Feedback Control System Unit 1_2
Transfer function of physical systems

The Transfer Function

Consider a n-th-order, linear, time-invariant continuous time system describe by the differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the a_n, $a_{n-1}, \ldots$, a_0 and b_m, b_{m-1}, …, b_0 are the coefficients. Taking the Laplace transform of both sides,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) =$$
$$b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

If we assume that all initial conditions are zero, then

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

The Transfer Function

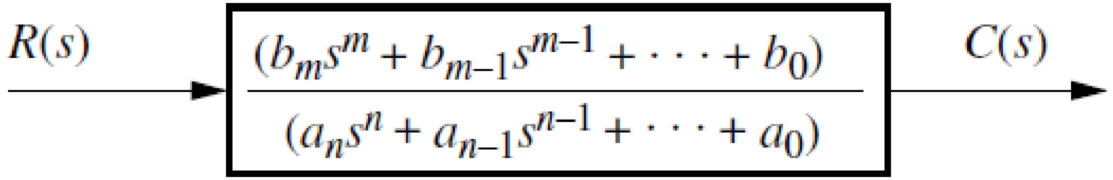
The ratio of the output transform, C(s), divided by the input transform, R(s) is:

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

The transfer function G(s) is defined as the ratio of Laplace Transform of output C(s) to the Laplace Transform of input R(s) with zero initial conditions

The transfer function can be represented as a block diagram, as shown in Figure, with the input on the left, the output on the right, and the system transfer function inside the block.

Also, we can find the output, C(s) by using C(s) = R(s)G(s)



The Transfer Function

Find the transfer function of a system represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s+2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

The Transfer Function

Find step response of the linear time invariant system described by the differential equation

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial

$$sC(s) + 2C(s) = R(s)$$

$$(s + 2)C(s) = R(s)$$

The transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s + 2}$$

Now the we know C(s) = R(s) G(s)

The Transfer Function

Now the we know C(s) = R(s) G(s)

r(t) = u(t) therefore R(s) = 1/s

$$C(s) = \frac{1}{s}\frac{1}{s+2}$$

Using partial fraction

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Electrical Network Transfer Functions

- the mathematical modeling of electric circuits
- three passive linear components: resistors, capacitors, inductors, and OPAMP.
- combine electrical components into circuits, decide on the input and output, and find the transfer function.
- Our guiding principles are Kirchhoff's laws.
- Sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the
- result to zero.
- From these relationships write the differential equations for the circuit.
- Then find the Laplace transforms of the differential equations and finally solve for the transfer function

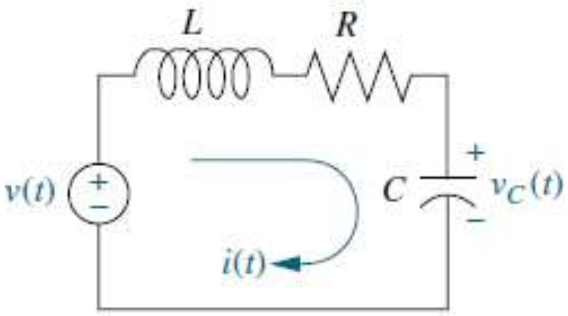
Electrical Network Transfer Functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance Z(s)=V(s)/I(s)	Admittance Y(s)=I(s)/V(s)
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical Network : RLC Circuit (Mesh Analysis)

Transfer functions can be obtained using **Kirchhoff's voltage law** and summing voltages around loops or meshes

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s).



In any problem, the designer must first decide what the input and output should be.

In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current.

In this problem it is stated as statement the capacitor voltage is the output the applied voltage as the input.

Electrical Network : RLC Circuit (Mesh Analysis)

Summing the voltages around the loop, assuming zero initial conditions, we get

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \ldots 1$$

Changing variables from current to charge using $i(t) = dq(t)/dt$

$$\therefore L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \ldots 2$$

From the voltage-charge relationship for a capacitor from the Table $q(t) = Cv_C(t)$

$$\therefore LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \ldots 3$$

Electrical Network : RLC Circuit (Mesh Analysis)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \ldots 4$$

Hence the transfer function is

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \ldots 5$$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

First, take the Laplace transform of the equations in the voltage-current assuming zero initial conditions.

Component	**Voltage current**	**LT of voltage current**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$V(s) = \frac{1}{Cs}I(s)$
Resistor	$v(t) = Ri(t)$	$V(s) = RI(s)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$V(s) = LsI(s)$

Now define the transfer function $\frac{V(s)}{I(s)} = Z(s)$

Electrical Network : RLC Circuit (Mesh Analysis)

Alternative:

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current.

But this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function impedance. The impedance for each of the electrical elements is

Component	**Impedance Z(s)=V(s)/I(s)**
Capacitor	$\frac{1}{Cs}$
Resistor	R
Inductor	Ls

Electrical Network : RLC Circuit (Mesh Analysis)

Let us use the concept of impedance for simplified solution

for the transfer function. The Laplace transform of $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$,

assuming zero initial conditions, is

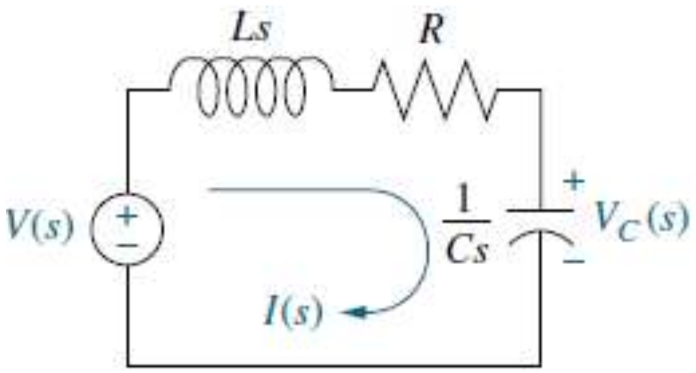
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \ldots 6$$

This equation is in the form

(Sum of impedances) I(s) = (Sum of applied voltages)

Electrical Network : RLC Circuit (Mesh Analysis)

From equation 6 we can have the series circuit as shown. This circuit could have been obtained immediately from the original RLC circuit simply by replacing each element with its impedance. Let us call this altered circuit the transformed circuit.



From the transformed circuit we can write Eq. (6) immediately, if we add impedances in series as we add resistors in series.

Electrical Network : RLC Circuit (Mesh Analysis)

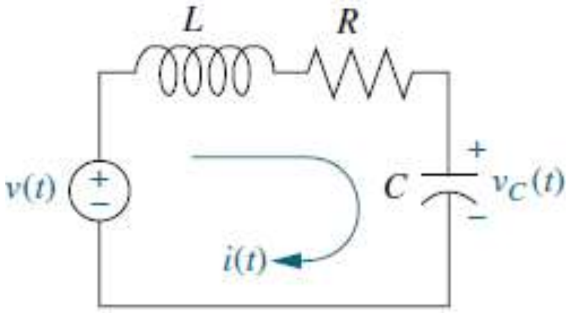
So instead of writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit.

We conclude this discussion as follows

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

Electrical Network : RLC Circuit (Mesh Analysis)

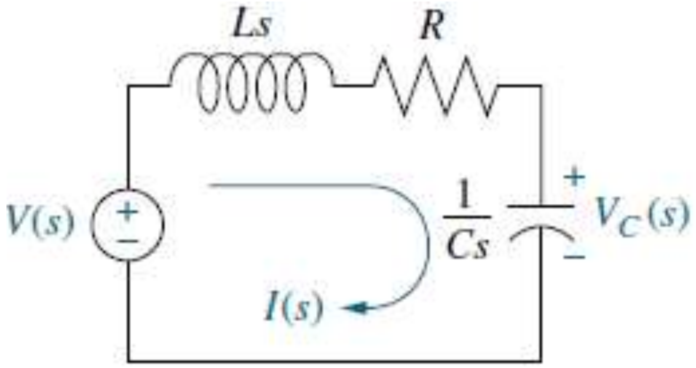
Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s)



by transform method.

Steps

1. Draw the transform network



Electrical Network : RLC Circuit (Mesh Analysis)

2. Write the equation

$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$ solving for I(s)/V(s)

$$\frac{I(s)}{V(s)} = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor.

$$V_{C(s)} = I(s)\frac{1}{Cs} = \frac{1}{Cs}\frac{V(s)}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{V(s)}{(LCs^2 + RCs + 1)} = \frac{(1/LC)V(s)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Electrical Network : RLC Circuit (Mesh Analysis)

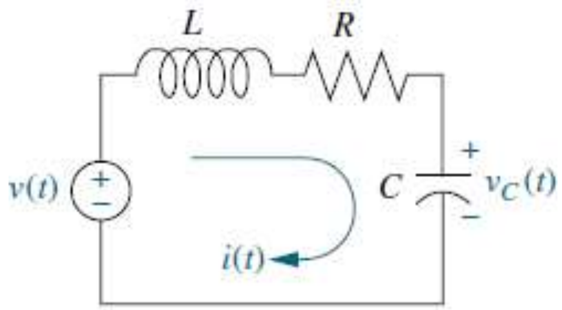
$$\frac{V_{C(s)}}{V(s)} = \frac{(1/LC)}{\left(s^2 + \left(\frac{R}{L}\right)Cs + (1/LC)\right)}$$

Same as equation 5.

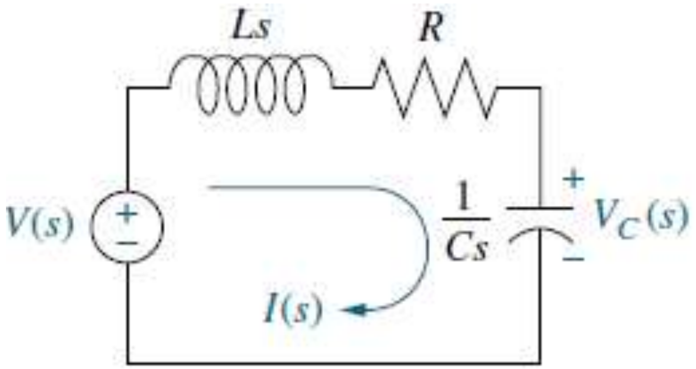
Electrical Network : RLC Circuit (Nodal Analysis)

Transfer functions also can be obtained using **Kirchhoff's current law** and summing currents flowing from nodes. This is nodal analysis method.

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by nodal analysis method and without writing a differential equation i.e. using transform method



transform to



Electrical Network : RLC Circuit (Nodal Analysis)

The transfer function is obtained by summing currents flowing out of the node whose voltage is VC(s) transformed figure with assumption that currents leaving the node are positive and currents entering the node are negative.

The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. Therefore for each I(s) = V(s)/Z(s). Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \ldots 1$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls$ is the current flowing out of the node through the series resistor and inductor. Solve eq. (1) for the transfer function, $V_C(s)/V(s)$

Electrical Network : RLC Circuit (Nodal Analysis)

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = \frac{V_C(s)}{1/Cs} + \frac{V_C(s)}{R + Ls} - \frac{V(s)}{R + Ls} = 0$$

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s)}{R + Ls} = \frac{V(s)}{R + Ls}; \rightarrow \frac{(R + Ls)V_C(s)}{1/Cs} + VC(s) = V(s);$$

$$V_C(s)\left(1 + \frac{(R + Ls)}{1/Cs}\right) = V(s); \rightarrow V_C(s)\left(1 + \frac{(R + Ls)}{1/Cs}\right) = V(s)$$

$$V_C(s)\left(\frac{\frac{1}{Cs} + R + Ls}{1/Cs}\right) = V(s); \rightarrow \frac{V_C(s)}{V(s)} = \frac{1/Cs}{\frac{1}{Cs} + R + Ls}$$

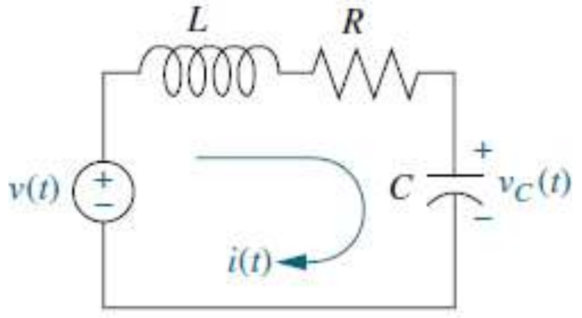
Electrical Network : RLC Circuit (Nodal Analysis)

$$V_C(s)\left(\frac{\frac{1}{Cs}+R+Ls}{1/Cs}\right)=V(s);\rightarrow\frac{V_C(s)}{V(s)}=\frac{1/Cs}{\frac{1}{Cs}+R+Ls}=\frac{1}{LCs^2+RCs+1}$$

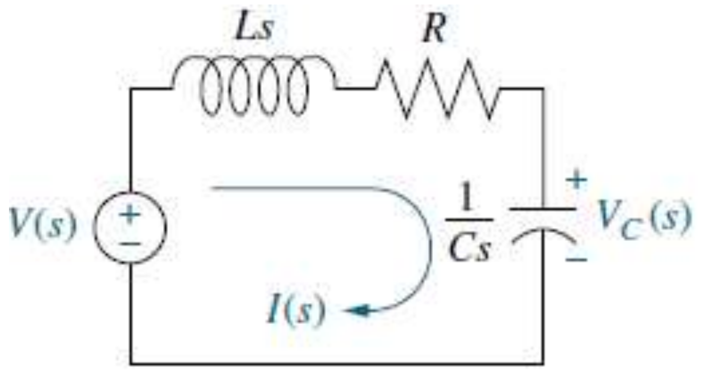
$$=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Electrical Network : RLC Circuit (Voltage Division)

Find the transfer function relating the capacitor voltage VC(s) to the input voltage V(s) by votage division method



transformed to



The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus

$$V_C(s) = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} V(s)$$

Solve this equation for the transfer function, $V_C(s)\ /V(s)$

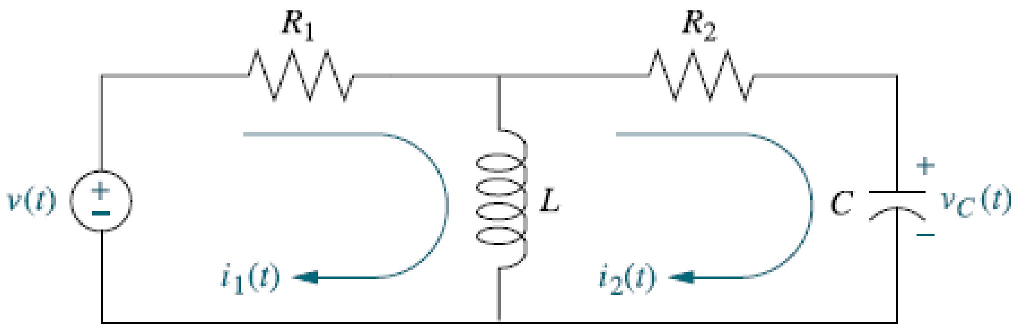
Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

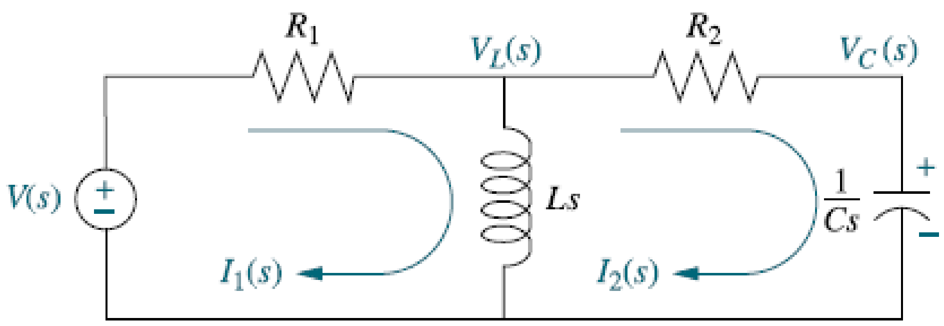
1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh
4. Write Kirchhoff’s voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function

Complex Circuits via Mesh Analysis

Find the transfer function $I_2(s)/V(s)$ for the circuit given below.



The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.



Complex Circuits via Mesh Analysis

Find the two simultaneous equations for the transfer function by summing voltages around each mesh.

For mesh 1

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \ldots 1$$

For mesh 2

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \ldots 2$$

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.

Complex Circuits via Mesh Analysis

Combining terms, equation 1 & 2 become simultaneous equat

$$(R_1+Ls)I_1(s) \qquad - LsI_2(s) = V(s) \ldots 3$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \ldots 4$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve equation (3 & 4) for $I_2(s)$. Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1+Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

Complex Circuits via Mesh Analysis

Where

$$\Delta = \begin{vmatrix} (R_1+Ls) & - \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Therefore, the transfer function $I_2(s)/V(s)$ is

$$G(s) = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Translational Mechanical System: Transfer Function

Mechanical systems are analogous to electrical networks. They mechanical systems also have passive components, energy storage components etc.

The symbols and units are:

$f(t)$: force, N (newtons),

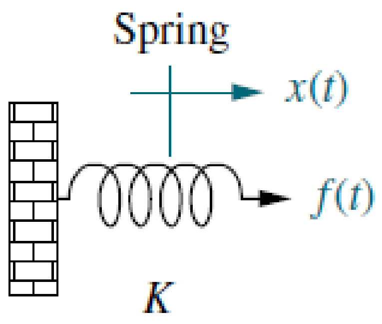
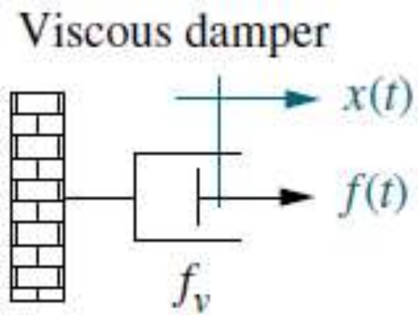
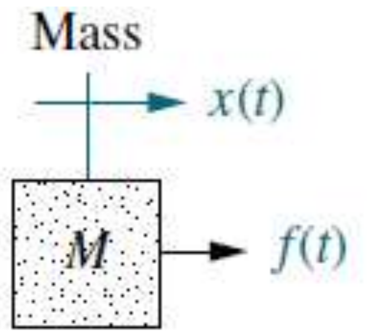
$x(t)$: displacement, m (meters), $v(t)$: velacity m/s (meters/second),

K: spring constant, N/m (newtons/meter),

f_v: coefficient of viscous friction, N-s/m (newton-seconds/meter),

M: Mass, kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System: Transfer Function

Component	Force-velocity	Force-displacement	Impedance ZM(s) =F(s)/X(s)
	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = K\text{x}(\text{t})$	K
	$f(t) = fv\, v(t)$	$f(t) = fv\,\frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Translational Mechanical System: Transfer Function

Component	**Voltage Current**	**Component**	**Force velocity**
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	Spring	$f(t) = K\int_0^t v(\tau)d\tau$
Resistor	$v(t) = Ri(t)$	Viscous Damper	$f(t) = fv\, v(t)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	Mass	$f(t) = M\frac{dv(t)}{dt}$

The mechanical **force** is analogous to electrical **voltage** and mechanical **velocity** is analogous to electrical **current**.

The **spring** is analogous to the **capacitor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **inductor**.

Translational Mechanical System: Transfer Function

Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations.

If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations.

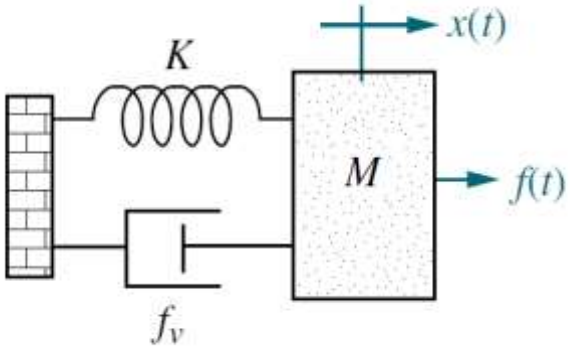
Translational Mechanical System: Transfer Function

Component	Current voltage	Component	Force velocity
Capacitor	$i(t) = C\dfrac{dv(t)}{dt}$	Mass	$f(t) = M\dfrac{dv(t)}{dt}$
Resistor	$i(t) = \dfrac{1}{R}v(t)$	Viscous Damper	$f(t) = fv\,v(t)$
Inductor	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	Spring	$f(t) = K\displaystyle\int_0^t v(\tau)d\tau$

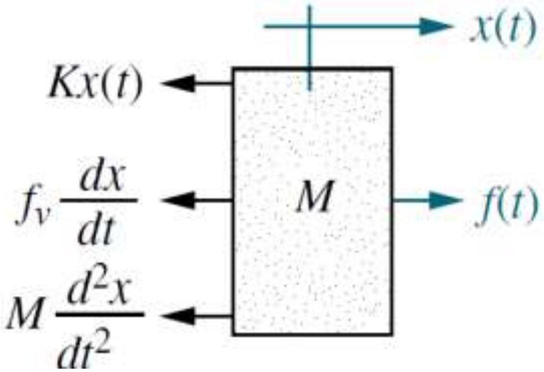
Here the **analogy** is between **force** and **current** and between **velocity** and **voltage**. The **spring** is analogous to the **inductor**, the **viscous damper** is analogous to the **resistor**, and the **mass** is analogous to the **capacitor**. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical **differential equations** are analogous to **nodal** equations.

Translational Mechanical System: Transfer Function

Find the transfer function, X(s)/F(s) for the system as shown



→ draw the free body diagram



Place all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

Write the differential equation of motion using Newton's law to sum of all the forces on the mass is zero

Translational Mechanical System: Transfer Function

$$M\frac{d^2x(t)}{dt^2} + f v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f v\, sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f v\, s + K)X(s) = F(s)$$

Hence the transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + f v\, s + K)}$$

Translational Mechanical System: Transfer Function

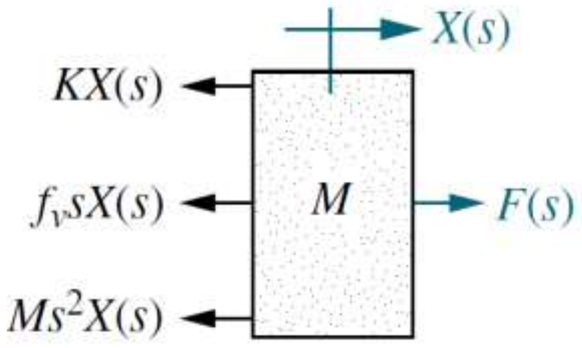
Component	Force-displacement	LT of Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring	$f(t) = K\mathrm{x}(\mathrm{t})$	$F(s) = K\mathrm{X}(\mathrm{s})$	K
Viscous Damper	$f(t) = fv\,\frac{dx(t)}{dt}$	$F(s) = fv\,sX(s)$	$f_v s$
Mass	$f(t) = M\frac{d^2x(t)}{dt^2}$	$F(s) = Ms^2X(s)$	Ms^2

Last column is for the impedance of mechanical components.

Replacing each force in free body diagram by its Laplace transform, which is in the format

$F(s) = Z_M(s)X(s)$

Translational Mechanical System: Transfer Function



we obtain the figure as shown, from which we could have obtained the equation immediately without writing the differential equation.

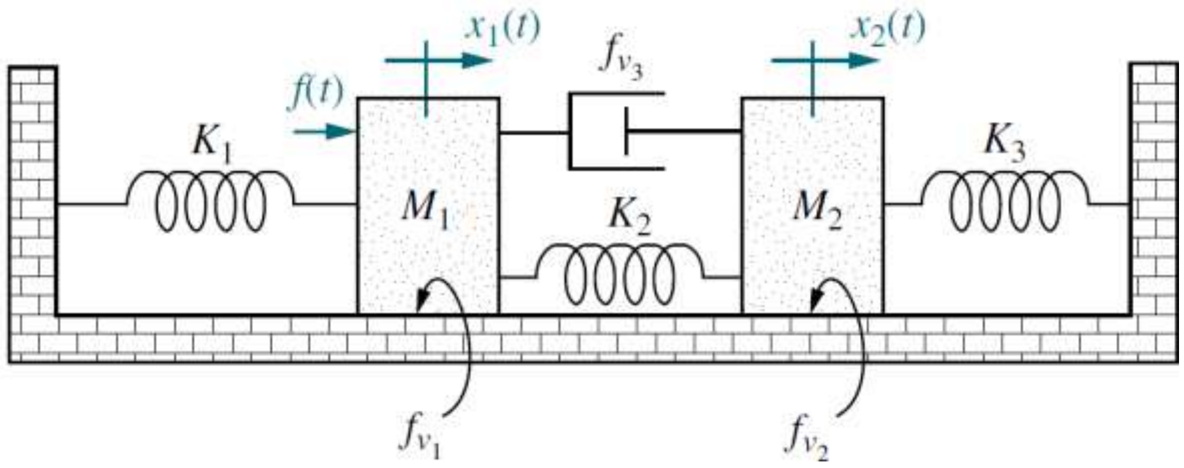
$$Ms^2X(s) + fv\, sX(s) + KX(s) = F(s)$$

And the equation $(Ms^2 + fv\, s + K)X(s) = F(s)$ is of the form of

{Sum of impedances} X(s) = {Sum of applied force}

Translational Mechanical System: Transfer Function

Find the transfer function, X2(s)/F(s), for the system as shown Figure 1.



The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still.

Thus, two simultaneous equations of motion will be required to describe the system.

The two equations come from free-body diagrams of each mass.

Translational Mechanical System: Transfer Function

Forces acting on mass M_1

only due to motion of M_1

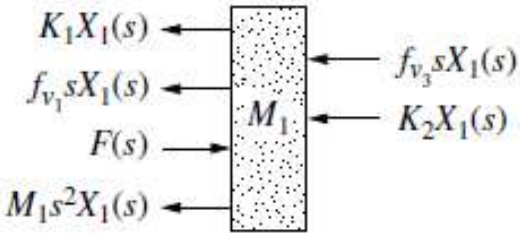


figure 2 a

Forces acting on mass M_1

only due to motion of M_2

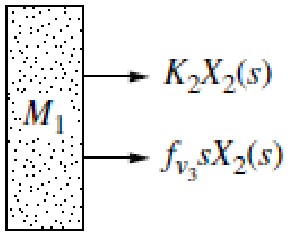


figure 2 b

All forces acting on mass M_1

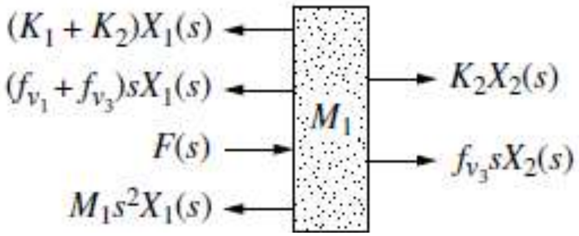


figure 2 c

Translational Mechanical System: Transfer Function

Forces acting on mass M_2

only due to motion of M_2

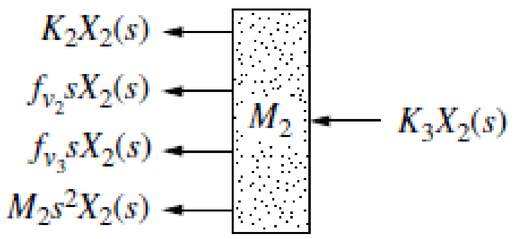


figure 3 a

Forces acting on mass M_2

only due to motion of M_1

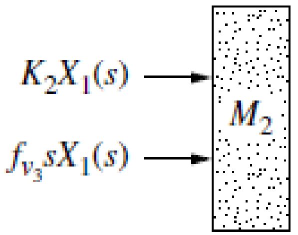


figure 3 b

All forces acting on mass M_2

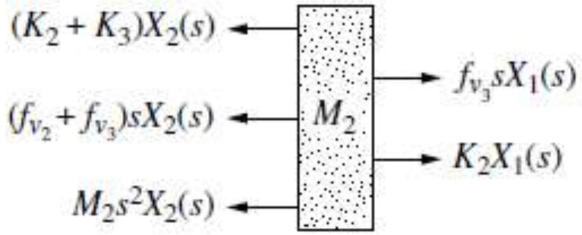


figure 3 c

Translational Mechanical System: Transfer Function

The Laplace transform of the equations of motion can now be written from figures 2 (c) and 3(c)

$$[M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) \qquad - (f_{v3}s + K_2)X_2(s) = F(s)$$

$$-(f_{v3}s + K_2)X_{1(s)} + [M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

From this, $X_2(s)$is

$$X_2(s) = \frac{\begin{vmatrix} [M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & F(s) \\ -(f_{v3}s + K_2) & 0 \end{vmatrix}}{\Delta}$$

Translational Mechanical System: Transfer Function

and the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = \frac{(f_{v3}s + K_2)}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} M_1 s^2 (f_{v1} + f_{v3})s + (K_1 + K_2) & -f_{v3}s + K_2 \\ -f_{v3}s + K_2 & M_2 s^2 (f_{v2} + f_{v3})s + (K_2 + K_3) \end{vmatrix}$$