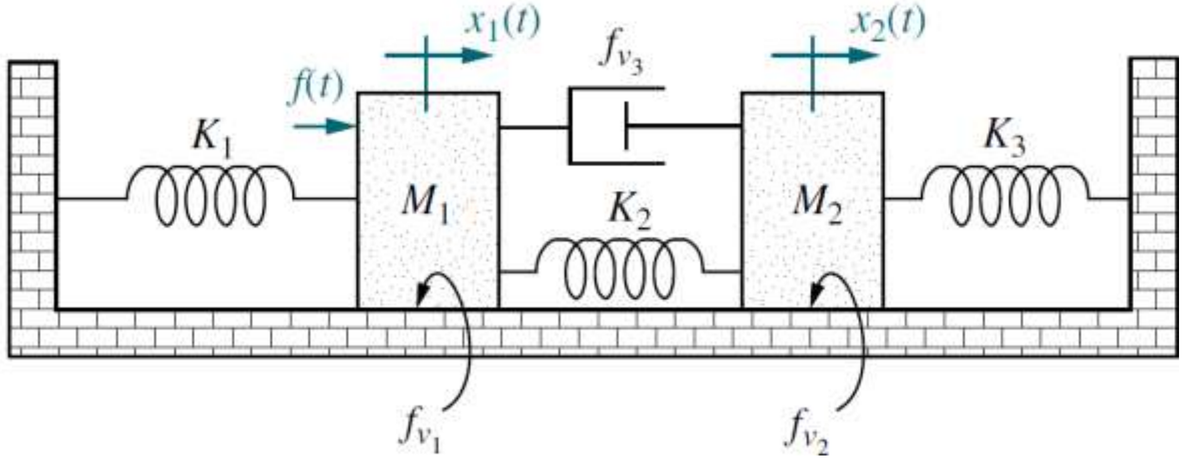
$$Ms^2X(s) + fv\,sX(s) + KX(s) = F(s)$$

And the equation $(Ms^2 + fv\,s + K)X(s) = F(s)$ is of the form of

{Sum of impedances} X(s) = {Sum of applied force}

Translational Mechanical System: Transfer Function

Find the transfer function, X2(s)/F(s), for the system as shown Figure 1.



The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.

Thus, two simultaneous equations of motion will be required to describe the system.

The two equations come from free-body diagrams of each mass.

Translational Mechanical System: Transfer Function

Forces acting on mass M_1

only due to motion of M_1

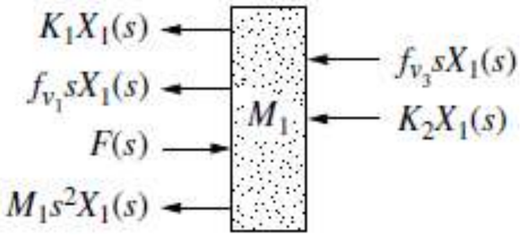


figure 2 a

Forces acting on mass M_1

only due to motion of M_2

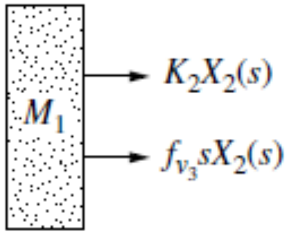


figure 2 b

All forces acting on mass M_1

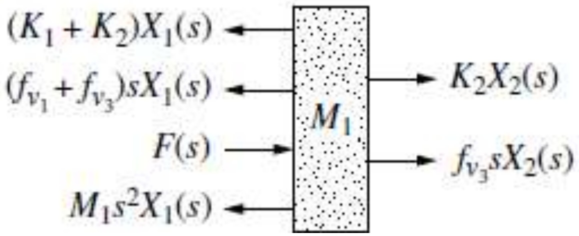


figure 2 c

Translational Mechanical System: Transfer Function

Forces acting on mass M_2

only due to motion of M_2

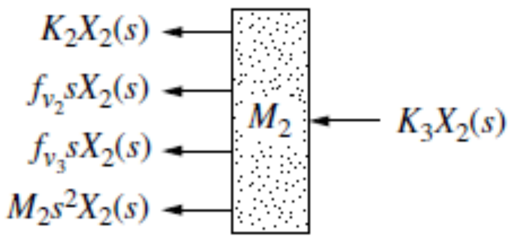


figure 3 a

Forces acting on mass M_2

only due to motion of M_1

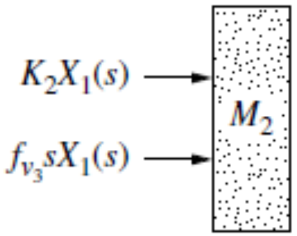


figure 3 b

All forces acting on mass M_2

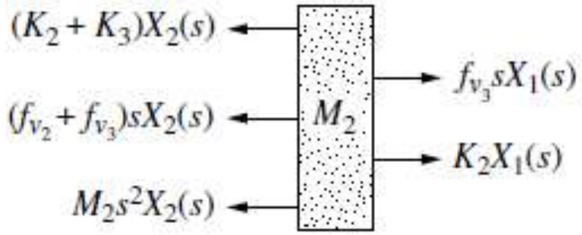


figure 3 c

Translational Mechanical System: Transfer Function

The Laplace transform of the equations of motion can now be written from figures 2 (c) and 3(c)

$$[M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) \qquad - (f_{v3}s + K_2)X_2(s) = F(s)$$

$$-(f_{v3}s + K_2)X_{1(s)} + [M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

From this, $X_2(s)$is

$$X_2(s) = \frac{\begin{vmatrix} [M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & F(s) \\ -(f_{v3}s + K_2) & 0 \end{vmatrix}}{\Delta}$$

Translational Mechanical System: Transfer Function

and the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = \frac{(f_{v3}s + K_2)}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & -f_{v3}s + K_2 \\ -f_{v3}s + K_2 & M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3) \end{vmatrix}$$