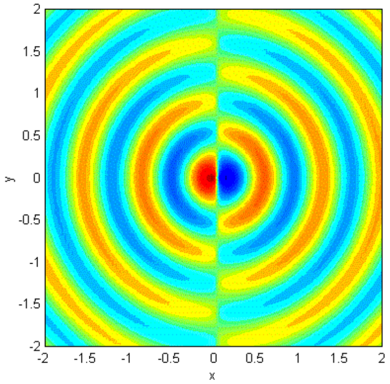
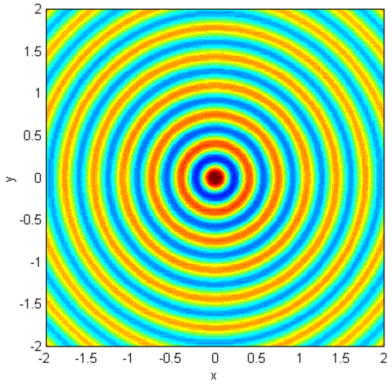
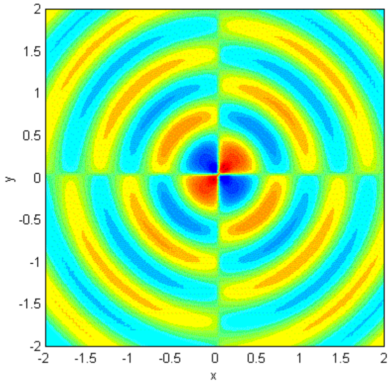
Chapter 2

Linear Acoustics







Linear acoustics

- The theory is based on **small oscillatory pressures** (p') of sound that are being generated and propagated in a medium at the speed of sound (c_o)
- **<u>The general assumptions</u>:**
  - No significant flow in the medium – **flow is considered at rest** ($\mathbf{v}_o = 0$)
    - Time-average properties of the fluid $()_o$ are uniform throughout the region of interest
    - **Acoustic waves are the only source of pressure and velocity fluctuations**
  - A ***weak* sound wave** propagating in a medium (*small perturbation approximation*)
    - $SPL < 140$ dB (re 20 μPa) in air or $SPL < 220$ dB (re 1 μPa) in water
    - Density fluctuations ρ' are **much less** than the mean density ρ_o
    - Acoustic particle velocity $\mathbf{v}'$ is **much less** than the speed of sound c_o
    - Acoustic pressure p' is **much less** than the mean background pressure p_o
  - *Inviscid flow*, $\sigma_{ij} = 0$
  - *No heat addition*, $Q_i = 0$

$$\boxed{\mathbf{v} = \overset{0}{\cancel{\mathbf{v}_o}} + \mathbf{v}' = \mathbf{v}'}$$

$$\boxed{\rho = \rho_o + \rho'}$$

$$\boxed{p = p_o + p'}$$

Linear acoustics

$$\boxed{p' \ll p_o \qquad \mathbf{v}' \ll c_o \qquad \rho' \ll \rho_o}$$

Governing equations

- Mass conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0}$$

- Momentum conservation

$$\boxed{\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial (\rho v_i v_j + p_{ij})}{\partial x_j} = 0}$$

- Energy conservation

$$\boxed{\rho T \frac{Ds}{Dt} = \sigma_{ij} \frac{\partial v_j}{\partial x_i} - \frac{\partial Q_i}{\partial x_i}}$$

Substitute

$\rho = \rho_o + \rho'$

$p = p_o + p'$

$v_i = v_i'$

→ *Linearization*

Linearization

$$\rho = \rho_o + \rho'$$
$$p = p_o + p'$$
$$v_i = v_i'$$

$$\frac{\partial(\rho' v_i)}{\partial x_i} \ll \frac{\partial(\rho_o v_i)}{\partial x_i}$$

Governing equations

- **Mass conservation:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

Product of two *small* quantities → *negligible*

linerization

$$\frac{\partial \rho'}{\partial t} + \frac{\partial(\rho_o v_i)}{\partial x_i} + \frac{\partial(\cancel{\rho' v_i})}{\partial x_i} = 0 \quad \Longrightarrow \quad \boxed{\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} \approx 0}$$

- v_i - acoustic particle velocity (velocity fluctuations due to acoustic waves)

Governing equations

Linearization

$$\rho = \rho_o + \rho'$$
$$p = p_o + p'$$
$$v_i = v_i'$$

$$\frac{\partial(\rho' v_i v_j)}{\partial x_j} \ll \frac{\partial(\rho_o v_i v_j)}{\partial x_j}$$

$$\frac{\partial(\rho' v_i)}{\partial t} \ll \frac{\partial(\rho_o v_i)}{\partial t}$$

- **Momentum conservation:**

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p_{ij})}{\partial x_j} = 0$$

$$\sigma_{ij} = 0$$

$$p_{ij} = p\delta_{ij} - \cancel{\sigma_{ij}} \approx p\delta_{ij}$$

Neglecting viscous effects

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0$$

linerization

$$\frac{\partial(\rho_o v_i)}{\partial t} + \underbrace{\frac{\partial(\cancel{\rho' v_i})}{\partial t}}_{\text{negligible}} + \frac{\partial(\rho_o v_i v_j)}{\partial x_j} + \underbrace{\frac{\partial(\cancel{\rho' v_i v_j})}{\partial x_j}}_{\text{negligible}} + \frac{\partial(p_o + p')}{\partial x_i} = 0 \Rightarrow \rho_o \frac{\partial v_i}{\partial t} + \frac{\partial(p_o + p')}{\partial x_i} + \rho_o \frac{\partial(v_i v_j)}{\partial x_j} \approx 0$$

- In a **stationary fluid**, the **mean pressure gradient is zero** ($\nabla p_o = 0$) and only matched by gravitational forces, which have been *ignored*

$$\frac{\partial(p_o + p')}{\partial x_i} \approx \frac{\partial p'}{\partial x_i}$$

- To determine the importance of the nonlinear $\partial(v_i v_j)/\partial x_j$ term (product of two small quantities), we consider a perturbation with a time scale T and length scale λ, so:

$$c_o \sim \frac{\lambda}{T} \Rightarrow \frac{\partial v_i}{\partial t} \sim \frac{v_i c_o}{\lambda} \quad \text{and} \quad \frac{\partial(v_i v_j)}{\partial x_j} \sim \frac{v_i^2}{\lambda} \Rightarrow v_i^2 \ll v_i c_o$$

$$\rho_o \frac{\partial(v_i v_j)}{\partial x_j} \ll \rho_o \frac{\partial v_i}{\partial t}$$

$$\boxed{\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} \approx 0}$$

Governing equations

Linearization

$$\rho = \rho_o + \rho'$$

$$p = p_o + p'$$

$$v_i = v_i'$$

❑ **Energy conservation:**

$$\rho T \frac{Ds}{Dt} = \sigma_{ij} \frac{\partial v_j}{\partial x_i} - \frac{\partial Q_i}{\partial x_i}$$

Ignoring viscous effects
& heat addition
= ***Isentropic***

$$\boxed{\frac{Ds'}{Dt} \approx 0} \qquad s = s_o + s'$$

Governing equations - Linearized

Linearization

$$\rho = \rho_o + \rho'$$

$$p = p_o + p'$$

$$v_i = v_i'$$

- Mass conservation

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} = 0} \quad \text{(I)}$$

- Momentum conservation

$$\boxed{\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0} \quad \text{(II)}$$

- Energy conservation

$$\boxed{\frac{Ds'}{Dt} = 0} \quad \text{(III)}$$

Conservation of acoustic energy

- **Momentum conservation** $\boxed{\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0}$ (II) Multiply by v_i

$$\rho_o v_i \frac{\partial v_i}{\partial t} + v_i \frac{\partial p'}{\partial x_i} = 0 \Rightarrow \rho_o \frac{\partial}{\partial t}\left(\frac{v_i^2}{2}\right) + \frac{\partial (p' v_i)}{\partial x_i} - p' \underbrace{\frac{\partial v_i}{\partial x_i}} = 0 \Rightarrow \rho_o \frac{\partial}{\partial t}\left(\frac{v_i^2}{2}\right) + \frac{\partial (p' v_i)}{\partial x_i} + \frac{p'}{\rho_o}\frac{\partial \rho'}{\partial t} = 0$$

Mass conservation: $\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} = 0$

$$\xRightarrow{\rho' = p'/c_o^2} \rho_o \frac{\partial}{\partial t}\left(\frac{v_i^2}{2}\right) + \frac{p'}{\rho_o c_o^2}\frac{\partial p'}{\partial t} + \frac{\partial (p' v_i)}{\partial x_i} = 0$$

$$\frac{\partial}{\partial t}\left(\rho_o \frac{v_i^2}{2} + \frac{p'^2}{2\rho_o c_o^2}\right) + \underbrace{\frac{\partial}{\partial x_i}(p' v_i)}_{\text{Flux of acoustic intensity}} = 0$$

or

$$\frac{\partial E}{\partial t} + \frac{\partial \mathbf{I}}{\partial x_i} = 0$$

$$\mathbf{I} = p' v_i$$

Conservation of acoustic energy

(sum of *kinetic* and *potential* acoustic energy)

Concept of acoustic potential

- **Momentum conservation** $\boxed{\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0}$ (II)

Take the curl
$\nabla \times$(II)

$$\nabla \times \left(\rho_o \frac{\partial v_i}{\partial t}\right) = \underbrace{-\nabla \times \left(\frac{\partial p'}{\partial x_i}\right)}_{\textit{Identity: } \nabla \times (\nabla p') \equiv 0} \Longrightarrow \rho_o \frac{\partial}{\partial t}(\nabla \times v_i) = 0 \Longrightarrow \boxed{\nabla \times v_i = 0}$$

- Thus, there is an **acoustic velocity potential**: $\boxed{v_i = \nabla\phi}$ ϕ – perturbation potential

- **NOTE** – **acoustic excitation of an inviscid fluid DOES NOT produce rotational flow**
  - No boundary layer, shear, turbulence…

- Plugging the acoustic velocity potential back into the momentum conservation eq. (II) yields:

$$\rho_o \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial p'}{\partial x_i} = 0 \Longrightarrow \frac{\partial p'}{\partial x_i} = -\rho_o \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}\right) \Rightarrow \boxed{p' = -\rho_o \frac{\partial \phi}{\partial t}} \text{ or } \boxed{v_i = \nabla\phi = -\int \frac{1}{\rho_o} \nabla p' dt}$$

Bernoulli's eq.

Governing equations - Linearized

Linearization

$$\rho = \rho_o + \rho'$$
$$p = p_o + p'$$
$$v_i = v_i'$$

- Mass conservation

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} = 0} \quad \text{(I)}$$

- Momentum conservation

$$\boxed{\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0} \quad \text{(II)}$$

- Energy conservation

$$\boxed{\frac{Ds'}{Dt} = 0} \quad \text{(III)}$$

Perform
$\nabla \cdot \text{(II)} \; - \frac{\partial}{\partial t}\text{(I)}$

→ *Linear Acoustic Wave Equation*

Linearized acoustic wave equation

$$\nabla \cdot \left(\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} \right) - \frac{\partial}{\partial t}\left(\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} \right) = 0$$

$$\Longrightarrow \cancel{\rho_o \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial t} \right)} + \frac{\partial^2 p'}{\partial x_i^2} - \frac{\partial^2 \rho'}{\partial t^2} - \cancel{\rho_o \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial t} \right)} = 0 \quad \Longrightarrow \quad \frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0$$

- ❑ The above equation involves both ρ' and p', which is inconvenient to work with…
- ❑ Recall the compressibility relation we obtained: $p' = c_o^2 \rho'$
- ❑ Substituting this relation into the above equation yields:

Wave equation $\boxed{\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0}$ or $\boxed{\frac{1}{c_o^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_i^2} = 0}$ or $\boxed{\Box p' = 0}$

Wave operator

$$\Box = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

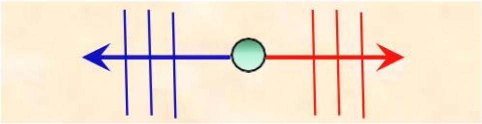
- ➢ ***Linear* 2nd order PDE** → Solution *depends* on a set of **boundary** and **initial conditions**

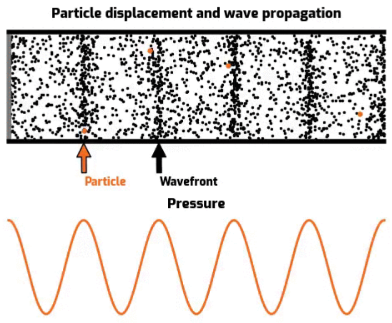
One-dimensional plane wave

$$\boxed{\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_1^2} = 0}$$

- The simplest example of an acoustic wave is a **one-dimensional plane wave**
  - *Example* – sound propagation along a thin tube (semi-infinite tube with a piston at one end) – only a function of *distance* in the direction of propagation and *time*
- The solution can be obtained from the ***method of characteristics***:
  - We introduce two new variables: $\xi = x_1 - c_o t$ and $\eta = x_1 + c_o t$
  - In terms of (ξ, η), the wave equation simplifies to:

$$\frac{\partial^2 p'}{\partial \xi \partial \eta} = 0 \xrightarrow{\text{integrating twice}} p'(\xi, \eta) = f(\xi) + g(\eta)$$



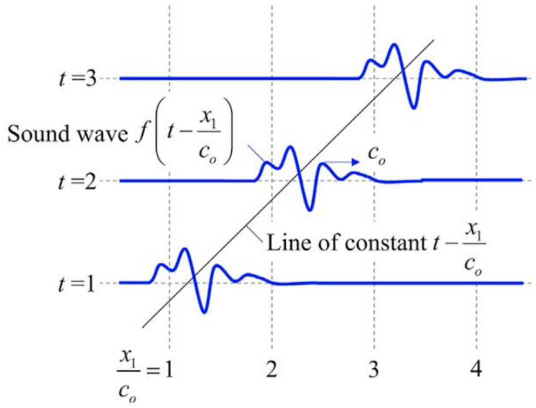


2 Wave travelling in $-x_1$

1 Wave travelling in $+x_1$

$$\boxed{p'(x_1, t) = \underbrace{f(t - x_1/c_o)}_{1} + \underbrace{g(t + x_1/c_o)}_{2}}$$

d'Alembert's solution



$$\boxed{v(x_1, t) = \frac{1}{z_o} f\left(t - \frac{x_1}{c_o}\right) + \frac{1}{z_o} g\left(t + \frac{x_1}{c_o}\right)}$$

$z_o = \rho_o c_o = p'/v$

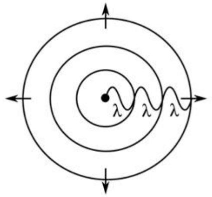
Complex harmonic plane wave

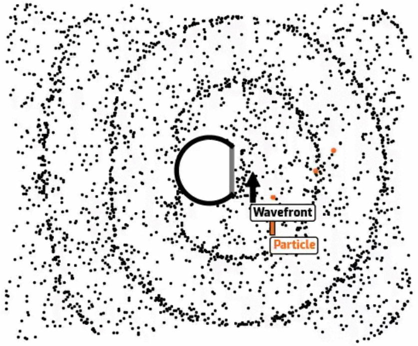
$$\boxed{p' = \hat{p}^+ e^{-i(\omega t - k x_1)} + \hat{p}^- e^{-i(\omega t + k x_1)}}$$

$\omega = 2\pi f \quad f = \frac{c_o}{\lambda} \quad k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o}$

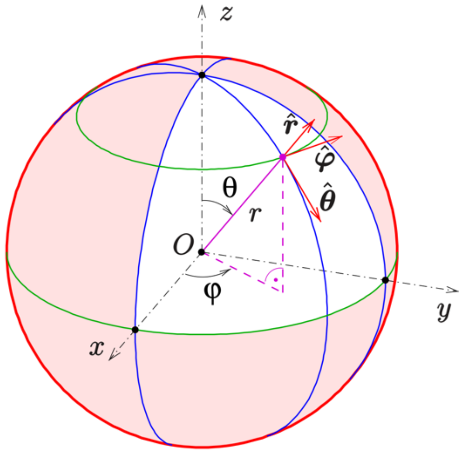
  - A pressure perturbation $f(x_1)$ or $g(x_1)$ at $t = 0$ will be repeated at the location $f(x_1 = d)$ or $g(x_1 = -d)$ at a time $t = d/c_o$ later

Spherical waves





- Greater *practical* importance – solution to the wave equation in *spherical coordinates*
- We shall limit the analysis to waves that are **only a function of the radial distance r** from the center of the coordinate system ($r_0 = 0$)
  - *Example* – sound propagation from a speaker



- Recalling:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right)} + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)$$

- Therefore, we can write:

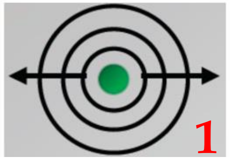
$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_1^2} = 0 \implies \frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p'}{\partial r}\right) = 0 \implies \boxed{\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp') = 0}$$

- Multiplying through by r gives a 1D wave equation in terms of the variable rp':

Amplitude decays as $\frac{1}{r}$

$$\boxed{p'(r,t) = \frac{f(t - r/c_o)}{r}}$$

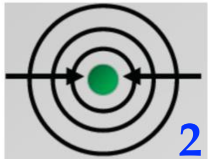
For most cases, we only care about outwardly propagating waves



$$\boxed{rp'(r,t) = \underbrace{f(t - r/c_o)}_{1} + \underbrace{g(t + r/c_o)}_{2}}$$

1 Wave travelling outwards

2 Wave travelling inwards



Frequency domain analysis

$$\boxed{\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp') = 0}$$

- Generally, we are interested in evaluating the sound field as a function of frequency rather than time
- To do so, we decompose the time signal into a single frequency with **harmonic time dependence**:
  - $A \rightarrow$ wave amplitude
  - $\omega = 2\pi f \rightarrow$ angular frequency [rad/sec] $\quad f = \frac{c_o}{\lambda}$
  - $\phi \rightarrow$ wave phase [rad]

$$\boxed{f(t) = A\cos(\omega t - \phi)}$$

- The solution to the wave equation for **outwardly propagating spherical waves (<u>monopole</u>)** will be:

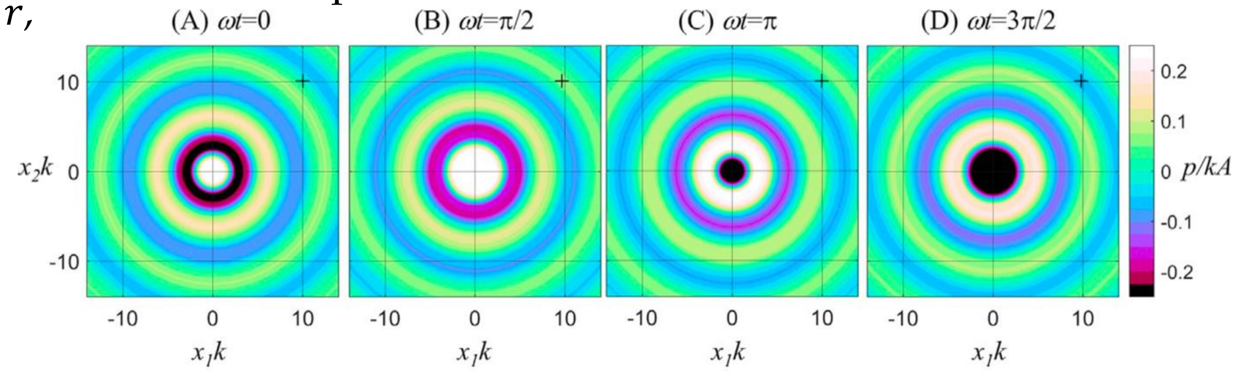
$$p'(r,t) = \frac{f(t - r/c_o)}{r} \implies \boxed{p'(r,t) = \frac{A\cos(\omega t + \omega r/c_o - \phi)}{r}}$$

Acoustic wavenumber [rad/m]

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o}}$$

- **At fixed time**
  - Harmonic *spatial* dependence of the sound field with r, oscillating at the acoustic *wavelength* λ
- **At fixed position (+)**
  - Harmonic *time* dependence, oscillating at the acoustic *frequency* ω
- **Sound field repeat** $\rightarrow$ when the phase $\omega r/c_o$ is incremented by 2π, via $\lambda = 2\pi c_o/\omega$

Spherical waves at different times



Frequency domain analysis

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp') = 0$$

- In *complex form*, the harmonic time series is the real part of a complex exponential:

Monopole

$$\boxed{p'(r,t) = \mathbb{Re}\left[\hat{p}(r)e^{-i\omega t}\right] = \mathbb{Re}\left[\frac{\hat{A}e^{-i(\omega t - kr)}}{r}\right]}$$

$$\hat{p}(r) = \frac{\hat{A}e^{ikr}}{r}$$

$$\hat{A} = Ae^{i\phi}$$

- ^ denotes the complex amplitude
- Acoustic wavenumber k – expresses the phase of the complex pressure amplitude $\hat{p}$
- Since the field is harmonic, the phase shift increases by $k\lambda = 2\pi$ when r increases by one wavelength, so $k = 2\pi/\lambda$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o}$$

Helmholtz equation

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o}$$

$$p'(\mathbf{x},t) = \mathbb{Re}\left[\hat{p}(\mathbf{x})e^{-i\omega t}\right]$$

$$\mathbf{v}(\mathbf{x},t) = \mathbb{Re}\left[\hat{\mathbf{v}}(\mathbf{x})e^{-i\omega t}\right]$$

$$\phi(\mathbf{x},t) = \mathbb{Re}\left[\hat{\phi}(\mathbf{x})e^{-i\omega t}\right]$$

- The linear acoustic wave equation can be solved in the **frequency domain**
  - *Significant simplification!*
- We substitute a harmonic fluctuation $p'(\mathbf{x},t)$ to express the acoustic wave equation in terms of the complex pressure amplitude $\hat{p}(\mathbf{x})$:

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0 \implies \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\left(\mathbb{Re}\left[\hat{p}(\mathbf{x})e^{-i\omega t}\right]\right) - \frac{\partial^2}{\partial x_i^2}\left(\mathbb{Re}\left[\hat{p}(\mathbf{x})e^{-i\omega t}\right]\right) = 0$$

$$\implies \frac{\hat{p}}{c_o^2}\left(\omega^2 e^{-i\omega t}\right) + e^{-i\omega t}\frac{\partial^2 \hat{p}}{\partial x_i^2} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial^2 \hat{p}}{\partial x_i^2} + k^2\hat{p} = 0} \quad \text{or} \quad \boxed{\nabla^2\hat{p} + k^2\hat{p} = 0}$$

Helmholtz equation

- Recall the acoustic momentum conservation:

$$\rho_o\frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0 \implies i\omega\rho_o e^{-i\omega t}\hat{\mathbf{v}} = e^{-i\omega t}\frac{\partial \hat{p}}{\partial x_i} \quad \Rightarrow \quad \boxed{i\omega\rho_o\hat{\mathbf{v}} = \frac{\partial \hat{p}}{\partial x_i}} \quad \text{or} \quad \boxed{i\omega\rho_o\hat{\mathbf{v}} = \nabla\hat{p}}$$

Relates the velocity perturbation amplitude to gradients of pressure perturbation amplitude

- And the acoustic potential:

$$\boxed{p' = -\rho_o\frac{\partial \phi}{\partial t}} \qquad \boxed{i\omega\rho_o\hat{\phi} = \hat{p}}$$

Helmholtz number

- ❑ **Helmholtz number**

$$\boxed{\mathbf{He} = kL = \frac{2\pi L}{\lambda} = \frac{2\pi f L}{c_o} = 2\pi \mathbf{St} \cdot \mathbf{M}}$$

- ❑ **Dimensionless number to quantify acoustic fluctuations**
- ❑ Can be used to define the time discretization or the sampling frequency
- ❑ Can be used to define if a body is acoustically compact
  - ➢ **He $\ll 1$ – acoustically compact body ($kL \ll 1$)**
  - ➢ **He $\geq O(1)$ – non-acoustically compact body ($kL \gg 1$)**
- ❑ A flow instability (e.g., vortex with size L) will radiate sound at a wavelength scaled by $1/\mathrm{M}$
  - ➢ At low Mach numbers, a very small vortex can radiate a sound with a long wavelength…
- ❑ *Example*:
  - ➢ Consider a small 4″ Ø loudspeaker ($L = 0.106$ m) emitting a tone at $f_1 = 50$ Hz and another at $f_2 = 3000$ Hz

$$\mathbf{He}_1 = \frac{2\pi \cdot 50\mathrm{Hz} \cdot 0.106\mathrm{m}}{340.3\mathrm{m/s}} = 0.1 \ll 1 \implies$$ Nearly **omnidirectional sound radiation**

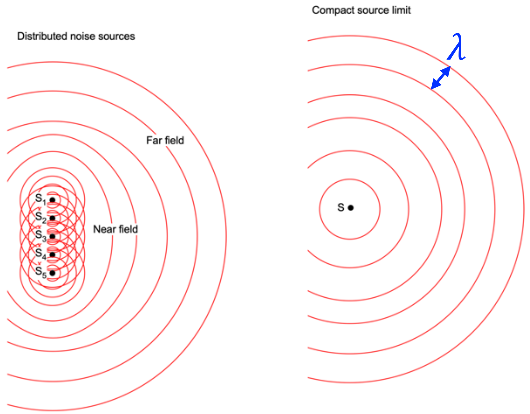
$$\mathbf{He}_2 = \frac{2\pi \cdot 3000\mathrm{Hz} \cdot 0.106\mathrm{m}}{340.3\mathrm{m/s}} = 5.9 \gg 1 \implies$$ Complex waveforms – **strong directivity pattern**

Mach number $\mathbf{M} = \dfrac{U_\infty}{c_o}$

Strouhal number $\mathbf{St} = \dfrac{fL}{U_\infty}$

Acoustic wavenumber [rad/m]

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o} = \frac{2\pi f}{c_o}$$



Summary of plane and spherical waves

$$z_o = \rho_o c_o$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c_o} = \frac{2\pi f}{c_o}$$

❑ Plane harmonic waves

Wave Eq.		Harmonic sound signal	Acoustic velocity	Acoustic potential
Time-domain	*Frequency-domain*			
$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0$ $\frac{1}{c_o^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_i^2} = 0$	$\frac{\partial^2 \hat{p}}{\partial x_i^2} + k^2\hat{p} = 0$	$p'(x_1,t) = \mathbb{Re}\left[\hat{p}(x_1)e^{-i\omega t}\right]$ $\hat{p}(x_1) = \hat{A}e^{ikx_1}$ $\hat{A} = Ae^{i\phi}$	$v(x_1,t) = \mathbb{Re}\left[\hat{v}(x_1)e^{-i\omega t}\right]$ $\hat{v}(x_1) = \frac{1}{i\omega\rho_o}\frac{\partial \hat{p}}{\partial x_1} = \frac{\hat{p}(x_1)}{z_o} = \frac{\hat{A}e^{ikx_1}}{z_o}$	$\phi(x_1,t) = \mathbb{Re}\left[\hat{\phi}(x_1)e^{-i\omega t}\right]$ $\hat{\phi}(x_1) = \frac{\hat{p}(x_1)}{i\omega\rho_o} = \frac{\hat{A}e^{ikx_1}}{ikz_o}$

❑ Spherical harmonic waves

Wave Eq.		Harmonic sound signal	Acoustic velocity	Acoustic potential
Time-domain	*Frequency-domain*			
$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp') = 0$ $\frac{1}{c_o^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) = 0$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rp') + k^2\hat{p} = 0$	$p'(r,t) = \mathbb{Re}\left[\hat{p}(r)e^{-i\omega t}\right]$ $\hat{p}(r) = \frac{\hat{A}e^{ikr}}{r}$ $\hat{A} = Ae^{i\phi}$	$v(r,t) = \mathbb{Re}\left[\hat{v}(r)e^{-i\omega t}\right]$ $\hat{v}(r) = \frac{1}{i\omega\rho_o}\frac{\partial \hat{p}}{\partial r} = \frac{\hat{A}e^{ikr}}{z_o r}\left(1 + \frac{i}{kr}\right)$	$\phi(r,t) = \mathbb{Re}\left[\hat{\phi}(r)e^{-i\omega t}\right]$ $\hat{\phi}(r) = \frac{\hat{p}(r)}{i\omega\rho_o} = \frac{\hat{A}e^{ikr}}{ikz_o r}$

Elementary acoustic sources

- Elementary acoustic sources can be solved in two different ways:
  - **Inhomogeneous wave equation**
    - Sources are introduced *locally* either in the form of a mass flow q or in the form of a volume force **F**
    - Involves **Green functions** (will be discussed later…)

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2}-\frac{\partial^2 p'}{\partial x_i^2}=\frac{\partial q}{\partial t}-\frac{\partial F_i}{\partial x_i}=Q(\mathbf{x},t)}$$

or

$$\boxed{\frac{\partial^2 \hat{p}}{\partial x_i^2}+k^2\hat{p}=-i\omega q-\frac{\partial F_i}{\partial x_i}=Q(\mathbf{x},\omega)}$$

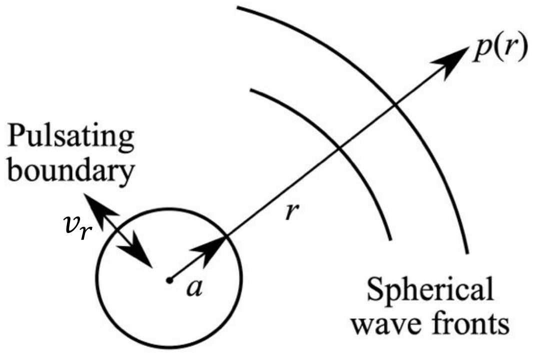
  - **Homogeneous wave equation**
    - Solution is achieved by setting boundary conditions to determine the unknown constants introduced by the sources
    - **Will be discussed next…**

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2}-\frac{\partial^2 p'}{\partial x_i^2}=0}$$

or

$$\boxed{\frac{\partial^2 \hat{p}}{\partial x_i^2}+k^2\hat{p}=0}$$

Case 1 - Pulsating sphere



- Consider the sound radiation from a small sphere of radius a, whose surface oscillates radially with a normal surface velocity $[v_r]_{r=a}$, pulsating with ω

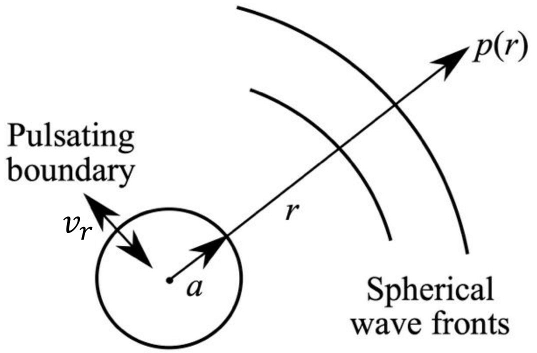
$$\textbf{BC:} \begin{cases} v_r = v_o e^{-i\omega t}, \qquad \hat{v}_r = v_o, & r = a \\ \dfrac{\partial^2 \hat{p}}{\partial x_i^2} + k^2 \hat{p} = 0, & r \geq a \end{cases}$$

- The solution to the wave equation matching the BC is: $\boxed{\hat{p}(r) = \dfrac{\hat{A} e^{ikr}}{r}}$ $\boxed{\hat{v}(r) = \dfrac{\hat{p}(r)}{z_o}\left(1 + \dfrac{i}{kr}\right)}$

- At $r = a$ we have: $[\hat{v}]_{r=a} = v_o = \left[\dfrac{\hat{p}(r)}{r z_o}\left(1 + \dfrac{i}{kr}\right)\right]_{r=a} \Rightarrow v_o = \dfrac{\hat{A} e^{ika}}{a z_o}\left(1 + \dfrac{i}{ka}\right) \Rightarrow \boxed{\hat{A} = -\dfrac{i\omega\rho_o a^2 v_o e^{-ika}}{(1 - ika)}}$

- The **acoustic field** is thus: $\boxed{\hat{p}(r) = -\dfrac{i\omega\rho_o a^2 v_o}{(1 - ika)} \dfrac{e^{ik(r-a)}}{r}}$

Case 1 - Pulsating sphere



- If the sphere is acoustically compact ($ka \ll 1$), we approximate: $e^{-ika} \approx 1 - ika$
- Therefore, the solutions approximates to:

$$e^{-x} \approx 1 - x + O(x^2)$$

$$\boxed{\hat{p}(r) = -\frac{i\omega\rho_0 a^2 v_0 e^{ikr}}{r}} \propto \frac{1}{r}$$

- **Note**: no initial conditions were required because a *harmonic time dependence* was assumed
- Given a sphere's surface area of $S = 4\pi a^2$, we can define the **rate of change of the sphere's volume** (also known as its **volumetric flow rate**):

$$Q = \int_S [v_r]_{r=a} dS = \int_0^{2\pi}\int_0^{\pi} v_0 e^{-i\omega t} a^2 \sin\theta \, d\theta d\varphi \implies \boxed{\begin{matrix} Q(t) = v_0 S e^{-i\omega t} \\ \hat{Q} = v_0 S \end{matrix}}$$

$$\boxed{M(t) = \rho_0 \hat{Q} e^{-i\omega t}}$$

mass flow rate

- **Monopole source**

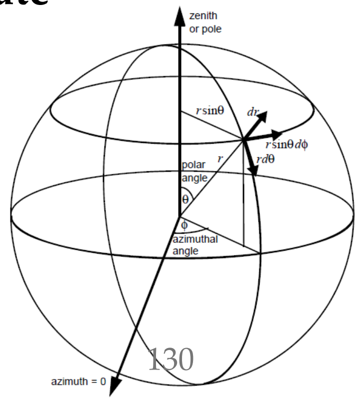
$$\boxed{\hat{p}(r) = -i\omega\rho_0\hat{Q}\frac{e^{ikr}}{4\pi r}}$$

Volume displacement source
(*Omni-directional*)

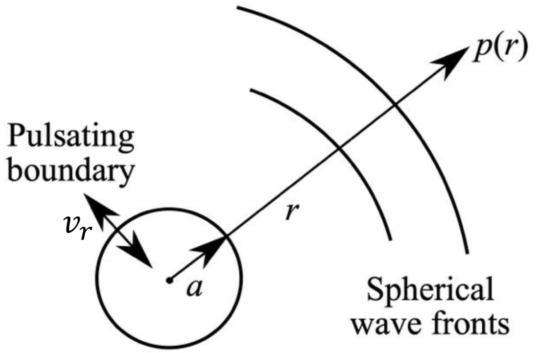
Harmonic monopole source strength

Free-field Green's function

$$G = \frac{e^{ikr}}{4\pi r}$$



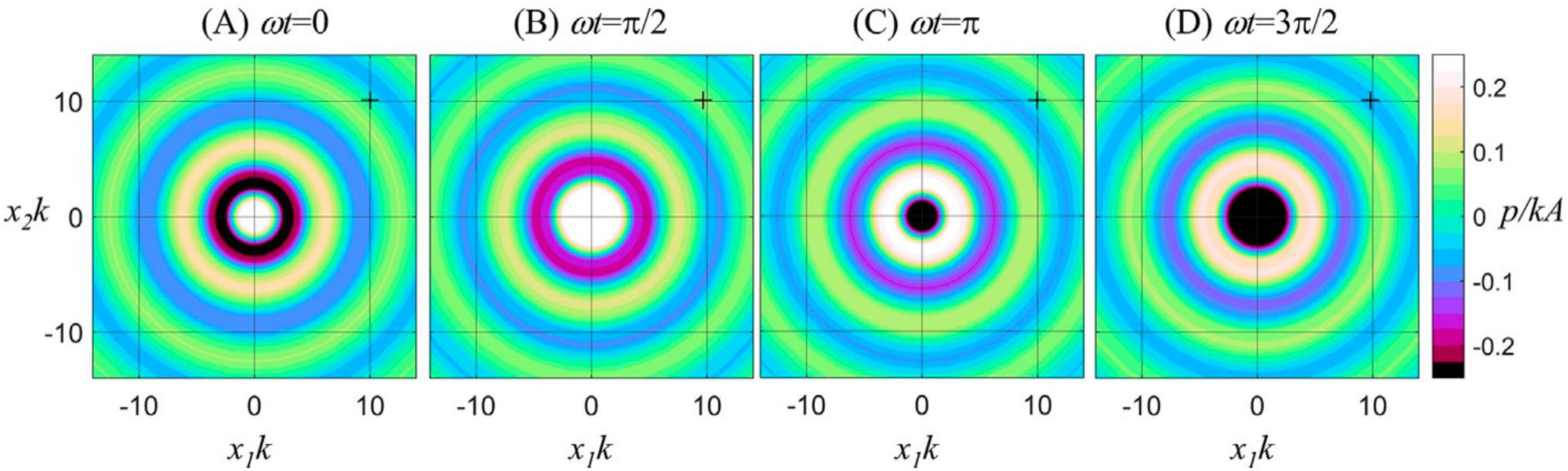
Case 1 - Pulsating sphere



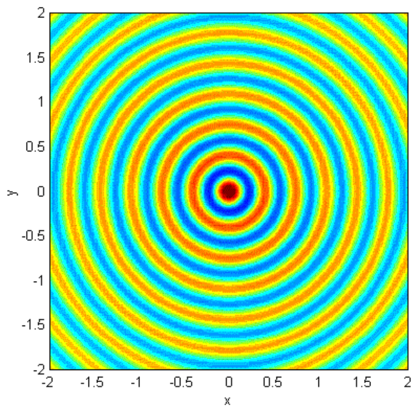
- Monopole source

$$\boxed{\frac{\hat{p}(r)}{k\hat{A}} = \frac{e^{ikr}}{kr}}$$

$$p'(r,t) = \mathbb{Re}\left[\hat{p}(r)e^{-i\omega t}\right]$$



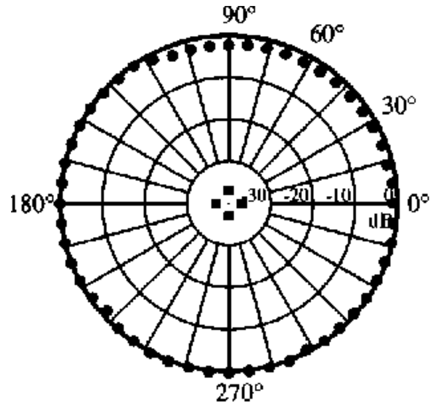
Monopole noise sources



- **Examples**

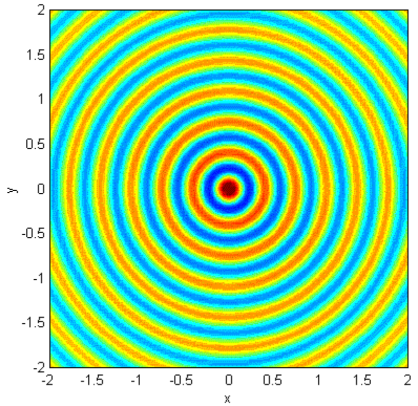
$$\boxed{\hat{p}(r) = -i\omega\rho_o\hat{Q}\,\frac{e^{ikr}}{4\pi r}}$$

- **Boxed loudspeaker** acts as an omnidirectional monopole source at low frequencies



Measured directivity patterns at 250 Hz for sound radiation from a monopole 4″ boxed loudspeaker

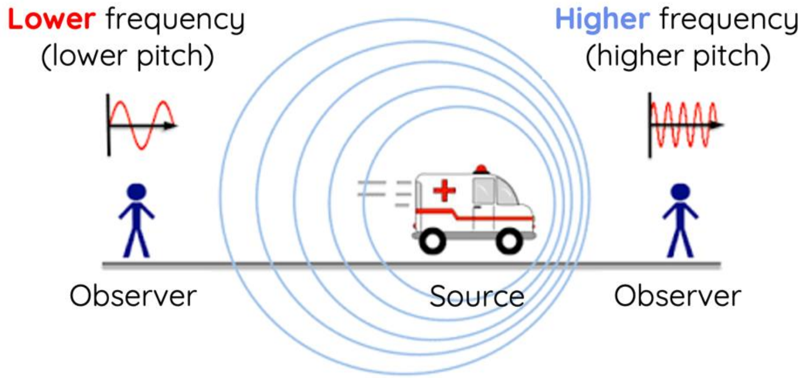
Monopole noise sources

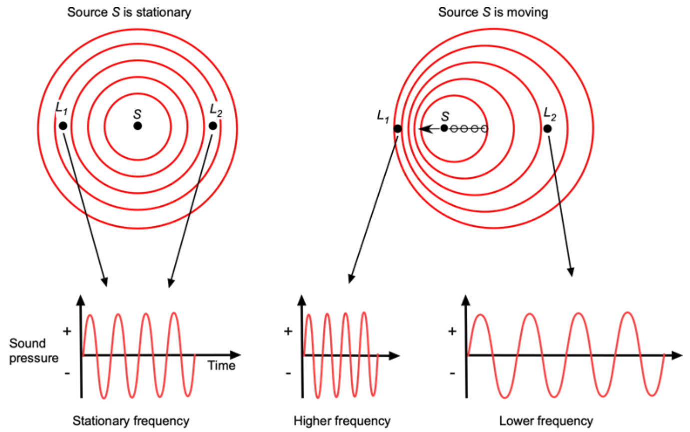


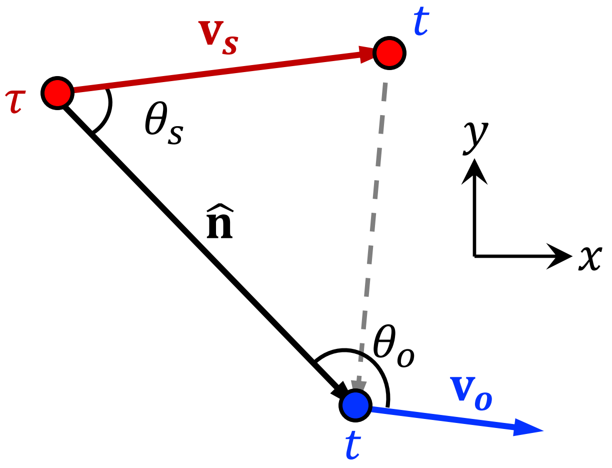
❑ Examples

$$\hat{p}(r) = -i\omega\rho_o\hat{Q}\,\frac{e^{ikr}}{4\pi r}$$

➢ **Siren**







$$f' = \frac{c_o - \mathbf{v_o}\cdot\hat{\mathbf{n}}}{c_o - \mathbf{v_s}\cdot\hat{\mathbf{n}}}\,f = \frac{c_o + |\mathbf{v_o}|\cos\theta_o}{c_o - |\mathbf{v_s}|\cos\theta_s}\,f$$

$|\mathbf{v_o}|\cos\theta_o$ → negative when *observer* is moving **away** the *source*

$|\mathbf{v_s}|\cos\theta_s$ → *negative* when *source* is moving **away** from the *observer*

$\hat{\mathbf{n}}$ – unit vector from *source* **to** *observer* at the emission time

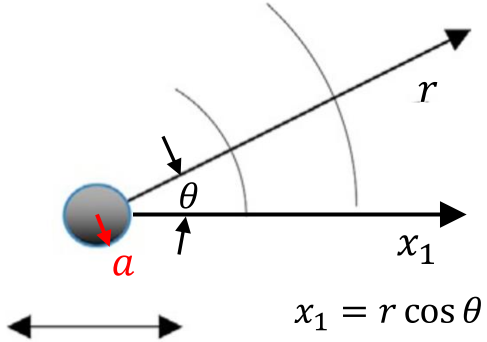
$\mathbf{v_o}$ → *observer* velocity vector

$\mathbf{v_s}$ → *source* velocity vector

τ – sound source emission time

t – observer time when hearing the sound emitted at time τ

Case 2 - Translating sphere



$v_1 = v_o e^{-i\omega t}$

$x_1 = r\cos\theta$

$r = \sqrt{x_i^2}$

$\dfrac{\partial r}{\partial x_1} = \dfrac{x_1}{r} = \cos\theta$

- Consider a sphere of radius a that translates back and forth in the x_1 direction with velocity v_1, pulsating with ω
- *Viscous effects are ignored* → **only the radial velocity of the surface of the sphere is considered** (*tangential velocity* to the surface is neglected)

BC: $\begin{cases} v_r = v_o \cos\theta \, e^{-i\omega t}, \quad \hat{v}_r = v_o \cos\theta\,, & r = a \\ \partial^2 \hat{p}/\partial x_i^2 + k^2 \hat{p} = 0, & r \geq a \end{cases}$

- The solution to the wave equation matching the BC is: $\boxed{\hat{p}(r) = \dfrac{\partial}{\partial x_1}\left(\dfrac{\hat{A}e^{ikr}}{r}\right)}$

Note: *any derivative of a solution* to the wave equation is also a solution to the wave equation!

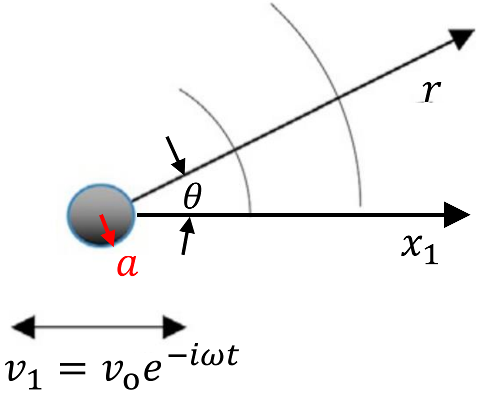
- Evaluating the derivative above yields: $\hat{p}(r) = \dfrac{\partial r}{\partial x_1}\dfrac{\partial}{\partial r}\left(\dfrac{\hat{A}e^{ikr}}{r}\right) \Longrightarrow$ $\boxed{\hat{p}(r) = ik\cos\theta\left(\dfrac{\hat{A}e^{ikr}}{r}\right)\left(1 - \dfrac{1}{ikr}\right)}$

Directivity term
can be used to match BC
for the translating sphere

- **Note** – near-field sound amplitude is *proportional* to $\hat{p} \propto \dfrac{1}{r}\left(1 - \dfrac{1}{ikr}\right)$
  - **In acoustic far-field ($kr \gg 1$)**, we obtain the regular $\hat{p} \propto 1/r$:

$$\boxed{\boldsymbol{kr \gg 1} \quad \Rightarrow \quad \hat{p}(r) = ik\cos\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)}$$

Case 2 - Translating sphere



$$\hat{p}(r) = ik\cos\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)$$

$$\frac{\partial\hat{p}}{\partial r} = ik\cos\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)$$

- From the acoustic momentum conservation, we obtain: $\hat{\mathbf{v}}\cdot\mathbf{n} = \dfrac{1}{i\omega\rho_o}\dfrac{\partial\hat{p}}{\partial r}$
  - $\mathbf{n}$ – unit vector on the sphere's surface, $\mathbf{n} = (n_r, n_\theta, n_\varphi)$
- The BC at sphere surface states: $[\hat{\mathbf{v}}\cdot\mathbf{n}]_{r=a} = [\hat{v}_r]_{r=a} = v_o\cos\theta$
- Thus, substituting the BC into the momentum conservation yields:

$$[\hat{\mathbf{v}}\cdot\mathbf{n}]_{r=a} = \left[\frac{1}{i\omega\rho_o}\frac{\partial\hat{p}}{\partial r}\right]_{r=a} \quad\Longrightarrow\quad v_o\cos\theta = \frac{1}{i\omega\rho_o}\left[\frac{\partial\hat{p}}{\partial r}\right]_{r=a}$$

$$\Longrightarrow \boxed{v_o\cos\theta = \frac{\cos\theta}{c_o\rho_o}\left(\frac{\hat{A}e^{ika}}{a}\right)\left(ik - \frac{2}{a} + \frac{2}{ika^2}\right)}$$

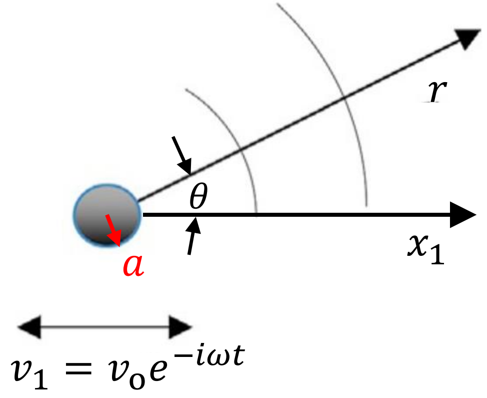
Need to be solved for the unknown $\hat{A}$

- If the **sphere is acoustically compact ($ka \ll 1$)**, we find that:

$$v_o = \frac{1}{c_o\rho_o}\left(\frac{\hat{A}e^{ika}}{a^2}\right)\left[\cancel{ika} - 2\left(\cancel{1} - \frac{1}{ika}\right)\right] \Longrightarrow \hat{A} \approx \frac{1}{2}ika^3c_o\rho_o v_o e^{-ika} \xRightarrow{e^{-ika}\approx 1-ika} \hat{A} \approx \frac{1}{2}ika^3\,ika\,c_o\rho_o v_o\left(\frac{1}{ika} - \cancel{1}\right)$$

$$\Longrightarrow \boxed{\hat{A} \approx \frac{1}{2}ika^3c_o\rho_o v_o}$$

Case 2 - Translating sphere

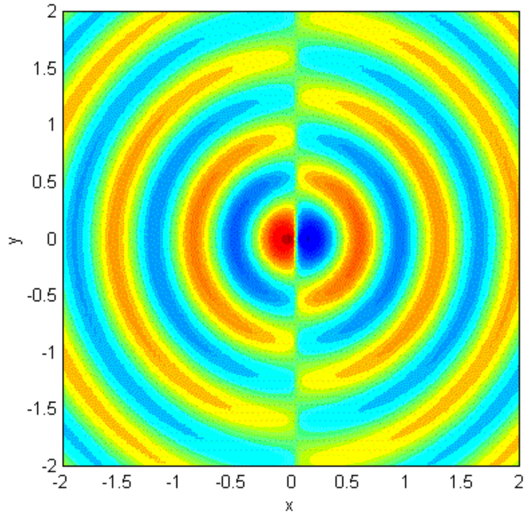


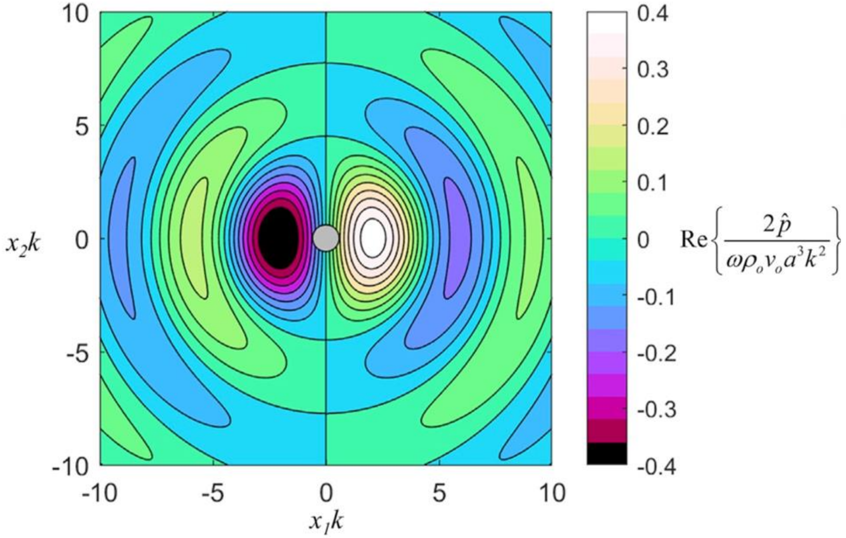
- Substituting $\hat{A}$ into the $\hat{p}(r)$ expression results in the following acoustic field:

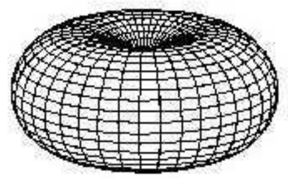
$$\hat{A} \approx \frac{1}{2} ika^3 c_o \rho_o v_o \quad \Longrightarrow \quad \hat{p}(r) = ik\cos\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)$$

- **Dipole source**

$$\boxed{\hat{p}(r) = \frac{ika\cos\theta}{2}\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)}$$





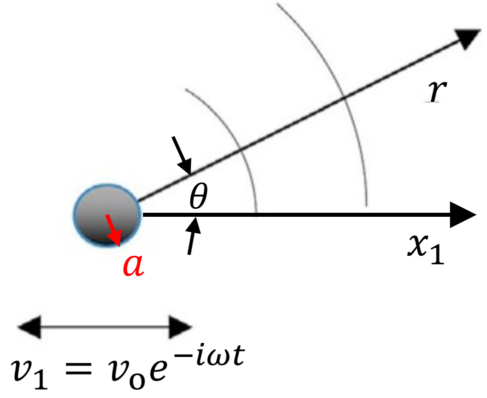


Directivity term – $\cos\theta$

Maximum *amplification* of the acoustic field along the x_1-axis

Zero *amplification* along the dipole principle axis

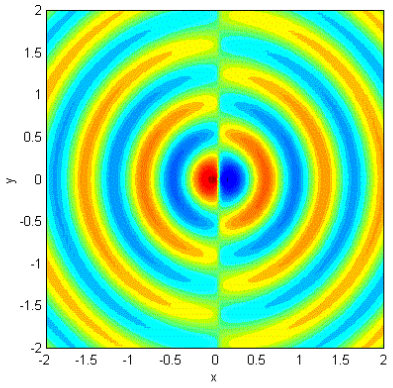
Case 2 - Translating sphere

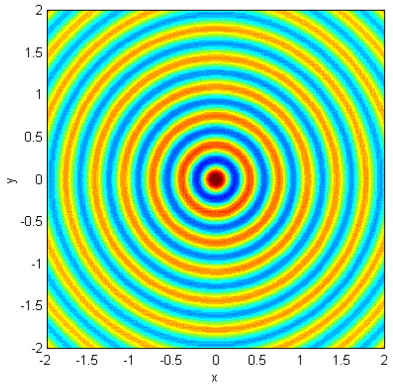


- ❑ **Dipole source**

$$\boxed{\hat{p}(r) = \frac{ika\cos\theta}{2}\left(\frac{i\omega\rho_0 v_0 a^2 e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)}$$

- ❑ Pressure amplitude of translating sphere (**dipole**) is less than a pulsating sphere (**monopole**) by factor of $ka/2$
  - ➢ **Pulsating sphere displaces mass** during each cycle, and so the medium has nowhere to go apart from propagating away as an acoustic wave
  - ➢ **Translating sphere causes no net displacement of mass**, and the fluid can adjust in the near field to accommodate the motion; yet some energy still escapes as sound and propagates to the acoustic far field

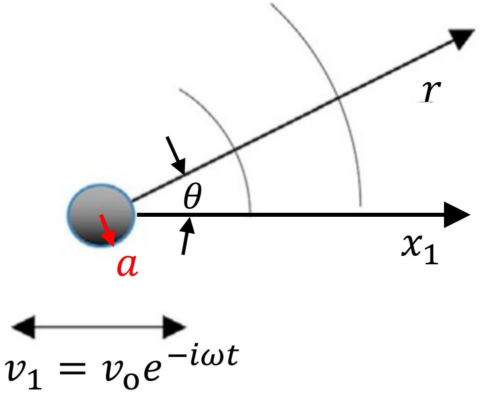




Monopole source

$$\boxed{\hat{p}(r) = -\frac{i\omega\rho_0 a^2 u_0 e^{ikr}}{r}}$$

Case 2 - Translating sphere



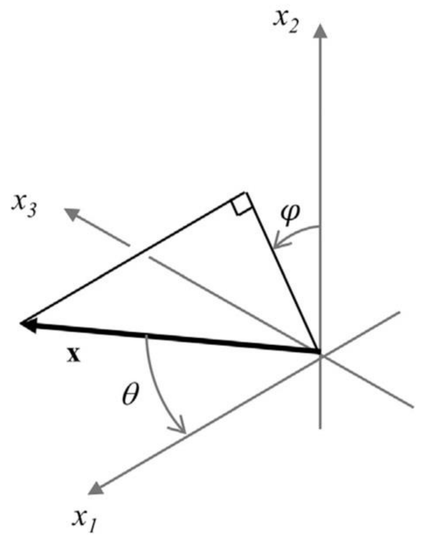
- What about the **force** needed to move the translating sphere?
  - For the *translating sphere*, the surface pressure depends on $\cos\theta \Rightarrow$ **spatial variation**
  - The **total force** $\hat{\mathbf{F}}$ applied to the fluid will be the **sum of:**
    - *Force* required to **move the mass of fluid displaced by the sphere** $(\rho_o 4\pi a^3/3)$
    - *Force* required to **overcome the net surface pressure** $\hat{p}$

$$\hat{\mathbf{F}} = -(i\omega\hat{\mathbf{v}})\rho_o\left(\frac{4\pi a^3}{3}\right) + \int_S [\hat{p}]_{r=a}\hat{\mathbf{n}}dS$$

Note: for the *pulsating sphere*, no net force is required since the surface pressure is constant over the sphere's surface and integrates to zero

  - The **only force to develop will be in the direction of the sphere's motion (x_1)**, and so we can limit our analysis to that direction only
  - In spherical coordinates we have $dS = ad\theta\, a\sin\theta\, d\varphi = a^2\sin\theta\, d\theta d\varphi$, and so:

$$\hat{F}_1 = -i\omega\rho_o v_o\left(\frac{4\pi a^3}{3}\right) + \int_S [\hat{p}]_{r=a}\hat{n}_1 dS \;\Rightarrow\; \hat{F}_1 = -i\omega\rho_o v_o\left(\frac{4\pi a^3}{3}\right) + \int_0^{\pi}\int_0^{2\pi}([\hat{p}]_{r=a}\cos\theta)a^2\sin\theta\, d\theta d\varphi$$

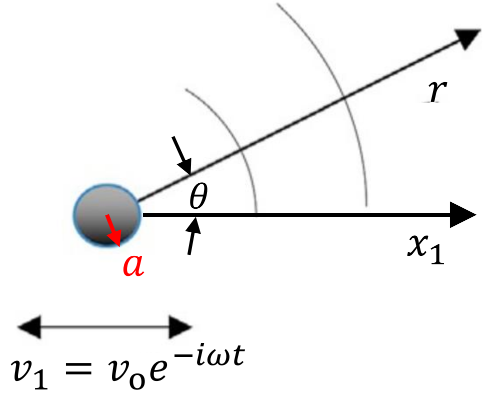


  - Where at $r = a$ for acoustically compact sphere ($ka \ll 1$):

$$[\hat{p}]_{r=a} = \left[\frac{ika\cos\theta}{2}\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)\right]_{r=a} \Rightarrow \boxed{[\hat{p}]_{r=a} = -\frac{i\omega\rho_o v_o a\cos\theta}{2}}$$

$e^{ika} \approx 1$

Case 2 - Translating sphere



- What about the **force** needed to move the translating sphere?
  - We therefore obtain:

$$\hat{F}_1 = -i\omega\rho_o v_o a^3 \left\{ \left(\frac{4\pi}{3}\right) + \pi \int_0^{\pi} (\cos^2\theta) \sin\theta \, d\theta \right\} \implies \boxed{\hat{F}_1 = -2\pi i \omega \rho_o v_o a^3}$$

$$\int_0^{\pi} (\cos^2\theta) \sin\theta \, d\theta = \frac{2}{3}$$

- The acoustic field can therefore be expressed as:

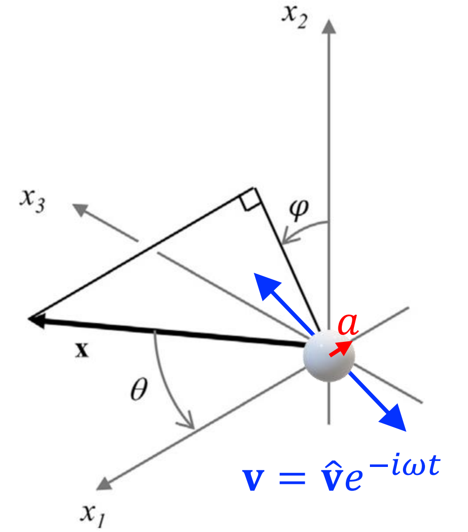
$$\hat{p}(r) = \frac{ika\cos\theta}{2} \left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right) \implies \hat{p}(r) = -ik\cos\theta \, e^{ikr} \left(\frac{-2\pi i\omega\rho_o v_o a^3}{4\pi r}\right)\left(1 - \frac{1}{ikr}\right)$$

$$\boxed{\hat{p}(r) = -ik\cos\theta \, e^{ikr} \left(\frac{\hat{F}_1}{4\pi r}\right)\left(1 - \frac{1}{ikr}\right)}$$

IMPORTANT RESULT
The acoustic source strength for the **transversely oscillating sphere** is determined by the force applied to the fluid

  - Later in the course we will see that this turns outs to be the most important mechanism of sound radiation in low Mach number flows

Case 3 - General spherical surface motion



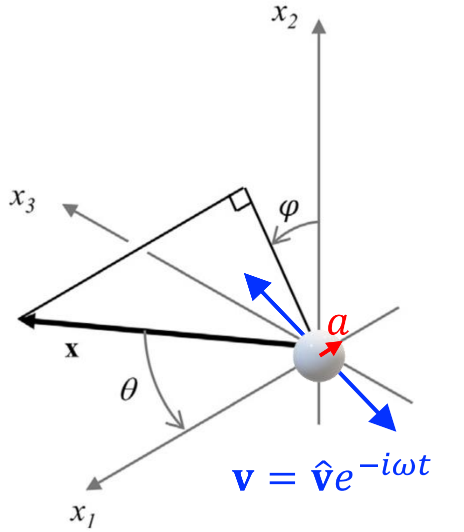
$\mathbf{x} = (r\cos\theta\,, r\sin\theta\cos\varphi\,, r\sin\theta\sin\varphi)$

- Consider the following acoustic field solution to the wave equation:

$$\boxed{\hat{p}(r) = \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\hat{A}e^{ikr}}{r}\right) = \delta_{ij}\left[\frac{\hat{A}e^{ikr}}{r^2}\left(ik - \frac{1}{r}\right)\right] + \frac{x_i x_j}{r^2}\left[\frac{\hat{A}e^{ikr}}{r}\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right)\right]}$$

  - δ_{ij} – Kronecker delta function
- We can match acoustic field to surface motion using acoustic momentum equation on the surface ($r = a$)
  - Leads to *complex expressions* that **can be simplified if limiting to acoustically compact surfaces ($ka \ll 1$)**

Case 3 - General spherical surface motion



$\mathbf{x} = (r\cos\theta, r\sin\theta\cos\varphi, r\sin\theta\sin\varphi)$

- Consider the surface velocity given in spherical coordinates by:

  v_o – maximum surface velocity $\quad \boxed{\hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = 2v_o \cos\theta \sin\theta \cos\varphi}$

  - On the surface ($r = a$) we can express the surface velocity as: $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = 2v_o \dfrac{x_1 x_2}{a^2}$
  - We can now match that velocity field to the pressure field expression we have with $i = 1$ and $j = 2$, yielding:

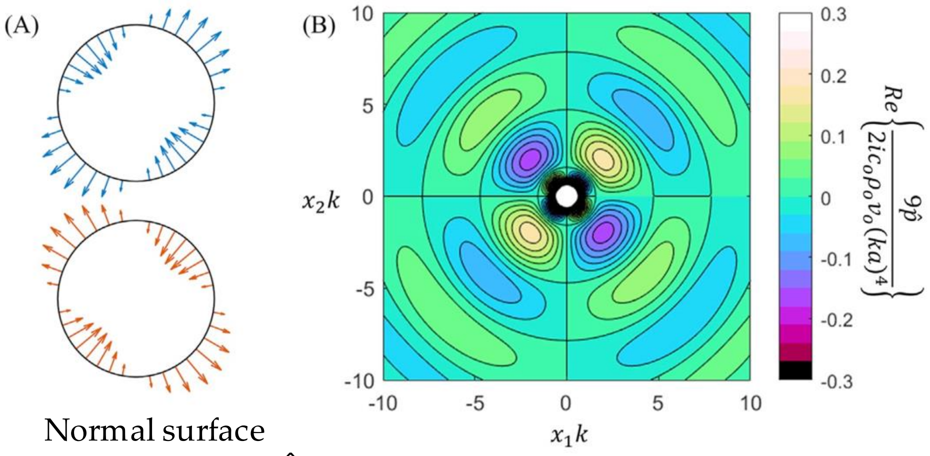
$$\hat{p}(r) = \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\hat{A}e^{ikr}}{r}\right) = \frac{x_1 x_2}{r^2}\left[\frac{\hat{A}e^{ikr}}{r}\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right)\right]$$

- As before, we solve for the constant $\hat{A}$ by matching the surface velocity to the pressure gradient in the radial direction:

$$[\hat{\mathbf{v}} \cdot \mathbf{n}]_{r=a} = \left[\frac{1}{i\omega\rho_o}\frac{\partial \hat{p}}{\partial r}\right]_{r=a} \quad \Longrightarrow \quad 2v_o \frac{x_1 x_2}{a^2} = \frac{1}{i\omega\rho_o}\left[\frac{\partial \hat{p}}{\partial r}\right]_{r=a}$$

$$\Rightarrow \boxed{2i\omega\rho_o v_o \frac{x_1 x_2}{a^2} = \left[\frac{x_1 x_2}{r^2}\frac{\hat{A}e^{ikr}}{r}\left\{ik\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right) - \left(\frac{9}{r^3} - \frac{6ik}{r^2} - \frac{k^2}{r}\right)\right\}\right]_{r=a}}$$

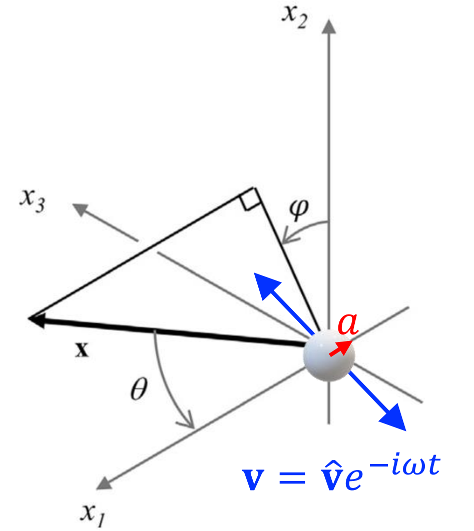
Lateral quadrupole



Normal surface motion amplitude $\hat{A}$ (positive & negative)

Far-field sound, $\hat{p}$

Case 3 - General spherical surface motion



$\mathbf{x} = (r\cos\theta\,, r\sin\theta\cos\varphi\,, r\sin\theta\sin\varphi)$

- Limiting to acoustically compact sphere ($ka \ll 1$) results in: $\boxed{\hat{A} = -\dfrac{2i\omega\rho_o v_o a^4}{9}}$

- Therefore, in the far-field ($kr \gg 1$) we obtain:

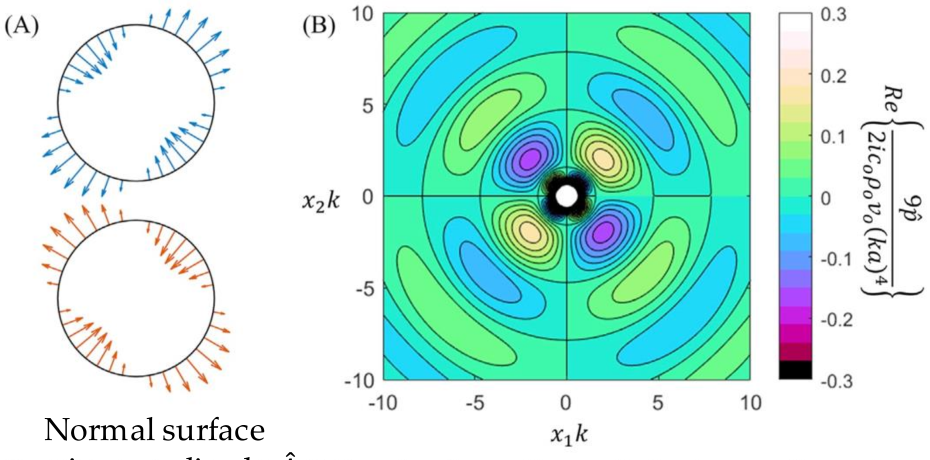
$$\boxed{\hat{p}(r) = \underbrace{\frac{2(ka)^2}{9}}_{\text{Scaling factor}} \underbrace{\left(\frac{x_1 x_2}{r^2}\right)}_{\text{Directivity}} \left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)}$$

$$\frac{x_1 x_2}{r^2} = \cos\theta\sin\theta\cos\varphi$$

Lateral quadrupole

- **More directional than a dipole source**
- **Its radiation is greatly reduced when reducing source size**
  - For comparison, a dipole is proportional to $ka/2$
- **No net volume velocity**
- **No net pressure applied on the fluid**

Lateral quadrupole



Normal surface motion amplitude $\hat{A}$ (positive & negative)

Far-field sound, $\hat{p}$