Journal of the Physical Society of Japan
Vol. 56, No. 12, December, 1987, pp. 4210–4213

Lyapunov Spectrum of a Chaotic Model of Three-Dimensional Turbulence

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(Received September 21, 1987)

A model equation of fully developed three-dimensional turbulence is proposed which exhibits the Kolmogorov's similarity law in its chaotic state. The structure of the chaotic attractor is investigated by examining the Lyapunov spectrum for several values of viscosity. The Lyapunov spectrum has a scaling property in the interior of the attractor. It appears that the distribution function of the Lyapunov exponents has a singularity at a null Lyapunov exponent in the inviscid limit.

Fully developed three-dimensional Navier-Stokes turbulence may be regarded as the manifestation of a strange (chaotic) attractor of very large dimensions in an infinite dimensional phase space (function space) from a viewpoint of a theory of chaotic dynamical systems.[1-3] Recent advances in the chaos theory give several methods of characterization of the strange attractors, which are now indispensable in the research of weak (low-dimensional) turbulence, *e.g.*, in convection systems.[4] It is, therefore, interesting to examine the properties of fully developed turbulence, including the similarity law, from the viewpoint of a strange attractor. However, the methods of the chaos theory cannot be applied directly to fully developed turbulence, mostly because of the insufficient capacity of present computers and the large dimensions of the strange attractor, estimated to be more than 10^6.

Thus, at the present stage, one possible way to study the turbulence is to treat a tractable model equation which shares some important properties with the Navier-Stokes equation.[5] Recently, Grappin *et al.*[6] proposed a new cascade model of MHD turbulence in which the solution is not only unsteady, but also chaotic, and Kolmogorov's $k^{-5/3}$-form[7] is observed in the time averaged spectrum. They computed the Lyapunov spectrum and verified that the Kaplan-Yorke dimension of the strange attractor is compatible with the Kolmogorov similarity law. Also, in this regard, the chaotic solutions of Gledzer's model equation[8] of two-dimensional Navier-Stokes turbulence was investigated,[9] and it was found that the time-averaged energy spectrum has the same similarity form as that derived in the enstrophy cascade theory by Batchelor, Kraichnan and Leith.[10] It was also found that the Lyapunov spectrum of the Gledzer's model has a similarity law and that the distribution function of the Lyapunov exponents appears to have a singularity at a null Lyapunov exponent.

It this letter, we propose a model equation of fully developed three-dimensional Navier-Stokes turbulence, which realizes, in its chaotic state, the Kolmogorov similarity law and $k^{-5/3}$-form of the energy spectrum in the inertial subrange. This model equation is not intended to be one which can be derived from the Navier-Stokes equation on a more or less ad hoc hypothesis, but is devised to investigate the characteristics of the strange attractor that can be related to the Kolmogorov similarity law. Therefore, no special attention will be paid to the justification of our model equation.

The model is constructed in the wavenumber space, defined as $k_n = k_0 q^n$ $(q>1, 1\leqq n\leqq N)$. The velocity is expressed by a set of complex variables, $\{u_n\}$, where u_n associated with k_n stands for the velocity components whose wavenumbers, k, lie between k_n and k_{n+1}; $k_n < |k| < k_{n+1}$. In Gledzer's model of two-dimensional turbulence, each u_n that appeared to be a complex variable in the Navier-Stokes equation was taken as a real variable. In the present model, however, we assume u_n to be the complex variable,[6] the inner product of velocities $\{u_n=u_n^{\mathrm{R}}+\mathrm{i}\,u_n^{\mathrm{I}}\}$ and $\{v_n=v_n^{\mathrm{R}}+\mathrm{i}\,v_n^{\mathrm{I}}\}$ being $\Sigma_{j=1}^{N}(u_j^{\mathrm{R}}v_j^{\mathrm{R}}+u_j^{\mathrm{I}}v_j^{\mathrm{I}})$, where $\mathrm{i}=\sqrt{-1}$ and the superscripts R and I denote the real and the imaginary parts, respectively. Therefore, for each k_n there are two degrees of freedom, and in this sense it is considered to be a two-component model.[11] The phase space of the system is constructed from $2N$ real variables, $\{u_n^{\mathrm{R}}, u_n^{\mathrm{I}}\}$. The energy, E, the entrophy, Q, and the energy spectrum, $E(k)$, are defined as $E=\Sigma_n\,|u_n|^2/2$, $Q=\Sigma_n\,k_n^2|u_n|^2/2$ and $E(k_n)=|u_n|^2/(2k_n)$, respectively.

Each evolution equation for u_n is assumed to be quadratically nonlinear and connected with u_{n-1}, u_{n-2} and u_{n+1}, u_{n+2}. The conservation of the phase volume, $\Sigma_n\,\{\partial\dot{u}_n^{\mathrm{R}}/\partial u_n^{\mathrm{R}}+\partial\dot{u}_n^{\mathrm{I}}/\partial u_n^{\mathrm{I}}\}=0$, is also assumed to hold in the inviscid unforced case, where $\dot{u}$ denotes the time derivative. The following equation for $\{u_n\,|\,1\leqq n\leqq N\}$ is adopted satisfying the above properties:

$$(\mathrm{d}/\mathrm{d}t+\nu k_n^2)u_n=\mathrm{i}\,[c_n^{(1)}u_{n+1}^*u_{n+2}^*+c_n^{(2)}u_{n-1}^*u_{n+1}^*+c_n^{(3)}u_{n-1}^*u_{n-2}^*]+f\delta_{n,4}, \qquad (1)$$

where * denotes the complex conjugate, f is the time-independent force, ν the kinematic viscosity, δ Kronecker's delta, and t is time. This equation may be considered as a complex version of Gledzer's model. The real constants $c_n^{(1)}$, $c_n^{(2)}$, $c_n^{(3)}$ are given as follows:

$$c_n^{(1)}=k_n,\ c_n^{(2)}=-\beta k_{n-1},\ c_n^{(3)}=(\beta-1)k_{n-2},$$
$$c_1^{(2)}=c_1^{(3)}=c_2^{(3)}=c_{n-1}^{(1)}=c_n^{(1)}=c_n^{(2)}=0, \qquad (2)$$

where β is a real parameter, so that the nonlinear terms of eq. (1) conserve the energy, E. Here we choose $\beta=1/2$. We note that for a value of β larger than unity, eq. (1) conserves $\Sigma_{j=1}^{N}k_j^{\alpha}|v_j|^2$ together with energy, where α is given by the relation $\beta=1+1/q^{\alpha}$.*

We discuss below the numerical solutions of eq. (1) with parameters $k_0=2^{-4}$, $q=2$, $f=5\times(1+\mathrm{i})\times10^{-3}$ and $(\nu, N)=(10^{-3}, 12)$, $(10^{-4}, 14)$, $(10^{-5}, 17)$, $(10^{-6}, 19)$, $(10^{-7}, 22)$, $(10^{-8}, 24)$, $(10^{-9}, 27)$, where N was large enough so that a plausible convergence was obtained with respect to the statistical quantities. Time marching was performed by the fourth order Runge-Kutta method slightly modified to handle the viscous term efficiently. The computation was carried out on the vectorial computer, VP-200, at Kyoto University. The initial condition of the numerical integration was chosen so that $E(k)=k^2\exp[-k^2]$. After an initial transient period, an unsteady but statistically stationary state is observed. All the quantities discussed below were obtained in this stationary state.

According to the Kolmogorov's universal equilibrium theory (K41),[7] the energy dissipation wavenumber, k_d, and the energy spectrum in the inertial range are expressed as $k_d=\varepsilon^{1/4}\nu^{-3/4}$, $E(k)=\varepsilon^{1/4}\nu^{5/4}E_e(k/k_d)$, where ε, ν and k denote the energy dissipation rate, kinematic viscosity, and wavenumber, respectively, and E_e is a nondimensional function. In the inertial subrange, the energy spectrum takes the well-known power form of $E(k)\sim\varepsilon^{2/3}k^{-5/3}$. In Fig. 1, the time-averaged energy spectra for several values of viscosity which are normalized following the K41 theory is shown. Good agreement of the spectra is obtained and they exhibit[11] the $k^{-5/3}$ spectrum in the inertial subrange. Moreover, the energy flux in the inertial subrange is fairly constant and equal to the energy dissipation rate

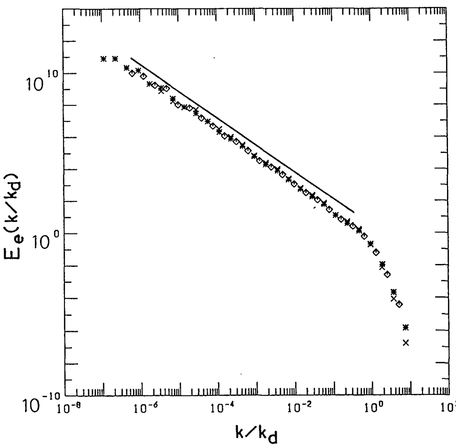

Fig. 1. The time averaged energy spectrum normalized using the Kolmogorov similarity law for $\nu=(\times)\,10^{-7}$, $(\diamond)10^{-8}$, $(*)10^{-9}$. The straight line shows the slope of $-5/3$.

* The relation between the constant of motion and the power law of the energy spectrum in the inertial subrange will be discussed in a separate paper.

$\varepsilon=2\nu Q$ (Fig. 2). Thus the K41 scaling law of the energy spectrum is embodied in the present model.

We computed the Lyapunov spectrum, $\{\lambda_j | 1 \leqq j \leqq N\}$, and its corresponding Lyapunov vectors, $v_n^{(j)}$ $(1 \leqq n, j \leqq N)$, by the use of the Gram-Schmidt orthogonalization method,[2,3] where $\sqrt{\Sigma_n |v_n^{(j)}|^2}$ was the norm of $v_n^{(j)}$, consistent with the inner product introduced previously, and the Lyapunov exponents are ordered as $\lambda_j \geqq \lambda_{j+1}$. The numerical integration was performed until a plausible convergence was obtained with respect to the Lyapunov exponents.

In every case, we calculated that some of the Lyapunov exponents are positive, which means the motion is due to a strange attractor. The magniude of the first (maximum) Lyapunov exponent λ_1 behaves as $\nu^{-1/2}$ (Fig. 3), similar to the results of the MHD model turbulence.[6] We note that the energy type spectrum of the Lyapunov vector associated with the first Lyapunov exponent exhibits a power form $k^{-1.25}$ less steep than Kolmogorov's $k^{-5/3}$-form in the inertial subrange.

In Fig. 4 it is shown that the Kaplan-Yorke dimension of the attractor, $D=p+\Sigma_{j=1}^{p} \lambda_j / |\lambda_{p+1}|$, $(p=\max\{m | \Sigma_{j=1}^{m} \lambda_j \geqq 0\})$, and the dissipative wavenumber are related as $2^{D/2} \propto k_d$. This means that the dimension of the attractor increases in proportion to the size of the inertial range, since $k_n=2^{n-4}$, and there are two degrees of freedom for each k_n. The Kolmogorov entropy H, where $H=\Sigma_{j=1}^{q} \lambda_j$, $(\lambda_q>0,\ \lambda_{q+1} \leqq 0)$, also behaves as $H \propto \nu^{-1/2}$ (Fig. 3), which agrees with the results of MHD

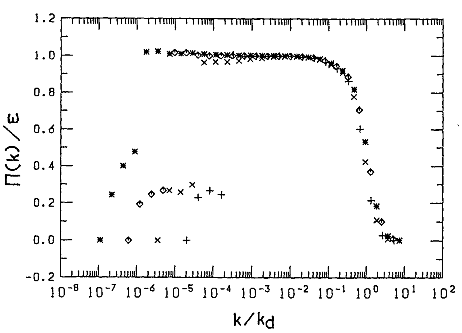

Fig. 2. The energy flux function, $\Pi(k)$, normalized by the energy dissipation rate, ε, the value of which is typically 1.5×10^{-3}. The values of the viscosity are $\nu=(+)10^{-6}$, $(\times)10^{-7}$, $(\diamond)10^{-8}$ and $(*)10^{-9}$.

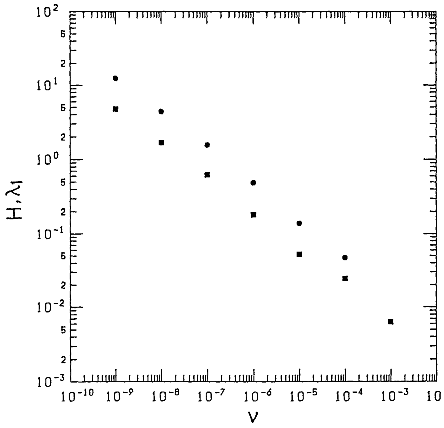

Fig. 3. The ν-dependence of the first Lyapunov exponent, λ_1, and the Kolmogorov entropy, H.

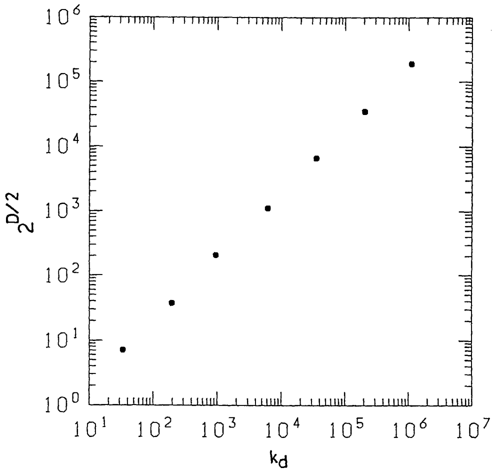

Fig. 4. The relation between the Kaplan-Yorke dimension, D, and the energy dissipation wavenumber, k_d.

model turbulence. Further the ν-dependence of H and λ_1 agrees with that of the reciprocal of the time scale at the highest wavenumber in the inertial subrange,[12] $\sqrt{\varepsilon/\nu}$, which is expected from the dimensional analysis of Kolmogorov type. The values of H and λ_1 themselves are, however, about 10^2 times smaller than $\sqrt{\varepsilon/\nu}$. This will be discussed in a subsequent paper.

In Fig. 5, $\Sigma_{i=1}^{j} \lambda_i$ normalized by H is plotted, taking j/D as the ordinate for $\nu=10^{-7}$, 10^{-8}, 10^{-9}. There is a good agreement between the plots for $j<D$ (the interior of the attractor[3]) except where j/D is a very small value. This agreement is not so clear for the larger values of viscosity (graphs omitted). Thus, the Lyapunov spectrum has a definite scaling law

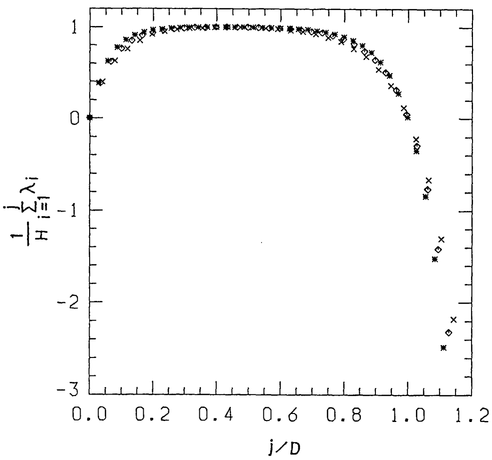

Fig. 5. The similarity of the Lyapunov spectrum.

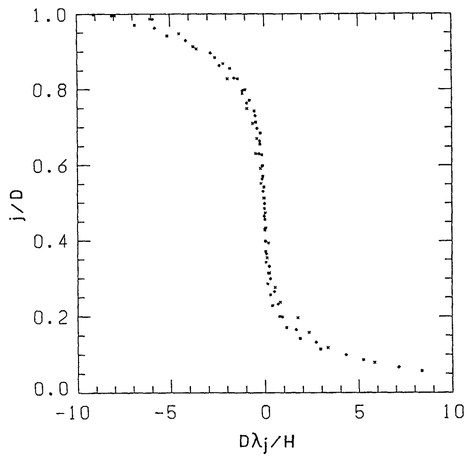

Fig. 6. The nondimensional function, f, such that $j/D=f(D\lambda_j/H)$. The distribution function of the Lyapunov exponent is proportional to $-\mathrm{d}f/\mathrm{d}\lambda$.

in the interior of the attractor in the inviscid limit. On the other hand, the Lyapunov exponents, λ_j, in the exterior of the attractor[3] (here $j \gg D$) coincide with a linear damping rate, namely $\lambda_{2j}=\lambda_{2j+1}=-\nu k_j^2$, where the degeneracy is due to the two-component character. These two properties of the Lyapunov exponents are perfectly analogous to those in the two-dimensional case,[9] while ν-dependence of H and λ_1 are not.

Finally, the distribution function of the Lyapunov exponents is considered. In Fig. 6, we show a nondimensional function, f, such that $j/D=f(D\lambda_j/H)$, where the j is the index of the Lyapunov exponent and is also the number of the Lyapunov exponents between λ_j and λ_1, $^{\#}\{\lambda_i \mid \lambda_j \leqq \lambda_i \leqq \lambda_1\}$. It is interesting that as in the Gledzer's model,[8,9] the distribution function of the Lyapunov exponent, which is proportional to $-\mathrm{d}f/\mathrm{d}\lambda$, appears to diverge in the inviscid limit. This type of singularity was suggested analytically by Ruelle[13] for three-dimensional Navier-Stokes turbulence in connection with an intermittency phenomenon, although the validity of his approximation has not yet been verified. Further analysis on this singularity is now under investigation and will be reported elsewhere together with a detailed study of this model.

Acknowledgement

The authors wish to express their cordial thanks to Professors T. Tatsumi and S. Kida for their valuable discussions.

References

1) U. Frisch: Phys. Scr. **T9** (1985) 137.
2) J. P. Eckmann and D. Ruelle: Rev. Mod. Phys. **57** (1985) 617.
3) K. Ikeda and K. Matsumoto: J. Stat. Phys. **44** (1986) 955.
4) T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. I. Schraiman: Phys. Rev. A**33** (1986) 114.
5) V. N. Desnyansky and E. A. Novikov: Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **10** (1974) 127; T. L. Bell and M. Nelkin: Phys. Fluids **20** (1977) 345; T. L. Bell and M. Nelkin: J. Fluid Mech. **88** (1978) 8; E. D. Siggia: Phys. Rev. A**17** (1978) 1166; T. Nakano and M. Nelkin: Phys. Rev. A**31** (1985) 1980.
6) R. Grappin, J. Leorat and A. Pouquet: J. Phys. (France) **47** (1986) 1127; C. Gloaguen, J. Leorat, A. Pouquet and R. Grappin: Physica **17D** (1985) 154.
7) A. N. Kolmogorov: C. R. Acad. Sci. USSR **30** (1941) 301.
8) E. B. Gledzer: Sov. Phys. Dokl. **18** (1973) 216.
9) M. Yamada and K. Ohkitani: in preparation.
10) G. K. Batchelor: Phys. Fluids Suppl. **12** (1969) II233; R. H. Kraichnan: Phys. Fluids **10** (1967) 1417; C. E. Leith: Phys. Fluids **1** (1968) 671.
11) U. Frisch: *Turbulence and Predictability of Geophysical Fluid Dynamics*, Varenna Summer School, 1983 (North-Holland, 1985).
12) D. Ruelle: Phys. Lett. **72A** (1979) 81.
13) D. Ruelle: Commun. Math. Phys. **87** (1982) 287.