Brief review of conservation laws

- Notation: vectors, matrices, and tensors are in **bold** fonts
  - Example, position **x**, velocity **v**
- Cartesian coordinates are x_1, x_2, and x_3
  - $\mathbf{x} = (x_1, x_2, x_3), \quad \mathbf{v} = (v, v_2, v_3)$
- $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ – Cartesian unit vectors in $i = 1,2,3$ directions
- *Einstein* notation
  - Examples:

$$q_i v_i \equiv \sum_{i=1}^{3} q_i v_i \qquad \frac{\partial v_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

- **Kronecker delta:**

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

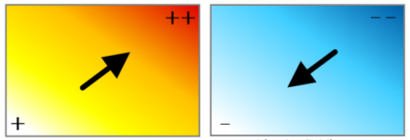
Vector analysis

- **Vector** $\mathbf{v} = v_i\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$
- **Magnitude squared of a vector** $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x} = x_i x_i = x_i^2 = \sum_{i=1}^{3} x_i^2 = x_1^2 + x_2^2 + x_3^2$

- **Gradient of a scalar field**
  - Produces a **vector** that **points in the direction where the scalar field is** ***increasing the fastest***

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{i} + \frac{\partial\phi}{\partial x_2}\mathbf{j} + \frac{\partial\phi}{\partial x_3}\mathbf{k} \equiv \frac{\partial\phi}{\partial x_i}$$

Gradient

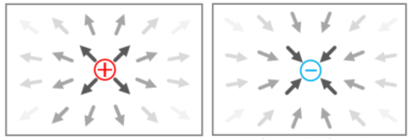


grad (scalar field) = vector

- **Divergence of a vector field**
  - Produces a *scalar* that is **a measure of how much 'something' is** ***expanding*** **or** ***contracting***

$$\nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \equiv \frac{\partial v_i}{\partial x_i}$$

Divergence



div (vector field) = scalar

- **Gradient of a vector field**
  - Produces a <u>**tensor**</u> that **measures the change of the vector field in each direction**

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \equiv \frac{\partial v_i}{\partial x_j}$$

Vector analysis

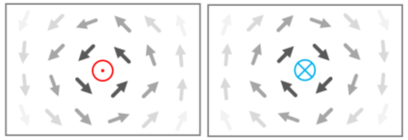
- **Convective derivatives**

$$(\mathbf{v}\cdot\nabla)\mathbf{v} \equiv v_j\frac{\partial v_i}{\partial x_j} \qquad (\mathbf{v}\cdot\nabla)\phi \equiv v_j\frac{\partial \phi}{\partial x_j}$$

- **Laplacian of a scalar/vector field**

$$\nabla^2\mathbf{v} = (\nabla\cdot\nabla)\mathbf{v} \equiv \frac{\partial^2 v_i}{\partial x_j^2} \qquad \nabla^2\phi = (\nabla\cdot\nabla)\phi \equiv \frac{\partial^2\phi}{\partial x_i^2}$$

- **Curl of a vector field**
  - Produces a **vector** that **measures the rotation in the vector field**
  - ε_{ijk} is the *alternating tensor*



$$\nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)\mathbf{k}$$

$$\nabla\times\mathbf{v} \equiv \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ are cyclic,} & (1,2,3),(2,3,1),(3,1,2) \\ -1, & \text{if } (i,j,k) \text{ are reverse cyclie order,} & (1,3,2),(2,1,3),(3,2,1) \\ 0, & \text{if } i=j, i=k \text{ or } k=j. \end{cases}$$

- **Cross product of two vectors**

$$\mathbf{a}\times\mathbf{b} \equiv \varepsilon_{ijk}a_j b_k$$

- **Outer (dyadic) product of two vectors**

$$\mathbf{S} = \boldsymbol{a}\otimes\boldsymbol{b} = \mathbf{a}\mathbf{b}^{\mathrm{T}}$$

$$S_{ij} = a_i b_j$$

$$S_{ij} = v_i v_j = \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2^2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3^2 \end{bmatrix}$$

$$S_{ii} = \sum_{i=1}^{3} S_{ii} = S_{11}+S_{22}+S_{33}$$

Vector analysis

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

- **Examples:**

$$p\delta_{ij} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$\delta_{ik}\delta_{kj} = \sum_{\mathrm{k}=1}^{3} \delta_{ik}\delta_{kj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$S_{ij}\delta_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}\delta_{ij} = S_{ii}$$

Helmholtz's Decomposition Theorem

- Any sufficiently smooth, rapidly decaying vector field can be written as the sum of an **irrotational field** (compressible) and a **solenoidal field** (incompressible)

$$\boxed{\mathbf{v} = \nabla\phi + \nabla \times \boldsymbol{\psi}}$$

$$\nabla \times \nabla\phi = 0$$
$$\nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0$$

- Taking divergence from the above equation yields:

$$\nabla \cdot \mathbf{v} = \nabla \cdot \nabla\phi + \underbrace{\nabla \cdot (\nabla \times \boldsymbol{\psi})}_{0} \quad \Rightarrow \quad \boxed{\nabla \cdot \mathbf{v} = \nabla^2\phi}$$

Velocity potential field
$\mathbf{v} = \nabla\phi$

- Taking curl from the above equation yields:

$$\nabla \times \mathbf{v} = \underbrace{\nabla \times \nabla\phi}_{0} + \nabla \times (\nabla \times \boldsymbol{\psi}) \Rightarrow \quad \boxed{\nabla \times \mathbf{v} = -\nabla^2\boldsymbol{\psi}}$$

For irrotational flow
$0 = \nabla^2\boldsymbol{\psi}$

Vector identity: $\nabla \times (\nabla \times \boldsymbol{\psi}) = \nabla(\nabla \cdot \boldsymbol{\psi}) - \nabla^2\boldsymbol{\psi}$

For 2D: $w = 0, \frac{\partial}{\partial z} = 0, \boldsymbol{\psi} = (0,0,\psi_z) \ \Rightarrow \ \nabla \cdot \boldsymbol{\psi} = 0$

For 3D: $\boldsymbol{\psi}$ is arbitrary, and so we choose it to be divergence-free $\Rightarrow \ \nabla \cdot \boldsymbol{\psi} = 0$

Vector calculus identities

(a)	$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$
(b)	$\nabla \cdot (\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi$
(c)	$\nabla \times (\phi\mathbf{a}) = \phi\nabla \times \mathbf{a} + \nabla\phi \times \mathbf{a}$
(d)	$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$
(e)	$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$
(f)	$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b}$
(g)	$\mathbf{a} \cdot \nabla\mathbf{a} = \frac{1}{2}\nabla(\mathbf{a} \cdot \mathbf{a}) - \mathbf{a} \times (\nabla \times \mathbf{a})$
(h)	$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$
(i)	$\nabla \cdot (\nabla \times \mathbf{a}) = 0$
(j)	$\nabla \times (\nabla\phi) = 0$

$\mathbf{a}$, $\mathbf{b}$ – any two **vector** fields

ϕ , ψ – any two *scalar* fields

Useful theorems

F – any **vector** field

ϕ – any *scalar* field

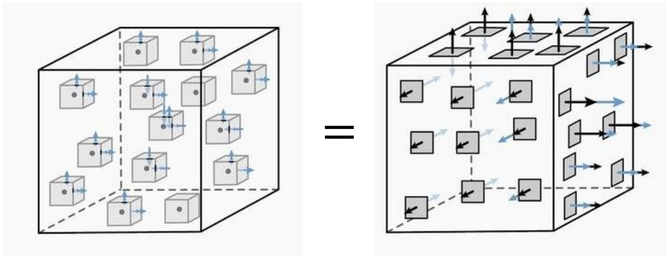
$\mathbf{n}^{(o)}$ – **unit vector** normal to the surface S pointing out of volume V

Gauss's Divergence Theorem

$$\boxed{\iiint_V (\nabla \cdot \mathbf{F})dV = \oiint_S \mathbf{F} \cdot \mathbf{n}^{(o)} dS}$$

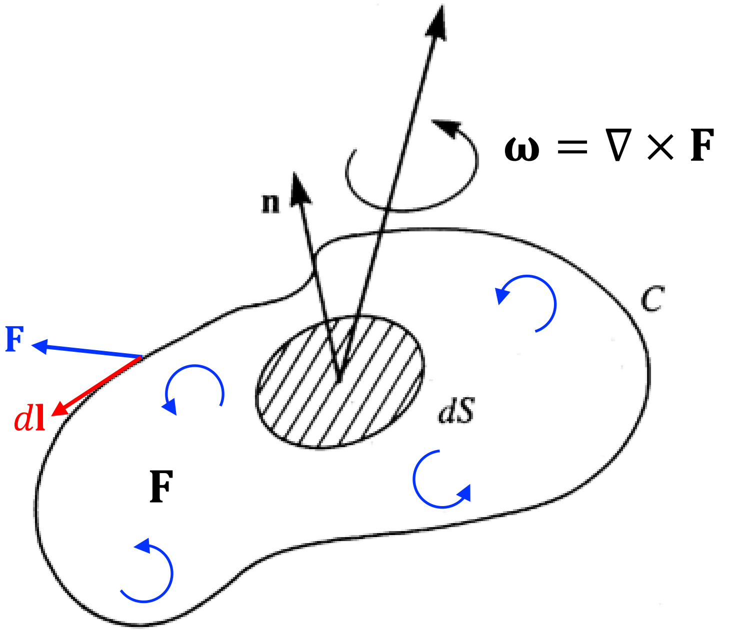
$$\boxed{\iiint_V \nabla\phi dV = \oiint_S \phi\mathbf{n}^{(o)} dS}$$

$$\boxed{\iiint_V (\nabla \times \mathbf{F})dV = -\oiint_S \mathbf{F} \times \mathbf{n}^{(o)} dS}$$

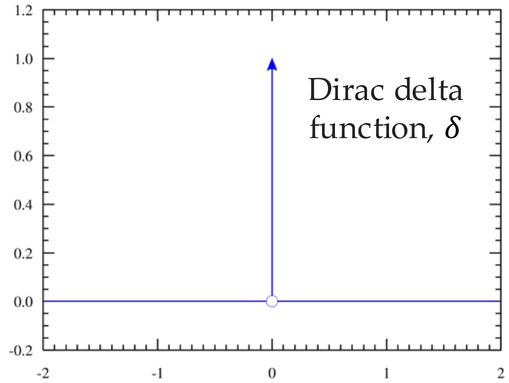


Stokes's Theorem

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}^{(o)} dS = \oint_C \mathbf{F} \cdot d\mathbf{l}} = \iint_S \boldsymbol{\omega} \cdot \mathbf{n}^{(o)} dS$$



Dirac delta function (time)



- A generalized function that is very helpful for solving dynamic problems
  - It is sometimes defined as the ***unit impulse function***

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \qquad \text{such that:} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

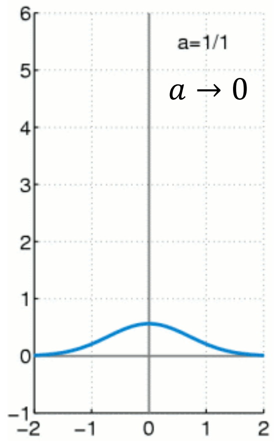
- In aeroacoustics, we often use this **time-delayed Dirac delta function**:

$$\delta(t-\tau) = \begin{cases} \infty, & \text{if } t = \tau, \\ 0, & \text{if } t \neq \tau. \end{cases} \qquad \text{such that:} \quad \int_{-\infty}^{\infty} \delta(t-\tau) d\tau = 1, \qquad -\infty < t < \infty$$

- **Sifting property** – for any function $G(t)$ *continuous* at τ:

$$\int_{-\infty}^{\infty} G(t)\delta(t-\tau) dt = G(\tau)$$

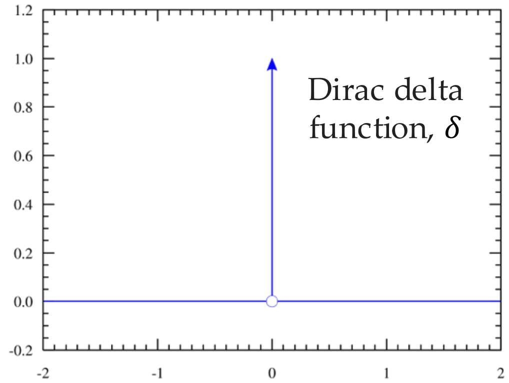
  - The sifting property of the Dirac delta function allows to measure the value of $G(t)$ at $t = \tau$



$$\delta_a(x) = \frac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2}$$

Approx. as a sequence of zero-centered Gaussian distributions with variance tending to zero ($a \to 0$)

Dirac delta function (space)



- When the argument of the Dirac is a vector, we define it so that:

$$\delta(\mathbf{x}-\mathbf{y}) = \begin{cases} \infty, & \text{if } \mathbf{x}=\mathbf{y}, \\ 0, & \text{if } \mathbf{x}\neq\mathbf{y}. \end{cases} \quad \text{such that:} \quad \int_V \delta(\mathbf{x}-\mathbf{y})\,dV(\mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{x}\in V, \\ 0, & \text{if } \mathbf{x}\notin V. \end{cases}$$

- **Sifting property** – for any function $G(\mathbf{x})$ *continuous* at $\mathbf{y}$:

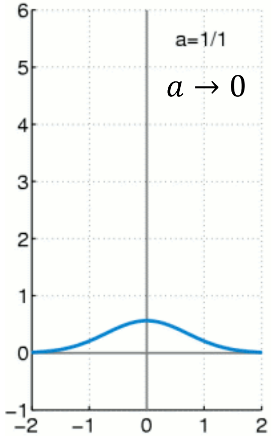
$$\int_{\mathbb{R}^3} G(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,d^3\mathbf{x} = G(\mathbf{y})$$

- **Isotropic Gaussian** ***approximation***: $\delta_\varepsilon(\mathbf{x}-\mathbf{y}) \approx \dfrac{1}{(\epsilon\sqrt{\pi})^3} e^{-(|\mathbf{x}-\mathbf{y}|/\epsilon)^2}$

- For **Cartesian coordinates**, we have: $\delta(\mathbf{x}-\mathbf{y}) = \delta(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3)$

- For **spherical coordinates**, we have: $\delta(\mathbf{x}-\mathbf{y}) = \lim_{\epsilon^+\to 0}\left(\dfrac{e^{-(r/\epsilon)^2}}{2\pi^{3/2}\epsilon r^2}\right)$

$$r = |\mathbf{x}-\mathbf{y}|$$
$$\epsilon > 0$$
$$\delta(\mathbf{x}-\mathbf{y}) = \frac{\delta(r)}{4\pi r^2}$$

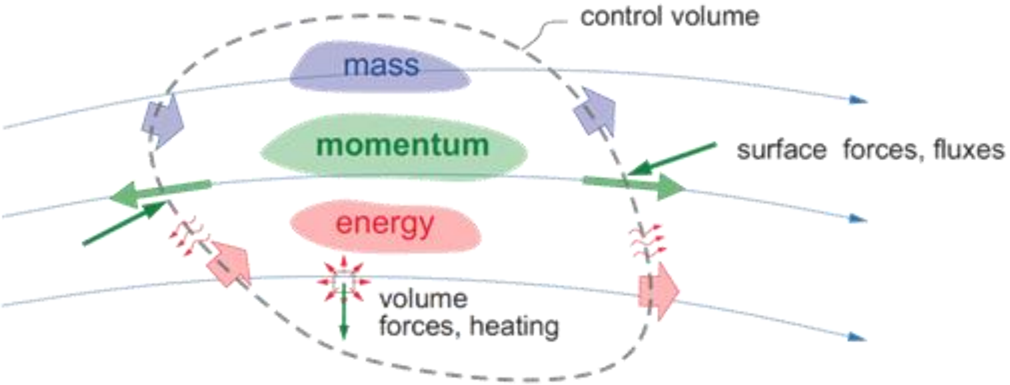


$\delta_a(x) = \dfrac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2}$

Approx. as a sequence of zero-centered Gaussian distributions with variance tending to zero ($a \to 0$)

Governing Equations

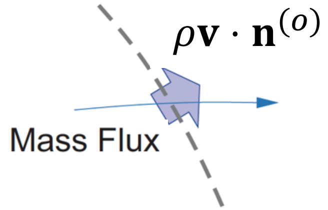
- Any fluid motion within a control volume, fixed in space, can be described by:
  - Conservation of <u>mass</u> (scalar eq.)
  - Conservation of <u>momentum</u> (vector eq.)
  - Conservation of <u>energy</u> (scalar eq.)
- The governing eq. in fluid mechanics are analogous to the principles of balancing your bank account:
  - The same idea works in fluids although we generally reorder, combine inflow/outflow, and speak of rates:



$$Accumulation = inflow - outflow + production$$

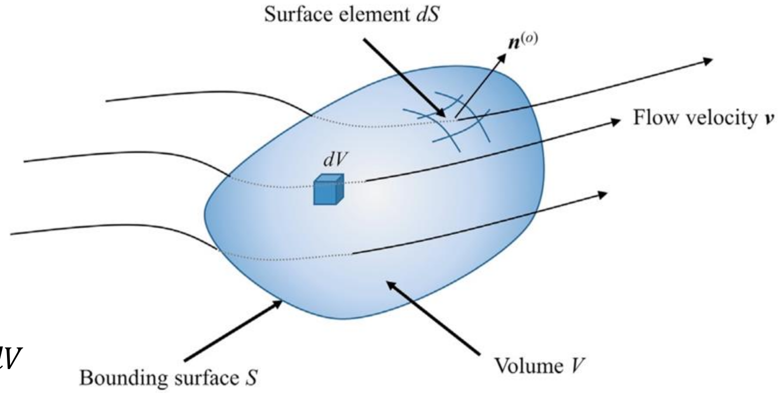
Rate of Accumulation	+	Rate of Net Outflow	=	Rate of production

- The main difficulty in solving the complete equations
  - **2nd order**, **non-linear**, and ***coupled* PDEs**



Mass conservation (Continuity equation)

- A *fixed* and *stationary* control volume (CV) containing fluid, with volume V bounded by the surface S
  - $\mathbf{n}^{(o)}$ – outward pointing normal to the surface S
  - $\mathbf{v}(\mathbf{x},t)$ – fluid velocity
  - $\rho(\mathbf{x},t)$ – fluid density
  - Change in mass balanced by mass flux through volume boundary



- **Integral form:**

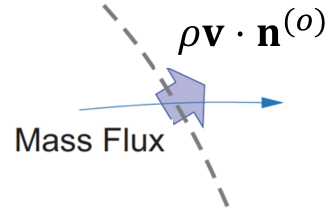
$$\boxed{\int_V \frac{\partial\rho}{\partial t}dV + \int_S \rho\mathbf{v}\cdot\mathbf{n}^{(o)}dS = 0}$$

$$\int_V \frac{\partial\rho}{\partial t}dV = \frac{d}{dt}\int_V \rho dV$$

- **Differntial form:**
  - Applying Gauss's theorem on the RHS: $\int_V \nabla\cdot(\rho\mathbf{v})dV = \int_S \rho\mathbf{v}\cdot\mathbf{n}^{(o)}dS \quad\Rightarrow\quad \int_V \frac{\partial\rho}{\partial t}dV + \int_V \nabla\cdot(\rho\mathbf{v})dV = 0$

$$\Rightarrow \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right]dV = 0 \quad\Rightarrow\quad \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0} \quad \text{or} \quad \boxed{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0}$$

Holds for any CV, thus integrand must be zero everywhere



Mass conservation – Lagrangian frame

- The material (total) derivative Df/Dt can compute the rate of change of mass in a frame of reference moving with a small piece of the fluid; i.e., a fluid particle

- **Material derivative:** $\dfrac{D}{Dt}() = \dfrac{\partial}{\partial t}() + \mathbf{v}\cdot\nabla()$

- **Mass conservation (non-conservative form):** $\dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad\Rightarrow\quad \boxed{\dfrac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho} + \rho\nabla\cdot\mathbf{v} = 0$

$\nabla\cdot(\alpha\mathbf{F}) = \mathbf{F}\cdot\nabla\alpha + \alpha\nabla\cdot\mathbf{F}$

Material derivative: $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho$

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0} \quad \text{or} \quad \boxed{\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0}$$

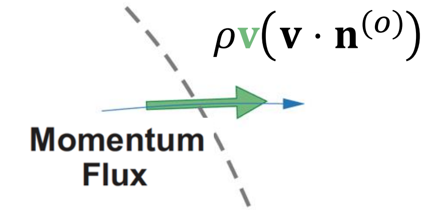
Incompressible **flow**

$\rho = \text{const.} \;\Rightarrow\; D\rho/Dt = 0$

$\Rightarrow\; \nabla\cdot\mathbf{v} = 0$

- **Specific volume:** $\boxed{\dfrac{D(1/\rho)}{Dt} = -\dfrac{1}{\rho^2}\dfrac{D\rho}{Dt} = \dfrac{1}{\rho}\nabla\cdot\mathbf{v}}$

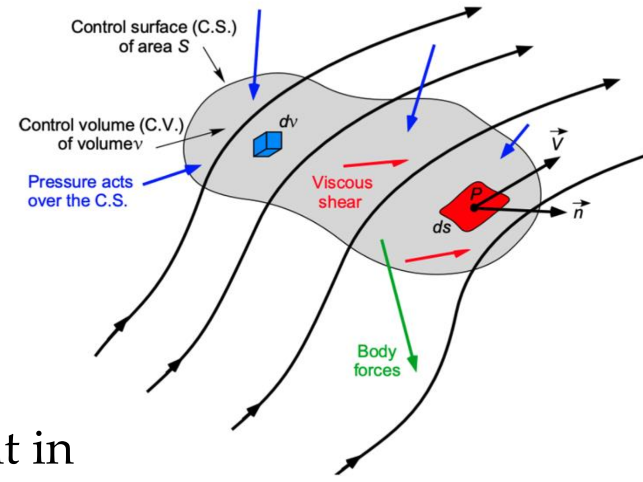
Acoustic waves are **compressible**! Thus, this form **cannot** be used for sound analysis in low-Mach flows, or even for incompressible fluids (water)!



Momentum conservation (N-S equations)

- **Integral form:** $\dfrac{d}{dt}(m\mathbf{v}) = \mathbf{F}$

$$\boxed{\underbrace{\int_V \frac{\partial(\rho\mathbf{v})}{\partial t}dV}_{\text{Unsteady}} + \underbrace{\int_S \rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{n}^{(o)}\right)dS}_{\text{convection}} = \underbrace{\int_V \rho\mathbf{F}_e dV}_{\text{Ext. volume force}} + \underbrace{\int_S \bar{\bar{\boldsymbol{\sigma}}}\cdot\mathbf{n}^{(o)}dS - \int_S \left(p\bar{\bar{\mathbf{I}}}\right)\cdot\mathbf{n}^{(o)}dS}_{\text{Surface forces internal to the fluid (shear stress and pressure)}}}$$

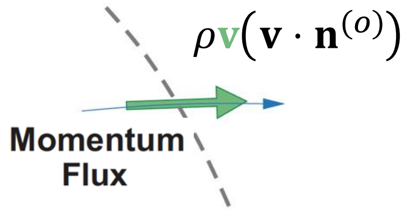


- **External forces** (e.g., body forces, such as gravity) are almost never important in aeroacoustics and so **will be ignored** – $\mathbf{F}_e = 0$
- **Viscous shear stress** tensor – $\bar{\bar{\boldsymbol{\sigma}}} = \sigma_{ij}$ ← Shear in direction i applied on surface with normal in j direction

  Symmetric tensor – $\sigma_{ij} = \sigma_{ji}$
- **Pressure force** applies normal to surface – $p\bar{\bar{\mathbf{I}}} = p\delta_{ij}$
- We shall define the **compressive stress tensor**: $\boxed{p_{ij} = p\delta_{ij} - \sigma_{ij}}$

In fluid dynamics, text, $\tau_{ij} = -p_{ij}$

Momentum conservation (N-S equations)

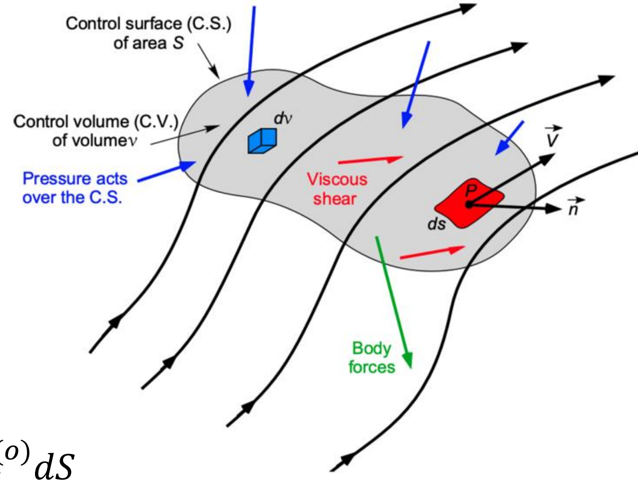


❑ Differntial form:

➢ Substituting all results with the momentum conservation in index notation:

$$\int_V \frac{\partial(\rho v_i)}{\partial t} dV + \int_S \rho v_i v_j n_j^{(o)} dS = -\int_S p_{ij} n_j^{(o)} dS$$

$$\Rightarrow \int_V \frac{\partial(\rho v_i)}{\partial t} dV = -\int_S (\rho v_i v_j + p_{ij}) n_j^{(o)} dS$$



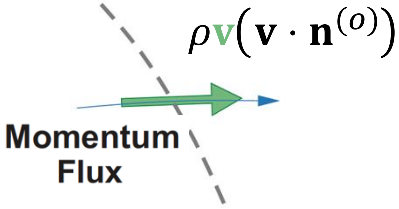
➢ Applying Gauss's theorem: $\int_V \frac{\partial(\rho v_i v_j)}{\partial x_j} dV = \int_S \rho v_i v_j n_j^{(o)} dS$ $\quad \int_V \frac{\partial p_{ij}}{\partial x_j} dV = \int_S p_{ij} n_j^{(o)} dS$

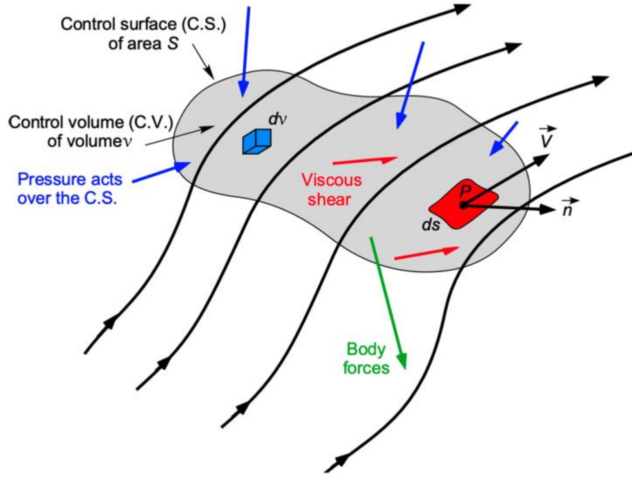
$$\Rightarrow \int_V \left[\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p_{ij})}{\partial x_j}\right] dV = 0$$

$$\boxed{\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p_{ij})}{\partial x_j} = 0}$$

Holds for any CV, thus integrand
must be zero everywhere

Momentum conservation (N-S equations)





- **Non-conservative form** (*Newton's second law*):
  - Expanding the LHS and using the continuity equation, we obtain:

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p_{ij})}{\partial x_j} = 0$$

$$\Rightarrow \frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} + v_i\frac{\partial(\rho v_j)}{\partial x_j} + \rho v_j\frac{\partial v_i}{\partial x_j}$$

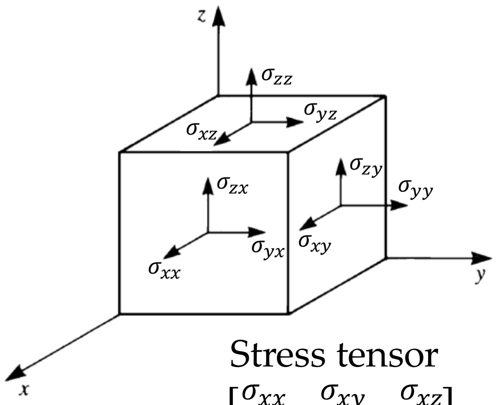
$$\Rightarrow \rho\underbrace{\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right]} + v_i\underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j}\right]}_{0} = \rho\frac{Dv_i}{Dt}$$

$$\boxed{\rho\frac{Dv_i}{Dt} + \frac{\partial p_{ij}}{\partial x_j} = 0}$$

Material derivative:
$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \mathbf{v}\cdot\nabla v_i$

$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$
continuity equation

Viscous stresses



Stress tensor

$$\bar{\bar{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

- ❑ Viscous stresses are due to molecular diffusion across the CV boundary S
- ❑ Macroscopically, for a Newtonian fluid (e.g., water, air), the stress components σ_{ij} are proportional to the derivatives $\partial v_i/\partial x_j$ via viscosity factor:
  - ➢ μ – dynamic viscosity

$$\boxed{\sigma_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)} \qquad \frac{\partial v_k}{\partial x_k} = \nabla\cdot\mathbf{v}$$

- ❑ For *incompressible flow* ($\nabla\cdot\mathbf{v} = 0$), the above is simplified to:

$$\boxed{\sigma_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}$$

$$\bar{\bar{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_{22} & \sigma_{23} \\ \cdot & \cdot & \sigma_{33} \end{bmatrix} = \mu\begin{bmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \cdot & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \cdot & \cdot & 2\frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \mu\left(\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right) + \cancel{\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right)}\right) \Longrightarrow \boxed{\frac{\partial \sigma_{ij}}{\partial x_j} = \mu\left(\frac{\partial^2 v_i}{\partial x_j^2}\right) = \mu\nabla^2 v_i}$$

0

$$\frac{\partial v_j}{\partial x_j} = \nabla\cdot\mathbf{v} = 0$$

assuming constant viscosity

Importance of viscosity via the Reynolds number

- The importance of the viscous term can be assessed by writing the N-S equations in terms of **dimensionless variables**:
  - $\tilde{v}_i = v_i/U$ – velocity
  - $\tilde{x}_i = x_i/L$ – distance
  - $\tilde{t} = Ut/L$ – time
  - $\tilde{\rho} = \rho/\rho_\infty$ – density
  - $\tilde{p} = p/\rho_\infty U^2$ – pressure

$$\rho\frac{Dv_i}{Dt} + \frac{\partial p_{ij}}{\partial x_j} = 0 \qquad \Rightarrow \rho\frac{Dv_i}{Dt} + \frac{\partial(\tilde{p}\delta_{ij} - \sigma_{ij})}{\partial x_j} = 0$$

$$\Rightarrow \frac{\rho_\infty U^2}{L}\tilde{\rho}\frac{D\tilde{v}_i}{D\tilde{t}} + \frac{\rho_\infty U^2}{L}\frac{\partial(\tilde{p}\delta_{ij})}{\partial\tilde{x}_j} - \frac{1}{L}\frac{\partial\sigma_{ij}}{\partial\tilde{x}_j} = 0$$

$$\Rightarrow \frac{\rho_\infty U^2}{L}\tilde{\rho}\frac{D\tilde{v}_i}{D\tilde{t}} + \frac{\rho_\infty U^2}{L}\frac{\partial(\tilde{p}\delta_{ij})}{\partial\tilde{x}_j} - \frac{\mu U}{L^2}\frac{\partial}{\partial\tilde{x}_j}\left(\frac{\partial\tilde{v}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{v}_j}{\partial\tilde{x}_i} - \frac{2}{3}\frac{\partial\tilde{v}_k}{\partial\tilde{x}_k}\delta_{ij}\right) = 0 \qquad / \ \frac{\rho_\infty U^2}{L}$$

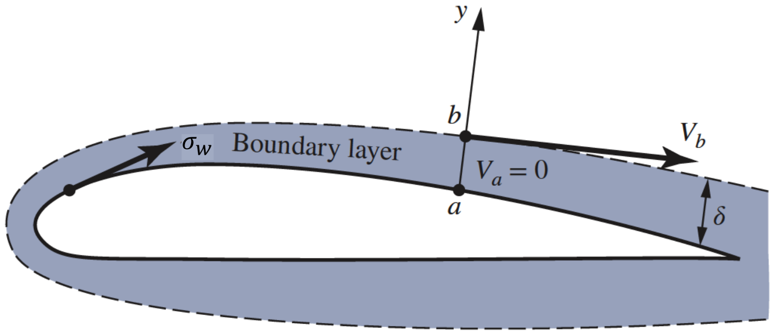
$$\Rightarrow \tilde{\rho}\frac{D\tilde{v}_i}{D\tilde{t}} + \frac{\partial(\tilde{p}\delta_{ij})}{\partial\tilde{x}_j} - \frac{\mu}{\rho_\infty UL}\frac{\partial}{\partial\tilde{x}_j}\left(\frac{\partial\tilde{v}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{v}_j}{\partial\tilde{x}_i} - \frac{2}{3}\frac{\partial\tilde{v}_k}{\partial\tilde{x}_k}\delta_{ij}\right) = 0$$

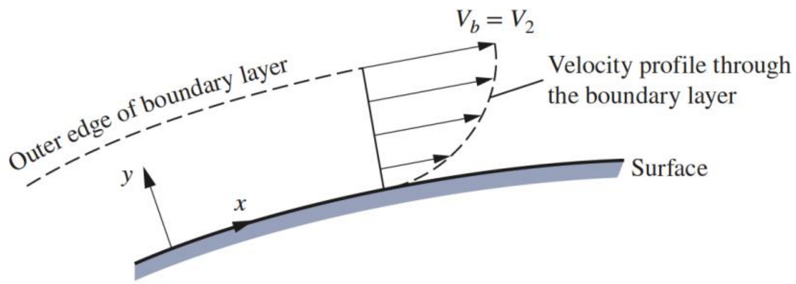
Reynolds number

$$\boxed{\mathbf{Re} = \frac{\rho_\infty UL}{\mu} \propto \frac{\text{inertial forces}}{\text{viscous forces}} = \left(\frac{\rho_\infty U^2}{L}\right)\left(\frac{\mu U}{L^2}\right)^{-1}}$$

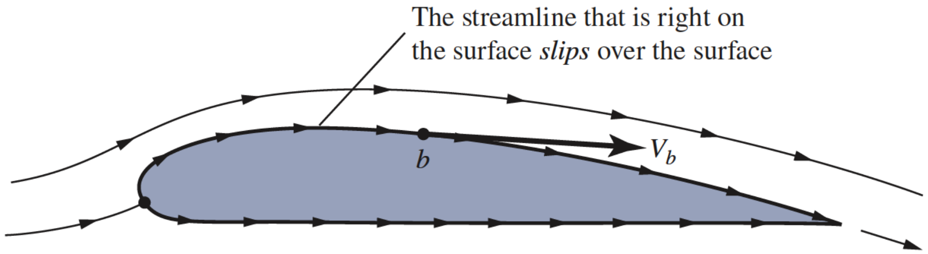
Boundary conditions

- Viscous flow - *no-slip condition*: $\mathbf{v}_{\text{wall}} = 0$





- Inviscid flow - *flow-tangency (slip) condition:* $\mathbf{v}_{\text{wall}} \cdot \mathbf{n}^{(o)} = 0$



High Reynolds number flows

Momentum eq.

$$\boxed{\rho \frac{D v_i}{Dt} + \frac{\partial p_{ij}}{\partial x_j} = 0}$$

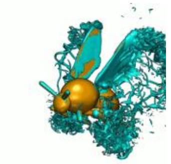
$$p_{ij} = p\delta_{ij} - \sigma_{ij}$$

- *Notice that:*

$$\sigma_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) \sim \frac{\mathbf{1}}{\mathbf{Re}}$$

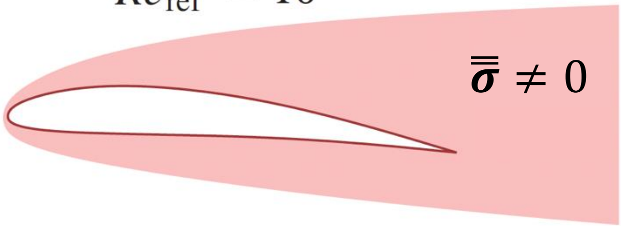
- *Typical* aerodynamic flows of interest have very large Reynolds numbers, or $\mathbf{Re} \gg 1$
- Because the viscous term $\bar{\bar{\boldsymbol{\sigma}}}$ is scaled by $1/\mathbf{Re}$, it is *negligible* over most of the flow field
- ***The exception***: close to a body surface, where $\mathbf{v} \to 0$ due to the no-slip condition ($\mathbf{v}_{\text{wall}} = 0$)
  - In the momentum equation, only ∇p remains to balance the viscous term, so $\bar{\bar{\boldsymbol{\sigma}}}$ must remain significant sufficiently close to a wall

The action of viscosity is confined to *boundary layers* and *wakes*
collectively termed the "*shear layers*" or "*viscous regions*"



Small insect

$Re_{\text{ref}} \simeq 10^3$



$\bar{\bar{\boldsymbol{\sigma}}} \neq 0$

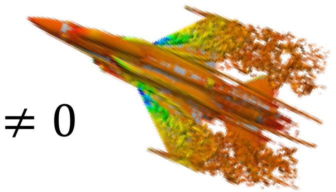
Small UAV

$Re_{\text{ref}} \simeq 10^5$

$\bar{\bar{\boldsymbol{\sigma}}} \neq 0$

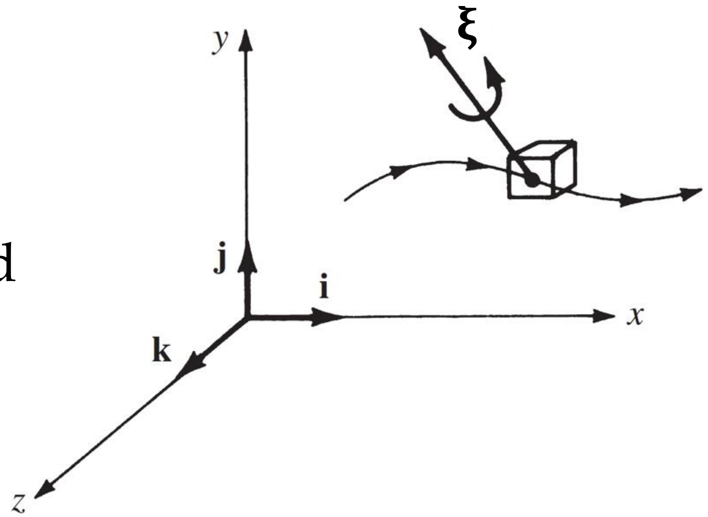
Full-size aircraft

$Re_{\text{ref}} \simeq 10^7$



$\bar{\bar{\boldsymbol{\sigma}}} \neq 0$

Vorticity



- Consider an infinitesimal fluid element moving in a 3D flow field
- As it translates along a streamline, it may **rotate** (its shape may also become distorted)
- **The amount of rotation** depends on the velocity field
- The **angular velocity** of the fluid element in three-dimensional space is a vector $\boldsymbol{\xi}$ that is oriented in some general direction: $\boldsymbol{\xi} = \xi_x \mathbf{i} + \xi_y \mathbf{j} + \xi_z \mathbf{k} = \frac{1}{2}\nabla \times \mathbf{v}$
- It is convenient to define the **vorticity $\boldsymbol{\omega}$ as twice the angular velocity**:

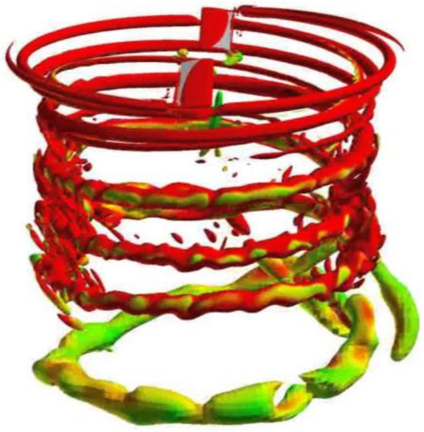
$$\boxed{\boldsymbol{\omega} = 2\boldsymbol{\xi} = \nabla \times \mathbf{v}} \quad \text{or} \quad \boxed{\omega_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}}$$

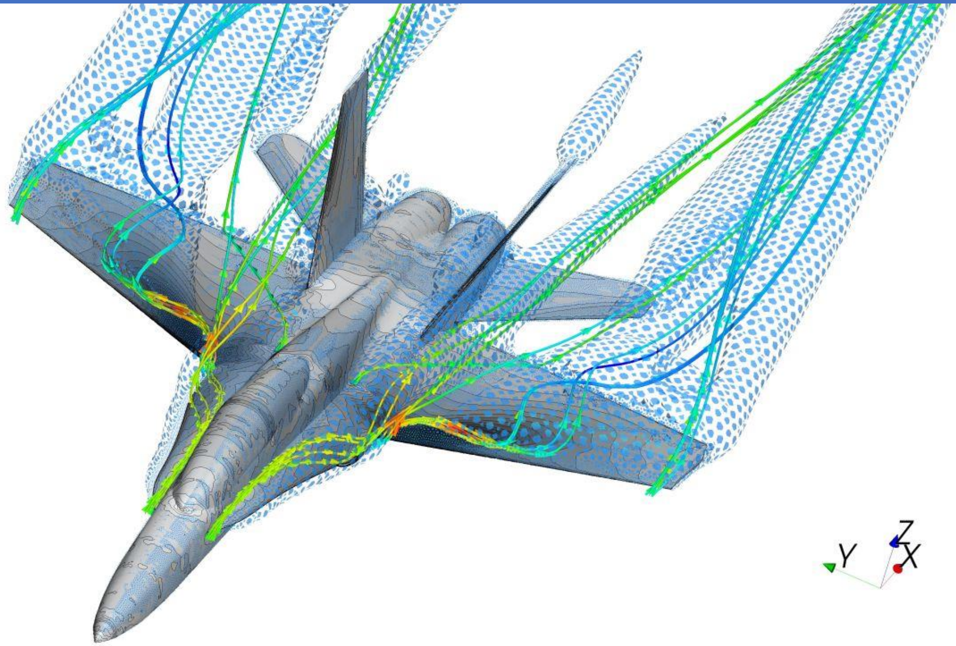
The **vorticity field expresses the angular velocity of the fluid field in terms of the velocity field**, or more precisely, in terms of **velocity derivatives**

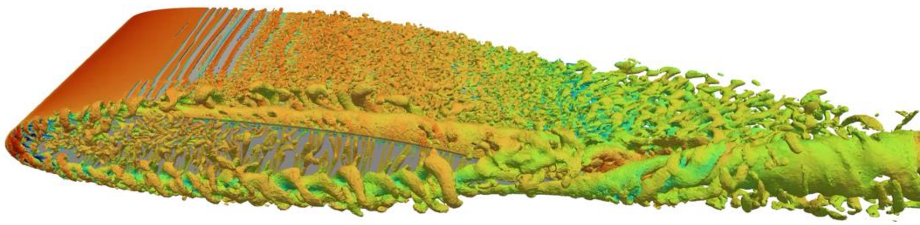
- In Cartesian coordinates:

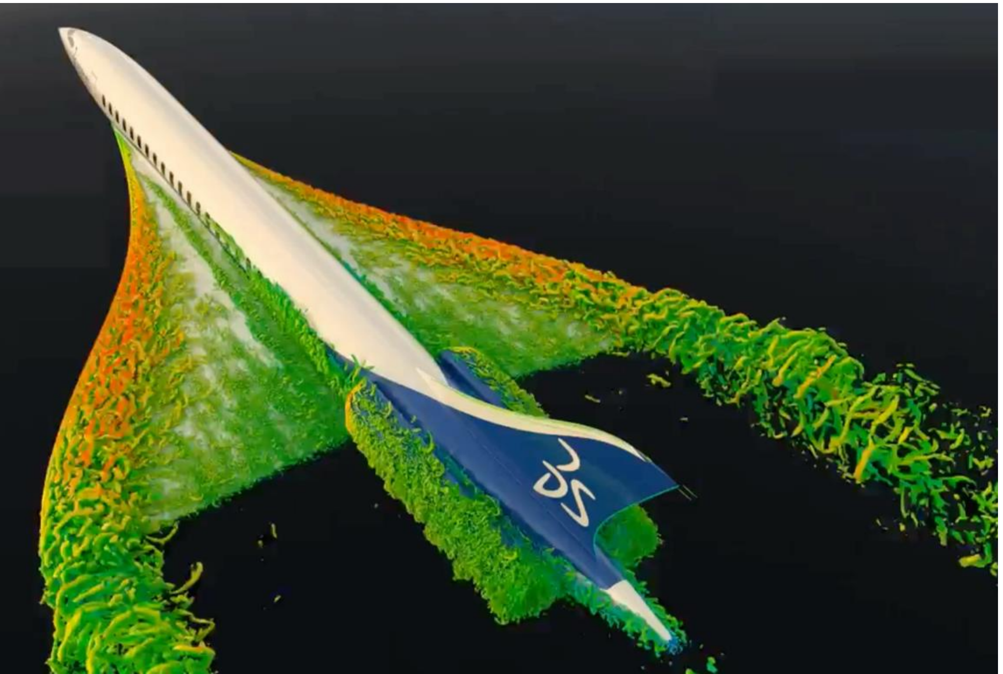
$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{k}$$

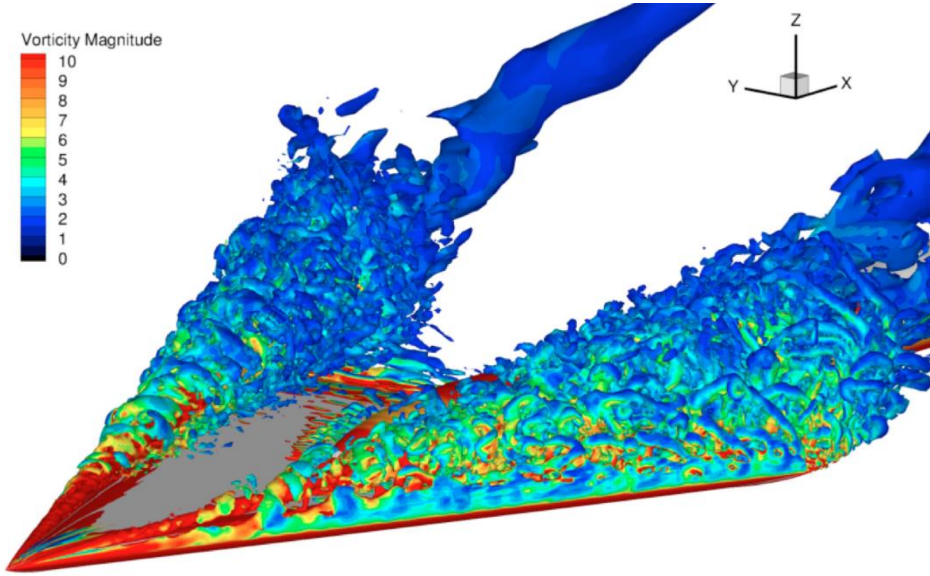
Vorticity









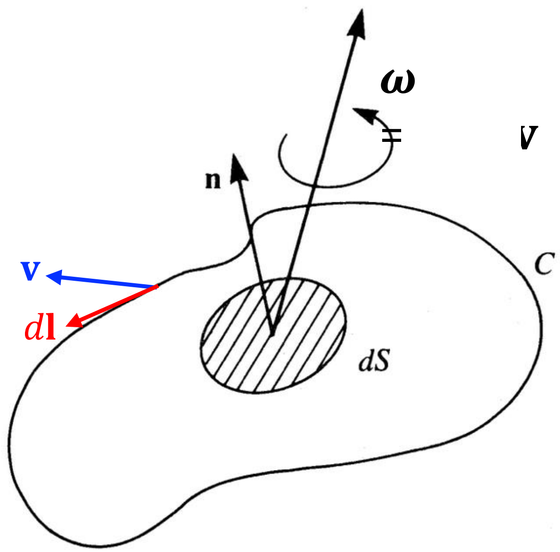


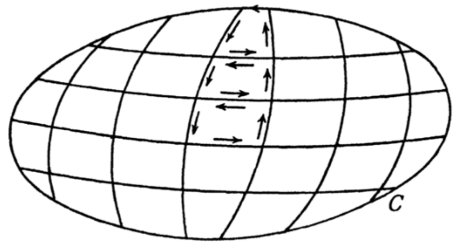
Circulation

- An open surface S has the closed curve C as its boundary
- With the use of ***Stokes's theorem***, the **vorticity $\boldsymbol{\omega}$** on the surface S can be related to the line integral around C:

$$\boxed{\textit{Circulation} \qquad \Gamma = \oint_C \mathbf{v}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{v})\cdot\mathbf{n}dS = \iint_S \boldsymbol{\omega}\cdot\mathbf{n}dS}$$

- When circulation exists in a flow, it simply means that the line integral in the eq. is **finite**
  - For example, if an airfoil is generating lift, the circulation taken around a closed curve enclosing the airfoil will be finite
- **The circulation is tied to the rotation in the fluid (e.g., to the angular velocity of a solid body type rotation)**





$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$$

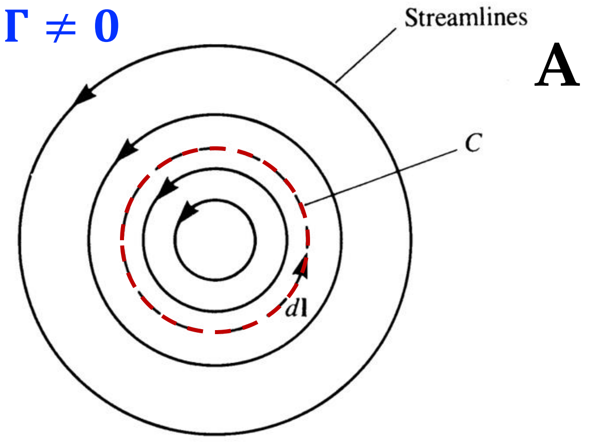
* Some use the minus sign in the definition of circulation since *clockwise circulation generates lift*

Rotational vs. irrotational flow

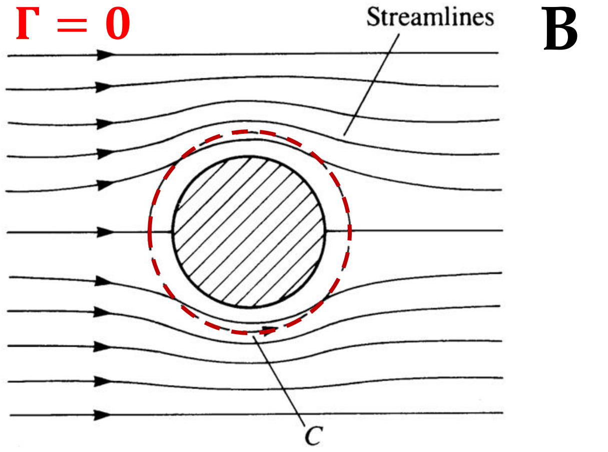
$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{v})\, d\boldsymbol{S} = \iint_S \boldsymbol{\omega}\, d\boldsymbol{S}$$

- Two examples are shown to illustrate the concept of circulation
  - The curve C (dashed lines) is taken to be a circle in each case
- **A** - **Rotational flow** $\boldsymbol{\omega} = \nabla \times \mathbf{v} \neq 0$
  - Flow field consists of concentric circular streamlines in the counter-clockwise direction
  - Along the circular integration path C, $\mathbf{v}$ and $d\mathbf{l}$ are positive for all $d\mathbf{l}$ and therefore C has a **positive circulation**
  - Fluid is rotating as a **rigid body** while following the path $\mathbf{l}$
- **B** - **Irrotational flow** $\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$
  - Flow field is a symmetric flow of a uniform stream past a circular cylinder
  - It is clear from the symmetry that the **circulation is zero** for this case
  - For this example, the shear forces in the fluid are negligible, and the fluid will not be rotated by the shear force of the neighboring fluid elements

$\Gamma \neq 0$

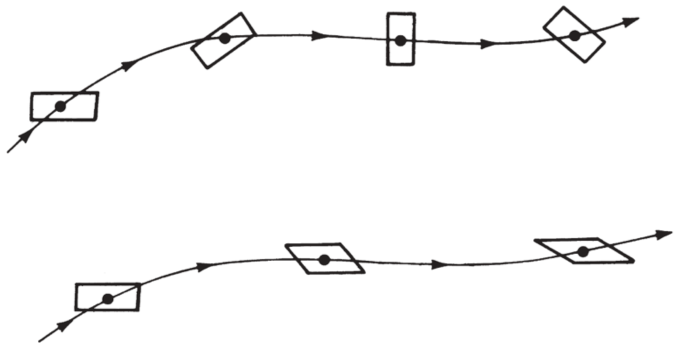


$\Gamma = 0$



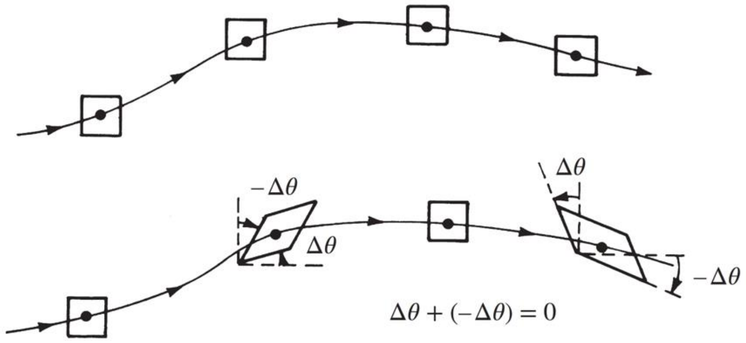
Rotational vs. irrotational flow

- $\boxed{\textbf{Rotational flow} \quad \boldsymbol{\omega} = \nabla \times \mathbf{v} \neq 0}$



Fluid elements in a rotational flow

- $\boxed{\textbf{Irrotational flow} \quad \boldsymbol{\omega} = \nabla \times \mathbf{v} = 0}$



Fluid elements in an irrotational flow

Crocco’s equation - derivation

Kinematic viscosity
$$\nu = \frac{\mu}{\rho}\left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]$$

Viscous force per unit mass
$$e_i = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

For *incompressible* flow:
$$\mathbf{e} = \nu\nabla^2\mathbf{v}$$

$$\boxed{\boldsymbol{\omega} = \nabla \times \mathbf{v}}$$

- We rearrange the momentum equation into Crocco’s **form**
  - This relates the rate of change of velocity to terms such as the vorticity
  - Will help to highlight some of the most important features of inviscid, irrotational flow

$$\rho\frac{Dv_i}{Dt} + \frac{\partial p_{ij}}{\partial x_j} = 0 \xRightarrow{\text{Expand}} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\rho}\nabla p - \mathbf{e} = 0$$

- Given the vector identity: $\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$
  - By setting $\mathbf{a} = \mathbf{b} = \mathbf{v}$ we obtain:

$$\frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\times\boldsymbol{\omega} \Rightarrow \mathbf{v}\cdot\nabla\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) + \boldsymbol{\omega}\times\mathbf{v}$$

- Substituting the above into the momentum equation yields: $\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) + \boldsymbol{\omega}\times\mathbf{v} + \frac{1}{\rho}\nabla p - \mathbf{e} = 0$

Crocco's equation - derivation

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) + \boldsymbol{\omega}\times\mathbf{v} + \frac{1}{\rho}\nabla p - \mathbf{e} = 0$$

$$\frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\nabla v_i^2$$

Viscous force per unit mass

$$e_i = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

For *incompressible* flow:

$$\mathbf{e} = \nu\nabla^2\mathbf{v}$$

$$\boxed{\boldsymbol{\omega} = \nabla\times\mathbf{v}}$$

- Introducing the **stagnation enthalpy**: $\boxed{H = h + \frac{1}{2}v_i^2}$
- We then write the 1st law in terms of gradients:

$$dh = Tds + dp/\rho \quad \Longrightarrow \quad \boxed{\nabla h = T\nabla s + \frac{1}{\rho}\nabla p}$$

  - The above assumes that at some upstream point, **the fluid properties are initially uniform in space**, so that their values at adjacent points can be thought of as being connected by a single thermodynamic process
- Substituting the definition of the stagnation enthalpy into the 1st law:

$$\nabla H = \nabla h + \frac{1}{2}\nabla v_i^2 \quad \Longrightarrow \quad \boxed{\nabla H = T\nabla s + \frac{1}{\rho}\nabla p + \frac{1}{2}\nabla v_i^2}$$

Substituting into the *momentum eq.*

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + \nabla H - T\nabla s + \boldsymbol{\omega}\times\mathbf{v} - \mathbf{e} = 0}$$

Crocco's equation

Crocco's equation - interpretation

Crocco's equation $\boxed{\frac{\partial \mathbf{v}}{\partial t} + \nabla H - T\nabla s + \boldsymbol{\omega} \times \mathbf{v} - \mathbf{e} = 0}$

Viscous force per unit mass

$$e_i = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

For *incompressible* flow:

$$\mathbf{e} = \nu \nabla^2 \mathbf{v}$$

- **Given:**
  - **Irrotational flow**: $\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0 \xrightarrow{\text{given } \nabla\times(\nabla\phi)=0} \mathbf{v} = \nabla\phi$
    - ϕ – *velocity potential*
  - **Homentropic flow**: $\nabla s = 0$
  - **Inviscid flow**: $\mathbf{e} = 0$
- Therefore, Crocco's equation reduces to: $\frac{\partial(\nabla\phi)}{\partial t} + \nabla H - \underbrace{T\nabla s}_{0} + \underbrace{\boldsymbol{\omega}}_{0} \times (\nabla\phi) - \underbrace{\mathbf{e}}_{0} = 0 \Longrightarrow \nabla\left(\frac{\partial \phi}{\partial t} + H\right) = 0$

$$\boxed{\frac{\partial \phi}{\partial t} + H = f(t)}$$

$f(t)$ is constant across all space

- ***Steady*** **flow:** $\boxed{H = \text{const.} = H_o}$
- ***Unsteady*** **flow over a limited region:** $\boxed{H - H_o = -\frac{\partial \phi}{\partial t} \equiv H'}$
  - H' – unsteady stagnation enthalpy
- ***Incompressible*** **flow:** $\boxed{\nabla h = \underbrace{T\nabla s}_{0} + \frac{1}{\rho}\nabla p} \Longrightarrow \nabla h = \frac{1}{\rho}\nabla p \xrightarrow{\rho=\text{const.}} h = \frac{p}{\rho} + \text{const.}$

$$H = \frac{p}{\rho} + \frac{1}{2}v_i^2$$

Bernoulli's equation

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}v_i^2 = \mathbb{C}}$$

(everywhere)

Vorticity transport equation

$\boxed{\boldsymbol{\omega} = \nabla \times \mathbf{v}}$

- The vorticity transport equation will be developed by manipulating Crocco's equation:

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + \nabla H - T\nabla s + \boldsymbol{\omega} \times \mathbf{v} - \mathbf{e} = 0}$$

  - The manipulation will use the following identity: $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + \mathbf{b} \cdot \nabla\mathbf{a} - \mathbf{a} \cdot \nabla\mathbf{b}$
    - Valid for any vector fields **a** and **b**
  - By setting $\mathbf{a} = \boldsymbol{\omega}$ & $\mathbf{b} = \mathbf{v}$ we obtain:

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) - \cancel{\mathbf{v}(\nabla \cdot \boldsymbol{\omega})} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} \Longrightarrow \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$$

Vector identity, $\nabla \cdot (\nabla \times \mathbf{v}) \equiv 0$

- Taking the curl, $\nabla \times ()$, of Crocco's equation: $\nabla \times \left[\frac{\partial \mathbf{v}}{\partial t} + \nabla H - T\nabla s + \boldsymbol{\omega} \times \mathbf{v} - \mathbf{e}\right] = 0$

$$\Longrightarrow \frac{\partial(\nabla \times \mathbf{v})}{\partial t} + \cancel{\nabla \times (\nabla H)} - \underbrace{\nabla \times (T\nabla s)}_{\nabla T \times \nabla s} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) - \nabla \times \mathbf{e} = 0 \Longrightarrow \boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - \nabla T \times \nabla s - \nabla \times \mathbf{e} = 0}$$

Vector identity $\nabla \times (\nabla f) \equiv 0$

Vector identity: $\nabla \times (\phi\mathbf{a}) \equiv \phi\nabla \times \mathbf{a} + \nabla\phi \times \mathbf{a}$

$\nabla \times (T\nabla s) = \cancel{T\nabla \times (\nabla s)} + \nabla T \times \nabla s$

0

Recall mass conservation: $\boxed{\frac{D(1/\rho)}{Dt} = \frac{1}{\rho}\nabla \cdot \mathbf{v}}$

Vorticity transport equation

- Starting with: $$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} + \rho\boldsymbol{\omega}\frac{D(1/\rho)}{Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} - \nabla T\times\nabla s - \nabla\times\mathbf{e} = 0$$

- Given: $$e_i = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{and} \quad \sigma_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)$$

- We obtain: $$\boxed{\nabla\times\mathbf{e} = \nu\nabla^2\boldsymbol{\omega}}$$ (assuming constant viscosity)

* Note that the *curl commutes* with the $\partial(\quad)/\partial t$ operation or with the $\nabla^2(\quad)$ operator for fixed reference frame

$$\nabla\times\mathbf{e} = \nabla\times\left(\frac{1}{\rho}\nabla\cdot\bar{\bar{\boldsymbol{\sigma}}}\right) = \nabla\times\left\{\frac{1}{\rho}\nabla\cdot\left(\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}} - \frac{2}{3}(\nabla\cdot\mathbf{v})\mathbf{I}\right]\right)\right\} = \nabla\times\left\{\frac{\mu}{\rho}\left(\nabla\cdot(\nabla\mathbf{v}) + \underbrace{\nabla\cdot(\nabla\mathbf{v})^{\mathrm{T}}}_{\nabla(\nabla\cdot\mathbf{v})} - \frac{2}{3}\underbrace{\nabla\cdot[(\nabla\cdot\mathbf{v})\mathbf{I}]}_{\nabla(\nabla\cdot\mathbf{v})}\right)\right\}$$

$$\nabla\times\mathbf{e} = \nabla\times\left\{\nu\left(\nabla^2\mathbf{v} + \nabla(\nabla\cdot\mathbf{v}) - \frac{2}{3}\nabla(\nabla\cdot\mathbf{v})\right)\right\} = \nu\nabla\times\left\{\nabla^2\mathbf{v} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{v})\right\} = \nu\nabla\times(\nabla^2\mathbf{v}) + \frac{1}{3}\nu\nabla\times[\nabla(\nabla\cdot\mathbf{v})] = \nu\nabla^2\boldsymbol{\omega}$$

Vector identity $\nabla\times(\nabla f)\equiv 0$

Vector identity: $\nabla\cdot(\phi\mathbf{a}) \equiv \phi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\phi$

$$\nabla\cdot[(\nabla\cdot\mathbf{v})\mathbf{I}] = (\nabla\cdot\mathbf{v})\underbrace{(\nabla\cdot\mathbf{I})}_{0} + \mathbf{I}\cdot\nabla(\nabla\cdot\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v})$$

$$\nabla\cdot(\nabla\mathbf{v})^{\mathrm{T}} = \frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) = \nabla(\nabla\cdot\mathbf{v})$$

Vorticity transport equation

- Starting with: $\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} + \rho\boldsymbol{\omega}\frac{D(1/\rho)}{Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} - \nabla T \times \nabla s - \nu\nabla^2\boldsymbol{\omega} = 0$
- Given the material derivative: $\frac{D(\boldsymbol{\omega}/\rho)}{Dt} = \frac{\partial(\boldsymbol{\omega}/\rho)}{\partial t} + (\mathbf{v}\cdot\nabla)(\boldsymbol{\omega}/\rho) = \frac{1}{\rho}\left[\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega}\right] + \boldsymbol{\omega}\left[\frac{\partial(1/\rho)}{\partial t} + \mathbf{v}\cdot\nabla\left(\frac{1}{\rho}\right)\right]$
- We obtain (after dividing by ρ):

Compressible **vorticity equation**

$$\boxed{\underbrace{\frac{D(\boldsymbol{\omega}/\rho)}{Dt}}_{\text{Transport}} - \underbrace{\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{v}}_{\text{Deformation}} - \underbrace{\frac{1}{\rho}\nabla T \times \nabla s}_{\text{Generation}} - \underbrace{\frac{\nu}{\rho}\nabla^2\boldsymbol{\omega}}_{\text{Diffusion}} = 0}$$

Transport unsteady / convection / compression

Deformation stretching / tilting

Generation *Baroclinic term* when ∇p & $\nabla\rho$ are not alinged, $\nabla p \times \nabla\rho \neq 0$

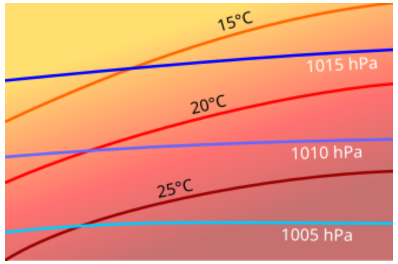
Diffusion

Incompressible **vorticity equation**

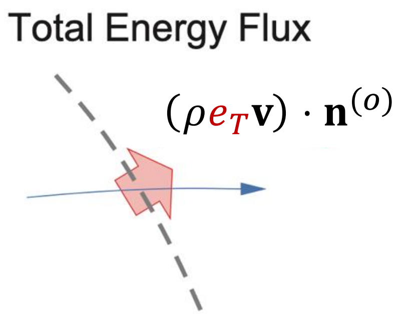
$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} - \frac{\nu}{\rho}\nabla^2\boldsymbol{\omega} = 0}$$

Inviscid flow, no heating effects

In 2$\mathbb{D}$ flow, vorticity and velocity are orthogonal, $\boldsymbol{\omega} \perp \mathbf{v}$
→ *no deformation*



Energy conservation



- **Total energy per unit mass:** $\boxed{e_T = e + \frac{1}{2} v_i^2}$

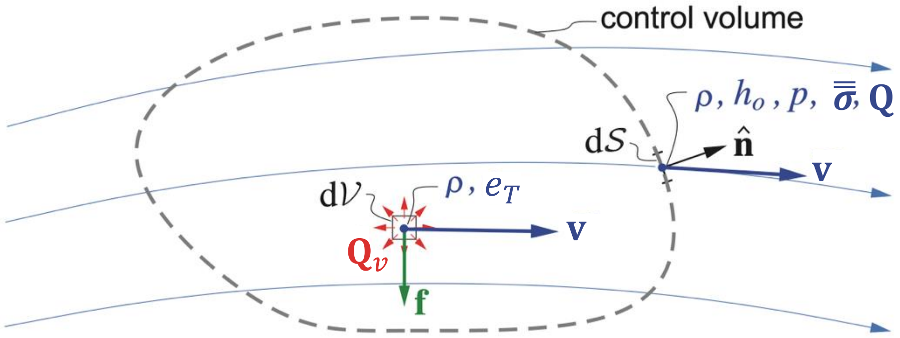
  e – Specific internal energy
  $\frac{1}{2} v_i^2$ – Kinetic energy per unit mass

- The 1st Law of Thermodynamics asserts that the time rate of change of total energy, plus its net outflow rate, equals the sum of heat and work sources in the interior, plus heat inflow and work at the boundary

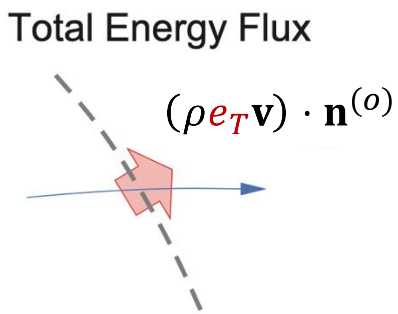
- **Integral form:**

$$\underbrace{\int_V \frac{\partial(\rho e_T)}{\partial t} dV}_{\text{Unsteady energy}} + \underbrace{\int_S (\rho e_T \mathbf{v}) \cdot \mathbf{n}^{(o)} dS}_{\text{Energy convective flux}} = -\underbrace{\int_S \mathbf{Q} \cdot \mathbf{n}^{(o)} dS}_{\text{Heat flux}} + \underbrace{\int_S (\bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{v}) \cdot \mathbf{n}^{(o)} dS - \int_S (p\bar{\bar{\mathbf{I}}} \cdot \mathbf{v}) \cdot \mathbf{n}^{(o)} dS}_{\text{Work done by surface forces internal to the fluid (shear stress and pressure)}}$$

  - **Work done by external forces** (e.g., body forces, such as gravity) **is ignored** – $\mathbf{F_e} = 0$
  - Heat flux per unit area through the fluid – $\mathbf{Q}$
  - No interior heat sources – $\mathbf{Q}_v = 0$
  - Recall the **compressive stress tensor**: $p_{ij} = p\delta_{ij} - \sigma_{ij}$
    - **Viscous shear stress** tensor – $\bar{\bar{\boldsymbol{\sigma}}} = \sigma_{ij}$
    - **Pressure force** applies normal to surface – $p\bar{\bar{\mathbf{I}}} = p\delta_{ij}$



Energy conservation



❑ Differntial form:

➢ Substituting all results with the energy conservation in index notation:

$$\int_V \frac{\partial(\rho e_T)}{\partial t} dV + \int_S \rho e_T v_i n_i^{(o)} dS = -\int_S Q_i n_i^{(o)} dS - \int_S v_j p_{ij} n_i^{(o)} dS \;\Longrightarrow\; \int_V \frac{\partial(\rho e_T)}{\partial t} dV + \int_S \rho e_T v_i n_i^{(o)} dS = -\int_S (v_j p_{ij} + Q_i) n_i^{(o)} dS$$

➢ Applying Gauss's theorem: $\int_V \frac{\partial(\rho e_T v_i)}{\partial x_i} dV = \int_S \rho e_T v_i n_i^{(o)} dS$ $\quad \int_V \frac{\partial(v_j p_{ij} + Q_i)}{\partial x_i} dV = \int_S (v_j p_{ij} + Q_i) n_i^{(o)} dS$

$$\Longrightarrow \int_V \left[\frac{\partial(\rho e_T)}{\partial t} + \frac{\partial(\rho e_T v_i)}{\partial x_i} + \frac{\partial(v_j p_{ij} + Q_i)}{\partial x_i}\right] dV = 0$$

Holds for any CV, thus integrand must be zero everywhere

$$h = e + \frac{p}{\rho} \qquad e_T = e + \frac{1}{2} v_i^2$$

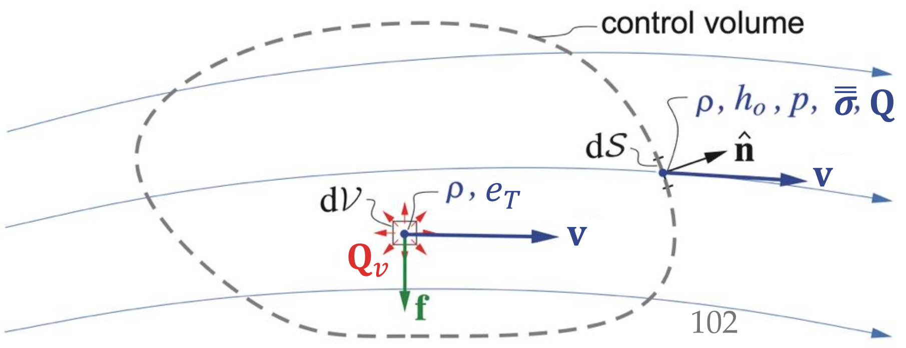
$$H = h + \frac{1}{2} v_i^2 \qquad p_{ij} = p\delta_{ij} - \sigma_{ij}$$

$$\boxed{\frac{\partial(\rho e_T)}{\partial t} + \frac{\partial}{\partial x_i}(\rho e_T v_i + v_j p_{ij} + Q_i) = 0}$$

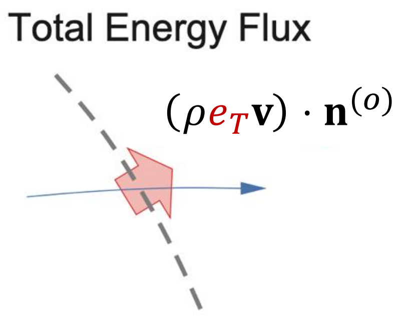
Introducing the **stagnation enthalpy**

$$H = e_T + \frac{p}{\rho}$$

$$\boxed{\frac{\partial(\rho e_T)}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i H - v_j \sigma_{ij} + Q_i) = 0}$$



Energy conservation



❑ Non-conservative form:

➢ Expanding the two terms in the LHS and using the continuity equation, we obtain:

$$\frac{\partial(\rho e_T)}{\partial t} + \frac{\partial(\rho e_T v_i)}{\partial x_i} + \frac{\partial}{\partial x_i}(v_j p_{ij} + Q_i) = 0$$

continuity equation
$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

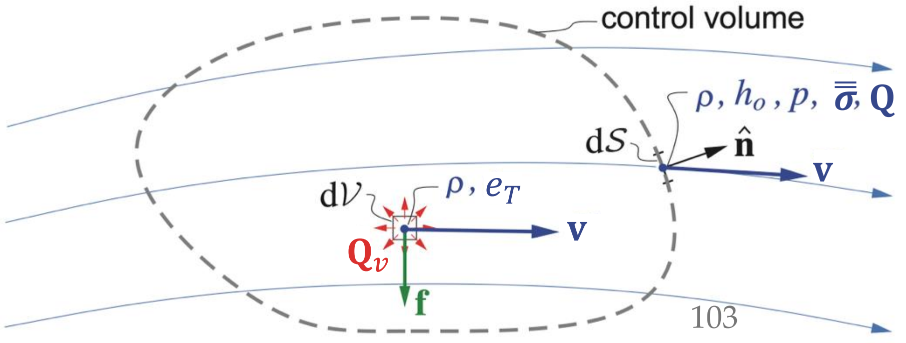
$$\Rightarrow \frac{\partial(\rho e_T)}{\partial t} + \frac{\partial(\rho e_T v_i)}{\partial x_i} = \rho \frac{\partial e_T}{\partial t} + e_T \frac{\partial \rho}{\partial t} + e_T \frac{\partial(\rho v_i)}{\partial x_i} + \rho v_i \frac{\partial e_T}{\partial x_i} \Rightarrow \rho \underbrace{\left[\frac{\partial e_T}{\partial t} + v_i \frac{\partial e_T}{\partial x_i}\right]}_{\text{Material derivative}} + \cancel{e_T \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i}\right]} = \rho \frac{De_T}{Dt}$$

Material derivative:
$\frac{De_T}{Dt} = \frac{\partial e_T}{\partial t} + \mathbf{v} \cdot \nabla e_T$

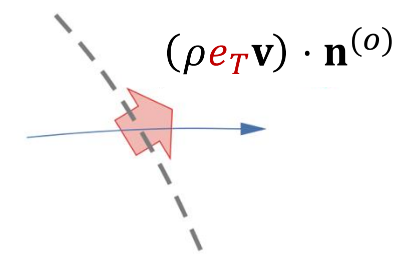
$e_T = e + \frac{1}{2} v_i^2$

$p_{ij} = p\delta_{ij} - \sigma_{ij}$

$$\boxed{\rho \frac{De_T}{Dt} + \frac{\partial}{\partial x_i}(v_j p_{ij} + Q_i) = 0}$$



Energy conservation



❑ "Thermodynamic" form:

- We can also derive the rate of change of energy directly from the thermodynamic relationships
- Rewriting the 1st law in terms of material derivatives:

$$\boxed{\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right)}$$

Assuming the rates of change in the thermodynamic variables experienced by the fluid over time constitute a thermodynamic process

- Incorporating the specific total energy and multiplying by density gives:

$$\Rightarrow \rho\frac{De_T}{Dt} = \rho T\frac{Ds}{Dt} - \rho p\underbrace{\frac{D}{Dt}\left(\frac{1}{\rho}\right)} + \rho\underbrace{\frac{D}{Dt}\left(\frac{v_i^2}{2}\right)}$$

$$\frac{D(1/\rho)}{Dt} = \frac{1}{\rho}\nabla\cdot\mathbf{v} = \frac{1}{\rho}\frac{\partial v_i}{\partial x_i}$$

Mass conservation

$$\rho v_i\frac{Dv_i}{Dt} = -v_i\frac{\partial p_{ij}}{\partial x_j}$$

Momentum conservation

$$\Rightarrow \left.\begin{array}{l} \rho\dfrac{De_T}{Dt} = \rho T\dfrac{Ds}{Dt} - p\dfrac{\partial v_i}{\partial x_i} - v_i\dfrac{\partial p_{ij}}{\partial x_j} \\ \rho\dfrac{De_T}{Dt} = -\dfrac{\partial}{\partial x_i}(v_j p_{ij} + Q_i) \end{array}\right\} -$$

$$e_T = e + \frac{1}{2}v_i^2$$

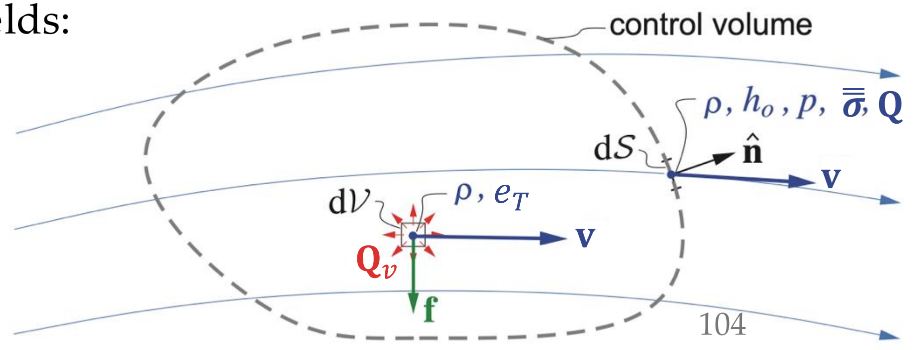
$$p_{ij} = p\delta_{ij} - \sigma_{ij}$$

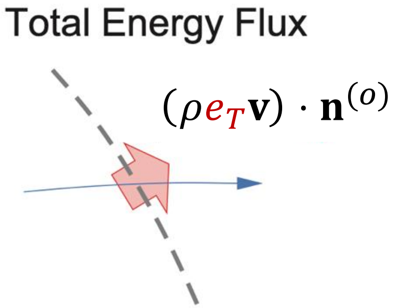
- Subtracting the non-conservative energy conservation equation yields:

$$\boxed{\rho T\frac{Ds}{Dt} = \sigma_{ij}\frac{\partial v_j}{\partial x_i} - \frac{\partial Q_i}{\partial x_i}}$$

Ignoring viscous effects
& heat addition
Isentropic

$$\boxed{\frac{Ds}{Dt} = 0}$$



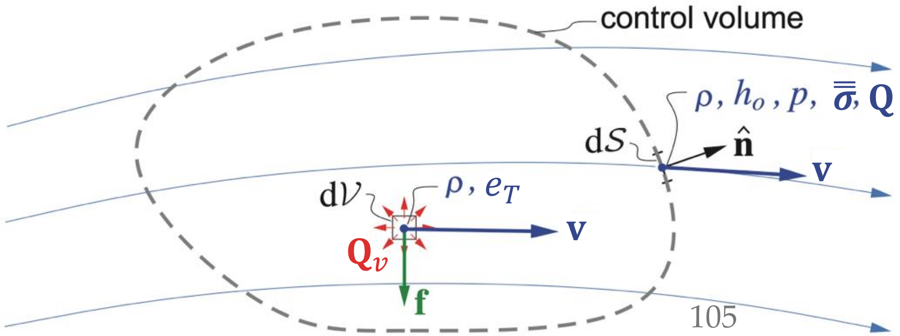


Energy conservation

- ❑ *Isentropic flow* assumption **APPLIES** in acoustic waves for most ***low-Mach*** aeroacoustics applications:
  - ➢ Momentum flux is almost completely balanced by pressure perturbations
    → ***viscous* effects on acoustic propagation are found to be *very small***
  - ➢ Temperature is uniform (or varies only slowly on the scale of wave propagation)
    → ***heat addition* effects are *not significant***
    - ▪ Large entropy fluctuations can be caused by a burst of heat, such as a combustion event in the flow

$$\boxed{\frac{Ds}{Dt} = 0}$$

- ❑ **For *inviscid unheated* flow - entropy is a convected quantity -** $s = s_\infty$
  - ➢ Upstream entropy level remains throughout the flow



Sound power

Application of energy conservation

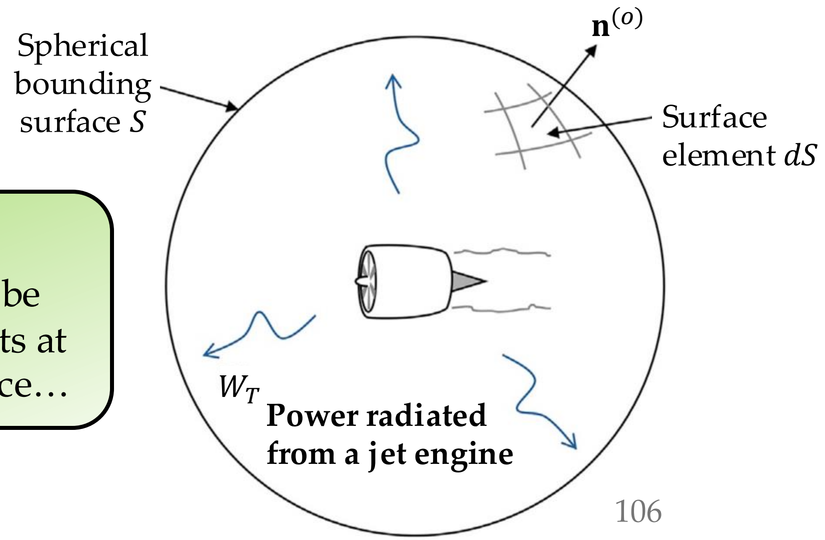
- Integral energy conservation is valuable in many applications
- Consider a **jet engine** that generates instantaneous power W_T radiating out of a CV with a spherical surface of **very large** diameter enclosing the engine and its exhaust
- At **steady state**, the rate of energy flux out of the CV must be equal to the instantaneous power generated by the engine W_T
- **Viscous effects and heat conduction** can be ***neglected*** on the control surface if it is **far enough**
- Taking the differential form of the energy conservation (with enthalpy), and integrate over the CV yields the total power generated by the engine:

$$\int_V \frac{\partial(\rho e_T)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i H - \underbrace{\cancel{v_j\sigma_{ij}}}_{0} + \underbrace{\cancel{Q_i}}_{0}\right) = 0$$

$$\Rightarrow \boxed{W_T = \int_V \nabla\cdot(\rho H\mathbf{v})\,dV = \int_S (\rho H\mathbf{v})\cdot\mathbf{n}^{(o)}\,dS}$$

Applying Gauss's divergence theorem

Valuable result!
Acoustic source strength can be computed from measurements at large distances from the source…



Sound power

$$\boxed{W_T = \int_S (\rho H \mathbf{v}) \cdot \mathbf{n}^{(o)}\, dS}$$

- ***Note*** – $W_T(t)$ – power generated by a system is an ***instantaneous*** quantity that includes:
  - Power needed to **drive the system**
  - Power required to maintain the **unsteady flow** and the **acoustic waves** (may be a small fraction of the total)
- In aeroacoustics, we wish to *separate* the mechanical power from the "power" required to feed the acoustic motion of the fluid at large distances from the source
- To do so, we **average the power generated by the system over *a period of time***: $\boxed{E[W_T] = W_s + W_a}$
  - $E[W_T]$ – *expected value* of the instantaneous power, with **steady** (W_s) and **unsteady** (W_a) contributions
- Denoting steady quantities with subscript $()_o$ and unsteady with prime $()'$:
  - Stagnation enthalpy – $H = H_o + H'$
  - Mass flux per unit area – $\rho\mathbf{v} = E[\rho\mathbf{v}] + (\rho\mathbf{v})'$
  - Mean of fluctuations is zero, but not correlations:
    - $E[(\rho\mathbf{v})'] \equiv 0$, $E[(\rho\mathbf{v})'H'] \neq 0$

$$(\rho\mathbf{v})H = E[\rho\mathbf{v}]H_o + E[\rho\mathbf{v}]H' + (\rho\mathbf{v})'H_o + (\rho\mathbf{v})'H'$$

$$E[(\rho\mathbf{v})H] = E[\rho\mathbf{v}]H_o + E[\rho\mathbf{v}]\underbrace{E[H']}_{0} + \underbrace{E[(\rho\mathbf{v})']}_{0}H_o + E[(\rho\mathbf{v})'H']$$

$$\Rightarrow E[W_T] = \int_S E[(\rho\mathbf{v})H] \cdot \mathbf{n}^{(o)}\, dS$$

$$E[(\rho\mathbf{v})H] = E[\rho\mathbf{v}]H_o + E[(\rho\mathbf{v})'H']$$

$$\boxed{E[W_T] = \overbrace{\int_S E[\rho\mathbf{v}]H_o \cdot \mathbf{n}^{(o)}\, dS}^{W_s} + \overbrace{\int_S E[(\rho\mathbf{v})'H'] \cdot \mathbf{n}^{(o)}\, dS}^{W_a}}$$

Sound power

$$E[W_T] = \int_S E[\rho\mathbf{v}]H_o \cdot \mathbf{n}^{(o)}\, dS + \int_S E[(\rho\mathbf{v})'H'] \cdot \mathbf{n}^{(o)}\, dS$$

- **For *steady homentropic potential (irrotational inviscid) flow*:**
  - According to Crocco's equation:

Applying Gauss's divergence theorem

$$\frac{\partial\phi}{\partial t} + H = f(t) \implies \boxed{H = \text{const.} = H_o} \implies W_s = H_o \int_S E[\rho\mathbf{v}] \cdot \mathbf{n}^{(o)}\, dS = H_o \int_V E[\underbrace{\nabla \cdot (\rho\mathbf{v})}_{}]\, dV = -H_o \int_V E\left[\frac{\partial\rho}{\partial t}\right] dV = \mathbf{0}$$

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0$$

Mass conservation

  - **Steady power is zero** ($W_s = 0$)
  - No power source is needed to maintain the flow…
- In contrast, for ***real flow***, which includes **turbulent wakes and viscous effects**, would require a source of power to maintain it in a steady state, so $W_s \neq 0$
  - We define W_s as the **mean power** and W_a as the power from **unsteady enthalpy**
  - This allows us to measure the sound power output W_a of an acoustic system:
    - **I** – acoustic intensity in the region outside of the turbulent flow

$$W_a = \int_S \mathbf{I} \cdot \mathbf{n}^{(o)}\, dS$$

$$\mathbf{I} = E[(\rho\mathbf{v})'H']$$

Measuring **I** on a surface enclosing sources of sound determines their **sound power output,** W

Sound power

$$\boxed{W_s = \int_S \mathbf{I} \cdot \mathbf{n}^{(o)}\, dS \qquad \mathbf{I} = E[(\rho \mathbf{v})' H']}$$

- For *unsteady homentropic* **flow** ($\nabla s = 0$):
  - Recall the expression we obtained for the stagnation enthalpy: $H = h + \frac{1}{2} v_i^2$
  - Then, using the 1st law:

$$\boxed{\nabla h = T \cancel{\nabla s}_{0} + \frac{1}{\rho} \nabla p} \implies dh = \frac{dp}{\rho} \xrightarrow{\text{linerization}} h' = \frac{p'}{\rho_o} + 0\left(\frac{1}{\rho_o^2}\right) \implies H' = h' + \frac{1}{2}(v_i^2)' = \frac{p'}{\rho_o} + \frac{1}{2}(v_i^2)'$$

  - We denote the instantaneous velocity component as: $v_i = U_i + v_i'$ or $\mathbf{v} = \mathbf{U} + \mathbf{v}'$
  - We do a 1st order approximation: $v_i^2 = (U_i + v_i')(U_i + v_i') \approx U_i^2 + 2U_i v_i' + \ldots \implies {}^1\!/_2 (v_i^2)' \approx U_i v_i'$

$$\rho = \rho_o + \rho' \implies \rho \mathbf{v} = (\rho_o + \rho')(\mathbf{U} + \mathbf{v}') \approx \rho_o \mathbf{U} + \rho_o \mathbf{v}' + \rho' \mathbf{U} + \ldots$$

  - We then obtain: $\boxed{H' = \frac{p'}{\rho_o} + U_i v_i' = \frac{p'}{\rho_o} + \mathbf{U} \cdot \mathbf{v}'}$ $\boxed{(\rho \mathbf{v})' = \rho_o \mathbf{v}' + \rho' \mathbf{U}}$
  - The **acoustic intensity** is: $\boxed{\mathbf{I} = E\left[(\rho_o \mathbf{v}' + \rho' \mathbf{U})\left(\frac{p'}{\rho_o} + \mathbf{U} \cdot \mathbf{v}'\right)\right]}$
  - *Easier* to measure in **stationary medium** ($\mathbf{U} = 0$): $\boxed{\mathbf{I} = E[p' \mathbf{v}']}$

CAUTION!!!
In turbulent flow, **not all unsteady pressure and velocity fluctuations propagate as acoustic waves!!**