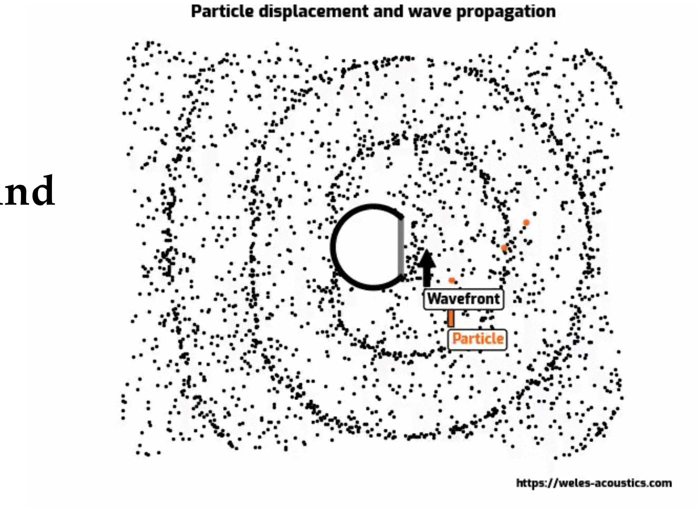
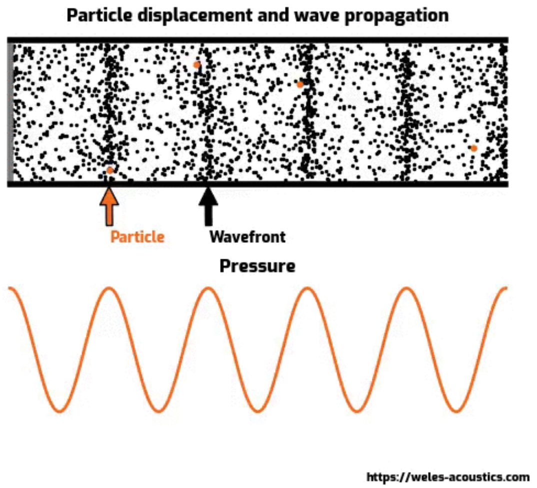
Acoustic particle velocity

- Consider an **infinitely long tube (filled with air) with a piston at one end of the tube moving at a fixed audible frequency**
  - A *plane sound pressure wave* is formed, traveling along the tube and creating local areas of high/low pressure/density
  - Since the tube is filled with air particles, *the moving piston will cause the air particles to move as well*
  - However, the air particles will **not** be displaced along the tubes' length
  - The **air particles will oscillate around their respective points of origin**
  - The **velocity of this oscillation is known as the** **acoustic particle velocity**
  - The same behavior of air particles is also observed in free-field conditions

Spherical speaker sound radiation



A piston moving at a fixed audible frequency, causing a plane wave to travel in a tube



Acoustic particle velocity

- **Acoustic particle velocity** - physical speed of a parcel of fluid as it moves back and forth in the direction the sound wave is travelling as it passes
- Knowing the pressure perturbation p' in a one-dimensional acoustic wave, we can evaluate the acoustic particle velocity u' or density perturbation ρ'
- Consider a **one-dimensional acoustic wave** that propagates at the speed of sound c_o with the following **assumptions**:
  - *Small* pressure perturbations, $p' \ll p_o$
  - *Small* density perturbations, $\rho' \ll \rho_o$
  - *Small* acoustic particle velocity, $u' \ll c_o$

Linear acoustics

- Applying the **continuity equation (non-linear) on a bulk of the medium (*stationary*):**

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

OR

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0}$$

Index notation

Linearization is needed…

Acoustic particle velocity

- The air particle density is given by: $\boxed{\rho = \rho_o + \rho'}$
- Since the medium is stationary ($u_o = 0$) - **no fluid motion** – we can write: $\boxed{u = u'}$
  - Velocity perturbations are only due to the acoustic wave propagation
- Plugging the above into the continuity equation (**1D** analysis) results in:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial(\cancel{\rho_o} + \rho')}{\partial t} + \frac{\partial((\rho_o + \rho')u')}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}(\rho_o u' + \cancel{\rho' u'}) = 0$$

$$\frac{\partial \rho_o}{\partial t} = 0$$

The cross-product of small quantities is neglected in the linearization

$$\Rightarrow \boxed{\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x_i} = 0}$$

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x_i} = 0}$$

Continuity Eq.

Acoustic particle velocity

- Consider a general plane acoustic wave traveling to the right: $\boxed{p' = f(x - c_o t)}$
- Previously, we derived the relation: $\boxed{p' = c_o^2 \rho'}$
- Therefore, we can define the density perturbations as: $\boxed{\rho' = \frac{1}{c_o^2} f(x - c_o t)}$
- We define: $\mathrm{X} = x - c_o t$, and so: $\boxed{\rho' = c_o^{-2} f(\mathrm{X})}$
- Thus, the time derivative of the density can be expressed as:

$$\frac{\partial \rho'}{\partial t} = \frac{1}{c_o^2}\frac{\partial f(X)}{\partial t} = \frac{1}{c_o^2}\frac{\partial f}{\partial \mathrm{X}}\frac{\partial \mathrm{X}}{\partial t} \Rightarrow \frac{\partial \rho'}{\partial t} = \frac{1}{c_o^2}\frac{\partial f}{\partial \mathrm{X}}(-c_o) \Rightarrow \boxed{\frac{\partial \rho'}{\partial t} = -\frac{1}{c_o}\frac{\partial f}{\partial \mathrm{X}}}$$

- Plugging the above the *continuity equation*, yields:

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} = 0 \Rightarrow \frac{1}{c_o}\frac{\partial f}{\partial \mathrm{X}} = \rho_o \frac{\partial u'}{\partial x} \Rightarrow \frac{\partial u'}{\partial x} = \frac{1}{\rho_o c_o}\frac{\partial f}{\partial \mathrm{X}}$$

$$\Rightarrow \frac{\partial u'}{\partial x} = \frac{1}{\rho_o c_o}\frac{\partial f}{\partial x}\frac{\partial x}{\partial \mathrm{X}} \Rightarrow \frac{\partial u'}{\partial x} = \frac{1}{\rho_o c_o}\frac{\partial f(x - c_o t)}{\partial x} \Rightarrow \frac{\partial u'}{\partial x} = \frac{1}{\rho_o c_o}\frac{\partial p'}{\partial x}$$

integration

For a plane wave: u' and p' are <u>in-phase</u>

$$\boxed{u' = \frac{p'}{\rho_o c_o}}$$

$$\boxed{p' = \rho_o c_o u'}$$

Acoustic particle velocity - *example*

$$\boxed{u' = \frac{p'}{\rho_o c_o}}$$

$$\boxed{p' = \rho_o c_o u'}$$

- The threshold of human hearing is normally 0 dB
- Calculate the acoustic particle velocity and displacement of a plane wave at 1 kHz at SL

$$u' = \frac{p'}{\rho_o c_o} \quad \overset{p' \approx p_{\text{rms}}}{\underset{u' \approx u_{\text{rms}}}{\Rightarrow}} \quad u_{\text{rms}} = \frac{p_{\text{rms}}}{\rho_o c_o} = \frac{20\mu\text{Pa}}{1.225\text{kg/m}^3 \cdot 340\text{m/s}} \quad \Rightarrow \quad u_{\text{rms}} = \mathbf{4.8 \cdot 10^{-8} m/s}$$

- **Particle displacement** ξ - how far a tiny fluid parcel moves from its equilibrium position:

$$\boxed{\boldsymbol{\xi}(\mathbf{x}, t) = \int_0^t \mathbf{u}(\mathbf{x}, \tau) d\tau} \quad \Rightarrow \quad \boxed{\mathbf{u}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\xi}(\mathbf{x}, t)}{\partial t}}$$

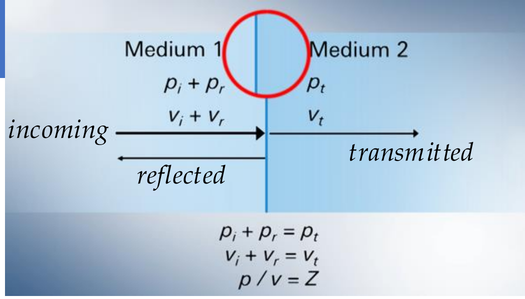
- For a single-frequency (harmonic) plane wave with angular frequency $\omega = 2\pi f$:

$$\boldsymbol{\xi}(\mathbf{x}, t) = \Re\left\{\hat{\boldsymbol{\xi}} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\right\}, \quad \mathbf{u}(\mathbf{x}, t) = \Re\left\{\hat{\mathbf{u}} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\right\}$$

- Where: $\hat{\mathbf{u}} = i\omega\hat{\boldsymbol{\xi}} \quad \Rightarrow \quad \boxed{|\hat{\mathbf{u}}| = \omega|\hat{\boldsymbol{\xi}}|}$ **u** lags $\boldsymbol{\xi}$ by 90°

$$\xi_{\text{rms}} \approx \mathbf{0.1A}$$

$$u_{\text{rms}} = \frac{u_{\max}}{\sqrt{2}} \quad \Rightarrow \quad \boxed{u_{\text{rms}} = \omega \xi_{\text{rms}}} \quad \Rightarrow \quad \xi_{\text{rms}} = \frac{u_{\text{rms}}}{2\pi f} = \frac{4.8 \cdot 10^{-8}\text{m/s}}{2\pi \cdot 1000\text{Hz}} \quad \Rightarrow \quad \xi_{\text{rms}} = \mathbf{7.6 \cdot 10^{-12} m}$$



Specific acoustic impedance

- Ratio of acoustic pressure to acoustic particle velocity in a one-dimensional sound wave
- Specific acoustic impedance [Pa·s/m] or [Rayl] - **medium resistance to the transmission of sound waves** – what acoustic pressure is required to produce a certain acoustic particle velocity?

Analogous to $R = \frac{V}{I}$

$$\boxed{\frac{p'}{u'} = \pm z = \pm \rho_o c_o}$$

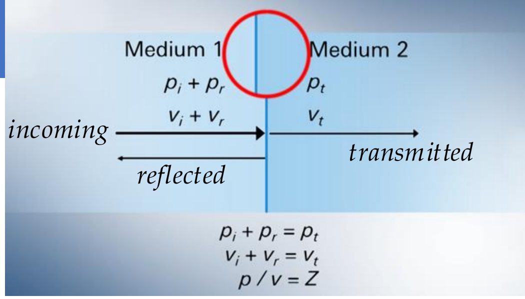
\+ forward traveling wave ($+x$ direction)
− backward traveling wave ($-x$ direction)

- At atmospheric pressure and 20°C, the specific impedances of air and water are:

Fluid	Density (ρ)	Speed of sound (c)	Specific impedance (ρc)
Air	1.204 kg/m^3	343.4 m/s	413.4 kg/m^2-s
	0.07516 lb/ft^3	1127. ft/s	2.633 lb-s/ft^3
Water	998.2 kg/m^3	1482. m/s	1.479×10^6 kg/m^2-s
	62.32 lb/ft^3	4863. ft/s	9420 lb-s/ft^3

$$\Rightarrow \frac{Z_{\text{water}}}{Z_{\text{air}}} = 3775$$

29dB loss during air–water transmission

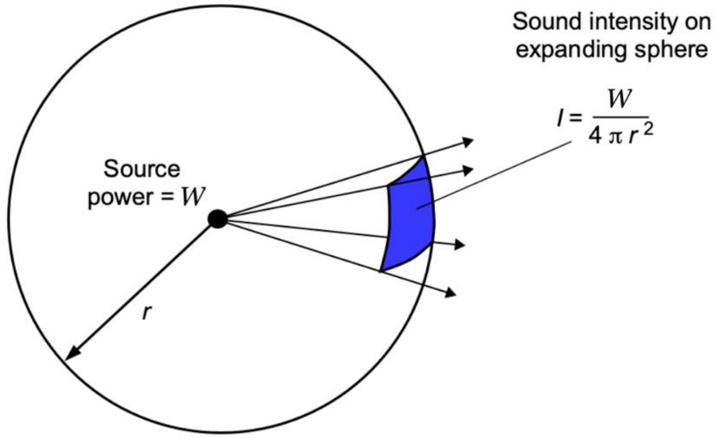


Specific acoustic impedance

Material	ρ_0 Density (10^3 kg m^{-3})	c_0 Velocity (m s^{-1})	Z (MRayl)
Air	0.0012	330	0.0004
Water	1	1430	1.43
Soft tissue	1.1	1540	1.69
Liver	1.05	1570	1.65
Fat	0.95	1450	1.38
Bone	1.91	4080	7.8
Aluminium	2.7	6420	17

Acoustic waves travel faster… Why?

Sound power and intensity



- A sound disturbance from a point source (compact) with a specific **sound power, *W***, produces a spherical pressure wavefront that spreads out in time as it propagates away from the location of the original disturbance, a phenomenon known as **spherical spreading**

- **Instantaneous sound intensity, $\mathbf{I}(t)$** – **rate of acoustic energy flowing through area in unit time**, or the acoustic power W transmitted per unit area; units of [W/m^2]
  - If in a point in space the acoustic pressure $p'(\mathbf{x},t)$ produces at the same point a particle velocity $\mathbf{u}'(\mathbf{x},t)$, $\mathbf{I}(t)$ is the rate at which work is done on the fluid per unit area at time t

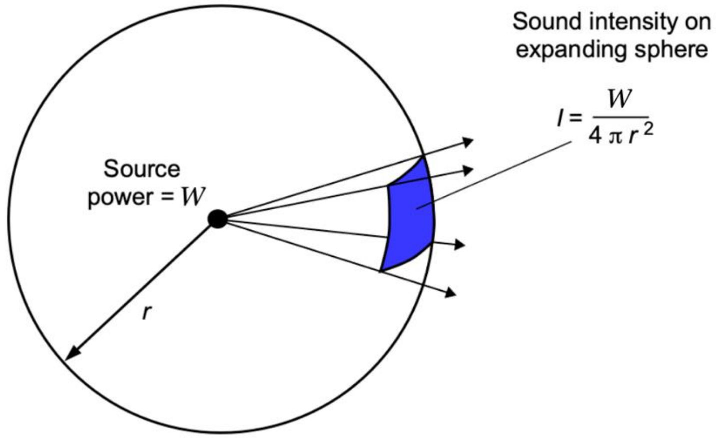
$$\boxed{\mathbf{I}(t) = p'(\mathbf{x},t)\mathbf{u}'(\mathbf{x},t) = \frac{\text{acoustic energy}}{\text{time} \times \text{area}}}$$

I – vector quantity in the direction of the particle velocity

- **Time-averaged sound intensity (I)** - **time-averaged rate of acoustic energy flow per unit surface area at a given vector direction**; units of [W/m^2]

$$\boxed{\mathbf{I} = \frac{1}{2T}\int_{-T}^{T} p'(\mathbf{x},t)\mathbf{u}'(\mathbf{x},t)\,dt}$$

Sound power and intensity



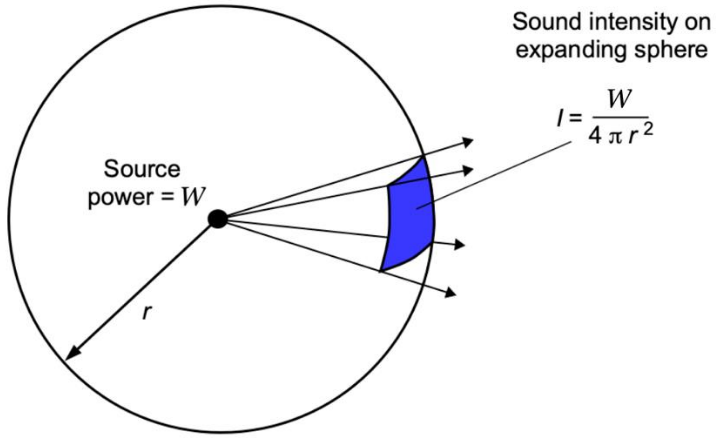
- **Sound power (W) - flux of acoustic energy through a surface S** (the rate at which the wavefront transfers sound energy); units of [W]

$$\boxed{W = \int_S \mathbf{I} \cdot \hat{\mathbf{n}} \mathrm{d}S = \frac{S}{2T} \int_{-T}^{T} p'(\mathbf{x},t)\mathbf{u}'(\mathbf{x},t)\, dt = \frac{\text{acoustic energy}}{\text{time}}}$$

$\hat{\mathbf{n}}$ - unit vector in the direction of wave propagation, normal to S

$$\boxed{\mathbf{I} = \frac{1}{2T} \int_{-T}^{T} p'(\mathbf{x},t)\mathbf{u}'(\mathbf{x},t)\, dt}$$

Sound power and intensity



- **Assuming the acoustic spreading occurs in the far field:**
  - No bounds or obstacles can affect the propagation
  - The acoustic wave can be treated as a locally plane wave, for which: $u' = \dfrac{p'}{\rho_o c_o}$
- Thus, we can write:

$$\frac{1}{2T}\int_{-T}^{T} p'\mathbf{u}'\,dt = \frac{1}{2T}\int_{-T}^{T} \frac{p'^2}{\rho_o c_o}dt = \frac{1}{\rho_o c_o 2T}\int_{-T}^{T} p'^2\,dt = \frac{\overline{p'^2}}{\rho_o c_o} = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o}$$

$$p_{\mathrm{rms}}^2 = \frac{1}{2T}\int_{-T}^{T}[p'(t)]^2\,dt = \overline{[p'(t)]^2}$$

- Therefore, the sound intensity and sound power can be reduced to:

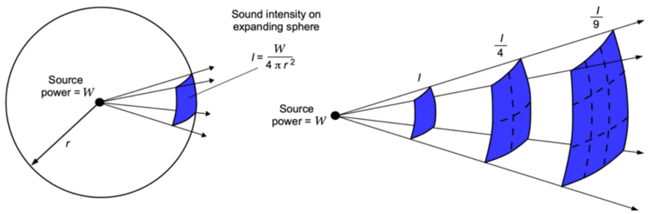
$$\boxed{\mathbf{I} = \frac{1}{2T}\int_{-T}^{T} p'\mathbf{u}'\,dt = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o}\hat{\mathbf{n}}}$$

$$\boxed{W = \int_S \mathbf{I}\cdot\hat{\mathbf{n}}dS = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o}S}$$

p_{rms}^2 is **identical over the entire integration surface** (e.g., constant on a sphere)

$\mathbf{u}'$ - acoustic particle velocity vector
$\hat{\mathbf{n}}$ - unit vector in the direction of wave propagation, normal to S

Sound power and intensity



- Sound levels are also given by **sound power levels (SWL)** in [dB]:

$$\boxed{\mathrm{SWL} = 10\log_{10}\left(\frac{W}{W_{\mathrm{ref}}}\right) \quad [\mathrm{dB}(\mathrm{re}\ W_{\mathrm{ref}})]} \qquad W_{\mathrm{ref}} = \frac{p_{\mathrm{ref}}^2}{\rho_o c_o}$$

- **Human threshold** sound pressure is $p_{\mathrm{ref}} = 20$ [μPa], characterized by:
  - Intensity of $\mathrm{I_{ref}} = 10^{-12}[\mathrm{W/m^2}]$
  - Sound power of $W_{\mathrm{ref}} = 10^{-12}$ [W] through a 1 [m^2] area perpendicular to the propagation direction
- If the sound pressure is **identical over the entire integration surface** (e.g., constant on a sphere – plane wave) we obtain:

$$\boxed{\mathrm{SWL} = \mathrm{SPL} + 10\log_{10}(4\pi r^2)}$$

$$\mathrm{SWL} = 10\log_{10}\left(\frac{W}{W_{\mathrm{ref}}}\right) = 10\log_{10}\left(\frac{p_{\mathrm{rms}}^2 S}{p_{\mathrm{ref}}^2}\right)$$
$$= \mathrm{SPL} + 10\log_{10}(S)$$
$$S = 4\pi r^2$$

  - For example, at 1 m, we have: SWL = SPL + 11dB
  - More generally, we have an **additional directivity factor ($4\pi r^2$)**

$$\mathrm{I} = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o} \qquad W = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o} S$$

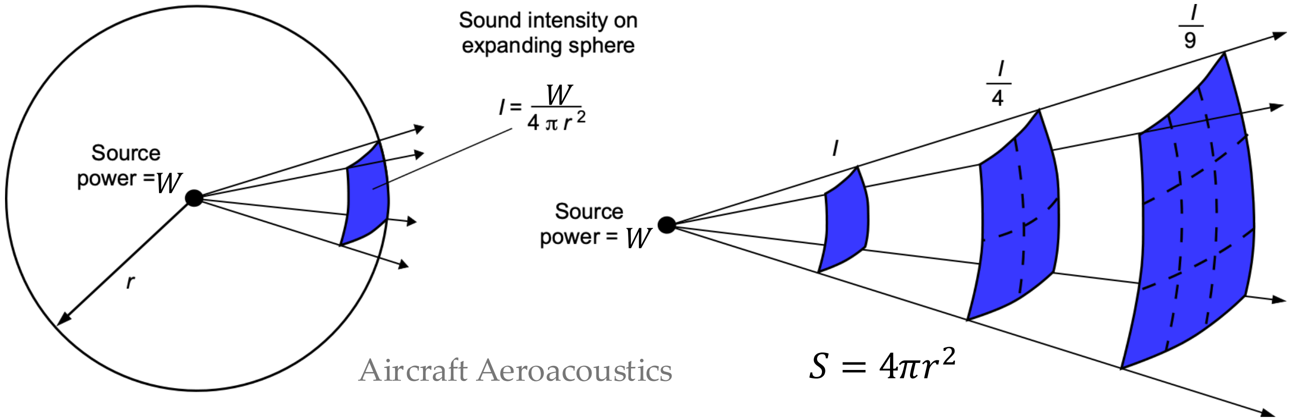
Spherical spreading

- Assuming p_{rms}^2 is identical over the entire integration surface, and substituting $S = 4\pi r^2$ into the definition for the sound power definition, we can write:

$$\boxed{p_{\mathrm{rms}} = \sqrt{\frac{W \rho_o c_o}{4\pi}} \left(\frac{1}{r}\right) \propto \frac{1}{r}} \quad \Longrightarrow \quad \frac{p_{\mathrm{rms},2}}{p_{\mathrm{rms},1}} = \frac{r_1}{r_2} \qquad \boxed{\mathrm{I} \propto p_{\mathrm{rms}}^2} \quad \Longrightarrow \quad \boxed{\mathrm{I} \propto \frac{1}{r^2}} \quad \Longrightarrow \quad \frac{\mathrm{I}_2}{\mathrm{I}_1} = \left(\frac{r_1}{r_2}\right)^2$$

  - Therefore, for a spherically traveling sound wave from a point source, the sound pressure will be proportional to $1/r$; i.e., **acoustic pressure *decreases* inversely with distance**

- **Inverse-square law of sound - sound intensity *decreases inversely* with the *square* of the distance**



Spherical spreading

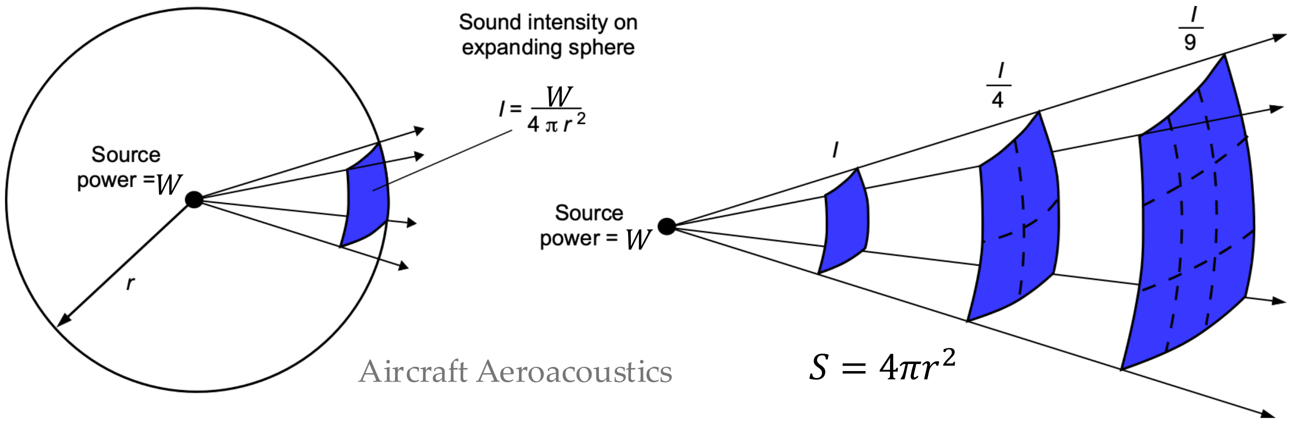
$$\mathrm{I} = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o} \qquad W = \frac{p_{\mathrm{rms}}^2}{\rho_o c_o} S$$

- Recalling the SPL definition: $\boxed{\mathrm{SPL} = 10\log_{10}\left(\frac{p_{\mathrm{rms}}^2}{p_{\mathrm{ref}}^2}\right) = 10\log_{10}\left(\frac{\mathrm{I}}{\mathrm{I}_{\mathrm{ref}}}\right) \quad [\mathrm{dB}]}$

$p_{\mathrm{ref}} = 20\ [\mu\mathrm{Pa}]$
$\mathrm{I}_{\mathrm{ref}} = 10^{-12} [\mathrm{W/m^2}]$
$W_{\mathrm{ref}} = 10^{-12}\ [\mathrm{W}]$

- We notice that at a specific r_i distance we have: $\mathrm{SPL}_i = 10\log_{10}\left(\frac{p_{\mathrm{rms},i}^2}{p_{\mathrm{ref}}^2}\right) = 10\log_{10}\left(\frac{\mathrm{I}_i}{\mathrm{I}_{\mathrm{ref}}}\right)$
- Therefore, the difference in SPL between r_2 and r_1 is as follows: $\mathrm{SPL}_2 = \mathrm{SPL}_1 + 10\log_{10}\left(\frac{\mathrm{I}_2}{\mathrm{I}_1}\right)$
- Substituting the proportion with distance yields:

$$\Rightarrow \boxed{\mathrm{SPL}_2 = \mathrm{SPL}_1 - 10\log_{10}\left(\frac{r_2^2}{r_1^2}\right)}$$



$S = 4\pi r^2$

$$\boxed{\mathrm{I} \propto \frac{1}{r^2}}$$

Example

- Suppose you are in an open-air concert, where sound is radiated from an arrangement of loudspeakers. At a distance of 20 m from the speakers, the SPL measured was 110 dB (re p_{ref} = 20μPa). Assuming that all sound waves incident on the ground are absorbed and that the sound field is omni-directional, what is the mean power output of the speakers (SL conditions)?

  - At 20 m, the rms of the pressure fluctuations can be computed as follows:

$$p_{\mathrm{rms}} = p_{\mathrm{ref}} 10^{\mathrm{SPL}/20} \implies p_{\mathrm{rms}} = 20\mu\mathrm{Pa} \times 10^{110/20} \implies p_{\mathrm{rms}} = 6.32\mathrm{Pa}$$

  - At $r = 20\mathrm{m}$ the sound wave can be treated as a plane wave
  - The time-averaged sound intensity at $r = 20\mathrm{m}$ is therefore:

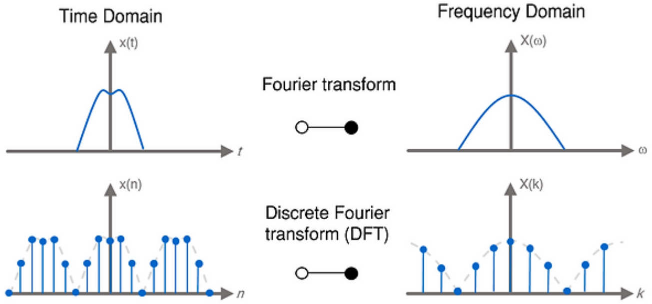
$$\mathrm{I} = \frac{p_{\mathrm{rms}}^2}{\rho_{\mathrm{o}} c_{\mathrm{o}}} \implies \mathrm{I} = \frac{6.32^2\mathrm{Pa}}{1.225\mathrm{kg/m^3} \cdot 340\mathrm{m/s}} \implies \mathrm{I} = 0.096[\mathrm{W/m^2}]$$

  - The energy flux through a **hemisphere** of $r = 20\mathrm{m}$ is:

$$W = \int_S \mathbf{I} \cdot \hat{\mathbf{n}} dS = \frac{p_{\mathrm{rms}}^2}{\rho_{\mathrm{o}} c_{\mathrm{o}}} S = \mathrm{I} \cdot 2\pi r^2 \implies W = 0.096\,\mathrm{W/m^2} \cdot 2\pi (20\mathrm{m})^2 \implies W = 241[\mathrm{W}]$$

  - Since the sound field at the source is omni-directional, an equal amount of energy has been absorbed by the ground, and so the power output of the speakers is $\boldsymbol{W} = \mathbf{483[W]}$

Signal analysis in acoustics



❑ Continuous-time Fourier transform (CTFT)

➢ Maps an analog signal $y(t)$ from ***continuous-time*** domain to the ***continuous frequency*** domain

$$\boxed{\tilde{p}(\omega) = \frac{1}{2\pi}\int_{-T}^{+T} p'(t)\, e^{i\omega t} dt} \qquad \boxed{p'(t) = \int_{-\infty}^{+\infty} \tilde{p}(\omega) e^{-i\omega t} d\omega} \quad inverse$$

➢ where T tends to infinity and ω is the angular frequency

❑ Discrete Fourier transform (DFT)

➢ Maps a digital signal $y[k]$ (e.g., microphone data) from the ***discrete-time*** domain to the ***discrete frequency*** domain

$$\text{For } n = 1 \text{ to } N \quad \boxed{A[n] = \sum_{k=1}^{N} a[k] e^{-\frac{2\pi i(k-1)(n-1)}{N}}} \qquad \boxed{a[k] = \frac{1}{N}\sum_{n=1}^{N} A[n] e^{\frac{2\pi i(k-1)(n-1)}{N}}} \quad \text{For } k = 1 \text{ to } N \;\; inverse$$

➢ Explicit DFT computation is an expensive calculation requiring $O(N^2)$ operations

➢ Instead, the **Fast Fourier transform (FFT)** algorithm can be used, which takes advantage of efficiencies that become possible when N is a composite number, and particularly when it is a power of 2

- Reduces the computational effort to $O(N\log N)$
- In MATLAB: fft() & ifft()

Signal analysis in acoustics

Auto-spectral density

- The Fourier transform $\tilde{p}(\omega)$ is not by itself a very useful measure since, just like $p'(t)$, it will vary *stochastically*[1]
- The average measure of the frequency content is given by the **auto-spectral density** of p' (often referred to as the auto-spectrum, power spectrum), defined as:

$$\boxed{S_{pp}(\omega) = \frac{1}{2\pi}\int_{-T}^{+T} R_{pp}(\tau)e^{i\omega t}d\tau}$$

$$\boxed{R_{pp}(\tau) = \int_{-\infty}^{+\infty} S_{pp}(\omega)e^{-i\omega\tau}d\omega}$$

inverse

- $R_{pp}(\tau)$ is the **time-delay auto-correlation function**
  - Average of the signal multiplied by itself at a later time

$$R_{pp}(\tau) = E[p'(t)p'(t+\tau)]$$

- **E – expected value** – mean value of a stochastic variable taken over many repeated realizations N (under identical conditions)
  - Example – multiple independent samples of the flow quantity p' at the same defined position and time
  - Please note, $\boldsymbol{R_{pp}(\tau)}$ **is symmetric about** $\boldsymbol{\tau = 0}$ since $E[a(t)a(t+\tau)] = E[a(t-\tau)a(t)] = E[a(t)a(t-\tau)]$
  - Therefore, $\boldsymbol{S_{pp}(\omega)}$ ***is real and even function of frequency***

$$E[p'(t)] = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} p'_n(t)$$

$p'_n(t)$ – n-th sample of p'
N – total samples taken

[1] A stochastic process is a collection of random variables indexed by time, representing a system that evolves randomly over time

Signal analysis in acoustics

Parseval's theorem

- *'Energy conservation'*
  - The **total energy of a time-domain signal** (calculated by summing power-per-sample across time) **should be equal** to the **total energy in the frequency-domain** (calculated by summing spectral power across frequency)

- In CTFT & DFT formulations:

$$\boxed{\int_{-T}^{+T} |p'(t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |\tilde{p}(\omega)|^2 d\omega}$$

$$\boxed{\sum_{k=1}^{N} |a[k]|^2 = \frac{1}{N} \sum_{n=1}^{N} |A[n]|^2}$$

- In auto-spectral density formulations, we obtain:

$$\boxed{R_{pp}(\tau = 0) = \overline{p'^2} = p_{\mathrm{rms}}^2 = \int_{-\infty}^{+\infty} S_{pp}(\omega) d\omega \xrightarrow[S_{pp}(f) \,=\, 2\pi S_{pp}(\omega)]{d\omega/df = 2\pi} \int_{-\infty}^{+\infty} S_{pp}(f) df}$$

$$R_{pp}(\tau) = E[p'(t)p'(t+\tau)]$$

$$E[p'(t)] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} p'_n(t)$$

- $S_{pp}(\omega)$ has units of [Pa²/rad·Hz]
- $S_{pp}(f)$ has units of [Pa²/Hz]

Signal analysis in acoustics

From double-sided spectrum to single-sided spectrum

- Note that the definition of $S_{pp}(\omega)$ includes both *positive* and *negative* frequencies – **double-sided spectrum**
  - This is the norm in mathematical analysis
  - In the context of one-dimensional spectra, these mean the same thing…
- However, from Parseval's theorem, the **energy is spread over both the positive and negative domains**
- For predictions/measurements of sound spectra, **it is normal to consider only positive frequencies and to double the spectral values**
- Therefore, the **acoustic spectrum should be treated as single-sided**, referred to as $G_{pp}(\omega)$

$$G_{pp}(\omega) = 2S_{pp}(\omega), \qquad \text{for } \omega > 0 \; (G_{pp}(\omega < 0) = 0)$$

- In the frequency domain (f), we take only the single-sided spectrum and write:

$$\boxed{\overline{p'^2} = p_{\mathrm{rms}}^2 = \int_0^\infty G_{pp}(f)df}$$

- $G_{pp}(f)$ – **single-sided spectral density**, [Pa²/Hz]

Signal analysis in acoustics

Computing the acoustic pressure spectra from spectral density $G_{pp}(f)$

- Using the spectral density definition, one can present acoustic pressure spectra in [dB] using the *narrow-band* sound pressure level, as follows:

$$\boxed{L_p(f) = 10\log_{10}\left(\frac{G_{pp}(f)\Delta f}{p_{\mathrm{ref}}^2}\right) \quad [\mathrm{dB}(\mathrm{re}\ p_{\mathrm{ref}})]}$$

$$p_{\mathrm{rms}}^2 = \int_0^\infty G_{pp}(f)df$$

- Alternatively, we can compute the **power spectra density (PSD)** in [dB/Hz] according to:

$$\boxed{\mathrm{PSD}(f) = 10\log_{10}\left(\frac{G_{pp}(f)}{p_{\mathrm{ref}}^2}\right) \quad \left[\frac{\mathrm{dB}}{\mathrm{Hz}}\right]}$$

- Integrating $G_{pp}(f)$ across the full frequency domain yields the overall sound pressure level (OASPL):

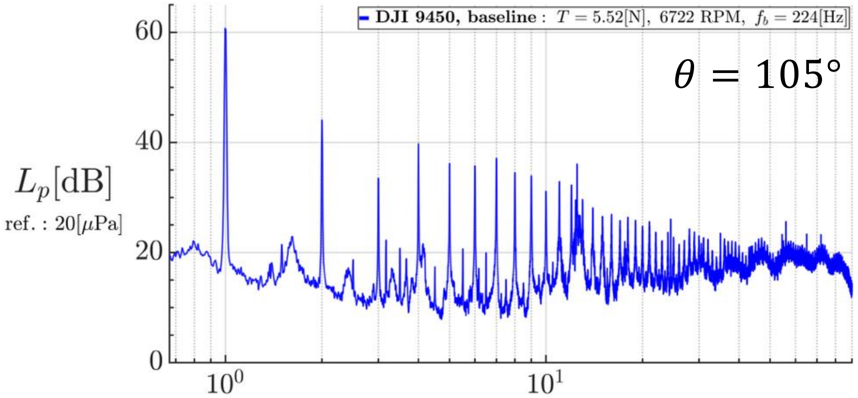
$$\boxed{\mathrm{OASPL} = 10\log_{10}\left(\frac{1}{p_{\mathrm{ref}}^2}\int_0^\infty G_{pp}(f)df\right) = 10\log_{10}\left(\frac{p_{\mathrm{rms}}^2}{p_{\mathrm{ref}}^2}\right) \quad [\mathrm{dB}(\mathrm{re}\ p_{\mathrm{ref}})]}$$

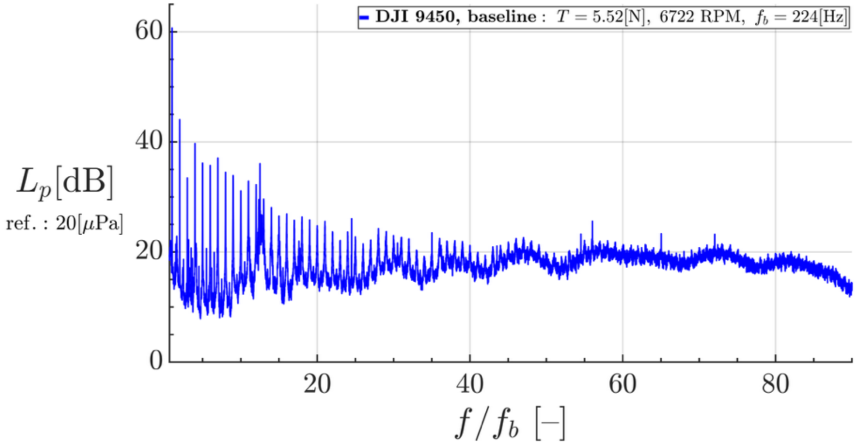
Example

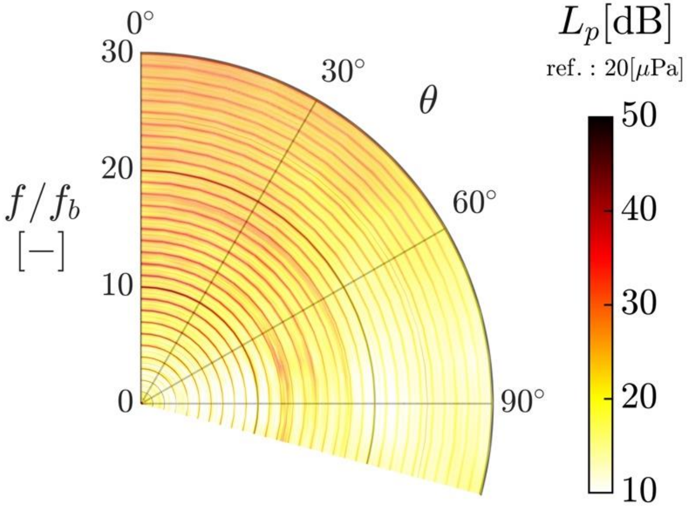
Blade passage frequency: $f_b = \frac{\Omega}{60} B$ [Hz]

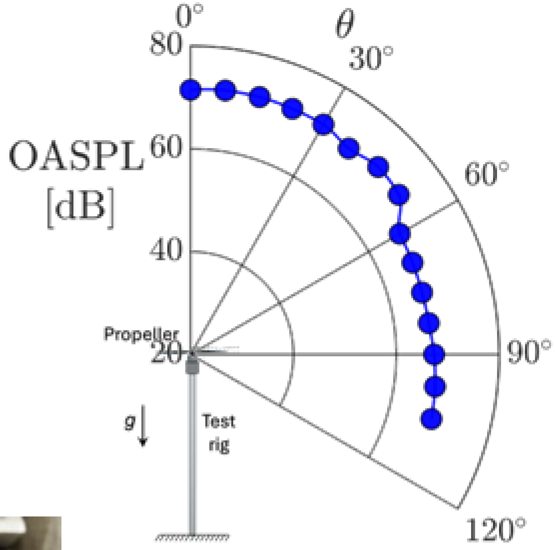
❑ DJI 9450 rotor at hover

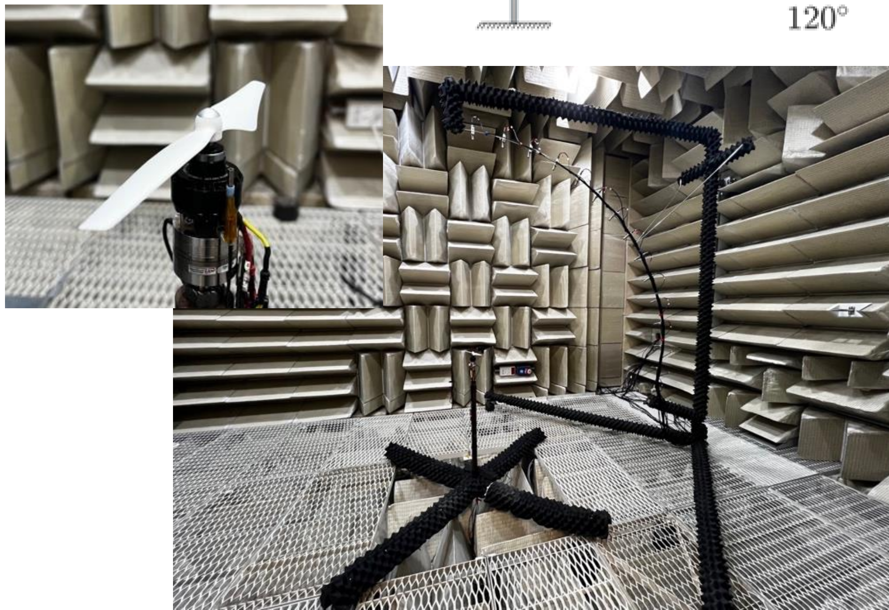
- ➢ Acoustic measurements in the Technion's anechoic chamber
- ➢ $\Omega = 6{,}722$ [rev/min], $\bar{T} = 5.52$ [N], 2 blades ($B = 2$)





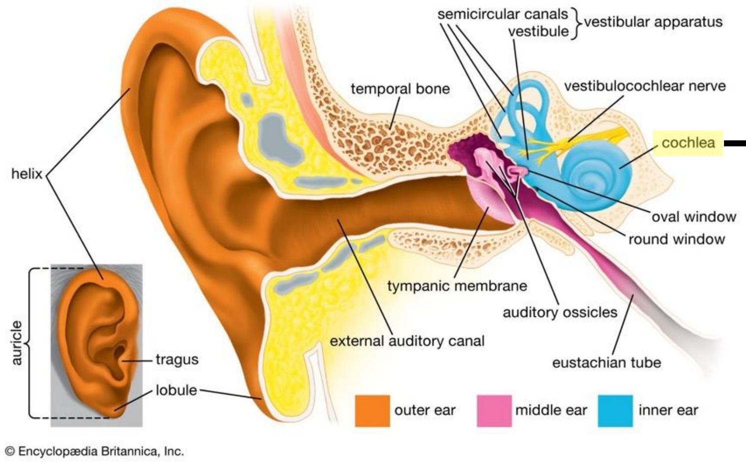




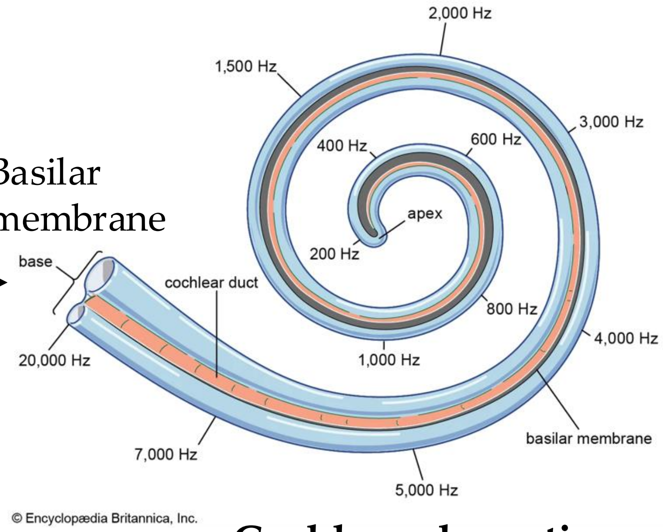


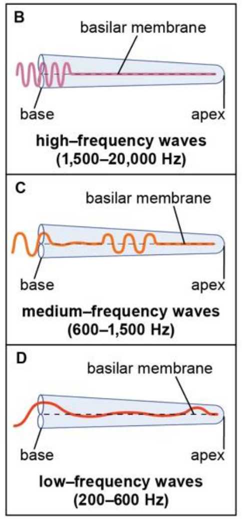
Human noise perception

- The ear responds to the *pressure fluctuations* of sound waves to cause the *sensation of hearing*
- When designing quiet systems, it is always important to ***keep the end goal in mind*** - the **level of annoyance experienced by a human observer**
- **The human ear is characterized by:**
  - A remarkable dynamic range (20 Hz to 20 kHz)
  - A **nonlinear (logarithmic) spectral sensitivity** in terms of frequency versus amplitude due to the varying physical properties of the basilar membrane within the cochlea
    - Each area of the basilar membrane vibrates preferentially to a particular sound frequency
  - The ear is **most sensitive to sounds between 1 kHz and 4 kHz**
    - This is unsurprisingly the frequency range of human speech.



Basilar membrane

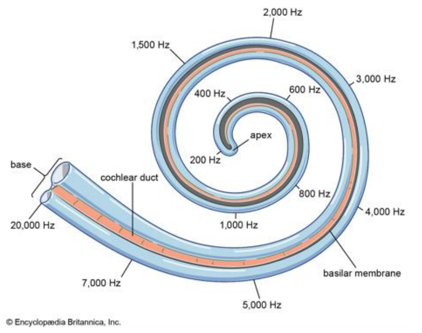


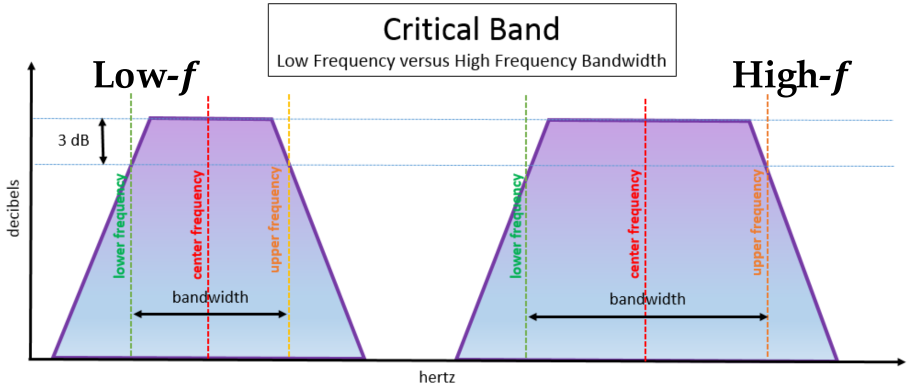


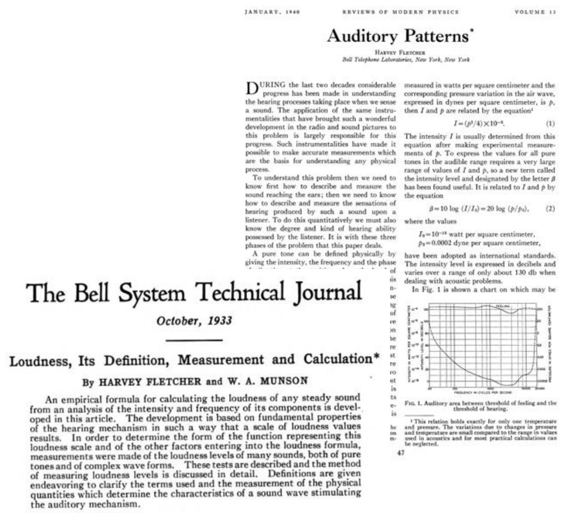
Cochlea schematics

Human noise perception

- In 1933-1940, Harvey Fletcher, an American physicist, introduced the concept of **critical bands**[1,2]
- Critical bands (CB) - frequency bandwidths of the "auditory filter" created by the cochlea
  - Quantify the ability of the human ear to **distinguish separate frequencies or tones**
  - At low frequencies, the human ear can distinguish changes in frequency more easily than at high frequencies (e.g., the ear can distinguish a 20Hz difference between 500 and 520Hz tones more readily than between 5000 and 5020Hz)
- The CB bands increase in bandwidth from low to high frequency (based on the cochlea)
  - The bands are defined similarly to bandpass filters, with a center frequency and a bandwidth
  - The bandwidth is defined by the lower and upper frequencies, where the amplitude falls -3 dB below the peak amplitude






Auditory Patterns
The Bell System Technical Journal
October, 1933
Loudness, Its Definition, Measurement and Calculation*
By HARVEY FLETCHER and W. A. MUNSON

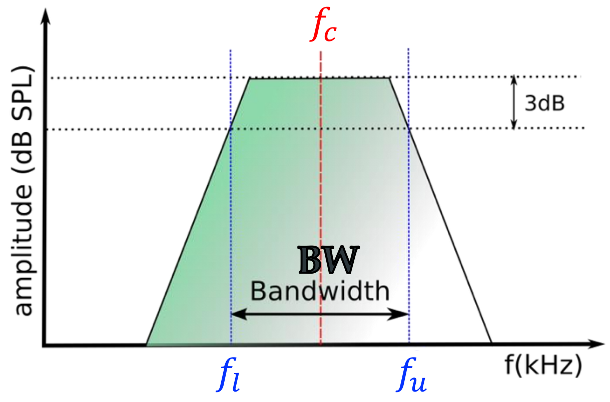
Harvey Fletcher
(Sep 11, 1884 – July 23, 1981)

[1] Bell System Technical Journal, October 1933, "Loudness, its Definition, Measurement and Calculation".
[2] Fletcher, Harvey (1940). "Auditory Patterns". Reviews of Modern Physics. 12 (1): 47–65.

Human noise perception

❑ Critical bands (CB)

- There are 24 critical bands in the human hearing range, where each band is referred to as a 'Bark' band
- Together, the 24 bands are called the '**Bark scale**'



Bark	f_l	f_c		f_u	BW
	Hz	Hz	Bark	Hz	Hz
1	0	50	0.5	100	100
2	100	150	1.5	200	100
3	200	250	2.5	300	100
4	300	350	3.5	400	100
5	400	450	4.5	510	110
6	510	570	5.5	630	120
7	630	700	6.5	770	140
8	770	840	7.5	920	150
9	920	1000	8.5	1080	160
10	1080	1170	9.5	1270	190
11	1270	1370	10.5	1480	210
12	1480	1600	11.5	1720	240

Bark	f_l	f_c		f_u	BW
	Hz	Hz	Bark	Hz	Hz
13	1720	1850	12.5	2000	280
14	2000	2150	13.5	2320	320
15	2320	2500	14.5	2700	380
16	2700	2900	15.5	3150	450
17	3150	3400	16.5	3700	550
18	3700	4000	17.5	4400	700
19	4400	4800	18.5	5300	900
20	5300	5800	19.5	6400	1100
21	6400	7000	20.5	7700	1300
22	7700	8500	21.5	9500	1800
23	9500	10500	22.5	12000	2500
24	12000	13500	23.5	15500	3500

E. Zwicker and H. Fastl, Psychoacoustics, Facts and Models, Berlin: Springer Verlag, 3rd edition, 1999

Human noise perception

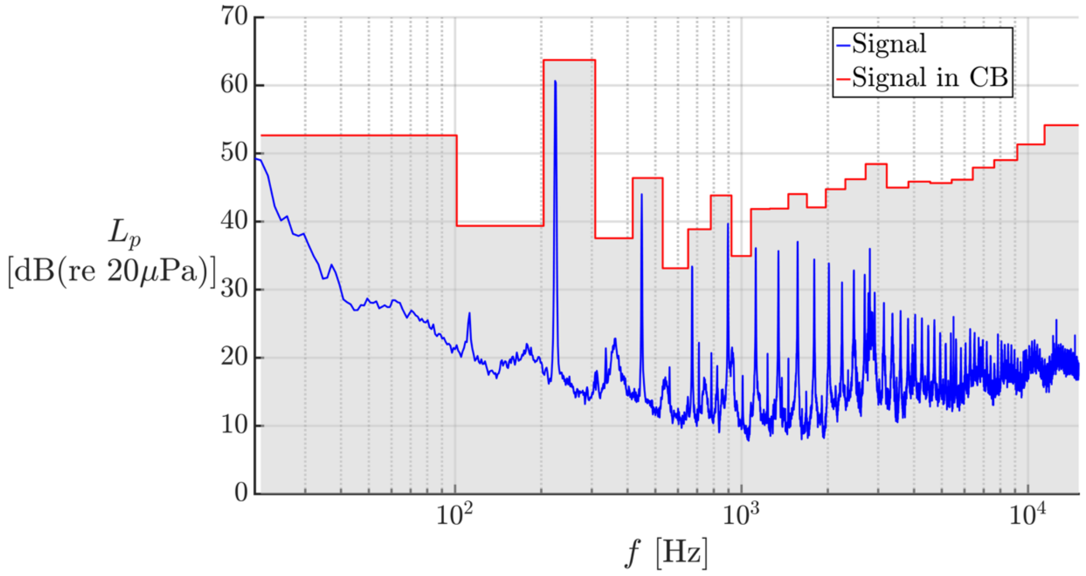
❑ Critical bands (CB)

➢ The spectrum critical bands can be calculated as follows:

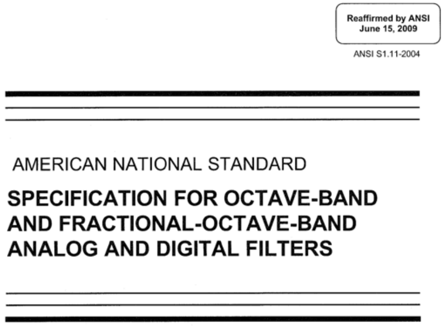
$$\frac{\mathrm{I}_{\mathrm{CB}}(z)}{\mathrm{I}_{\mathrm{ref}}} = \frac{1}{\mathrm{I}_{\mathrm{ref}}}\int_{f_l}^{f_\mathrm{u}}\left(\frac{d\mathrm{I}}{df}\right)df = \frac{1}{p_{\mathrm{ref}}^2}\int_{f_l}^{f_\mathrm{u}} G_{pp}(f)df \quad [\mathrm{W/m^2}]$$

$$L_{\mathrm{CB}}(\mathrm{z}) = 10\log_{10}\left(\frac{\mathrm{I}_{\mathrm{CB}}(z)}{\mathrm{I}_{\mathrm{ref}}}\right) \quad [\mathrm{dB}]$$

➢ $G_{pp}(f)$ is the single-sided spectral density spectrum, given in units of [Pa²/Hz]



Bark	f_l	f_c		f_u	BW
	Hz	Hz	Bark	Hz	Hz
1	0	50	0.5	100	100
2	100	150	1.5	200	100
3	200	250	2.5	300	100
4	300	350	3.5	400	100
5	400	450	4.5	510	110
6	510	570	5.5	630	120
7	630	700	6.5	770	140
8	770	840	7.5	920	150
9	920	1000	8.5	1080	160
10	1080	1170	9.5	1270	190
11	1270	1370	10.5	1480	210
12	1480	1600	11.5	1720	240
13	1720	1850	12.5	2000	280
14	2000	2150	13.5	2320	320
15	2320	2500	14.5	2700	380
16	2700	2900	15.5	3150	450
17	3150	3400	16.5	3700	550
18	3700	4000	17.5	4400	700
19	4400	4800	18.5	5300	900
20	5300	5800	19.5	6400	1100
21	6400	7000	20.5	7700	1300
22	7700	8500	21.5	9500	1800
23	9500	10500	22.5	12000	2500
24	12000	13500	23.5	15500	3500


Reaffirmed by ANSI June 15, 2009

ANSI S1.11-2004

AMERICAN NATIONAL STANDARD

SPECIFICATION FOR OCTAVE-BAND AND FRACTIONAL-OCTAVE-BAND ANALOG AND DIGITAL FILTERS

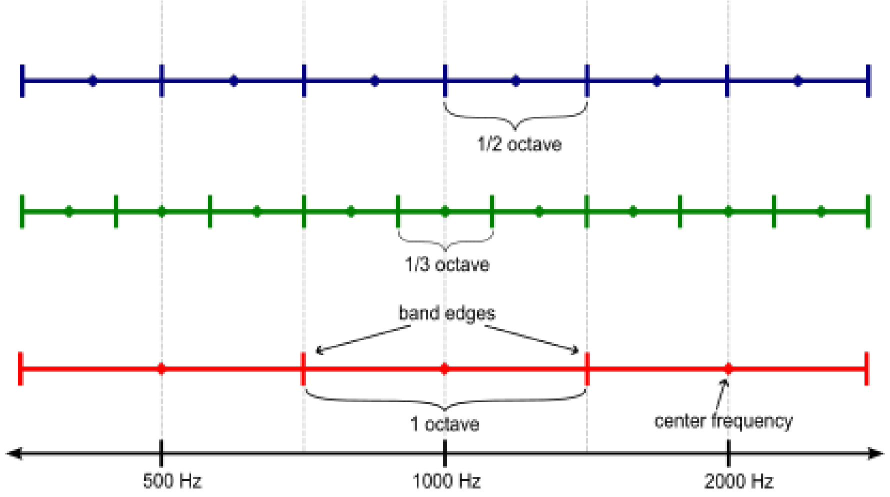
octaveFilterBank()

Octave-band and fractional-octave-band

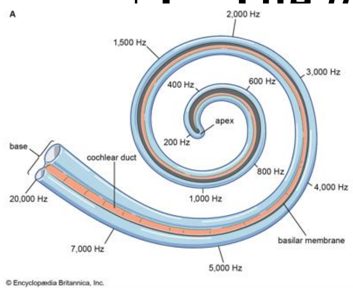
- CB bands are almost similar to 1/3-octave bands…
- **ANSI S1.11-2004 defines the midband frequencies (f_c) of the octave bands as:**

$$\boxed{f_c = \begin{cases} f_r \times G^{(x-30)/b}, & b \text{ is odd} \\ f_r \times G^{(2x-59)/2b}, & b \text{ is even} \end{cases}}$$

  - ***G* – nominal octave ratio**
    - Base 10 (preferred) – $G_{10} = 10^{3/10}$
    - Base 2 – $G_2 = 2$
  - **f_r – reference frequency** (commonly, $f_r = 1000$ Hz)
  - ***x* – band number** (Integer, positive, negative, or zero)
  - ***b* – no. of bands per octave** (e.g., 2, 3, 12,…)



- The *lower* and *upper* limits of each band are given by:



$$\boxed{f_c = \sqrt{f_l f_u}}$$

$$\boxed{\begin{aligned} f_l &= f_c \times G^{-1/2b} \\ f_u &= f_c \times G^{+1/2b} \\ f_u/f_l &= G^{1/b} \end{aligned}}$$

Octave-band and fractional-octave-band

- ANSI S1.11-2004
  - Example for 1/3-octave
- ***Nominal* midband frequency** (for $b < 24$)
  - When the most significant digit (left-most) of f_c is between 1 and 4 inclusive, round to the ***first* three *significant digits***
  - When the most significant digit (left-most) of f_c is between 5 and 9 inclusive, round to the ***first* two *significant digits***
  - *Example*: for 1/24-octave, using $\mathrm{G}_{10} = 10^{3/10}$ and $x = -81$, the exact midband frequency from is:

$$f_c = f_r \times G^{(2x-59)/2b}$$
$$\Rightarrow f_c = 10^3 \times 10^{3(2x-59)/(20\cdot 24)}$$
$$\Rightarrow f_c = 10^3 \times 10^{(-2\cdot 81-59)/(20\cdot 8)}$$
$$\Rightarrow f_c = 10^3 \times 10^{-221/160} = 41.567\text{ Hz}$$
$$\xRightarrow{\text{round}} f_c \approx 41.6\text{ Hz}$$

  - For $x = +105$, we get

$$f_c = 8{,}785.2\text{ Hz} \xRightarrow{\text{round}} f_c \approx 8800\text{ Hz}$$

Reaffirmed by ANSI June 15, 2009
ANSI S1.11-2004

AMERICAN NATIONAL STANDARD

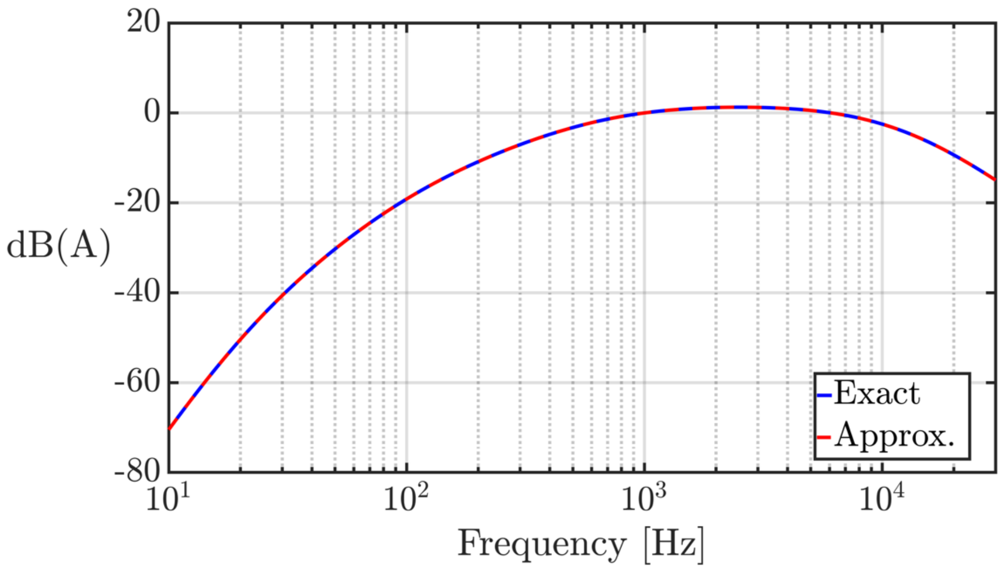
SPECIFICATION FOR OCTAVE-BAND AND FRACTIONAL-OCTAVE-BAND ANALOG AND DIGITAL FILTERS

Band number x	Base-ten exact f_c ($10^{x/10}$), Hz	Base-two exact f_c ($2^{(x-30)/3}$)(1000), Hz	Nominal midband frequency, Hz	Octave
14	25.119	24.803	25	
15	31.623	31.250†	31.5	*
16	39.811	39.373	40	
17	50.119	49.606	50	
18	63.096	62.500†	63	*
19	79.433	78.745	80	
20	100.00†	99.213	100	
21	125.89	125.00†	125	*
22	158.49	157.49	160	
23	199.53	198.43	200	
24	251.19	250.00†	250	*
25	316.23	314.98	315	
26	398.11	396.85	400	
27	501.19	500.00†	500	*
28	630.96	629.96	630	
29	794.33	793.70	800	
30	1,000.0†	1,000.0†	1 000	*
31	1,258.9	1,259.9	1 250	
32	1,584.9	1,587.4	1 600	
33	1,995.3	2,000.0†	2 000	*
34	2,511.9	2,519.8	2 500	
35	3,162.3	3,174.8	3 150	
36	3,981.1	4,000.0†	4 000	*
37	5,011.9	5,039.7	5 000	
38	6,309.6	6,349.6	6 300	
39	7,943.3	8,000.0†	8 000	*
40	10,000†	10,079	10 000	
41	12,589	12,699	12 500	
42	15,849	16,000†	16 000	*
43	19,953	20,159	20 000	

Human noise perception

dB(A)

- To measure the perceived loudness of a sound, the most commonly used metric is the dB(A) level
- The A-weighted sound pressure, dBA or dB(A), was designed to **mimic the human ear's sensitivity to different frequencies at lower sound levels**
- The weighting attenuates the contributions of low and high frequencies while **emphasizing the mid-range around 1 kHz**, where human hearing is most sensitive
- dB(A) can be easily measured directly with a sound level meter to provide a good measure of annoyance



$$R_A(f) = \frac{a_1^2 f^4}{(f^2 + a_2^2)\sqrt{(f^2 + a_3^2)(f^2 + a_4^2)}(f^2 + a_1^2)}$$

$$A(f) = 20\log_{10}\big(R_A(f)\big) - 20\log_{10}\big(R_A(1000)\big) \quad [\text{dB(A)}]$$

$$a_1 = 12194,\ a_2 = 20.6,\ a_3 = 107.7,\ a_4 = 737.9$$