Solving Multi-Facility Location Problem Using Convex Hull

A term paper submission for

Design of Production system

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1 Introduction

1.1 Facility location problem (FLP):

FLP is a problem in which we determine the locations of a set of facilities and assign customers to those locations so that total summed distance of customers to their assigned facility is minimized. Facilities may be considered as warehouses, plants, hospitals, schools, etc. whereas customers could be retailers, dealers, patients, students, etc. If there is only one facility required to be located then we say it is single-facility location problem (SFLP) and if there are two or more facilities to be located and assigned to then we call it multi-facility location problem (MFLP). If the capacity of facility serving number of customers is limited, then it is said to be capacitated facility location problem else it is an uncapacitated FLP.

1.2 MFLP:

In general sense, MFLP is to locate certain facilities so as to serve optimally a given set of customers, whose locations and requirements are known. The simplest MFLP is when there are n demand points (customers) and we are required to:

(a) determine the locations of the m facilities (location decision), and

(b) assign customers to facilities (allocation or assignment),

1.3 Solution for FLP:

SFLP can be solved optimally by Weiszfeld's algorithm. Whereas MFLP is a NP hard problem. The complexity increases exponentially with increase in number of facilities to be located. There are many heuristic methods to solve the MFLPs like clustering of the customer locations and then solving SFLP for each set of clusters. Some clustering techniques are K-means clustering, Agglomerative clustering, Location Allocation clustering,

There is also a method to solve MFLP optimally which utilizes convex hull approach. In this term paper, we will be discussing about that approach.

2 Literature review

Miehle (1958) was first to consider the MFLP. He suggested partitioning the full set of fixed points into sub-sets and then finding several minimizing points (one per set) simultaneously. **Cooper (1963)** was the first who formulated the MFLP and proved that the objective function (equation-(1)) is neither concave nor convex. He proposed a heuristic which he later developed more fully as ALTERNATE (Cooper, 1964).

Harris et al. (1970, 1972) published an algorithm which can solve the two sub-set problem. They recognized that in any feasible solution the subsets must be contained in non-overlapping convex hulls and exploited this convex hull property of the problem to generate all feasible solutions and then found the optimum by complete enumeration of these feasible solutions. Their algorithm involves projective geometry and operates on the dual of the line. Fisher (1958), when performing cluster analysis on a one-dimensional (temporal) data set, exploits the convex hull property even earlier without, however, recognizing what it is. He solves this problem by enumerating only feasible alternatives (non-overlapping sets on a line). EquaIly important is that, in one-dimensional space, he handles division into three or more sets simultaneously.

Ostresh (1975) developed a different algorithm to solve the two set problem than that utilized by Harris et al. He concentrated on obtaining the one optimum solution of the MFLP rather than studying the feasibility surface and local optima. He also (Ostresh, 1973a) provided a computer code for his algorithm which he named TWAIN.

Drezner (1984) has developed a similar technique for the two-centre problem which operates through consideration of ordered angles of lines connecting points to create a series of two disjoint hulls. Any of these approaches require the identification of a set of non-overlapping convex hulls which together cover the distribution of fixed points completely. In enumerating all such solutions the functional value must be evaluated for each. It is this total enumeration step which makes the approach, for problems where p > 2 'quite messy' (Drezner, 1984).

Rosing (1991) chose Ostresh's TWAIN for his investigation on convex hull because he had used it before in other work (Rosing 1991, 1992) and thus had the computer programme at hand. Rosing (1991) optimally solved models similar to the MFLP in an attempt to find optimal/ near optimal solutions to it. Finding the correct sub-sets was extremely important which was also solved by Rosing (1992).

3 General problem formulation of MFLP

In general, the multi-facility location problem with Euclidean distance can be formulated as

3.1 Parameters:

		index
Facility	{1,2,3,…..p}	j
Fixed points	{1,2,3,….n}	i

Number of fixed points = n

Number of facility locations = p

Weight associated with the fixed point-i = $w_i \quad \forall\, i$

Capacity of the facility = C

Horizontal and vertical coordinate, respective, of the fixed point-i : = $(a_i, b_i) \quad \forall\, i$

3.2 Variable:

Horizontal and vertical coordinate, respective, of the facility-j : = $(x_j, y_j) \quad \forall\, j$

$$z_{ij} = \begin{cases} 1 & if\ fixed\ point - i\ is\ assign\ to\ the\ facility - j \\ 0 & otherwise \end{cases}$$

3.3 Constraints and objective function:

$$Min \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \,.\, w_i \,.\, \sqrt{(x_j - a_i)^2 + (y_j - b_i)^2} \qquad (1)$$

$$s.t. \qquad \sum_{j=1}^{m} z_{ij} = 1 \qquad \forall\, i \qquad (2)$$

$$\sum\nolimits_{i=1}^{n} z_{ij} \,.\, w_i \leq C \qquad \forall\, j \qquad (3)$$

$$z_{ij} \in (1,0) \qquad \forall\, i, j$$

$$x_j, y_j \ \in \mathcal{R} \qquad \forall\, j$$

In the objective function (1), we are minimizing the summed Euclidean distance of each fixed point to the facility center of its sub-set. The decision variable, X_{ij} ensures that the cost of each fixed point is counted only to one center. Constraint (2) ensures that each fixed points is assigned to only one facility. We can clearly see that it is a Non Linear Programming problem. Formulating the problem in this way leads to great difficulty in solving it. If we consider the complexity of the problem, then it seems to be even more difficult in solving it optimally. This is the main reason why many heuristics have been introduced for a good solution. Is there any possible way to formulate the problem differently and in a more tractable manner so that we may solve it optimally? Let’s just explore that.

4 Convex hull

The convex hull of a set of points is defined as the smallest convex polygon that encloses all of the points in the set. Convex polygon is the polygon which has none of its corner bent inwards.

5 Solving methodology of MFLP by the convex hull

The two facility MFLP can be solved by generating all feasible solutions, a relatively small number, and performing a complete enumeration of these feasible solutions. It has been observed that in feasible solutions all points must be contained within non-overlapping convex hulls. The membership of the two groups is known and its objective function can be evaluated. Each time a pair of hulls is generated the objective function (sum of the two weighted aggregate distances) is evaluated and the solution saved if it is the best so far or discarded if it is not.

Also, an alternative approach might be to store the fixed points, create and save all possible convex hulls, and then match these convex hulls together to identify feasible solutions and, eventually, choose the optimal solution. If this could be done then problems where $p > 2$ could be solved optimally through the exploitation of the convex hull properties.

5.1 Algorithm to generate set of convex hull :

Our aim is to find a set of all possible unique convex hulls and then find the functional value associated with each convex hull. The value is sum of the weight times the Euclidean distance of each member of the convex hull from the facility location, which comes by solving SFLP for the fixed points in that hull.

We represent the associated value by C_i where i is the index of convex hull ($i = 1, 2, \dots, m$). Let there are m unique convex hulls.

STEPS:

Step 1. Divide the set of fixed points into two non-overlapping convex hulls ($k = 1, 2$), $k = 1$ here.

Step 2. Check all previously identified unique convex hulls and save convex hull k of this pair, if it is unique, as a member of list i. Solve the SFLP for that convex hull and save the functional value, then $i = i + 1$.

Step 3. Divide the set of fixed points contained in the convex hull k into two non-overlapping convex hulls ($l = 1, 2$).

Step 4. Perform Steps 2-3 on convex hull $l = 1$.

Step5. Continue to re-divide each successive, lower level, convex hull until the depth (number of times subdivided) equals $p - 1$. Perform Steps 2-3 each time.

Step 6. Perform Steps 2-3 and 5 on convex hull $l=2$.

Step 7. Perform Steps 3-6 until convex hull k is exhausted, if $k = 2$ then skip to Step 9.

Step 8. Perform Steps 2-6 for convex hull $k = 2$.

Step 9. Perform Steps 1-8 until the set of fixed points is exhausted, then stop.

5.2 How to check this convex hull is unique?

Convex hulls may be identified more than once. For this reason it is necessary to record each unique hull as it is found. Each new hull, as it is found has its membership compared to all other convex hulls. If it is unique (not generated up to this point) then it is recorded and its functional value-C_i is evaluated. Also, we can find its functional value-C_i and compare it with all previously found. If two are equal the memberships are compared. If these are equal then the new hull is non-unique and discarded. If they are unequal then it is unique and the functional value is recorded and the membership recorded in the 'history' matrix.

5.3 How to divide the set of fixed points into two non-overlapping convex hulls? (TWAIN's algorithm)

The algorithm works by passing lines between pairs of points. Any such line divides the points into two disjoint sets. There are ½n(n- 1) pairs of points. Therefore, at a maximum, there are ½n(n- 1) pairs of non-overlapping convex hulls (less if there is collinearity). When all pairs of points have been investigated in this way, all pairs of non-overlapping convex hulls have been found. This smaller set is completely enumerated and the pair with the minimum functional value declared the optimum. In conclusion he agrees with Harris et al. that finding more than two convex hulls simultaneously is, geometrically, too complex a problem to resolve.

5.4 Solving it by ILP

5.4.1 Parameter:

		index
Convex hulls	{1,2,3,…..m}	i
Fixed points	{1,2,3,….n}	k

Number of fixed points = n

Number of facility locations = p

Set of convex hulls of which k is a member = $M_k \quad \forall\, k$

C_i = The summed distances of all fixed points in that convex hull from the one Single-facility location of that convex hull, the point of minimum aggregate travel of that convex hull, the Single-Weber point, that is the cost;

5.4.2 Variable:

$$X_i = \begin{cases} 1 & if\ convex\ hull - i\ is\ choosen \\ 0 & otherwise \end{cases}$$

5.4.3 Constraints and objective function:

$$Min \qquad \sum_{i=1}^{m} X_i C_i \qquad (4)$$

$$s.t. \qquad \sum_{i=1}^{m} X_i = p \qquad (5)$$

$$\sum_{i \in M_k}^{m} X_i \geq 1 \qquad \forall\, k \qquad (6)$$

$$X_i \in (1,0) \qquad \forall\, i \qquad (7)$$

As is instantly recognizable this formulation is a small modification of the classical Set Covering Problem solved with LP. In the objective function (4), we are minimizing the sum of functional value of the each chosen convex hulls. Constraint (5) says, when operating together with (7), that the number of convex hulls chosen has to be equal to number of facilities to be located. Constraint (6) is the common covering constraint stating that each fixed point must be covered by one or more of the facilities chosen, in this case by convex hulls whose associated cost is included in the objective function and thus are minimized.

6 Proof of optimality

Galvani (1933) has proven that for any number of points the minimum distance point is unique i.e. SFLP has an optimal solution. Kuhn (1967) proves, in the SFLP, in Euclidean space that the optimal solution lies inside the convex hull of the fixed points. Wendell and Hurter (1973) show and infer that Kuhn's dominance applies to the MFLP. Juel and Love (1983) extend Kuhn's result to any L_p norm (p > 1) in two dimensions. Weiszfeld (1937) corrected by Kuhn and Kuenne (1962) corrected by Ostresh (1978a) prove descent of an iterative algorithm to identify the optimal facility site in a SFLP and Rado (1988) proves convergence.

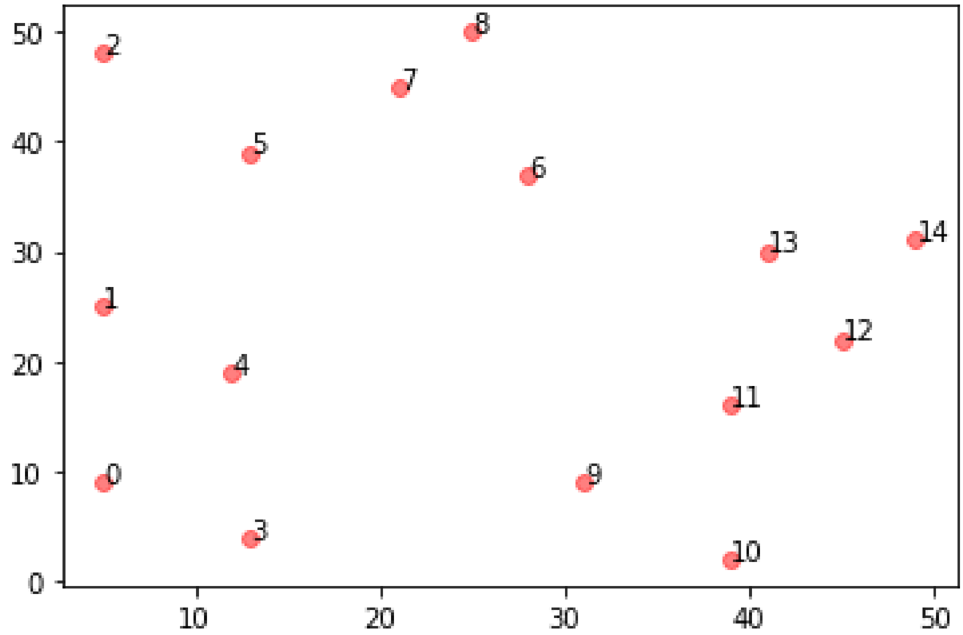
Harris et al. (1970, 1972) prove that for any number of points, on a plane, the maximum number of lines which can divide them into two disjoint groups is ½n(n - 1). Ostresh (1973c, 1975) and Drezner (1984) prove that a single straight line is sufficient to divide groups at the p = 2 level and also that one of the pairs of groups so found will be optimal. From this review it is obvious that all elements of a proof of optimality of the method presented here have been proven with the exception of the algorithm of divide and divide again (Steps 1-9, above). We address this point as follows.

Any two clusters of points allocated to each facility are always separated by the perpendicular bisector of the line segment joining the two facilities (Ostresh, 1975). This means that each convex hull which is an element of the optimal solution is obtainable by at most p- 1 successive linear cuts. Hence this will generate, among many others and with great repetition (which can be partially avoided), the individual elements of an optimal solution.

The improved Weiszfeld algorithm converges to an acceptable estimate of the location of the true Single facility location within each convex hull. The covering formulation (4)-(7) covers the full set of fixed points with minimum valued (summed distance from fixed points within each convex hull to the optimum single facility) convex hulls. Solutions to the MFLP found in this way are thus optimal.

7 Data and Results

We have taken the same data as taken by Rosing (1992) for 15 fixed points.



And for solving it for 2,

Problem n/p	Objective function		Convex hull		x	y
15/2	214.343	1	1,2,3,5,6,7,8,9	11	16.10925	93.98271
		2	4,10, t1,12,13,14,15	12	38.04637	120.3606

8 Conclusion

The first and most important conclusion is that the MFLP can be solved optimally at least in case of small and moderately sized problem. It should also be noted that when more than one value of p is required for a given data set, the method is even more economic if the deepest p approach is used. Larger problems than these can also be solved at the expense of rapidly increasing times and associated expense. In theory, this method can be expanded to larger n. In practice, the method is bound by the preparation step to small and moderate sized problems at this time.

One more point to be noted is that this method can also be used to solve when weights of fixed points are to be considered without any increase in time or CPU. Further, this method is not confined to the FLP with Euclidean distance but can also be used in case of FLP with rectangular distance or squared distance by calculating functional vale of convex hulls differently. Other variants for which it is suited can also be identified.

This method can also be used to solve the capacitated MFLP. All that is required is to discard from consideration all convex hulls that have fixed points more than the capacity. This method can be used to solve the MFLP constrained by maximum distance. All that is required is to discard from consideration all convex hulls that have its functional value (evaluated by solving minmax objective SFLP) greater than maximum distance permitted.

9 Reference

Rosing, K.E., (1992), " An optimal method for solving the (generalized) multi-Weber problem ", European Journal of Operational Research 58 (1992) 414-426.

Miehle, W. (1958), "Link-length minimization in networks", Operations Research 6, 232-243.

Cooper, L. (1963), "Location-allocation problems," Opera- tions Research 11, 331-343.

Cooper, L. (1964), "Heuristic methods for location-allocation problems", SIAM Review 6, 37-53.

Harris, B., Farhi, A., and Dufour, J. (1972), "Aspects of a problem in clustering", Institute for Environmental Stud- ies, University of Pennsylvania, Philadelphia, PA.

Drezner, Z. (1984), "The planar two-center and two median problems", Transportation Science 18, 351-361.

Ostresh Jr., L.M. (1975), "An efficient algorithm for solving the two-center location-allocation problem", Journal of Regional Science 15, 209-216.

Weiszfeld, E. (1937) "Sur le point pour lequel la somme des distances de n points donn6 est minimum", Tohoku Journal of Mathematics 32, 355-386.