UNIT- II Two Dimensional Random Variable.

* Joint distributions
* Marginal & conditional distributions
* Corelational & Linear Regression
* Transformation of random variable.

→ Joint probability Mass function:-

$P_{ij} = P(x = x_i, y = y_j)$

i) $P_{ij} \geq 0 \quad \forall i \ \& \ j$

ii) $\sum_{i=1}^{M} \sum_{j=1}^{N} P_{ij} = 1$

→ Joint probability Density function.

i) $f(x, y)$

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$

The conditional probability distribution of x & y.

$$P(x = x_i \mid y = y_i) = \frac{P(x = x_i \cap y = y_i)}{P(y = y_i)}$$

$$P(y = y_i \mid x = x_i) = \frac{P(x = x_i \cap y = y_i)}{P(x = x_i)}$$

Marginal density fn of x,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Marginal density fn of y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

1) The Joint probability mass function of (x, y) is given by $P(x,y) = k(2x+3y)$ $x = 0, 1, 2,$ ~~for~~, $y = 1, 2, 3$. Find marginal distribution of x & y. Also find the probability distribution of $x+y$ & $P(x+y>3)$.

Sol Given JPMF is,

$P(x,y) = k(2x+3y)$, $x = 0, 1, 2$. $y = 0, 1, 2, 3$

x \ y	1	2	3	Σx_i
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
Σy_i	15K	24K	33K	72K

WKT,

$$\sum_{i=0}^{2} \sum_{j=1}^{3} P(x_i, y_j) = 1$$

$$72K = 1$$

$$\boxed{K = 1/72}$$

$\therefore P(x,y) = \frac{1}{72}(2x+3y)$, $x = 0, 1, 2 \ldots$ $y = 1, 2, 3$

i) Marginal distribution of x.

x	0	1	2
P(xi)	18/72	24/72	30/72

Cummulative.

x	0	1	2
P(xi)	18/72	42/72	72/72

$P(X \leq x_i)$

Marginal distribution of y is,

y	1	2	3
$P(y_i)$	$15/72$	$\frac{24}{72}$	$\frac{33}{72}$

The conditional distribution of x is.

$$P(x=x_i \mid y=y_j) = \frac{P(x=x_i \cap y=y_j)}{P(y=y_j)} = \frac{p(x_i\, y_j)}{P(y=y_j)}$$

The conditional distribution of x given y=1 is,

$$P(x=0 \mid y=1) = \frac{P(x=0, y=1)}{P(y=1)} = \frac{\frac{3}{72}}{\frac{15}{72}} = \frac{1}{5}$$

$$P(x=1 \mid y=1) = \frac{P(x=1, y=1)}{P(y=1)} = \frac{\frac{5}{72}}{\frac{15}{72}} = 1/3$$

$$P(x=2 \mid y=1) = \frac{P(x=2, y=1)}{P(y=1)} = \frac{\frac{7}{72}}{\frac{15}{72}} = \frac{7}{15}$$

The conditional distribution of x given y=2 is

$$P(x=0 \mid y=2) = \frac{P(x=0, y=2)}{P(y=2)} = \frac{\frac{6}{72}}{\frac{24}{72}} = 1/4$$

$$P(x=1 \mid y=2) = \frac{P(x=1, y=2)}{P(y=2)} = \frac{8/72}{24/72} = 1/3$$

$$P(x=2 \mid y=2) = \frac{P(x=2, y=2)}{P(y=2)} = \frac{10/72}{24/72} = \frac{5}{12}$$

The conditional distribution of x given y=3 is,

$$P(x=0 \mid y=3) = \frac{P(x=0, y=3)}{P(y=3)} = \frac{9/72}{33/72} = 3/11$$

$$P(x=1 \mid y=3) = \frac{P(x=1, y=3)}{P(y=3)} = \frac{11/72}{33/72} = 1/3$$

$$P(x=2 \mid y=3) = \frac{P(x=2, y=3)}{P(y=3)} = \frac{13/72}{33/72} = \frac{13}{33}$$

The conditional distribution of y given x=0 is,

$$P(y=1 \mid x=0) = \frac{P(x=0, y=1)}{P(x=0)} = \frac{3}{18} = 1/6$$

$$P(y=2 \mid x=0) = \frac{P(x=0, y=2)}{P(x=0)} = \frac{6}{18} = 1/3$$

$$P(y=3 \mid x=0) = \frac{P(x=0, y=3)}{P(x=0)} = \frac{9}{18} = 1/2$$

CD of y x=1 is,

$$P(y=1 \mid x=1) = \frac{P(x=1, y=1)}{P(x=1)} = \frac{5}{24}$$

$$P(y=2 \mid x=1) = \frac{P(x=1, y=2)}{P(x=1)} = \frac{8}{24} = 1/3$$

$$P(y=3 \mid x=1) = \frac{P(x=1, y=3)}{P(x=1)} = \frac{11}{24}$$

CD of y x=2 is,

$$P(y=1 \mid x=2) = \frac{P(x=2, y=1)}{P(x=2)} = \frac{7}{30}$$

$$P(y=2 \mid x=2) = \frac{P(x=2, y=2)}{P(x=2)} = \frac{10}{30} = 1/3$$

$$P(y=3 \mid x=2) = \frac{P(x=2, y=3)}{P(x=2)} = \frac{13}{30}$$

$x = 0, 1, 2, \quad y = 1, 2, 3$

$x+y$	$P(x+y)$
1	$P(0,1) = 3/72$
2	$P(0,2) + P(1,1) = \frac{6}{72} + \frac{5}{72} = 11/72$
3	$P(0,3) + P(1,2) + P(2,1) = 9/72 + \frac{8}{72} + \frac{7}{72} = \frac{24}{72} = 1/3$
4	$P(1,3) + P(2,2) = 11/72 + \frac{10}{72} = \frac{21}{72} = \frac{7}{24}$
5	$P(2,3) = 13/72$

$P(x+y>3) = P(x+y=4) + P(x+y=5)$

$= \frac{21}{72} + \frac{13}{72} = \frac{34}{72} = \frac{13}{36}.$

Home work:-

Given $f(x,y) = \frac{x+2y}{27}$ $\quad x = 0, 1, 2; \quad y = 0, 1, 2.$

Find the conditional distribution of y given $x=1$.

also find the conditional distribution of x given $y=1$.

Covariance of x, y is.

$cov(x,y) = E(xy) - E(x).E(y).$

Correlation coefficient of x, y.

$r_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y}$ → Note:- range is -1 to $+1$.

where, $\sigma_x = \sqrt{var(x)}$

$\sigma_y = \sqrt{var(y)}$

Mean:-

$E(xy) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j P(x_i, y_j).$

16 Mrk. Qn.

1) Let x & y be discrete random variable with probability function $f(x,y) = \frac{x+y}{21}$, $x = 1, 2, 3$, $y = 1, 2$. Find the marginal distribution. also find mean, variance of x & y $E(xy)$. Find the correlation of x & y.

Sol Given,

$f(x,y) = P(x,y) = \frac{x+y}{21}$, $x = 1, 2, 3.$ $y = 1, 2.$

x \ y	1	2	Σx_i	
1	2/21	3/21	5/21	$P(x_1)$
2	3/21	4/21	7/21	$P(x_2)$
3	4/21	5/21	9/21	$P(x_3)$
Σy_j	9/21	12/21	21/21	
	$P(y_1)$	$P(y_2)$		

Marginal distribution of x,

X	1	2	3
$P(x_i)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$

Marginal distribution of Y is,

Y	1	2
$P(y_j)$	$\frac{9}{21}$	$\frac{12}{21}$

$$E(x) = \sum_{i=1}^{3} x_i\, p(x_i)$$

$$= 1\left(\frac{5}{21}\right) + 2\left(\frac{7}{21}\right) + 3\left(\frac{9}{27}\right)$$

$$= \frac{5+14+27}{21} = \frac{46}{21}$$

$$\therefore \boxed{E(x) = \frac{46}{21}}$$

$$E(x^2) = \sum_{i=1}^{3} x_i^2\, p(x_i)$$

$$= 1^2\left(\frac{5}{21}\right) + 2^2\left(\frac{7}{21}\right) + 3^2\left(\frac{9}{27}\right)$$

$$= \frac{5+28+81}{21} = \frac{114}{21}.$$

$$\boxed{E(x^2) = \frac{114}{21}}$$

$$E(y) = \sum_{j=1}^{2} y_j\, p(y_j)$$

$$= 1\left(\frac{9}{21}\right) + 2\left(\frac{12}{21}\right) = \frac{9+24}{21} = \frac{33}{21}.$$

$$\boxed{E(y) = \frac{33}{21}}$$

$$E(y^2) = 1^2\left(\frac{9}{21}\right) + 2^2\left(\frac{12}{21}\right)$$

$$= \frac{9}{21} + \frac{48}{21} = \frac{57}{21}.$$

$$\boxed{E(y^2) = \frac{57}{21}}$$

$$E(xy) = \sum_{i=1}^{3}\sum_{j=1}^{2} x_i y_j\, P(x_i, y_j)$$

$$= (1)(1)\left(\frac{2}{21}\right) + (1)(2)\left(\frac{3}{21}\right) + (2)(1)\left(\frac{3}{21}\right) + (2)(2)\frac{4}{21}$$

$$+ 3(1)\frac{4}{21} + (3)(2)\frac{5}{21}$$

$$= \frac{2}{21} + \frac{6}{21} + \frac{6}{21} + \frac{16}{21} + \frac{12}{21} + \frac{30}{21}$$

$$\boxed{E(xy) = \frac{72}{21}}$$

covariance of x, y.

$$cov(x_i y_j) = E(xy) - E(x).E(y)$$

$$= \frac{72}{21} - \left(\frac{46}{21}\right)\left(\frac{33}{21}\right)$$

$$= \frac{1512}{441} - \frac{1518}{441} = \frac{-6}{441}$$

$$\cancel{\boxed{cov(x,y) = \frac{-2}{147}}}$$

$$\therefore \boxed{cov(x,y) = \frac{-6}{441}}$$

Correlation coefficient of (x,y) is,

$$r_{xy} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{\operatorname{Var} x}$$

$$\operatorname{Var}(x) = E(x^2) - E(x)^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \frac{278}{441}$$

$$\operatorname{Var}(y) = E(y^2) - E(y)^2 = \frac{57}{21} - \left(\frac{33}{21}\right)^2 = \frac{108}{441}$$

$$\sigma_x = \sqrt{\frac{278}{441}} = \frac{\sqrt{278}}{21}, \quad \sigma_y = \sqrt{\frac{108}{441}} = \frac{\sqrt{108}}{21}$$

$$r_{xy} = \left(\frac{\frac{-6}{441}}{\frac{\sqrt{278}}{\cancel{21}} \frac{\sqrt{108}}{\cancel{21}}}\right) = \frac{-6}{(\sqrt{278})(\sqrt{108})}$$

$$= \frac{-6}{173.}$$

$$\boxed{r_{xy} = -0.034}$$

2) Two random variables x & y, have the joint probability density function.

$$f(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find i) The marginal & conditional density function of x & y.

ii) $\operatorname{Var}(x)$, $\operatorname{Var}(y)$.

Show that $\operatorname{Cov}(x,y) = \frac{-1}{144}$.

Sol given.

$$f(x,y) = \begin{cases} 2-x-y, & 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal density function of x,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^1 (2-x-y)\, dy$$

$$= \left[(2-x)y - \frac{y^2}{2}\right]_0^1$$

$$= \left(2-x-\frac{1}{2}\right)$$

$$\boxed{f(x) = \frac{3}{2} - x, \quad 0 \le x \le 1}$$

Marginal density function of y,

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^1 (2-x-y)\, dx$$

$$= \left[(2-y)x - \frac{x^2}{2}\right]_0^1$$

$$= 2-y-\frac{1}{2}$$

$$\boxed{f(y) = \frac{3}{2} - y, \quad 0 \le y \le 1}$$

Conditional density Function of x is given y is,

$$P(X=x \mid Y=y) = \frac{P(X=x \cap Y=y)}{P(Y=y)}$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy}{\int_{-\infty}^{\infty} f(y)\,dy}$$

$$f(x=x \mid y=y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{2-x-y}{\frac{3}{2}-y}$$

CD of y is given x,

$$f(y=y \mid x=x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{2-x-y}{\frac{3}{2}-x}.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 (2-x-y)\,dx\,dy$$

$$= \int_0^1 \cancel{(2-y)\,y - \frac{y^2}{2}}$$

$$= \int_0^1 \left[(2-y)x - \frac{x^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(2-y-\frac{1}{2}\right) dy$$

$$= \int_0^1 \left(\frac{3}{2}-y\right) dy$$

$$= \left(\frac{3}{2}y - \frac{y^2}{2}\right)_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$\boxed{P(X=x, Y=y = 1}$$

$$\int_{-\infty}^{\infty} f(y)\,dy = \int_0^1 \left(\frac{3}{2}y - y\right) dy$$

$$= \left(\frac{3}{2}y - \frac{y^2}{2}\right)_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$\boxed{P(Y=y) = 1}$$

$$\boxed{P(X=x \mid Y=y) = \frac{1}{1} = 1}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(\frac{3}{2}-x\right) dx$$

$$= \int_0^1 \left(\frac{3}{2}x - x^2\right) dx$$

$$= \left(\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{3}\right)$$

$$= \frac{3}{4} - \frac{1}{3}$$

$$\boxed{E(X) = \frac{5}{12}}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\cdot\left(\frac{3}{2}-y\right) dy$$

$$= \int_0^1 \left(\frac{3}{2}y - y^2\right) dy$$

$$= \left(\frac{3}{2}\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{3}$$

$$= \frac{3}{4} - \frac{1}{3}$$

$$\boxed{E(Y) = 5/12}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \left(\frac{3}{2} - x\right) dx = \int_0^1 \left(\frac{3}{2}x^2 - x^3\right) dx$$

$$= \left[\frac{3}{2}\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{3}\right) - \frac{1}{4}$$

$$= \frac{3}{6} - \frac{1}{4}$$

$$= \frac{6}{24}$$

$$\boxed{E(x^2) = \frac{1}{4}}$$

$$\therefore \boxed{E(y^2) = 1/4}$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy.$$

$$= \int_0^1\int_0^1 xy\,(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 y\left((2-y)x - x^2\right) dx\,dy$$

$$= \int_0^1 y\left[(2-y)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 dy$$

$$= \int_0^1 y\left(\frac{1}{2}(2-y) - \frac{1}{3}\right) dy$$

$$= \int_0^1 y\left(1 - \frac{y}{2} - \frac{1}{3}\right) dy.$$

$$= \int_0^1 y\left(\frac{2}{3} - \frac{y}{2}\right) dy$$

$$= \int_0^1 \left(\frac{2}{3}y - \frac{y^2}{2}\right) dy$$

$$= \left(\frac{2}{3}\frac{y^2}{2} - \frac{y^3}{6}\right)_0^1$$

$$= \left(\frac{2}{3}\left(\frac{1}{2}\right) - \frac{1}{6}\right)$$

$$= \frac{2}{6} - \frac{1}{6}$$

$$\boxed{E(xy) = \frac{1}{6}}$$

covarience of xy.

$$\mathrm{cov}(x_i, y_i) = E(xy) - E(x)E(y)$$

$$= \frac{1}{6} - \left(\frac{5}{12}\,\frac{5}{12}\right)$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$= \frac{24-25}{144}$$

$$\boxed{\mathrm{cov}(xy) = \frac{-1}{144}}$$

$$\frac{24}{\cancel{144}}\over{6}$$

$$\mathrm{Var}(x) = E(x^2) - E(x)^2$$

$$= \frac{1}{4} - \frac{25}{144}$$

$$= \frac{36-25}{144} = \frac{11}{144}$$

$$\frac{36}{\cancel{144}}\over{4}$$

$$\mathrm{Var}(y) = E(y^2) - E(y)^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144}$$

$$= 11/144$$

H.W.

Sol Given,

$$f(x,y) = \frac{x+2y}{27} \qquad x = 0,1,2 \;;\quad y = 0,1,2.$$

x \ y	0	1	2	Σx_i
0	0	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$
1	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{5}{27}$	$\frac{9}{27}$
2	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$	$\frac{12}{27}$
Σy_i	$\frac{3}{27}$	$\frac{9}{27}$	$\frac{15}{27}$	$\frac{27}{27}$

The conditional distribution of x given y = 1

$$P(x=0 \mid y=1) = \frac{P(x=0,\ y=1)}{P(y=1)} = \frac{\frac{2}{27}}{\frac{9}{27}} = \frac{2}{9}$$

$$P(x=1 \mid y=1) = \frac{P(x=1,\ y=1)}{P(y=1)} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{1}{3}$$

$$P(x=2 \mid y=1) = \frac{P(x=2,\ y=1)}{P(y=1)} = \frac{\frac{4}{27}}{\frac{9}{27}} = \frac{4}{9}$$

The conditional distribution of y given x = 1.

$$P(y=0 \mid x=1) = \frac{P(x=1,\ y=0)}{P(x=1)} = \frac{1}{27} \Big/ \frac{9}{27} = 1/9$$

$$P(y=1 \mid x=1) = \frac{P(x=1,\ y=1)}{P(x=1)} = \frac{\frac{3}{27}}{\frac{9}{27}} = 1/3$$

$$P(y=2 \mid x=1) = \frac{P(x=1,\ y=2)}{P(x=1)} = \frac{\frac{5}{27}}{\frac{9}{27}} = 5/9.$$

Correlation coefficient of x, y is.

$$r_{xy} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} = \frac{\left(\frac{-1}{144}\right)}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{\frac{-1}{144}}{\frac{11}{144}} = -1/11.$$

$$\boxed{r_{xy} = \frac{-1}{11}}$$

x given y,

$$f(x=x \mid y=y) = \frac{f(x,y)}{f(y)} = \frac{2-x-y}{\frac{3}{2}-y}$$

y given x,

$$f(y=y \mid x=x) = \frac{f(x,y)}{f(x)} = \frac{2-x-y}{\frac{3}{2}-x}.$$

If x & y are independent then,

$$\boxed{f(x,y) = f(x).f(y)}$$

1) The joint pdf of the random variable (x,y) is given by $f(x,y) = \cancel{kxy/e}\ kxy\, e^{-(x^2+y^2)}$, $x>0$, $y>0$. Find the value of k and also prove that x & y are independent.

Sol Given,

$f(x,y) = kxy\, e^{-(x^2+y^2)}$, $x>0, y>0$.

We know that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$k\int_0^{\infty}\int_0^{\infty} xy\, e^{-x^2}\cdot e^{-y^2}\,dx\,dy = 1$$

$$k\left[\left(\int_0^{\infty} xe^{-x^2}dx\right)\left(\int_0^{\infty} ye^{-y^2}dy\right)\right] = 1$$

Let $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

when $x=0$ $t=0$

$x\to\infty$ $t=\infty$

$$\int_0^{\infty} xe^{-x^2}dx = \int_0^{\infty} e^{-t}\frac{dt}{2}$$

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^{\infty}$$

$$= \frac{1}{2}(0+1)$$

$$\int_0^{\infty} xe^{-x^2}dx = 1/2$$

Illy for $\int_0^{\infty} ye^{-y^2}dy = 1/2$.

$$k\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$$

$$\frac{k}{4} = 1$$

$$\boxed{k=4}$$

$$\therefore \boxed{f(x,y) = 4xy\,e^{-(x^2+y^2)},\ x>0,\ y>0}$$

Marginal distribution of x is,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4\int_0^{\infty} xe^{-x^2}\, ye^{-y^2}\,dy$$

$$= 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}dy$$

$$= 4xe^{-x^2}\left(\frac{1}{2}\right)$$

$$\boxed{f(x) = 2xe^{-x^2},\ x>0}$$

Marginal distribution of y is,

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4ye^{-y^2}\int_0^{\infty} xe^{-x^2}dx$$

$$= 4ye^{-y^2}\left(\frac{1}{2}\right)$$

$$\boxed{f(y) = 2ye^{-y^2},\ y>0}$$

To prove: x & y are independent,

ie) $f(x,y) = f(x)\cdot f(y)$.

$$f(x)\cdot f(y) = (2xe^{-x^2})\cdot(2ye^{-y^2})$$

$$= 4xy\,e^{-(x^2+y^2)}$$

$$\therefore \boxed{f(x)\cdot f(y) = f(x,y)}$$

$\therefore$ x & y are independent.

H.W

The joint pdf of (x,y) is given by $f(x,y) = e^{-(x+y)}$, $0<x,y<\infty$. Are x & y are independent?

Sol given,

$f(x,y) = e^{-(x+y)} \quad 0 < x, y < \infty$

To prove X & Y are independent they should satisfy the condition :

$$f(x,y) = f(x) \cdot f(y).$$

Marginal distribution of x,

$$f(x) = \int_0^\infty e^{-x-y}\, dy$$

$$= e^{-x} \int_0^\infty e^{-y}\, dy$$

$$= e^{-x} \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= e^{-x}(0+1)$$

$$\boxed{f(x) = e^{-x}}$$

Marginal distribution of y

$$f(y) = \int_0^\infty e^{-x-y}\, dx$$

$$= e^{-y} \int_0^\infty e^{-x}\, dx$$

$$= e^{-y} \left(\frac{e^{-x}}{-1}\right)_0^\infty$$

$$= e^{-y}(0+1)$$

$$\boxed{f(y) = e^{-y}.}$$

To prove X & Y are independent

i.e) $f(x,y) = f(x) \cdot f(y).$

$$f(x) \cdot f(y) = e^{-x} \cdot e^{-y}$$

$$= e^{-(x+y)}$$

$$\boxed{f(x) \cdot f(y) = f(x,y)}$$

$\therefore$ X & Y are independent.

3) If X & Y are two random variables having joint density function $f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) \\ 0, & \text{otherwise} \end{cases}$

$0 < x < 2, \quad 2 < y < 4.$

find i) $P(X<1 \cap Y<3)$ or $P(X<1, Y<3)$

ii) $P(X+Y<3)$ iii) $P(X<1 \mid Y<3).$

Sol given that.

$f(x,y) = \frac{1}{8}(6-x-y) \quad 0 < x < 2 \;;\; 2 < y < 4.$

i) $$P(X<1, Y<3) = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\, dy\, dx$$

$$= \frac{1}{8}\int_0^1 \left[(6-x)y - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1 \left[(6-x)3 - \frac{9}{2} - (6-x)2 + \frac{4}{2}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left((6-x) - \frac{5}{2}\right) dx = \frac{1}{8}\int_0^1 \left(6-x-\frac{5}{2}\right) dx$$

$$= \frac{1}{8}\int_0^1 \frac{7}{2} - x\, dx.$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = \frac{1}{8}\left(\frac{7}{2} - \frac{1}{2} - 0 + 0\right)$$

$$= \frac{1}{8}\left(\frac{6}{2}\right)$$

$$\boxed{P(X<1, Y<3) = \frac{3}{8}}$$

~~ii) P(X+Y<3).~~

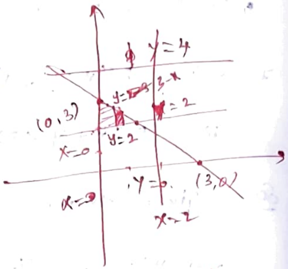
~~X = 0, 1   Y = 2, 3~~

~~X+Y   P(X+Y)~~

~~2~~

~~3~~

~~4~~



$x+y<3$

$x+y=3$

$x+y=3$

$x+2=3$

$x=1$

ii) $P(X+Y<3) = \int_2^3 \int_0^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy$

$x+y=3$

$x=3-y$

$x=0$ to $3-y$

$y=2$ to 3.

$= \frac{1}{8}\int_2^3 \left((6-y)x-\frac{x^2}{2}\right)_0^{3-y} dy.$

$= \frac{1}{8}\int_2^3 (6-y)(3-y)-\frac{(3-y)^2}{2}\,dy$

$= \frac{1}{8}\int_2^3 (3-y)\left((6-y)-\frac{(3-y)}{2}\right)$

$= \frac{1}{8}\int_2^3 (3-y)\left(\frac{12-2y-3+y}{2}\right)$

$= \frac{1}{8}\int_2^3 (3-y)\left(\frac{9-y}{2}\right)$

$= \frac{1}{8}\times\frac{1}{2}\int_2^3 27-3y-9y+y^2\,dy.$

$= \frac{1}{8}\times\frac{1}{2}\int_2^3 y^2-12y+27$

$= \frac{1}{16}\left[\frac{y^3}{3}-\frac{12y^2}{2}+27y\right]_2^3$

$= \frac{1}{16}\left[9-54+81-\frac{8}{3}+24-54\right]$

$= \frac{1}{16}\left(6-\frac{8}{3}\right)$

$= \frac{1}{\cancel{16}\,8}\left(\frac{\cancel{10}\,5}{3}\right)$

$= \frac{5}{24}$

64
54
108

90
24
114

114
108
6

iii) $P(X<1 \,/\, Y<3) = \frac{P(X<1,\ Y<3)}{P(Y<3)}$

$\therefore \boxed{P(X<1,\ Y<3) = \frac{3}{8}}$

To find $P(Y<3)$.

ie) $P(Y<3) = \int_2^3 f(y)\,dy.$

Marginal density function of Y is

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$

$= \frac{1}{8}\int_0^2 6-x-y\ \cancel{dy}\ dx.$

$12-2y-\frac{4}{2}$

$= \frac{1}{8}\left((6-y)x-\frac{x^2}{2}\right)_0^2$

$= \frac{1}{8}\left((6-y)2-\frac{4}{2}\right) = \frac{1}{8}(10-2y)$

$\boxed{f(y) = \frac{5-y}{4}}\quad 2<y<4.$

$$P(Y<3) = \frac{1}{4}\int_2^3 (5-y)\,dy$$

$$= \frac{1}{4}\left(5y - \frac{y^2}{2}\right)_2^3$$

$$= \frac{1}{4}\left(15 - \frac{9}{2} - 10 + \frac{4}{2}\right)$$

$$= \frac{1}{4}\left(5 - \frac{5}{2}\right)$$

$$= \frac{1}{4}\left(\frac{5}{2}\right)$$

$$P(Y<3) = \frac{5}{8}$$

$$\therefore P(X<1 \mid Y<3) = \frac{3/8}{5/8}$$

$$\boxed{P(X<1 \mid Y<3) = \frac{3}{5}}$$

4) If the joint distibution funuton of X & Y is given by $F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x>0,\ y>0 \\ 0, & \text{Otherwise.} \end{cases}$

i) Find the Marginal density fn. of X & Y.

ii) Are X & Y are independent.

iii) $P(1<X<3,\ 1<Y<2)$.

Sol Given

$$F(x,y) = (1-e^{-x})(1-e^{-y}), \quad x>0,\ y>0.$$

We know that,

$$f(x,y) = \frac{\partial^2}{\partial x\,\partial y} F(x,y)$$

$$= \frac{\partial^2}{\partial x\,\partial y}\left((1-e^{-x})(1-e^{-y})\right)$$

$$= \frac{\partial^2}{\partial x\,\partial y}\left(1 + e^{-x}e^{-y} - e^{-x} - e^{-y}\right)$$

$$= \frac{\partial}{\partial x}\left(-e^{-x}e^{-y} + e^{-y}\right)$$

$$= e^{-x}e^{-y}$$

$$\boxed{f(x,y) = \cancel{e^{-x}e^{-y}}\ e^{-(x+y)},\quad x>0,\ y>0}$$

Marginal density fun. of x,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} e^{-x}.e^{-y}\,dy$$

$$= e^{-x}\int_0^{\infty} e^{-y}\,dy$$

$$= e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty}$$

$$= e^{-x}(0+1)$$

$$f(x) = e^{-x}$$

Marginal density fn of y,

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} e^{-x}.e^{-y}\,dx$$

$$= e^{-y}\int_0^{\infty} e^{-x}\,dx$$

$$= e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$= e^{-y}(0+1)$$

$$f(y) = e^{-y}$$

W.K.T,

$$f(x,y) = f(x).f(y). \quad \therefore X \,\&\, Y \text{ are independent.}$$

$$\therefore f(x).f(y) = e^{-x}.e^{-y}$$

$$= e^{-(x+y)}$$

$$\therefore \boxed{f(x).f(y) = f(x,y)}$$

iii) $P(1<X<3,\ 1<Y<2)$.

$$= \int_1^3 \int_1^2 e^{-x} e^{-y}\, dy\, dx.$$

$$= \int_1^3 e^{-x} \left(\frac{e^{-y}}{-1}\right)_1^2 dx.$$

$$= \int_1^3 e^{-x} \left[-e^{-2} + e^{-1}\right] dx.$$

$$= e^{-1} - e^{-2} \left(\frac{e^{-x}}{-1}\right)_1^3$$

$$= e^{-1} - e^{-2} - e^{-3} + e^{-1}$$

$$= (e^{-1} - e^{-2})(e^{-1} - e^{-3})$$

$$= 0.0739$$

H.W

5) Suppose that the two dimensional RV (X,Y) has the joint pdf, $f(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$

i) obtain the correlation coefficient b/w X & Y

ii) check whether X & Y are independent

Sol given.

$f(x,y) = x+y \ ; \ 0 \leq x < 1,\ 0<y<1.$

Marginal density fn of X,

$$f(x) = \int_0^1 (x+y)\, dy. = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$$

Marginal density fn of Y,

$$f(y) = \int_0^1 (x+y)\, dx = \left[\frac{x^2}{2} + xy\right]_0^1 = \frac{1}{2} + y.$$

For independent of X & Y,

$$f(x,y) = f(x) \cdot f(y) = \left(x+\frac{1}{2}\right)\left(y + \frac{1}{2}\right)$$

$$= xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4}$$

$\therefore f(x,y) \neq f(x) f(y)$

$\therefore$ X & Y are not independent.

~~cov(x,y)~~ $r_{xy} = \dfrac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$

$\operatorname{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$E(XY) = \iint xy\, f(x,y)\, dx\, dy$

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx. \Rightarrow Y$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx \quad -\!\!\!\!\!\!\;\; y.$

$\operatorname{Var}(X) = E(X^2) - (E(X))^2$

$\operatorname{Var}(Y) = E(Y^2) - (E(Y))^2$

$\sigma_x = \sqrt{\operatorname{Var}(x)}$

$\sigma_y = \sqrt{\operatorname{Var} y}.$

Correlation & Regression for sample data:-

$$\operatorname{cov}(x,y) = \frac{1}{n}\sum xy - \bar{x}\bar{y}$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - \bar{x}^2}$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - \bar{y}^2}$$

Regression lines:-

i) Regression line of x on y,

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

$$(x-\bar{x}) = b_{xy}(y-\bar{y})$$

ii) Regression line of y on x.

$$(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

$$(y-\bar{y}) = b_{yx}(x-\bar{x})$$

b_{xy} & b_{yx} are called the regression coefficient.

Note:-

1) $$r\frac{\sigma_x}{\sigma_y} \cdot r\frac{\sigma_y}{\sigma_x} = b_{xy} \cdot b_{yx}$$

$$r^2 = b_{xy} \cdot b_{yx}$$

$$r = \pm\sqrt{b_{xy} \cdot b_{yx}}$$

2) If $r = \pm 1$ regression lines coincide.

3) If $r = 0$, regression lines are $\perp r$.

4) If the regression lines passing through the mean values $(\bar{x},\bar{y})$.

1) From the following data, find

i) Coefficient of Correlation between marks in Economics & statistics.

ii) The two regression lines.

iii) The most likely in statistics when the marks in Economics are 30

Marks in Economics :	25	28	35	32	31	36	29	38	34	32
Marks in Statistics :	43	46	49	41	36	32	31	30	33	39

Sol.

Let x be marks in Economics & y be marks in Statistics.

x	y	x^2	y^2	xy
25	43			
28	46			
35	49			
32	41			
31	36			
36	32			
29	31			
38	30			
34	33			
32	39			
320	380	10380	14838	12067

$\bar{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$

$\bar{y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$

$\frac{\Sigma xy}{n} - \bar{x}\bar{y}$

$\sigma^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2$

$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2}$

$cov(x,y) = \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$

$= \frac{1}{10}(12067) - (32)(38)$

$\boxed{cov(x,y) = -9.3}$

$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2} = \sqrt{\frac{1}{10}(10380) - (32)^2} = 3.742$

$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2} = \sqrt{\frac{1}{10}(14838) - (38)^2} = 6.309$

The correlation coefficient b/w x & y is

$r_{xy} = \frac{cov(x,y)}{\sigma_x \sigma_y}$

$= \frac{-9.3}{(3.742)(6.309)}$

$\boxed{r_{xy} = -0.394}$

Regression line of x on y is.

$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$

$(x-32) = (-0.394)\frac{(3.742)}{(6.309)}(y-38)$

$\boxed{x = -0.234y + 40.88}$

Regression lines of y on x is,

$(y-\bar{y}) = r \frac{\sigma y}{\sigma x}(x-\bar{x})$

$(y-38) = (-0.394)\left(\frac{6.309}{3.742}\right) x-32$

$\boxed{y = -0.664x + 59.25}$

iii) The Marks in economics is 30.

ie $x = 30$.

$y = -0.664(30) + 59.25$

$y = 39.33$

$\boxed{y \simeq 39}$

2) The height of fathers and sons are given in cms

Height of fathers :	150	152	155	157	160	161	164	166
Height of sons :	154	156	158	159	160	162	161	164

Find the two lines of regressions and calculate the expected any height of the son when the height of Father is 154 cm.

3) The regression equations are $3x+2y=26$ & $6x+y=31$. Find the correlation coefficient between x & y. Also find Mean values of x & y.

Sol Given.

$3x+2y=26 \rightarrow (1)$

$6x+y=31 \rightarrow (2)$

By solving (1) & (2).

$x = 4 \quad y = 7$

The regression lines passing through mean value $(\bar{x}, \bar{y})$.

$\therefore \bar{x} = 4 \quad \bar{y} = 7$

To find: Correlation Coefficient b/w x & y.

(1) ⇒ x on y

$3x = 26 - 2y$

$x = \frac{26}{3} - \frac{2}{3}y$

$x = -\frac{2}{3}y + \frac{26}{3}$

$(x-\bar{x}) = bxy + (y-\bar{y})$

$bxy = -2/3$

(2) ⇒ y on x

$y =$

(1) ⇒ y on x

$2y = 26 - 3x$

$y = -\frac{3}{2}x + \frac{26}{2}$

$byx = -\frac{3}{2}$

(2) ⇒ x on y.

$6x = 31 - y$

$x = -\frac{1}{6}y + \frac{31}{6}$

$bxy = -1/6$

W.K.T

$r = \pm\sqrt{bxy \cdot byx}$

$= \pm\sqrt{\left(-\frac{1}{6}\right)\cdot\left(-\frac{3}{2}\right)}$

$= \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$\therefore \boxed{r = -1/2}$ $\because$ bxy & byx are negative.

Transformation of 2-Dimensional Random variable.

$$f_{UV}(u,v) = |J|\, f_{XY}(x,y)$$

where $|J| = \dfrac{\partial(x,y)}{\partial(u,v)}$.

Note:-

If U is not given, then assume $U = x$

If V is not given then assume $V = y$

1) If the joint pdf of (x,y) is given by

$f(x,y) = x+y$, $0 \le x,y \le 1$. Find the pdf of

$U = xy$.

Sol.

Given $f(x,y) = x+y$ $\quad 0 \le x,y \le 1$.

$U = xy$ $\quad$ V ~~is~~ Let $V = y$.

$u = xy$ $\quad$ $v = y$

$x = \dfrac{u}{y}$ $\quad$ $\boxed{y = v}$

$\boxed{x = \dfrac{u}{v}}$

W.K.T.

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$x = \dfrac{u}{v}$ $\quad$ $y = v$

$\dfrac{\partial x}{\partial u} = 1/v$ $\quad$ $\dfrac{\partial y}{\partial u} = 0$

$\dfrac{\partial x}{\partial v} = \dfrac{-u}{v^2}$ $\quad$ $\dfrac{\partial y}{\partial v} = 1$

$$\therefore |J| = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$\boxed{|J| = 1/v}$

Pdf of (U,V) is,

$$f_{UV}(u,v) = |J|\, f_{xy}(x,y)$$

$$= \frac{1}{v}(x+y)$$

$$= \frac{1}{v}\left(\frac{u}{v} + v\right)$$

$\boxed{f_{UV}(u,v) = \dfrac{u}{v^2} + 1}$, $\quad 0 \le u \le v$, $\quad 0 \le v \le 1$

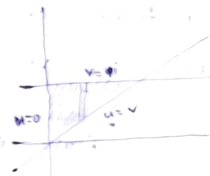
$\left[\dfrac{uv}{v^2} + v\right]_u^1$

Limits, $0 \le x \le 1$ $\quad$ $0 \le y < 1$

$0 \le \dfrac{u}{v} < 1$ $\quad$ $0 \le v < 1$

$0 \le u \le v$

To find $f_U(u)$



$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

$$= \int_{v=u}^{1} \frac{u}{v^2} + 1 \, dv$$

$$= \int_{v=u}^{1} (uv^{-2} + 1)\, dv$$

$$= \left(\frac{uv^{-1}}{-1} + v\right)_u^1$$

$$= \left(-\frac{u}{v} + v\right)_u^1$$

$$= \left[-u + 1 + \frac{u}{u} - u\right]$$

$$= 2 - 2u$$

$\boxed{f_U(u) = 2(1-u)}$ $\quad 0 \le u \le v$

2) If X & Y each follow an exponential distribution with parameter 1 and are independent. Find the pdf of $U = X - Y$.

Sol.

W.K.T., exponential distribution is,

$f(x) = \lambda e^{-\lambda x}, x \geq 0$ since X follows an exponential distribution with parameter 1, i.e. $\lambda = 1$

$$\therefore \boxed{f(x) = e^{-x}, x \geq 0}$$

Also, Y follows an exponential distribution with parameter 1, i.e. $\lambda = 1$

$$\therefore \boxed{f(y) = e^{-y}, y \geq 0}$$

X & Y are independent,

$$\therefore f(x,y) = f(x) \cdot f(y)$$
$$= e^{-x} \cdot e^{-y}$$
$$\boxed{f(x,y) = e^{-(x+y)}}$$

Given, $U = X - Y$, $V = Y$

$u = x - y$, $v = y$

$\boxed{x = u + v}$ $\boxed{y = v}$

$\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = 1$, $\frac{\partial y}{\partial v} = 1$

$$\therefore |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$\boxed{|J| = 1}$$

To find the limits:-

The joint pdf of (U,V) is,

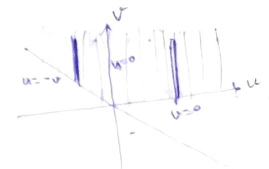
$$f(u,v) = |J| f(x,y)$$
$$= 1 \; e^{-(x+y)}$$
$$= e^{-(u+v+v)}$$
$$\boxed{f(u,v) = e^{-(u+2v)}}$$ $u \geq -v$, $v \geq 0$.

To find limits,

$x \geq 0$, $y \geq 0$

$u + v \geq 0$, $v \geq 0$

$u \geq -v$



To find $f_U(u)$,

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

Case i) $u < 0$

$$f(u) = \int_{v=-u}^{\infty} e^{-(u+2v)}\, dv$$
$$= e^{-u} \int_{-u}^{\infty} e^{-2v}\, dv$$
$$= e^{-u} \left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}$$
$$= e^{-u} \left(0 + \tfrac{1}{2} \cdot e^{2u}\right)$$
$$= e^{-u} \left(\frac{e^{2u}}{2}\right)$$

$$\boxed{f(u) = \frac{e^{u}}{2}, \; u < 0}$$

Case ii) $u \geq 0$

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

$$= \int_{0}^{\infty} e^{-(u+2v)}\, dv$$

$$= e^{-u} \int_{0}^{\infty} e^{-2v}\, dv$$

$$= e^{-u} \left[\frac{e^{-2v}}{-2}\right]_0^{\infty}$$

$$= e^{-u}\left(0 + \tfrac{1}{2}\right)$$

$$\boxed{f(u) = \frac{e^{-u}}{2}}$$

Rules for Covarience.

1) If x & y are independent, $\text{cov}(x,y)=0$

2) $\text{cov}(ax, by) = ab\,\text{cov}(x,y)$

3) $\text{cov}(x+a, y+b) = \text{cov}(x,y)$

4) $\text{cov}(x+y, z) = \text{cov}(x,z) + \text{cov}(y,z)$

5) $\text{var}(x+y) = \text{var}(x) + \text{var}(y) + 2\,\text{cov}(x,y)$

6) $\text{var}(x-y) = \text{var}(x) + \text{var}(y) - 2\,\text{cov}(x,y)$

7) $\text{cov}(x,x) = E(x^2) - (Ex)^2 = \text{var}(x)$

8) x & y are independent $\Rightarrow$ $E(xy) = E(x)\cdot E(y)$.

$\text{cov}(x,y)=0$, $r=0 \Rightarrow x$ & y are uncorreleated.

1) If the independent random variable x & y have varience 36 and 16 respecitively. Find the correlation coefficient between $x+y$ & $x-y$.

$r_{uv} = \dfrac{\text{cov}(u,v)}{\sigma_u \sigma_v}$.

Sol Let

$U = x+y$ $\quad V = x-y$

$\because$ Since x & y are independent, $\text{cov}(x,y)=0$

$$r_{uv} = \frac{\text{cov}(U,V)}{\sigma_U \sigma_V}$$

Given $\text{var}(x) = 36$ $\quad \text{var}(y) = 16$

$\sigma_x = 6$ $\quad \sigma_y = 4$

$$\text{cov}(u,v) = \text{cov}(x+y, x-y) = \text{cov}(x,x) - \text{cov}(x,y) + \text{cov}(x,y) + \text{cov}(y,y)$$

$$= \text{var}(x) - \text{var}(y) = 36 - 16 = 20.$$

$\therefore$ $\boxed{\text{cov}(U,V) = 20}$

$$\text{Var}(U) = \text{Var}(x+y) = \text{var}(x) + \text{var}(y) + 2\,\text{cov}(x,y)$$

$$= 36 + 16 + 0$$

$$\boxed{\text{Var}(U) = 52}$$

$$\text{Var}(V) = \text{var}(x-y) = \text{var}(x) + \text{var}(y) - 2\,\text{cov}(x,y)$$

$$= 36 + 16 - 0$$

$$\boxed{\text{Var}(V) = 52}$$

$\sigma_U = \sqrt{\text{var}(U)} = \sqrt{52}$

$\sigma_V = \sqrt{\text{Var}(V)} = \sqrt{52}$

$\therefore$ The correlation coefficient bet. $x+y$ & $x-y$

$$r_{uv} = \frac{\text{cov}(U,V)}{\sigma_U \sigma_V} = \frac{20}{\sqrt{52}\sqrt{52}} = \frac{20}{52}$$

$$\boxed{r_{uv} = 0.3846}$$