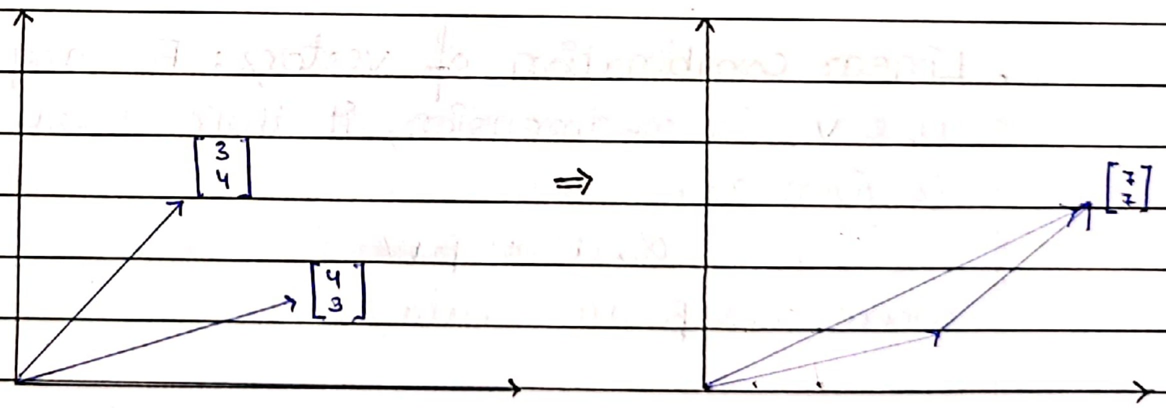
Vectors & Matrices

- Scalar: A magnitude, without any direction.
- Vector: Magnitude along w/ direction. Doesn't necessarily start from origin.

- Vector x Scalar / Stretching: When a vector is multiplied w/ a scalar value, it either gets stretched or shrinks, the direction however remains same when multiplying w/ a scalar $\alpha \geq 0$, & direction reverses w/ a scalar $\beta < 0$.

- Vector Addition: Two vectors of similar dimension can be added, by component-wise addition.

Ex.



- Dot Product: For two vectors, say u & v, of similar dimensions, there dot product is given as:

$$u \cdot v = u_1 \times v_1 + u_2 \times v_2 + \dots + u_n \times v_n$$

This operation results in a scalar.

Dot product can also be represented using matrix multiplication:

$$u \cdot v = u^T \times v$$

Dot product of vectors is a commutative operation.

- **Length of Vector:** For a vector u, $[u_1\ u_2\ u_3 \dots u_n]^T$, it's length is given by:

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots u_n^2}$$

A vector of length 1, in any direction is called as unit vector.
For a given vector u, to find the unit vector, we just need to multiply the vector u by its 1/length

$$\underbrace{\hat{u}}_{\text{unit vector}} = \underbrace{\frac{1}{\|u\|}}_{\text{scalar}} \times \underbrace{u}_{\text{vector}}$$

- **Linear combination of vectors:** For any two vectors u & v, in n-dimension, their linear combination is given as:

$$\alpha \times u + \beta \times v$$

where α & β are scalars.

- **Linearly Dependent Vectors:** ~~Two vectors, u & v are~~
A set of vectors, say, $\{u_1, u_2, u_3 \dots u_K\}$ each of n-dimension, are said to be linearly dependent iff "at least one" u_i, $1 \le i \le K$, can be represented as a linear combination of other vectors in the set. i.e.

$$u_i = \sum_{j=1,\, j\neq i}^{K} \underbrace{\alpha_j}_{\text{scalar}} \times u_j$$

$$\{\alpha_j \in \mathbb{R}\}$$

A set containing zero-vector is always linearly dependent, since there is at least one linear combination of all 0s which give zero-vector.

Ex. A superset of linearly dependent vectors is always linearly dependent. True / False?
True

Alternate Def^n of Linear Dependence: For a given set of vectors, iff we can obtain zero vector by a non-trivial combination (at least one α is non-zero) of vectors of the set, i.e.

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_k u_k = \underbrace{0}_{\text{zero vector}}$$

then the set of vectors is linearly dependent.
So, iff there is no non-trivial linear combination of vectors of the set which yields zero vector then the set is linearly independent.

Ex. Can there be more than two linearly independent vectors in a set of vectors of $\mathbb{R}^2$?
No. There can be atmost two linearly independent vectors in set of vectors of $\mathbb{R}^2$. Adding any third vector to the set makes it linearly dependent.
(We'll reason this later.)
For example $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ can produce all the vectors in $\mathbb{R}^2$, so we can say that this set covers $\mathbb{R}^2$.
There similarly are infinitely many such sets of two lin. indp. vectors in $\mathbb{R}^2$, which fill / cover $\mathbb{R}^2$.

Similarly, in $\mathbb{R}^n$, there can't be more than n linearly independent vectors.

Conversly, n or more vectors in $\mathbb{R}^n$ can fill the vector space $\mathbb{R}^n$, contingent, there are exactly n linearly independent vectors in the set.

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* **Matrix:** We'll interpret an $m \times n$ matrix as a collection of 'n' vectors in $\mathbb{R}^m$.
column

- **Column Picture of Matrix Multiplication**
In multiplication of $A \times B$ where A is $(l \times m)$ & B is $(m \times n)$, for each i^{th} column of $(A \times B)$ we'll just take linear combination of column vectors of A, using i^{th} column of B as coeff.

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 8 & 11 \\ -1 & 0 & -1 \end{bmatrix}$

* $Ax = b$

Whenever you see this formulation, the first thought should be that 'b' is a linear combination of column vectors of A, with coeff from x.

So, if b is not a linear combination of cols of A, then there can never be an x which satisfies $Ax = b$.

- There can be two pictures for $Ax = b$. Say,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Column Picture:

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Row Picture (Eq^n form)

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

- So, we can write the defⁿ of Linear Dependence as:

The column vectors of the matrix A are linearly dependent iff there exists a non-trivial X, such that:

$$A X = \vec{0}$$

- For a given system $Ax = b$, there can be exactly one of three cases:

(1) Unique solution

(2) Infinite solutions

(3) No solution

Ex. Suppose you have an $A_{3\times4}$ matrix w/ 3 linearly indp. vectors in $\mathbb{R}^3$. What can you say about the solⁿ of $Ax=b$ $(b \neq 0)$.

Since there are three linearly indp. vectors in $\mathbb{R}^3$ in A, then there would always be a solⁿ for any b, since the L.I. vectors would fill $\mathbb{R}^3$.

Ex. Same as above, but $A_{5\times4}$ w/ 3 L.I vectors in $\mathbb{R}^5$.

The 3 L.I. vectors in A can't cover $\mathbb{R}^5$, so we may or may not have solⁿ for a given b. If b is a linear combination of columns of A, then yes solⁿ exists, else no.

Ex. Same as above, $A_{2\times5}$, w/ 3 lin. indp. cols.
Not possible. In $\mathbb{R}^2$ at most 2 L.I cols can be there.

Ex. Consider the matrix $A_{m\times n}$. For a system $Ax = b$, what can you say when:

① $m < n$

n vectors in $\mathbb{R}^m$. May/may not have a sol^n.

- $Ax = \vec{0}$ will always have sol^n as more than 'm' vectors in $\mathbb{R}^m$ are allways L.Dep.
- For $b \neq \vec{0}$, if there are exactly m L.I. cols in A, then for any such b, sol^n exists as the L.I. cols of A cover $\mathbb{R}^m$ vector space.
- If there are $< m$ L.I. columns in A, then sol^n exists iff $b (\neq \vec{0})$ ~~belong~~ is a linear combination of the L.I. columns.

② If $m < n$, then there exist a system $Ax = b (b \neq \vec{0})$ which has sol^n. T/F

True. At least one L.I col. would be there, so we can simply have b multiple of that col.

③ $m > n$

- For $Ax = \vec{0}$ may have sol^n iff the cols are L.D.
- For $Ax = b (b \neq \vec{0})$ sol^n exists iff b is a linear combination of cols of A.

We already know that if b is not a linear combination of columns of A, $Ax=b$ has no sol.
Now we want to find the cond.^{n}s when the system will have unique & when it'll have infinite sol^n.

Firstly, b must be linear combination of A to have sol^n for $Ax=b$.

Now if the cols of A are L.D., then we'll have ∞ sol^n. because we can take different proportions of cols of A & still get b.

If cols of $A_{m\times n}$ are L.I. (possible only when $m \leq n$), then there must be a unique sol^n.

Proof: Suppose $A_{m\times n}$ has L.I. cols. & b is a linear combination of cols of A.
Say, there exist at least two such vectors $x_{n\times 1}$, c & d, such that $Ac = Ad = b$.
So,

$$b = \vec{a}_1 \times c_1 + \vec{a}_2 \times c_2 + \dots + \vec{a}_n \times c_n$$
$$b = \vec{a}_1 \times d_1 + \vec{a}_2 \times d_2 + \dots + \vec{a}_n \times d_n$$

Taking diff:

$$\vec{0} = \vec{a}_1 \times (c_1 - d_1) + \vec{a}_2 (c_2 - d_2) + \dots + \vec{a}_n (c_n - d_n)$$

But RHS can't be 0 since $\vec{c}$ must differ by at least one position from $\vec{d}$.
$\therefore$ Contradiction.

Sol^n of $Ax = b$

Is b a linear comb. of cols of A

yes	No
Sol^n exists	There is no sol^n

Are cols of A independent?

yes	No
Unique sol^n	Infinite sol^n

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Solving System of Linear Eqⁿ

* Gaussian Elemination

- Echelon Form: A matrix is in Echelon form iff:
  (1) All non-zero rows are above any rows of all zeroes.
  (2) All entries in a column below a ~~zero~~ leading entry (nonzero) are zero.
  (3) The leading entry of any row occurs to the right of the leading entry of row above it.

- A system of eqⁿs can be represented as:

say.

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$ $\Rightarrow$ $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$
$a_3x + b_3y + c_3z = d_3$

So each column corresponds to one variable, x, y, z,...etc.

- In Echelon form, variable assoc. w/ columns which have leading non zero entry are called ~~free~~ Pivot variables. Rest are free variables.

Ex. $\begin{array}{c} \begin{bmatrix} 0 & \boxed{1} & 2 & 4 & 6 \\ 0 & 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} x & y & z & w & v \end{array} \end{array}$

So,
Pivot var: y, w, v
Free var: x, z

- Augmented Matrix The system $Ax = b$ can be represented as $[A \vdots b]$.

- Elementry Row opⁿ: There are three types of elementary row opⁿs which may be performed on the rows of matrix:
  - Swap the positions of two rows.
  - Multiply a row w/ non-zero scalar.
  - Add to one row a scalar multiple of other.

Ex. Convert the following to Echelon form:

(1)
$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} \boxed{1} & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{array}{cccc} \begin{bmatrix} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \quad v_4 \end{array}$$

So we can say, v_1 & v_2 are Pivot vars.

(2)
$$\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} \boxed{3} & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2 \times \frac{3}{2}} \begin{array}{c} \begin{bmatrix} \boxed{3} & 1 & 1 \\ 0 & 0 & \boxed{2} \\ 0 & 0 & 0 \end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \end{array}$$

Pivot vars: v_1 & v_3.

- **Row Reduced Echelon Form:** A matrix in Echelon form w/ :
  (1) Leading non-zero value of a row should be 1.
  (2) Zero above & below the pivot element (leading non-zero entries).

Ex. In Echelon Form : (2 ex back)

$$\begin{bmatrix} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2/-2} \begin{bmatrix} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow R_1 \leftarrow R_1 - 3R_2$$

$$\begin{bmatrix} \boxed{1} & 0 & -2 & -3 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can easily observe that col 3 & col 4 (dependent cols) are linear combination of col 1 & col 2 (free cols).

$$\begin{bmatrix} 1 \\ -1 \\ +5 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 1 \\ 35 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (4)\begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$$

Ex. Convert to RREF :

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\downarrow R_2 \leftarrow R_2/2$$

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xleftarrow[-(R_2+R_3)]{R_1 \leftarrow R_1} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Ex.

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. write V_2 as lin. comb of V_1 & V_2 & V_3.

(a) $V_2 = 3V_3 + V_4$.

(b) $V_2 = -3V_3 - V_4$

(c) $V_2 = V_4 - 3V_3$

(d) $V_2 = -V_1 + V_4$

Col 1 & 2 are pivot cols.

$V_3 = 2V_1 + 3V_2$

$\Rightarrow V_2 = \frac{-2}{3}V_1 + \frac{1}{3}V_3$

$V_4 = V_1 + V_2$

$\Rightarrow V_2 = -V_1 + V_4$

* Rank.

- Rank is :
  - # independent rows
  - # independent cols.
  - # pivot elements of the matrix in echelon form.
  - # non-zero rows of a matrix in echelon form.
- 0 for only 0-matrix.

Ex. Find Rank.

$$\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} \boxed{3} & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\downarrow R_3 \leftarrow R_3 + \tfrac{3}{2} R_2$$

$$\begin{bmatrix} \boxed{3} & 1 & 1 \\ 0 & 0 & \boxed{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank $= 2$.

* Sol^n to System of Eq^n

- A system of linear eq^{n}s can be converted to Augmented matrix:

$$Ax = b \quad \Rightarrow \quad [A \,|\, b]$$

Ex.
$$5x_1 - 11x_2 = -2$$
$$-4x_1 + 9x_2 = 1$$
$$x_1 - 2x_2 = -1$$

$$\left[\begin{array}{cc|c} 5 & -11 & -2 \\ -4 & 9 & 1 \\ 1 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \boxed{5} & -11 & -2 \\ 0 & 17.8 & -0.6 \\ 0 & 0.2 & -0.6 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} \boxed{5} & -11 & -2 \\ 0 & \boxed{17.8} & -0.6 \\ 0 & 0 & \frac{-264}{445} \end{array}\right]$$

Now, the system is:

$$5x_1 - 11x_2 = -2$$
$$17.8x_2 = -0.6$$
$$0x_2 = \frac{-264}{445}$$

The third eq^n is not possible.

$\therefore$ System has no sol^n.

(Also the b col. vector is not in the column space of cols of A.)

Ex.

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 2x_1 - x_2 - 3x_3 &= 3 \\ 4x_1 + 5x_2 - x_3 &= 7 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

$$\xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 5 & 14 \\ 0 & \boxed{-7} & -13 & -25 \\ 0 & 0 & \boxed{-8} & -24 \end{array}\right]$$

Back Substitution:

$$-8x_3 = -24 \Rightarrow x_3 = 3.$$
$$+7x_2 + 13x_3 = 25 \Rightarrow x_2 = -2.$$
$$x_1 + 3x_2 + 5x_3 = 14 \Rightarrow x_1 = 5$$

$$\therefore X = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

- In case of $A x = \vec{0}$, we always have a trivial solⁿ of all 0s. There maybe other solⁿs as well.

Ex.
$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} \boxed{3} & 5 & -4 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, we get one free variable: x_3. Two pivot vars: x_1 & x_2.
Free vars can take any value.
Say, $x_3 = K$.
Eqⁿs:

$$3x_1 + 5x_2 - 4K = 0$$
$$3x_2 = 0$$

So, $x_2 = 0$ & $x_1 = \frac{4K}{3}$

- Thinking in terms of col vectors:

$$\frac{4K}{3} \times \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + 0 \times \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + K \times \begin{bmatrix} | \\ v_3 \\ | \end{bmatrix} = \vec{0}$$

So, this gives ∞ solⁿs as K can be adjusted.
- No. of LI colvectors is 2: v_1 & v_2. v_3 belongs to the column space of v_1 & v_2.

Ex. $\begin{array}{ccccc} x & y & z & w & \\ \end{array}$ $\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 3 & 0 \\ 0 & 0 & -1 & \boxed{-4} & 0 \end{array}\right]$ (columns x, y, z, w)

So, free vars : y & z pivot vars : x & w.

~~say~~

$\cancel{-}1\times z \cancel{-}4\times w = 0$ $\qquad x + y + 3w = 0$

$w = \frac{z}{4}$ $\qquad x = -\left(\frac{3}{4}z + y\right)$

$\therefore \quad X = \begin{bmatrix} -\frac{3}{4}z + y \\ y \\ z \\ \frac{z}{4} \end{bmatrix}$ We can decouple this by setting free vars as $\Rightarrow$ $\overset{y}{(1},\overset{z}{0)}$ & $\overset{y}{(0},\overset{z}{1)}$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3/4 \\ 0 \\ 1 \\ \frac{1}{4} \end{bmatrix} z$

Now the two obtained vectors are linearly independent. [Look at 2nd & 3rd rows of both vectors (rows corresponding to free vars)].

No. of sols = ∞.

All sols are linear combination of above two vectors.

So, the soln vectors of $Ax = \vec{0}$ also have their own vector space. (Nullspace).

- A system is consistent iff it has ≥ 1 solns. This can be checked by rank(A) == rank([A:b]).

- If a consistent system has atleast one free var, then there are ∞ solns. No. of independent cols in the vector space of solns = #free vars (nullity) = #L.I. solns to the system.

Ex. $A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 2 & 4 & -2 \end{bmatrix}$ $\quad RREF(A) = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$

Find $a+b+c+d$.

So, col 1 & col 2 are pivot cols & col 3 & 4 are free.

$V_3 = a V_1 + c V_2$

$\Rightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} a + 3c \\ 2c \end{bmatrix} \quad \Rightarrow c = 2$

$a = -5$

$V_4 = b V_1 + d V_2$

$\Rightarrow \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} b + 3d \\ 2d \end{bmatrix} \quad \Rightarrow d = -1$

$b = 3$

$\therefore a+b+c+d = -5 + 3 + 2 - 1 = -1$

- Elementary row opns don't change column dependency.

- Rank-Nullity Theorem: Rank + Nullity $= n$ ← no. of vars

  [dimension of null space / no. of L.I. cols in soln vector space of $A x = \vec{0}$ / No. of free vars.]

$$Ax = b$$

Homogeneous System

$b = 0$

(Always has trivial sol^n.
May also have non-triv sol^n)

Hetrogeneous System

$b \neq 0$

(Has sol^n iff b belongs to column space of A)

Ex. A homogeneous system w/ two eq^{n}s & ten unknowns always have ∞ sol^n. T/F?
True. We can have either one or two pivot cols. So, nine or eight free cols. So, always ∞ sol^n.

Ex. A homog. system w/ five eq^{n}s & five unknown always has unique non-trivial sol^n. T/F?
False. The system ~~has~~ can still have free vars.

Ex. Same as above w/ 10 eq^n & 3 unknowns.
False. The system can have free vars.

- Non-homog. system will have no sol^n iff there is a row in $[A \vdots b]$ such as: $[0\ 0 \ldots 0 \vdots \text{non-zero}]$.
This can also be shown by checking if rank (A) is not equal to rank $([A \vdots b])$.

- Non-homog. system has unique sol^n when:
  1. rank(A) = rank$([A \vdots b])$
  2. AND There are no free cols, i.e., rank$(A) = n$ $\{A_{m \times n}\}$
- Non homog system has ∞ sol. when free cols exist & rank(A) = rank$([A \vdots b])$.

Ex. $A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & -2 \\ 2 & 1 & -1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 3 & -2 \\ 0 & \boxed{-1} & -1 & -4 & +4 \end{array}\right]$$

So, free vars $\Rightarrow$ x_3 & x_4.
pivot vars $\Rightarrow$ x_1 & x_2.

Say, $x_3 = s$ $x_4 = t$

$\Rightarrow$ $-x_2 - s - 4t = +4$

$\Rightarrow$ $x_2 = -4 - s - 4t$

$\Rightarrow$ $x_1 + x_2 + 3x_4 = -2$

$\Rightarrow$ $x_1 + (-4) - s - 4t + 3t = -2$

$\Rightarrow$ $x_1 = 2 + s + t$

$$\therefore X = \begin{bmatrix} s+t+2 \\ -4-s-4t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}}_{\{s=t=0\}} + \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\{s=1,\ t=0\}} \times s + \underbrace{\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}_{\{s=0,\ t=1\}} \times t$$

Now, for $Ax = \vec{0}$, we didn't get particular sol^n. But we get Xparticular by putting all free vars as 0.
Hence,

$$X = X_{particular} + X_{nullspace}.$$

This can be seen from:

$$Ax = b + \vec{0}$$

Hence the system has ∞ sol^n.

So, for hetrog. system, the vector/vector space of sol. X is given as:

$$X = X_p + X_{homo.}$$

Determinant, Eigen Value & Vectors

Note: All matrix in this chapter can be assumed as Square, unless stated otherwise.

* Determinant.

- A scalar value associated w/ square matrix, $|A|$ or $\det(A)$.
- Property 1: The determinant of Identity matrix is 1.
- Property 2: For a matrix A if you obtain A' by either single row exchange or single column exchange, the sign of determinant changes, i.e.,

$$\det(A') = -\det(A)$$

- Property 3: Determinat is linear in nature. Linearity is one row (or col.) at a time.

i.e.

(1) $$\begin{vmatrix} at & bt \\ ci & di \end{vmatrix} = t \begin{vmatrix} a & b \\ ci & di \end{vmatrix}$$

(2) $$\begin{vmatrix} a+x & b+y \\ c+z & d+w \end{vmatrix} = \begin{vmatrix} a & b \\ c+z & d+w \end{vmatrix} + \begin{vmatrix} x & y \\ c+z & d+w \end{vmatrix}$$

Ex. $$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2^2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4$$

Ex. $$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab\begin{vmatrix} 1 & b \\ 1 & b \end{vmatrix} = ab\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \Rightarrow ab\left(\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right)$$

$$\Rightarrow ab\left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\right)$$

$$\Rightarrow ab\left(1 + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - 1 - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\right) = 0$$

Also, say, $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = t$. Exchange row 1 & row 2.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = -t$$

$$\Rightarrow 2 \times \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

Ex. $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} + (-l) \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Ex. $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ -a & -b \end{vmatrix} = (-1) \begin{vmatrix} a & b \\ a & b \end{vmatrix} = (-1) \times 0 = 0$

$R_2 \leftarrow R_2 + (-1) R_1$

Ex. $\begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots & 0 \\ 0 & 0 & a_{33} & & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = (a_{11})(a_{22})\ldots(a_{nn}) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$

$= (a_{11})(a_{22})\cdots(a_{nn})$

Ex. $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ & & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$

(Upper Triangular)

By subtracting multiple of rows from each other diag. matrix can be obtained.

So,

$(a_{11})(a_{22})(a_{33})\ldots(a_{nn})$.

- Calculating determinant of 2x2.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + (-1) \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix}$$

$$\Rightarrow ad - cb.$$

- Calculating det for 3x3.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix}$$

So, in total we'll get sum of 27 determinant. Each having exactly one non-zero entry in every row.

Out of the 27 possibilities, any possibility w/ a 0 col. won't survive.

#such possibilities = $27 - 3\times2\times1 = 21$

So, only 6 determinants survive.

$$\Rightarrow \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{vmatrix}$$

$\Rightarrow aei - ahf - bdi + bgf + cdh - cge.$

$\Rightarrow a(ei-hf) - b(di-gf) + c(dh-ge)$

Ex. In det. of a 4x4 matrix, what would be sign of $a_{13}a_{22}a_{34}a_{41}$

$$\begin{vmatrix} & & a_{13} & \\ & a_{22} & & \\ & & & a_{34} \\ a_{41} & & & \end{vmatrix} \Rightarrow (-1)\begin{vmatrix} a_{41} & & & \\ & a_{22} & & \\ & & & a_{34} \\ & & a_{13} & \end{vmatrix} \cancel{\Rightarrow(-1)(-1)}$$

$$\Rightarrow (-1)(-1)\begin{vmatrix} a_{41} & & & \\ & a_{22} & & \\ & & a_{13} & \\ & & & a_{34} \end{vmatrix}$$

$\therefore$ Sign should be +ve.

- So, we can derive the formula for determinant from this:

$$\det(A_{n\times n}) = \sum_{\text{all perm.}} \pm\, a_{1\alpha}\, a_{2\beta} \cdots a_{n\omega}$$

So, for 2x2

$\det(A) = a_{11}a_{22} - a_{21}a_{12}$

Ex. Find

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$-\underbrace{(1\times1\times1\times1)}_{\{a_{13}a_{22}a_{31}a_{44}\}} + \underbrace{(1\times1\times1\times1)}_{\{a_{14}a_{23}a_{32}a_{41}\}} = 0$ (rest don't survive)

Ex. For a matrix, $A_{n\times n}$, which of the following op^n doesn't change det?

(A) $R_i \leftarrow kR_i \pm R_j$ ~~(B)~~ $R_i \leftarrow R_i \pm kR_j$

(A) multiplies $|A|$ by K.

- **Another method to calculate determinant:**
  - Convert to Echelon form (by elementary row opⁿs.)
  - Take product of pivots involved
  (Keep track of row exchanges for sign of determinant)

- **Cofactor Formula**

The cofactor of a_{ij} is the determinant of matrix A without row i & column j, multiplied w/ $(-1)^{i+j}$

Say, $A_{3\times3}$ is:
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then $\text{cofactor}(a_{22}) = (-1)^{2+2} \times \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

So,
$$\det(A)_{n\times n} = \sum_j a_{xj} \times \text{cofactor}(a_{xj})$$
where j is one particular row/col.

So,
$$\det(A_{3\times3}) = a_{11}\times cof(a_{11}) + a_{12}\times cof(a_{12}) + a_{13}\times cof(a_{13})$$

Ex. Which of the following evaluates to 0? C_{ij} is cofactor of a_{ij}

(A) $\Delta = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$
(B) $\Delta = a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32}$
(C) $\Delta = a_{22}C_{11} + a_{21}C_{12} + a_{23}C_{13}$
(D) $\Delta = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(A) is equal to $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

So, 2 rows are same.

$\therefore \quad \Delta = 0$.

(B) is equal to $\begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix}$

$\therefore \quad \Delta = 0$

(C) is equal to $\begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

This isn't necessarily 0.

(D) is equal to $\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\therefore \quad \Delta = 0$.

Ex. Find product (C_{ij} is cof (a_{ij}))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

Say for row 1 col 2 we get 0, since

$$X_{12} = a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23}.$$

So this is
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$\therefore$ Multiplying a row (or col) w/ corresponding cofactor of any other row (or col), & then sum them up, it will give 0. Since its similar to copying the row (or col.) & have duplicate row (or col).

Ex.
$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

We can use this result:

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \times A = |A| \times I_{3\times 3}$$

{ w.k.t. $A^{-1} \times A = I = A \times A^{-1}$ }

$$\therefore \quad A^{-1} = \frac{1}{|A|} \times \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

we call:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$ as Cofactor matrix (C)

&

$$\text{adjoint}(A) = C^T$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \times \text{adj}(A)$$

Ex.

$$A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$ Solve for 3^{rd} column of A^{-1}.

$$A^{-1}[\,][3] = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} \times \frac{1}{|A|}$$

$$\Rightarrow \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} \times \frac{1}{a_{31} \times C_{31} + a_{32} \times C_{32} + a_{33} \times C_{33}}$$

$$\Rightarrow \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}.$$

OR

3rd col of $I_{3\times3}$

$$\left[\begin{array}{ccc|c} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 4 & 1 \\ 0 & \boxed{-1} & -2 & -2 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right]$$

$$\therefore X = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} = A^{-1}[\ ][3]$$

Cramer's Rule

For a matrix $A_{n \times n}$ for which A^{-1} exists (or $|A| \neq 0$), the solⁿ to

$$AX = b \quad \{\text{unique sol}^n\}$$

$$\Rightarrow X = A^{-1}b$$

$$\Rightarrow X = \frac{1}{|A|} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

what would $X[1]$ be?

$$\frac{1}{|A|} \times (C_{11} \times b_1 + C_{21} \times b_2 \ldots + C_{n1} \times b_n) = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots \\ b_2 & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots \end{vmatrix} \times \frac{1}{|A|}$$

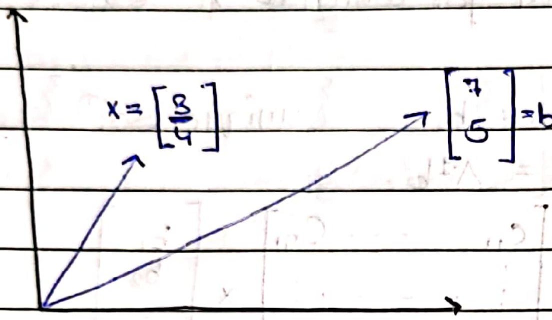
Similary for $X[i]$ we'll get det of 'A w/ i^{th} column replaced w/ '$\vec{b}$' $\times \frac{1}{|A|}$, let's call this $|A_b|_i$

$$\therefore X = \begin{bmatrix} |Ab|_1 \\ |Ab|_2 \\ \vdots \\ |Ab|_n \end{bmatrix} \times \frac{1}{|A|}$$

* Eigenvalue & Eigenvectors

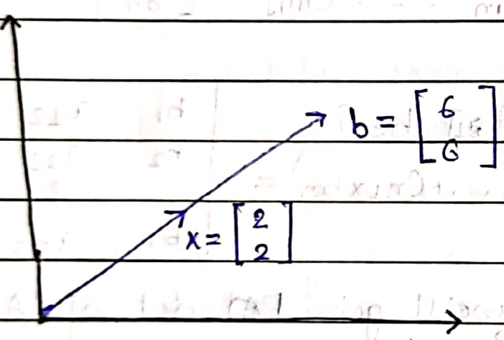
- What happens, when a vector ~~x~~ x is mult. by a matrix A & b is obtained.

C1.



So, b may be obtained from X by stretching, contracting, rotating or many other opⁿs.

C2.



In one specific case, the vector b may remain on the same line, i.e. $\vec{b} = X \times \alpha$ where α is a scalar term.

- For a given matrix $A_{n \times n}$ X {non zero vector} is called its eigenvector iff $Ax = \lambda X$, where λ is a scalar, which is termed as eigenvalue.

Ex For I_n, any X $\{\neq \vec{0}\}$ in $\mathbb{R}^n$ is an E.Vec w/ 1 as E.value.
But how many L.I. Evectors?
Since we are considering $\mathbb{R}^n$, we can have exactly n L.I Evectors.

Ex. Given that X is Evector of A, then any scalar (nonzero) multiple of X, is also EVec. of A.? T/F?

True.

Given : $AX = \lambda X$.

for any $\alpha X \Rightarrow A(\alpha X) = \alpha(AX) = \alpha(\lambda X)$

$\Rightarrow \lambda(\alpha X)$.

$\therefore$ αX is also Evector of A, w/ Same Evalue λ.

Calculating Eigenvalues & Eigenvectors

For a matrix $A_{n\times n}$: $AX = \lambda X$

then, $(A-\lambda I)X = \vec{0}$

w.r.t $X \neq \vec{0}$, so, $(A-\lambda I)$ must have some col. vector which is linearly dependent on other column.

So, with elementary transformations we can convert that col. to $\vec{0}$.

Hence $|A-\lambda I|$ must be 0 {due to 0 column}

$\therefore$ $|A-\lambda I| = 0$.

This eq^n is called Characteristic Eq^n.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Char eq^n: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)^2 - 4 = 0$

$\Rightarrow 1-\lambda = \pm 2$

$\Rightarrow \lambda = 3, -1$

For $\lambda = 3$:

$\Rightarrow (A - \lambda I)x = 0$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x = 0$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{ccc} x_1 & x_2 & \\ \left[\begin{array}{cc|c} \boxed{-2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

Free vars $\Rightarrow x_2$ Pivot vars $\Rightarrow x_1$

Say, $x_2 = K \Rightarrow x_1 = K$

$$\therefore \quad x = \begin{bmatrix} K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -1$:

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x = 0$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{ccc} x_1 & x_2 & \\ \left[\begin{array}{cc|c} \boxed{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

Say, $x_2 = K$, $x_1 = -K$.

$$\therefore \quad x = \begin{bmatrix} -K \\ K \end{bmatrix} = K \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So, corresp. to $\lambda = 3$ we get 1 L.I Evec.

corresp. to $\lambda = -1$ we get 1 L.I Evec.

Overall we get 2 L.I Evec.

Ex. $\begin{bmatrix} 9 & 0 & -8 \\ 15 & 3 & -15 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 9-\lambda & 0 & -8 \\ 15 & 3-\lambda & -15 \\ 0 & 0 & 1-\lambda \end{vmatrix} \Rightarrow (9-\lambda)(3-\lambda)(1-\lambda) = 0.$$

$\therefore \lambda = 9, 3 \; \& \; 1.$

For $\lambda = 9$

$$\begin{bmatrix} 0 & 0 & -8 \\ 15 & -6 & -15 \\ 0 & 0 & -8 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ 15 & -6 & -15 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} \left[\begin{array}{ccc|c} \boxed{15} & -6 & -15 & 0 \\ 0 & 0 & \boxed{-8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

Free vars $\Rightarrow x_2$

let $x_2 = K$

$x_3 = 0$, $x_1 = \frac{2}{5}K$

$$X = \begin{bmatrix} 2/5\,K \\ K \\ 0 \end{bmatrix} = K \begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, we'll get X for $\lambda = 1$ & $\lambda = 3$.

- Eigenvectors of A are linearly independent irrespective of their corresponding eigen value.

Proof:

Given that λ_1 & λ_2 are EVs of A, w/ corresponding Evectors e_1 & e_2 resp.

To show that e_1 & e_2 are L.I., we can show that no non-trivial sol. c exists for

$$\begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

i.e. $c_1 e_1 + c_2 e_2 = 0$ —(1) $\{$iff $c_1 = c_2 = 0\}$

(1) × A : $c_1 e_1 A + c_2 e_2 A = 0$ — (2)

(1) × λ_1 : $c_1 e_1 \lambda_1 + c_2 e_2 \lambda_1 = 0$ — (3)

(2) − (3) : ~~$e_2 e_2 (A - \lambda_1 I)$~~ ; $c_2 e_2 \lambda_2 - c_2 e_2 \lambda_1 = 0$

$\Rightarrow c_2 e_2 (\lambda_2 - \lambda_1) = 0$

$\because e_2 \neq 0$ & $\lambda_2 \neq \lambda_1 \Rightarrow \lambda_2 - \lambda_1 \neq 0$

$\therefore c_2 = 0.$

In (1) : $c_1 e_1 = 0$

$\because e_1 \neq 0$

$\therefore c_1 = 0.$

$\therefore$ Only solⁿ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix}$

$\therefore e_1$ & e_2 are L.I.

Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$\Rightarrow (1-\lambda)^3 = 0$

$\Rightarrow \lambda = 1$ (Repeating thrice)

$\{$or $1, 1, 1\}$

For $\lambda = 1$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Free vars $= x_1, x_2$ & x_3.

Say, $x_1 = s$ $\quad x_3 = u$.

$x_2 = t$

$$\therefore \quad X = \begin{bmatrix} s \\ t \\ u \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So, we get three LI eigenvectors corresponding to $\lambda = 1$.

Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$\Rightarrow (1-\lambda)^3 = 0 \quad \Rightarrow \quad \lambda = 1, 1, 1.$

For $\lambda = 1$: $(A - \lambda I)X = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Freevars $\Rightarrow x_2, x_3$

Pivot vars $\Rightarrow x_1$

Say, $x_2 = s$, $x_3 = t$ $\Rightarrow x_1 = -t$

$$X = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, we get 2 L.I evecs. corresponding to $\lambda = 1$.

Ex.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad |A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)^3 = 0$$

$\therefore \lambda = 1, 1, 1.$

For $\lambda = 1$:
Solving for X in
$(A - \lambda I)X = 0$

$$\left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cancel{\xrightarrow{R_2 \leftrightarrow R_1}} \cancel{\left[\begin{array}{ccc|c} 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

$\therefore$ Free vars $\Rightarrow x_1$ Pivot $\Rightarrow x_2$ & x_3.

$$\therefore X = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} \Rightarrow s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So, corresponding to $\lambda = 1$, we get one LI evector.

- For an eigenvalue λ (of $A_{n \times n}$), the no. of eigenvectors corresponding to it is bound by (upper) its multiplicity. Multiplicity of an eigenvalue $\lambda = \lambda_1$ is m_1, where its char. eqn looks like:

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots = 0$$

$\therefore$ #evectors corresponding to $\lambda = \lambda_1 \leq$ multiplicity(λ_1)

Ex. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ This is a rotation matrix. This rotates every vector in $\mathbb{R}^2$ by $90°$.

Intuitively we can see that no vector will remain in the same line when rotated. So this vector doesn't have evectors.

(If we solve we get evecs in Imaginary space).

- It's not necessary that every matrix has evectors & evalues

Ex. Suppose while solving for $\lambda=3$ in $(A-\lambda I)x=0$, no. of free vars $=5$. What could be the multiplicity of $\lambda=3$? $(A_{10\times10})$

Since #freevars $= 5$, we get 5 L.I. evec. corresponding to $\lambda=3$.

w.k.t. $\#\text{LI evec} \leq \text{mult}(\lambda) \leq n$.

$\therefore \quad 5 \leq \text{mult.} \leq 10$.

Ex. For A the char. eq^n is $-\lambda^2(\lambda-3)^4(\lambda-2)=0$. What could be possible value of rank(A)?

(A) 5 (C) 4

(B) 6 (D) 7

- A is 7×7 since $2+4+1=7$.
- For $\lambda=0$, mult. is 2.
- For $\lambda=0$, $A-\lambda I = A$. $\Rightarrow$ sol to $(A-\lambda I)x=0$ is same as sol to $Ax=0$.

So, no. of L.I. vectors in sol. to $Ax=0$ can be $\leq \text{mult}(\lambda=0)$

$\therefore$ nullity (A) can be 1 or 2.

$\therefore$ rank can be $7-1=6$ or $7-2=5$.

$\therefore$ (A) & (B).

• A symmetric matrix $A_{n\times n}$ always has n L.I. eigenvectors, irrespective of the multiplicity of the eigenvalues.

• A symmetric matrix is such that: $A^T = A$

• A symmetric (real) matrix always has real eigenvalues, i.e. eigenvalues always exist. (may / may not be repeating)

• Symmetric (real) matrix $A_{n\times n}$ ~~have~~ has n orthogonal (& hence L.I.) eigenvectors.

• So, in a symmetric matrix $A_{n\times n}$ for a $\lambda = \lambda_i$ w/ multiplicity m,

$$\#\text{LI evec corr. to } \lambda_i = m$$

• Theorem: All eigenvalues of real symmetric matrix are real.

Proof: $A x = \lambda x$.

{ For $c = a + ib$, $\bar{c} = a - ib$ (conjugate) }

We can get $\bar{x}^T A x$ in two ways:

① Mult. on both sides: $\bar{x}^T A x = \bar{x}^T \lambda x$ ⟶ (1)

② Take conjugate on both side: $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$\Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$ { A is real symm }

Take Transpose on both side: $\bar{x}^T A^T = \bar{\lambda}\bar{x}^T$

$\Rightarrow \bar{x}^T A = \bar{\lambda}\bar{x}^T$

Mult. w/ x: $\bar{x}^T A x = \bar{\lambda}\bar{x}^T x$ ⟶ (2)

So, from (1) & (2): $\bar{x}^T \lambda x = \bar{\lambda} \bar{x}^T x$.

$$\Rightarrow \lambda (\bar{x}^T x) = \bar{\lambda} (\bar{x}^T x).$$

So, $\lambda = \bar{\lambda}$.

This is possible only when λ is real.

$\therefore$ λ is real.

• Theorem: All eigenvectors of real symmetric matrix are orthogonal to each other.

Proof: Suppose that $A_{n\times n}$ has eigenvalue λ_1 & λ_2 w/ e_1 & e_2 eigenvectors respectively.

Case 1: $\lambda_1 \neq \lambda_2$.

$$e_1^T A e_2$$

$\Rightarrow e_1^T (Ae_2)$ $\qquad\qquad$ $\Rightarrow (e_1^T A) e_2$

$\Rightarrow e_1^T \lambda_2 e_2$ —(1) $\qquad\qquad$ $\Rightarrow (A^T e_1)^T e_2$

$\Rightarrow (\lambda_1 e_1)^T e_2$ $\qquad \{A = A^T\}$

$\Rightarrow \lambda_1 e_1^T e_2$ — (2)

from (1) & (2): $e_1^T e_2 \lambda_2 = e_1^T e_2 \lambda_1$

$$\Rightarrow e_1^T e_2 (\lambda_2 - \lambda_1) = 0.$$

w.r.t. $\lambda_2 \neq \lambda_1 \Rightarrow \lambda_2 - \lambda_1 \neq 0$.

$\therefore$ $e_1^T e_2 = 0$

$\therefore$ e_1 & e_2 are orthogonal.

Case 2: $\lambda_1 = \lambda_2$: This is complicated case.

• Multiplicity: For every eigenvalue λ of $A_{n\times n}$:

(1) Algebraic Multiplicity: The no. of times λ repeats. (In char eqn)

(2) Geometric Multiplicity: The no. of L.I. eigenvectors corresponding to λ.

$$GM(\lambda) \leq AM(\lambda)$$

Ex. Inverse of a matrix doesn't exist iff matrix has zero eigenvalue. Prove.

w.k.t $A^{-1} = \frac{1}{|A|} \times adj(A)$

Given that $\lambda = 0$ is e.value.

$(A - \lambda I)X = \vec{0}$

$\Rightarrow AX = \vec{0}$

Since e.vecs by def^n can't be $\vec{0}$, hence $X \neq \vec{0}$

$\therefore$ A has L.D. col vectors.

$\therefore |A| = 0$

$\therefore A^{-1}$ doesn't exist.

{we can similarly prove in reverse direction}

- For a matrix $A_{n \times n}$:
  - Its determinant is equal to product of its eigenvalues
  - Its trace (main diagonal) is sum of its eigenvalues.

- $A_{n \times n}$ is matrix w/ real coeff. If λ is an e.value of A w/ associated e.vector v, then $\bar{\lambda}$ is also an e.value w/ associated e.vector $\bar{v}$.

Proof:

Given: $Av = \lambda v$

Taking conjugate both sides: $\bar{A}\bar{v} = \bar{\lambda}\bar{v}$

$\Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$ {$\because$ A is real}

$\therefore$ Proved.

- Cayley-Hamilton Theorem: If $p(t)$ is the characteristic polynomial for a $n \times n$ matrix A, then the matrix $P(A)$ is the $n \times n$ O matrix.
{i.e. A satisfies $p(t) = 0$}.
{Char poly of A: $|A - \lambda I| =$ }

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Char eq^n: $\lambda^2 - 5\lambda - 2 = p(\lambda)$

$P(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex. The char. poly. of $B_{2\times 2}$ is $p(\lambda) = \lambda^2 - 3\lambda$. By C-H Theorem we can say that:

(a) $B^n = 3^{n-1} B$ (c) $B^n = 3^n B$ (e) NOTA.

(b) $B^n = 3^{n+1} B$ (d) $B^n = B^{98}$

$p(B) \Rightarrow B^2 - 3B = 0 \Rightarrow B^2 = 3B$.

$B^3 = 3B^2 = 3^2 B$

$B^4 = 3^2 B^2 = 3^2(3B) = 3^3 B$

$\vdots$

$B^n = 3^{n-1} B$.

Ex. $A = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$. Find A^{-1} by CHT.

$p(\lambda) = \lambda^2 - 4\lambda + 3$

$p(A) = A^2 - 4A + 3I = 0$

$\Rightarrow A(A - 4I) + 3 = 0$

$\Rightarrow \frac{-1}{3}(A - 4I) = A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix}$

Ex. Let $\lambda (\neq 0)$ be e.value of AB, w/ corresponding evector X. What will be e.value & evec of BA?
(Assume AB & BA exist).

Given: $ABX = \lambda X$ $\{\lambda \neq 0\}$
(Bx can't be 0 since then ABx will be 0)

$\Rightarrow BABX = B\lambda X$

$\Rightarrow (BA)(BX) = \lambda(BX)$.

So, evalue of BA is λ & evec. of BA is Bx.
$\therefore$ We can say that eigenvalue of AB & BA overlap.

Ex. Let λ be nonzero e.value of A^TA, & X be corr. evector.
What will be eval & evec. of AA^T?

Given: $A^TAX = \lambda X$

$\Rightarrow AA^TAX = A\lambda X$.

$\Rightarrow AA^T(AX) = \lambda(AX)$

$\therefore$ eval $= \lambda (\neq 0)$ & evec $= AX$.

- Given that evalue of A is λ, evalue of A^K is λ^K.

$$Ax = \lambda x$$
$$A^2 x = \lambda Ax = \lambda(\lambda x) = \lambda^2 x$$
$$\vdots$$
$$A^K x = \lambda^K x$$

Ex. If evalue of A is λ, then what will be eigenvalue of A^{-1}?

Given: $Ax = \lambda x$.

$$A^{-1} Ax = A^{-1} \lambda x$$
$$\Rightarrow \quad x = \lambda A^{-1} x$$
$$\Rightarrow \quad \frac{1}{\lambda} x = A^{-1} x$$
$$\Rightarrow \quad A^{-1} x = \lambda^{-1} x$$

$\therefore$ eval of A^{-1} is λ^{-1}, & evec. is λ.

- So, given that λ is eval & x is evec. of A, then for any $K \in \mathbb{Z}$, the eval. & evec. of A are λ^K & x respectively.

Ex. If for A eval & evec resp are λ & x, what are the eval & evec of $A + KI$?

Given: $Ax = \lambda x$. — (1)

also, $Ix = 1 \times x \Rightarrow KIx = Kx$ — (2)

$$\Rightarrow (A+KI)x = (\lambda + K)x$$

$\therefore$ eval is $\lambda + K$ & evec is x for $(A+KI)$.

* LU Decomposition

Given a matrix A, the idea is to divide ~~the~~ it into prod. of two matrices: Lower Triangular & Upper Triangular. The lower Triang. matrix further has special property that all diagonal entries are one.

i.e.,

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ & 1 & & 0 \cdots \\ * & \cdots & 1 & \\ * & * & * \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} * & * & * & \cdots \\ 0 & * & * & * \; * \; * \\ 0 & 0 \cdots & * & * \; * \\ 0 & 0 & \cdots & * \; * \; * \\ 0 & \cdots & & * \end{bmatrix}$$

$*$: any value.

Method 1: Convert A to echelon form, using only elementary row operations, without any row exchange. Compile the steps used for conversion into the matrix L.

Ex.

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

① $R_2 \leftarrow R_2 + 2R_1$
② $R_3 \leftarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & -8 & 16 \end{bmatrix}$$

③ $R_3 \leftarrow R_3 + \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & \frac{31}{2} \end{bmatrix}$$ This is U.

Compiling the steps into X (performed on A):

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$\therefore$

$$L = \left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & \frac{1}{2} & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{bmatrix}.$$

i.e. $XA = U$

$A = X^{-1}U$

$\therefore L = X^{-1}$

Ex. Find LU decomp. of following matrix, when it exists. For which real no. a & b does it exist ?

$$A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$$

① $R_2 \leftarrow R_2 - aR_1$

② $R_3 \leftarrow R_3 - bR_1$

③ $R_3 \leftarrow R_3 - \frac{b}{a}R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \frac{a}{b} & 1 \end{bmatrix}$$

So, $a \neq 0$ & $b \neq 0$ the solⁿ exists.

Types of Matrices

- **Identity Matrix:** I_n is a $n \times n$ matrix w/ all 0 entries except on the diagonal (main diagonal \\). All diagonal entries are 1.

- **Inverse Matrix:** For a square matrix $A_{(n\times n)}$, its inverse, $B_{(n\times n)}$ is such that:

$$AB = BA = I_n$$

  - $(A^{-1})^{-1} = A$
  - $(KA)^{-1} = K^{-1}A^{-1}$ for $K \neq 0$.
  - $(AB)^{-1} = B^{-1}A^{-1}$, given A & B both are invertible.

- **Transpose of a matrix:** For a matrix $A = [a_{ij}]$, $A^T = [a_{ji}]$.
  - $(A^T)^T = A$
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
  - $(cA)^T = c A^T$
  - $(A^{-1})^T = (A^T)^{-1}$

- **Lower Triangular Matrix:** A matrix $A_{n\times n}$ is said to be L.T. iff all entries above main diagonal (a_{ij}, $i<j$) are zeroes.

- **Upper Triangular Matrix:** A matrix $A_{n\times n}$ is said to be U.T. iff all entries below main diagonal (a_{ij}, $i>j$) are zeroes.

- **Diagonal Matrix:** A matrix $A_{n\times n}$ where all entries except the main diagonal are zero.

- **Symmetric Matrix:** A square matrix A such that $A = A^T$.
- **Skew Symmetric Matrix:** $A_{n\times n}$ such that $A = -A^T$. This can happen only when diagonal entries are zero.

Ex. For any matrix $A_{m \times n}$, AA^T is symmetric.

$$(AA^T)^T = (A^T)^T . A^T = AA^T$$

Ex. For any square matrix $A_{n \times n}$ w/ real entries, $A + A^T$ is symm & $A - A^T$ is skew-symmetric.

- $(A + A^T)^T = A^T + (A^T)^T = A^T + A$
- $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

Ex. Any sq. matrix A can be repr. as the sum of a symm. & skew symm. matrix.

$$A = \frac{1}{2}\left[\underbrace{(A + A^T)}_{\text{symm.}} + \underbrace{(A - A^T)}_{\text{skew symm.}}\right]$$

Ex. Show that if A^{-1} exists, A^{-1} is symm. iff A is symm.

$$[A^{-1} = (A^{-1})^T] \longleftrightarrow [A = A^T]$$

C1: Given $A = A^T$.

$A^{-1}A = A^{-1}A^T$

$\Rightarrow (A^{-1}A^T)^T = (I)^T$

$\Rightarrow A.(A^{-1})^T = I$.

$\therefore A^{-1} = (A^{-1})^T$

C2: Given $A^{-1} = (A^{-1})^T$

$A^{-1}A = I$

$\Rightarrow A^T.(A^{-1})^T = I$

$\Rightarrow A^T. A^{-1} = I$

$\therefore A = A^T$

- **Orthogonal Matrix:** A real square matrix whose columns & rows are ~~orthogonal~~ vectors.
orthonormal

i.e. $Q^TQ = QQ^T = I$

or $(Q^T = Q)$

Two vectors are orthonormal iff: $u.v = u^Tv = 0$ (orthogonal vecs) & $||u|| = ||v|| = 1$ (unit vecs)

- **Idempotent Matrix:** A matrix $A_{n\times n}$ is idempotent iff $A^2 = A$

Ex. If matrix A is idempotent then for any $k \geq 2$, $A^k = A$ $(k \in Z^+)$

Base: $A^2 = A$

for $k = n$; say $A^n = A$ is true.

for $k = n+1$; $A^{n+1} = A^n . A = A^2 = A$

Hence Proved.

Ex. If $AB = A$ & $BA = B$, then show that A & B are idempotent.

Given: $AB = A$
$\Rightarrow A(BA) = A^2$
$\Rightarrow AB = A^2$
$\Rightarrow A = A^2$

Given: $BA = B$
$\Rightarrow B(AB) = B^2$
$\Rightarrow BA = B^2$
$\Rightarrow B = B^2$

Ex. How many 3x3 invertible idempotent matrix are possible?

For any $A_{3\times3}$, it must satisfy:

① $A = A^2$

② B exists such that $AB = BA = I$.

$\Rightarrow A^2 - A = 0 \Rightarrow A(I - A) = 0$

either $A = [0]$ or $A - I = 0$

$\uparrow$ not invertible. $\Rightarrow A = I$.

$\therefore$ Only one matrix $A_{3\times3} = I_3$ exists.

- Involutory Matrix: A square matrix that is its own inverse.

i.e.

$$A \cdot A = I$$

or $A^2 = I$.

Ex. Show that $A_{n\times n}$ is involutory iff $(A+I)/2$ is idempotent.

C1: Given $A.A = I$.

$$\left[\frac{(A+I)}{2}\right]^2 = \frac{1}{4}\left[A^2 + AI + IA + I^2\right]$$

$$\Rightarrow \frac{1}{4}\left[I + A + A + I\right] = \frac{1}{2}\left[A + I\right].$$

C2: Given $\left[\frac{(A+I)}{2}\right]^2 = \frac{(A+I)}{2}$

$$\Rightarrow \frac{A^2 + AI + IA + I^2}{\cancel{4}\,2} = \frac{A+I}{\cancel{2}}$$

$$\Rightarrow A^2 + 2\cancel{A} + I = 2(\cancel{A} + I)$$

$$\Rightarrow A^2 = 2I - I$$

$$\Rightarrow A^2 = I.$$

classmate

Date

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Nilpotent Matrix: A ~~nilpotent~~ sq. matrix $A_{n \times n}$ such that for some $k \in \mathbb{Z}^+$, $A^k = 0$.

Conjugate Transpose: For any matrix $A_{m \times n}$, its conjugate transpose is given as

$$A^{\theta} = \bar{A}^{T}$$

{ $\bar{A}$ is complex conjugate of entries of A }

Hermitian Matrix: A sq. matrix $A_{n \times n}$ is Hermitian iff

$$A = A^{\theta}$$

{ Possible only if all main diagonal entries $\in \mathbb{R}$ }

- $(A^{\theta})^{\theta} = A$
- $(A+B)^{\theta} = A^{\theta} + B^{\theta}$
- $(AB)^{\theta} = B^{\theta} A^{\theta}$
- $(A^{\theta})^{-1} = (A^{-1})^{\theta}$
- $(zA)^{\theta} = \bar{z} A^{\theta}$

Ex. $A = \begin{bmatrix} 0 & a-ib & c-id \\ a+ib & 1 & m-id \\ c+id & m+id & 2 \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 0 & a+ib & c+id \\ a-ib & 1 & m+id \\ c-id & m-id & 2 \end{bmatrix}$

$$(\bar{A})^T = \begin{bmatrix} 0 & a-ib & c-id \\ a+ib & 1 & m-id \\ c+id & m+id & 2 \end{bmatrix}$$

Skew Hermitian Matrix: A sq. matrix $A_{n \times n}$ is Skew Hermitian iff:

$$A^{\theta} = -A$$

{ Diagonal entries must be pure imaginary }.