# Applied Aerodynamics 1

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# A Part A

## A.1

Let the complex potential of a point vortex $\Gamma$ at $z_0$ be

$$f(z) = \frac{i\Gamma}{2\pi} ln(z - z_0) \tag{1}$$

Then, placing 2 vortices at $z_0$, $\bar{z}_0$, and according to the circular theorem, which is

$$w = f(z) + \bar{f}(\frac{a^2}{z}) \tag{2}$$

Where if point $z$ is outside the circle, point $\frac{a^2}{z}$ is the corresponding point inside the circle to satisfy all the conditions of the cylinder. The two other vortices will be mirrored inside the cylinder as seen in Fig.1. We get our complex potential

$$f_1(z) = \frac{i\Gamma}{2\pi} ln(z - z_0) \tag{3}$$

$$f_2(z) = -\frac{i\Gamma}{2\pi} ln(z - \bar{z}_0) \tag{4}$$

Now, we can obtain the complex potential using the circular theorem 2. When $w_1$ is

$$w_1 = \frac{i\Gamma}{2\pi} ln(z - z_0) + \frac{i\Gamma}{2\pi} ln(\frac{a^2}{z} - z_0) \tag{5}$$

We can notice that

$$\begin{aligned} ln(\frac{a^2}{z} - z_0) = ln(\frac{a^2 - z_0 z}{z}) = -ln(z) \\ + ln(a^2 - z_0 z) = -ln(z) + ln(z - \frac{a^2}{z_0}) + \cancel{ln(-z_0)} \end{aligned} \tag{6}$$

Canceling $ln(-z_0)$ due to it being constant, thus $w_1$ is

$$w_1 = \frac{i\Gamma}{2\pi} \left( ln(z - z_0) - ln(z) + ln(z - \frac{a^2}{z_0}) \right) \tag{7}$$

Doing the same for $w_2$

$$\begin{aligned} w_2 = -\frac{i\Gamma}{2\pi} ln(z - \bar{z}_0) - \frac{i\Gamma}{2\pi} ln(z - \frac{a^2}{\bar{z}_0}) = \\ -\frac{i\Gamma}{2\pi} \left( ln(z - \bar{z}_0) - ln(z) + ln(z - \frac{a^2}{\bar{z}_0}) \right) \end{aligned} \tag{8}$$

And adding the complex potential according to the circular theorem for the uniform flow $w_3$

$$w_3 = U_\infty z + U_\infty \frac{a^2}{z} \tag{9}$$

thus, the full complex potential is

$$\begin{aligned} w = &\frac{i\Gamma}{2\pi} ln(z - z_0) + \frac{i\Gamma}{2\pi} ln(z - \frac{a^2}{z_0}) - \frac{i\Gamma}{2\pi} ln(z - \bar{z}_0) \\ &- \frac{i\Gamma}{2\pi} ln(z - \frac{a^2}{\bar{z}_0}) + \cancel{\frac{i\Gamma}{2\pi} ln(z)} - \cancel{\frac{i\Gamma}{2\pi} ln(z)} + U_\infty z + U_\infty \frac{a^2}{z} \end{aligned} \tag{10}$$

And after canceling terms and some rearranging

$$w = U_\infty z + U_\infty \frac{a^2}{\bar{z}} + \frac{i\Gamma}{2\pi} ln(z - z_0) - \frac{i\Gamma}{2\pi} ln(z - \bar{z}_0) - \frac{i\Gamma}{2\pi} ln(z - \frac{a^2}{\bar{z}_0}) + \frac{i\Gamma}{2\pi} ln(z - \frac{a^2}{z_0}) \tag{11}$$

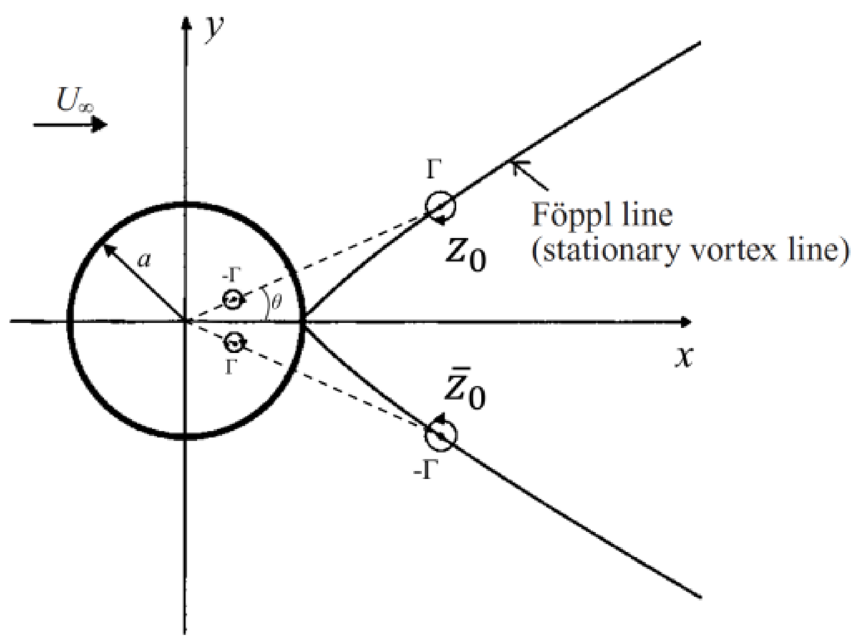



Figure 1: Vortices behind a circular cylinder and the Foppl line.

Now to find the complex velocity, we derive the potential $\frac{dw}{dz}$

$$V(z) = \frac{dw}{dz} = U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} (\frac{1}{z - z_0} - \frac{1}{z - \bar{z}_0} - \frac{1}{z - \frac{a^2}{\bar{z}_0}} + \frac{1}{z - \frac{a^2}{z_0}}) \tag{12}$$

## A.2

Now imposing the viscous boundary conditions $V(z_0) = V(\bar{z}_0) = 0$ thus that the two vortices that form behind the cylinder are stationary. Looking at point

$z_0$, the self-induced velocity of the vortex is $V(z) = \frac{i\Gamma}{2\pi}\frac{1}{z-z_0}$ creates a singularity and goes to infinity. Looking in a Polar coordinate system $V_r = 0, V_\theta = \frac{\Gamma}{2\pi(r-r_0)}$ and a plot of the velocity will be

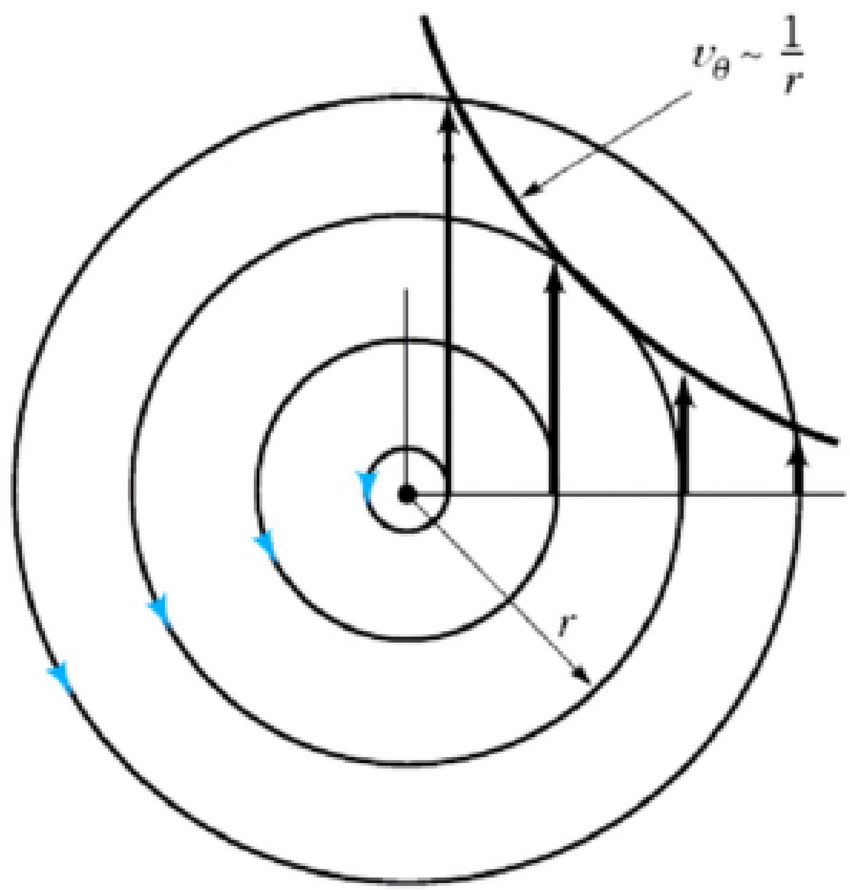



Figure 2: Velocity plot for a point vortex

Now, looking at the problem with an analogy to circular motion where $V = \omega r$, we can explain why neglecting the self-induced velocity is allowed. A point vortex exists in potential flow only. Looking into viscous flow, a point vortex can be replaced by a "Rankine vortex" as shown in fig 3.

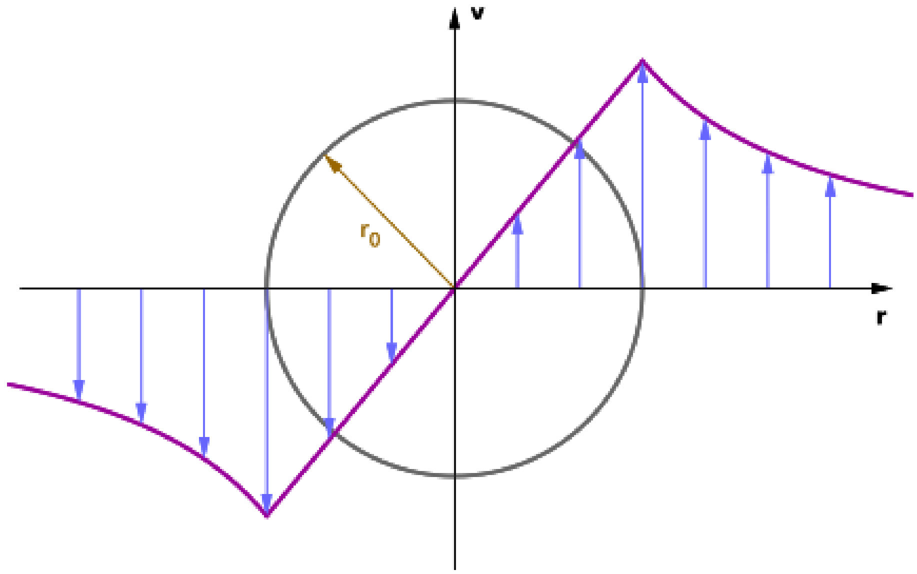


Figure 3: Velocity plot for a Rankine vortex

Where inside a certain radius, the vortex acts as a solid body, and the rate at which a particle spins near goes as $\omega \sim \frac{1}{a^2}$, and the velocity will be $V = \omega r$. Now, examining the velocity at $z_0$ where $r \equiv 0$ identically, thus even as the angular velocity goes to infinity, the velocity at the point vortex will remain zero ($V = \omega \times r = \omega \times 0 \equiv 0$).

## A.3

Now imposing $V(z_0) = 0$ we can derive equation I

$$V(z_0) = U_\infty(1 - \frac{a^2}{z_0^2}) + \frac{i\Gamma}{2\pi}(-\frac{1}{z_0 - \bar{z}_0} - \frac{1}{z_0 - \frac{a^2}{\bar{z}_0}} + \frac{1}{z_0 - \frac{a^2}{z_0}}) = 0 \tag{13}$$

$$U_\infty(1 - \frac{a^2}{z_0^2}) + \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{\bar{z}_0}{z_0\bar{z}_0 - a^2} + \frac{z_0}{z_0^2 - a^2}) = 0 \tag{14}$$

Knowing that $z_0 = x_0 + iy_0$, $\bar{z}_0 = x_0 - iy_0$ and $z_0\bar{z}_0 = x_0^2 + y_0^2 = r_0^2$ in polar coordinates we can get

$$U_\infty(1 - \frac{a^2}{(x_0 + iy_0)^2}) + \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0 + iy_0}{z_0^2 - a^2}) = 0 \tag{15}$$

looking only at the first term in equation 15

$$U_\infty(1 - \frac{a^2}{(x_0 + iy_0)^2}) = U_\infty(1 - \frac{a^2}{x_0^2 + 2ix_0y_0 - y_0^2}) \tag{16}$$

$$= U_\infty(1 - a^2 \frac{(x_0^2 - y_0^2) - 2ix_0y_0}{((x_0^2 - y_0^2) + 2ix_0y_0)((x_0^2 - y_0^2) - 2ix_0y_0))}) \tag{17}$$

$$= U_\infty(1 - a^2 \frac{(x_0^2 - y_0^2) - 2ix_0y_0}{(x_0^2 - y_0^2)^2 + 4x_0^2y_0^2}) \tag{18}$$

$$= U_\infty(1 - a^2 \frac{(x_0^2 - y_0^2) - 2ix_0y_0}{(x_0^2 + y_0^2)^2}) \tag{19}$$

$$= U_\infty(1 - a^2 \frac{(x_0^2 - y_0^2)}{r_0^4}) + iU_\infty a^2 \frac{2x_0y_0}{r_0^4} \tag{20}$$

With the real part being

$$U_\infty(1 - a^2 \frac{(x_0^2 - y_0^2)}{r_0^4}) \tag{21}$$

And the imaginary part being

$$iU_\infty a^2 \frac{2x_0y_0}{r_0^4} \tag{22}$$

And now doing the same for the second term in equation 15

$$\frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0 + iy_0}{z_0^2 - a^2}) = \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0 + iy_0}{(x_0 + iy_0)^2 - a^2}) \tag{23}$$

$$= \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0 + iy_0}{x_0^2 + 2ix_0y_0 - y_0^2 - a^2}) \tag{24}$$

$$= \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0 + iy_0}{(x_0^2 - y_0^2 - a^2) + 2ix_0y_0}) \tag{25}$$

$$= \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{(x_0 + iy_0)((x_0^2 - y_0^2 - a^2) - 2ix_0y_0)}{(x_0^2 - y_0^2 - a^2)^2 + 4x_0^2y_0^2}) \tag{26}$$

$$= \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{x_0(x_0^2 - y_0^2 - a^2) - 2ix_0^2y_0 + iy_0(x_0^2 - y_0^2 - a^2) + 2x_0y_0^2}{(x_0^2 - y_0^2)^2 + 4x_0^2y_0^2 - 2a^2(x_0^2 - y_0^2) + a^4}) \tag{27}$$

$$= \frac{i\Gamma}{2\pi}(-\frac{1}{2iy_0} - \frac{x_0 - iy_0}{r_0^2 - a^2} + \frac{(x_0(x_0^2 - y_0^2 - a^2) + 2x_0y_0^2) + i(y_0(x_0^2 - y_0^2 - a^2) - 2x_0^2y_0)}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4}) \tag{28}$$

The real part

$$-\frac{\Gamma}{4\pi y_0} - \frac{\Gamma y_0}{2\pi(r_0^2 - a^2)} - \frac{\Gamma}{2\pi}\frac{y_0(x_0^2 - y_0^2 - a^2) - 2x_0^2y_0}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4} \tag{29}$$

And the imaginary part be

$$i\left(-\frac{\Gamma x_0}{2\pi(r_0^2 - a^2)} + \frac{\Gamma}{2\pi}\frac{(x_0(x_0^2 - y_0^2 - a^2) + 2x_0y_0^2)}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4}\right) \tag{30}$$

now combining terms 21, 22, 29 and 30 we get a term that can be represented as $V(z) = u - iv = 0$. Now imposing the viscous conditions $u = 0, v = 0$ we get for the real part

$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} - \frac{\Gamma y_0}{2\pi(r_0^2 - a^2)} - \frac{\Gamma}{2\pi}\frac{y_0(x_0^2 - y_0^2 - a^2) - 2x_0^2 y_0}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4} = 0 \tag{31}$$

And the imaginary part

$$U_\infty a^2\frac{2x_0y_0}{r_0^4} + \frac{\Gamma}{2\pi}\left(-\frac{x_0}{r_0^2 - a^2} + \frac{(x_0(x_0^2 - y_0^2 - a^2) + 2x_0y_0^2)}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4}\right) = 0 \tag{32}$$

Or

$$-U_\infty a^2\frac{2x_0y_0}{r_0^4} - \frac{\Gamma}{2\pi}\left(\frac{x_0}{r_0^2 - a^2} + \frac{(x_0(x_0^2 - y_0^2 - a^2) + 2x_0y_0^2)}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4}\right) = 0 \tag{33}$$

Focusing on the imaginary part (eq. 33) and using the following identities

$$\begin{aligned}
&r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4 = r_0^4 - 2a^2x_0^2 + 2a^2y_0^2 + a^4 = \\
&\quad r_0^4 - 2a^2x_0^2 - 2a^2y_0^2 + a^4 + 4a^2y_0^2 = r_0^4 - 2a^2(x_0^2 + y_0^2) + a^4 + 4a^2y_0^2 \\
&\quad r_0^4 - 2a^2r_0^2 + a^4 + 4a^2y_0^2 = (r_0^2 - a^2)^2 + 4a^2y_0^2
\end{aligned} \tag{34}$$

And

$$(x_0^2 - y_0^2 - a^2) + 2y_0^2 = x_0^2 + y_0^2 - a^2 = r_0^2 - a^2 \tag{35}$$

Now plugin in identities 34 and 35 to equation 33 we get

$$-U_\infty a^2\frac{2x_0y_0}{r_0^4} + \frac{\Gamma}{2\pi}\left(\frac{x_0}{r_0^2 - a^2} - \frac{(x_0(r_0^2 - a^2))}{(r_0^2 - a^2)^2 + 4a^2y_0^2}\right) = 0 \tag{36}$$

Assuming $x_0 \neq 0$ we can divide equation 36 by $x_0$ and get

$$-U_\infty a^2\frac{2y_0}{r_0^4} + \frac{\Gamma}{2\pi}\left(\frac{1}{r_0^2 - a^2} - \frac{(r_0^2 - a^2)}{(r_0^2 - a^2)^2 + 4a^2y_0^2}\right) = 0 \tag{37}$$

Or

$$-U_\infty a^2\frac{2y_0}{r_0^4} + \frac{\Gamma}{2\pi}\left(\frac{(r_0^2 - a^2)^2 + 4a^2y_0^2 - (r_0^2 - a^2)^2}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2y_0^2)}\right) = 0 \tag{38}$$

$$-U_\infty a^2\frac{2y_0}{r_0^4} + \frac{\Gamma}{2\pi}\left(\frac{4a^2y_0^2}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2y_0^2)}\right) = 0 \tag{39}$$

We can isolate $U_\infty$

$$U_\infty = \frac{r_0^4}{\cancel{2}y_0\cancel{a^2}}\frac{\Gamma}{\cancel{2}\pi}\left(\frac{\cancel{4}\cancel{a^2}y_0^2}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2y_0^2)}\right) \tag{40}$$

Or

$$U_\infty = \frac{r_0^4\Gamma}{\pi}\left(\frac{y_0}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2y_0^2)}\right) \tag{41}$$

and plug into equation 31, but first we'll simplify the equation using identity 34

$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} - \frac{\Gamma y_0}{2\pi(r_0^2 - a^2)} - \frac{\Gamma}{2\pi}\frac{y_0(x_0^2 - y_0^2 - a^2) - 2x_0^2 y_0}{r_0^4 - 2a^2(x_0^2 - y_0^2) + a^4} =$$
$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-1}{r_0^2 - a^2} - \frac{x_0^2 - y_0^2 - 2x_0^2 - a^2}{(r_0^2 - a^2)^2 + 4a^2 y_0^2}\right) \tag{42}$$

Knowing that

$$(x_0^2 - y_0^2 - 2x_0^2 - a^2) = -x_0^2 - y_0^2 - a^2 = -(r_0^2 + a^2) \tag{43}$$

We get

$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-1}{r_0^2 - a^2} - \frac{x_0^2 - y_0^2 - 2x_0^2 - a^2}{(r_0^2 - a^2)^2 + 4a^2 y_0^2}\right) =$$
$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-1}{r_0^2 - a^2} - \frac{-(r_0^2 + a^2)}{(r_0^2 - a^2)^2 + 4a^2 y_0^2}\right) = 0 \tag{44}$$

$$U_\infty(1 - a^2\frac{(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-1}{r_0^2 - a^2} + \frac{(r_0^2 + a^2)}{(r_0^2 - a^2)^2 + 4a^2 y_0^2}\right) =$$
$$U_\infty(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-(r_0^2 - a^2)^2 - 4a^2 y_0^2 + (r_0^2 + a^2)(r_0^2 - a^2))}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) = 0 \tag{45}$$

$$U_\infty(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{-(r_0^2 - a^2)^2 - 4a^2 y_0^2 + (r_0^2 + a^2)(r_0^2 - a^2))}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) =$$
$$U_\infty(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{\cancel{-r_0^4} + 2r_0^2 a^2 - a^4 - 4a^2 y_0^2 + (\cancel{r_0^4} - a^4)}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) = 0 \tag{46}$$

$$U_\infty(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{2r_0^2 a^2 - 2a^4 - 4a^2 y_0^2}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) =$$
$$U_\infty(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0} + \frac{\Gamma}{2\pi} y_0\left(\frac{2a^2(2r_0^2 - 2y_0^2 - a^2)}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) = 0 \tag{47}$$

Now, plugging in $U_\infty$ from 41 into the equation above

$$\frac{r_0^4\Gamma}{\pi}\left(\frac{y_0}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right)(\frac{r_0^4 - a^2(x_0^2 - y_0^2)}{r_0^4}) - \frac{\Gamma}{4\pi y_0}$$
$$+ \frac{\Gamma}{2\pi} y_0\left(\frac{2a^2(2r_0^2 - 2y_0^2 - a^2)}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)}\right) = 0 \tag{48}$$

Starting to cancel out terms we can get

$$\frac{\cancel{r_0^4}}{1}\frac{\cancel{\Gamma}}{\cancel{\pi}}\left(\frac{y_0}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}\right)(\frac{r_0^4-a^2(x_0^2-y_0^2)}{\cancel{r_0^4}})- \\ \frac{\cancel{\Gamma}}{4\cancel{\pi}y_0}+\frac{\cancel{\Gamma}}{2\cancel{\pi}}y_0\left(\frac{2a^2(r_0^2-2y_0^2-a^2)}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}\right)=0 \tag{49}$$

$$\frac{y_0(r_0^4-a^2(x_0^2-y_0^2))}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}-\frac{1}{4y_0}+\frac{y_0}{2}\left(\frac{2a^2(r_0^2-2y_0^2-a^2)}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}\right)=0 \tag{50}$$

$$\frac{r_0^4-a^2(x_0^2-y_0^2)+a^2(r_0^2-2y_0^2-a^2)}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{51}$$

$$\frac{r_0^4-a^2x_0^2+a^2y_0^2+a^2r_0^2-a^22y_0^2-a^4)}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{52}$$

$$\frac{r_0^4+a^2r_0^2-a^2x_0^2-a^2y_0^2-a^4}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{53}$$

$$\frac{r_0^4+\cancel{a^2r_0^2}-\cancel{a^2r_0^2}-a^4}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{54}$$

$$\frac{r_0^4-a^4}{(r_0^2-a^2)((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{55}$$

$$\frac{\cancel{(r_0^2-a^2)}(r_0^2+a^2)}{\cancel{(r_0^2-a^2)}((r_0^2-a^2)^2+4a^2y_0^2)}=\frac{1}{4y_0^2} \tag{56}$$

$$\frac{(r_0^2+a^2)}{(r_0^2-a^2)^2+4a^2y_0^2}=\frac{1}{4y_0^2} \tag{57}$$

$$4y_0^2(r_0^2-a^2)=(r_0^2-a^2)^2+4a^2y_0^2 \tag{58}$$

$$4y_0^2r_0^2=(r_0^2-a^2)^2 \tag{59}$$

Taking a $\surd$ on both sides

$$\pm 2y_0r_0=r_0^2-a^2 \tag{60}$$

And now, **finally!!**, at equation 61 we have the wanted and elegant Foppl line:

$$\pm 2y_0=r_0-\frac{a^2}{r_0} \tag{61}$$

### A.4

Now that we have the Foppl line $y_0 = f(a, r_0)$, we can go back to equation 41, simplify it and plug in the simplified Foppl line with the plus sign as shown in equation 62

$$y_0 = \frac{1}{2}(r_0 - \frac{a^2}{r_0}) \tag{62}$$

$$U_\infty = \frac{r_0^4 \Gamma}{\pi} \left( \frac{y_0}{(r_0^2 - a^2)((r_0^2 - a^2)^2 + 4a^2 y_0^2)} \right) \tag{63}$$

$$\frac{\Gamma}{2\pi} = \frac{U_\infty}{2} \frac{r_0^2 - a^2}{r_0^4} \frac{1}{y_0} \left( (r_0^2 - a^2) + 4a^2 y_0^2 \right) \tag{64}$$

And now plugging in $y_0$ from equation 62

$$\frac{\Gamma}{2\pi} = U_\infty \frac{1}{r_0^3} \left( (r_0^2 - a^2)^2 + \frac{(r_0^2 - a^2)^2}{r_0^2} a^2 \right) \tag{65}$$

$$\frac{\Gamma}{2\pi} = U_\infty \frac{1}{r_0^5} \left( (r_0^2 - a^2)^2 r_0^2 + (r_0^2 - a^2)^2 a^2 \right) \tag{66}$$

$$\frac{\Gamma}{2\pi} = U_\infty (r_0^2 - a^2)^2 (r_0^2 + a^2) \frac{1}{r_0^5} \tag{67}$$

## B Part B

### B.1

As shown in figure 5, we can compute $r_0$ for both cases. For both cases, (a) for $Re = 20$, and (b) for $Re = 1.4 \times 10^5$, we'll need to move the origin point to the center of the cylinder as given in figure 1. Thus, the updated locations of the vortex will be:

$$\begin{cases} \frac{x}{D} = 0.83, \\ \frac{y}{D} = 0.23 \end{cases} \tag{68}$$

For $r_0$

$$\begin{aligned} r_0 &= \sqrt{x_0^2 + y_0^2} \\ \frac{r_0}{D} &= \sqrt{(\frac{x_0}{D})^2 + (\frac{y_0}{D})^2} \end{aligned} \tag{69}$$

Given the theoretical Foppl line as shown In equation 61, we can compute the theoretical $r_0$ for a given $y_0$

$$\begin{aligned} 2\frac{y_0}{D} &= \frac{r_0}{D} - \frac{\frac{a^2}{D^2}}{\frac{r_0}{D}} \\ (\frac{r_0}{D})^2 - 2\frac{y_0}{D}(\frac{r_0}{D}) - (\frac{a}{D})^2 &= 0 \end{aligned} \tag{70}$$

For $Re = 20$

$$\frac{r_0}{D} = \sqrt{0.83^2 + 0.23^2} = 0.8613 \tag{71}$$

And the theoretical value from 70

$$\frac{r_0}{D} = 0.7803 \tag{72}$$

Now, calculating the error between the literature value and theoretical value

$$Error = \frac{r_0(theory) - r_0(literature)}{r_0(theory)} = \frac{0.8613 - 0.7803}{0.7803} = 10.38\% \tag{73}$$

Now for $Re = 1.4 \times 10^5$

$$\begin{cases} \frac{x}{D} = 0.81, \\ \frac{y}{D} = 0.27 \end{cases} \tag{74}$$

So

$$\frac{r_0}{D} = \sqrt{0.81^2 + 0.27^2} = 0.8538 \tag{75}$$

And the theoretical value from 70

$$\frac{r_0}{D} = 0.8382 \tag{76}$$

Now, calculating the error between the literature value and theoretical value

$$Error = \frac{r_0(theory) - r_0(literature)}{r_0(theory)} = \frac{0.8538 - 0.8382}{0.8382} = 1.86\% \tag{77}$$

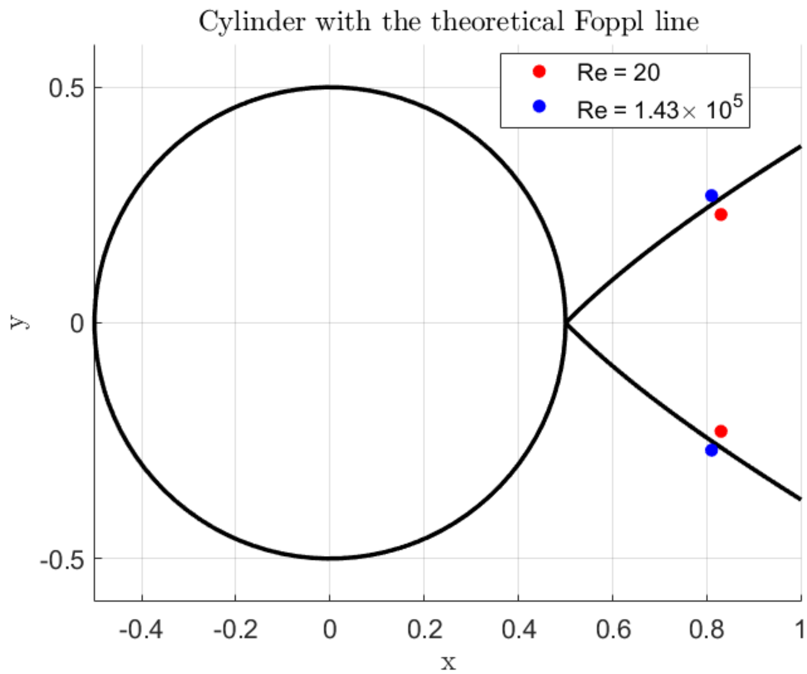
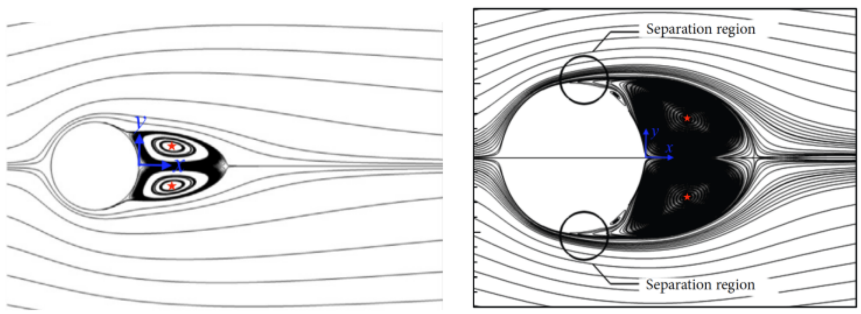



Figure 4: Foppl line with the published vortices locations

And the resulting streamlines for $r_0$ values from 71 and 75 were plotted as follows




(a) (b)

Figure 5: Literature streamlines: (a) Re = 20, (b) Re = $1.4 \times 10^5$

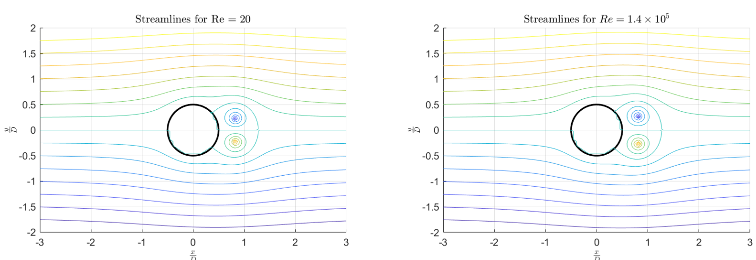



Figure 6: Theoretical streamlines

And also comparing the streamlines on the same figure

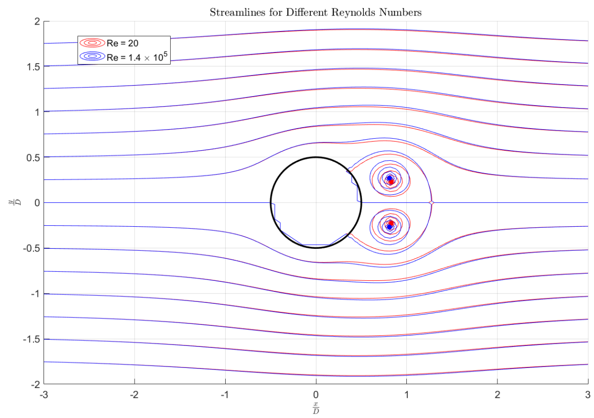



Figure 7: Theoretical streamlines on the same figure

We can also calculate the vorticity strength $\Gamma$ from equation 67

$$\begin{aligned}\frac{\Gamma}{2\pi} &= U_\infty(r_0^2 - a^2)^2(r_0^2 + a^2)\frac{1}{r_0^5} \\ &= U_\infty D^4((\frac{r_0}{D})^2 - (\frac{a}{D})^2)^2(r_0^2 + a^2)\frac{1}{r_0^5}\end{aligned} \tag{78}$$

$$= U_\infty D^4 D^2((\frac{r_0}{D})^2 - (\frac{a}{D})^2)^2((\frac{r_0}{D})^2 + (\frac{a}{D})^2)\frac{1}{r_0^5} \tag{79}$$

$$= U_\infty \frac{D^4 D^2}{D^5}((\frac{r_0}{D})^2 - (\frac{a}{D})^2)^2((\frac{r_0}{D})^2 + (\frac{a}{D})^2)\frac{1}{(\frac{r_0}{D})^5} \tag{80}$$

$$\frac{\Gamma}{2\pi} = U_\infty D((\frac{r_0}{D})^2 - (\frac{a}{D})^2)^2((\frac{r_0}{D})^2 + (\frac{a}{D})^2)\frac{1}{(\frac{r_0}{D})^5} \tag{81}$$

Now, we can compute the theoretical and literature $\Gamma$ values for each case using the $r_0$ values from before. For $Re = 20$

$$literature : \frac{\Gamma}{2\pi}(\frac{r_0}{D} = 0.8613) = 0.5061 U_\infty D \tag{82}$$

$$theoretical : \frac{\Gamma}{2\pi}(\frac{r_0}{D} = 0.7803) = 0.3823 U_\infty D \tag{83}$$

With the error being

$$Error = \frac{\Gamma(theory) - \Gamma(literature)}{\Gamma(theory)} = \frac{0.3823 - 0.5061}{0.3823} = 32.35\% \tag{84}$$

For $Re = 1.4 \times 10^5$

$$literature : \frac{\Gamma}{2\pi}(\frac{r_0}{D} = 0.8538) = 0.495 U_\infty D \tag{85}$$

$$theoretical : \frac{\Gamma}{2\pi}(\frac{r_0}{D} = 0.8382) = 0.4715U_\infty D \tag{86}$$

With the error being

$$Error = \frac{\Gamma(theory) - \Gamma(literature)}{\Gamma(theory)} = \frac{0.4715 - 0.495}{0.4715} = 4.98\% \tag{87}$$

To summarize the results, we know that the Foppl line was developed for potential, inviscid, and non-viscus flow, for which $Re \to \infty$. From that assumption, we can expect that the flow will be similar to the theoretical potential flow for higher Re numbers. Plotting the theoretical Foppl line and the point vortices published in the literature in figure 4, looking at the errors of the theoretical results from the published results as shown in 73, 77. We can see the same trend in figures 5 and 6 and in the errors for $\Gamma$ values. We can see that for higher Re numbers, the theoretical values are closer to the published literature values by an order of magnitude.

## B.2

Substituting the resulted $r_0, \Gamma$ values we obtained in 71, 75, 82 and 85 we can derive the pressure coefficient $C_p$ values at any given location on the cylinder as follows

$$Bernoulli : P_{cylinder} + \frac{1}{2}\rho U_Z^2 = P_\infty + \frac{1}{2}\rho U_\infty^2 \tag{88}$$

$$P - P_\infty = \frac{1}{2}\rho(U_\infty^2 - U_Z^2) \tag{89}$$

$$C_p = \frac{P - P_\infty}{\frac{1}{2}U_\infty^2} = 1 - (\frac{U}{U_\infty})^2 \tag{90}$$

Now, given all those values and the formula for $C_p$ above, we can calculate $U_Z$ from 13 and than plot the ideal $C_p$ values as a function of $\theta$, where The azimuth angle $\theta$ is measured from the trailing edge of the cylinder (counter-clockwise), where $\theta = 0°, 360°$ is the trailing edge, and $\theta = 180°$. is the leading edge

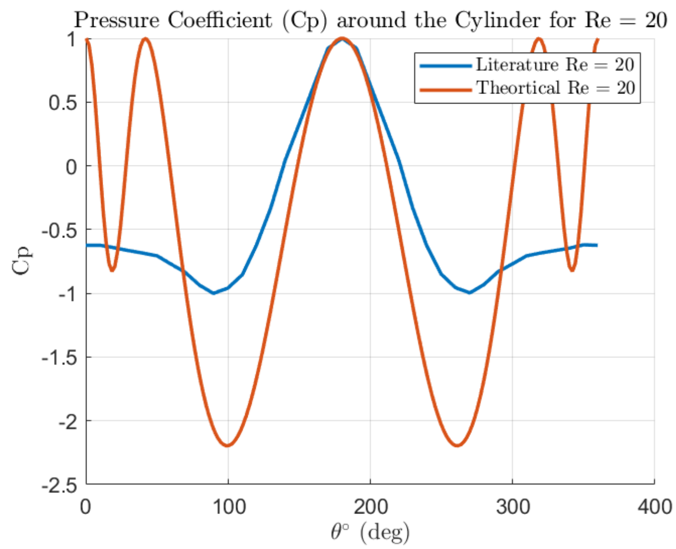



Figure 8: $C_p$ vs $\theta$ for $Re = 36 - 107$

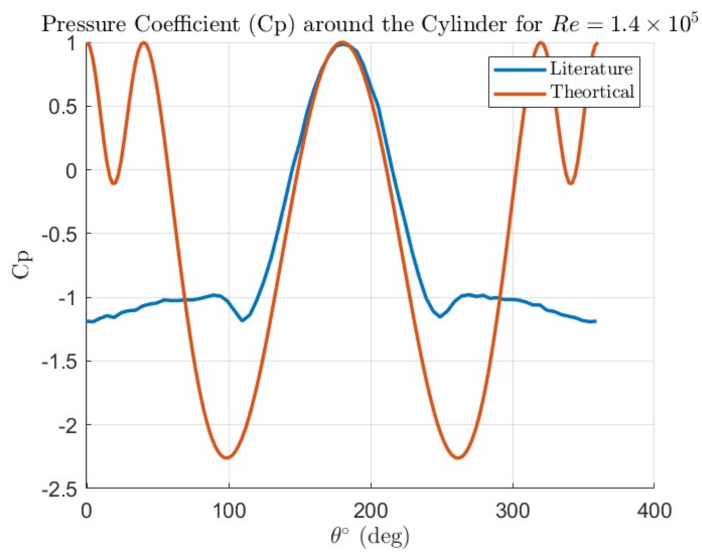



Figure 9: $C_p$ vs $\theta$ for $Re = 1.4 \times 10^5$

Looking at both graphs, we can see that the theoretical $C_p$ plot has a few more stagnation points ($C_p = 1$) than the graph from the literature. At $\theta = 0^\circ, 360^\circ$, the right point of the cylinder $z = a + 0i$, we get a stagnation in the potential flow, but due to the flow separating at the aft of the cylinder, the values from the literature are zero. At $\theta \approx 45°, 315°$, we get another stagnation point here, the free stream flow $U_\infty$, and the flow from the vortices cancel out, and the velocity is zero. At $\theta = 180°$, the "leading edge" of the cylinder where $z = -1 + 0i$, we get a stagnation point at both the theoretical and the literature graphs as expected.

# C Part C

## C.1

Blasius's theorem calculates the aerodynamic forces acting upon a given body as follows

$$X - iY = \frac{1}{2} i\rho \int_C (\frac{dw}{dz})^2 dz \tag{91}$$

In order to calculate the integral, we'll have to use Cauchy's integral theorem and Cauchy's residue theorem to simplify the integration process. According to Cauchy's integral theorem, if a function is holomorphic thus, the integral on a closed loop is zero

$$\oint_c f(z) dz = 0$$

Now, if the function isn't holomorphic, we can take the integral such as:

$$\oint_c f(z) dz = \sum_{n=1}^{N} \oint_{C_i} f(z) dz \tag{92}$$

where $C_i$ are the singular points. Then, calculate the integrals according to Cauchy's Residue Theorem, which is

$$\oint_c f(z) dz = 2\pi i (a_1 + a_2 + \cdots + a_n) \tag{93}$$

Where $(a_1 + a_2 + \cdots + a_n)$ are the residues, the coefficients of $\frac{1}{z-a}$ calculating around a. In our case, we have three different areas with singular points: the two vortices and our cylinder. Then, our integral will be:

$$\int_c (\frac{dw}{dz})^2 dz = \int_{c_1} (\frac{dw}{dz})^2 dz + \int_{c_2} (\frac{dw}{dz})^2 dz + \int_{c_3} (\frac{dw}{dz})^2 dz \tag{94}$$

And

$$\int_{c_3} (\frac{dw}{dz})^2 dz = \int_c (\frac{dw}{dz})^2 dz - \int_{c_1} (\frac{dw}{dz})^2 dz - \int_{c_2} (\frac{dw}{dz})^2 dz \tag{95}$$

With $C$, $C_1$, $C_2$ and $C_3$ as shown in figure 10

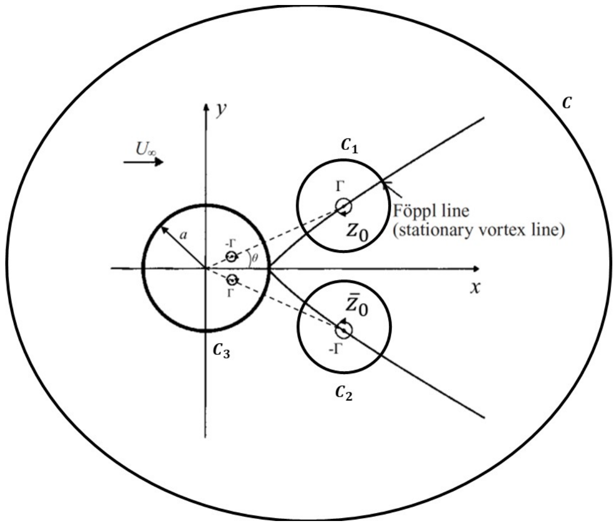



Figure 10: The integral's curves

Focusing on the far-field integral where $r \to \infty$

$$\int_c (\frac{dw}{dz})^2 dz = \int_c \left( U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} (\frac{1}{z - z_0} - \frac{1}{z - \bar{z}_0} - \frac{1}{z - \frac{a^2}{\bar{z}_0}} + \frac{1}{z - \frac{a^2}{z_0}}) \right)^2 dz \tag{96}$$

For simplicity, we can take each term in equation 96 and expand into a laurent series around $\frac{1}{z}$ knowing that $r \to \infty \Rightarrow z \to \infty$, taking only the relevant terms:

$$\frac{1}{z - z_0} = \frac{1}{z}\frac{1}{1 - \frac{z_0}{z}} = \frac{1}{z}\sum_{-\infty}^{\infty} z_0 z^n = \frac{1}{z}(\cdots + 1 + \frac{z_0}{z} + \frac{z_0^2}{z^2} + \cdots) = \frac{1}{z} + \frac{z_0}{z^2} \tag{97}$$

And the same for $\bar{z}_0$

$$\frac{1}{z - \bar{z}_0} = \frac{1}{z}\frac{1}{1 - \frac{\bar{z}_0}{z}} = \frac{1}{z}\sum_{-\infty}^{\infty} \bar{z}_0 z^n = \frac{1}{z}(\cdots + 1 + \frac{\bar{z}_0}{z} + \frac{\bar{z}_0^2}{z^2} + \cdots) = \frac{1}{z} + \frac{\bar{z}_0}{z^2} \tag{98}$$

Doing the same for $\frac{1}{z - \frac{a^2}{z_0}}$ and $\frac{1}{z - \frac{a^2}{\bar{z}_0}}$ we get

$$\frac{1}{z - \frac{a^2}{z_0}} = \frac{1}{z} + \frac{\frac{a^2}{z_0}}{z^2} \tag{99}$$

$$\frac{1}{z - \frac{a^2}{\bar{z}_0}} = \frac{1}{z} + \frac{\frac{a^2}{\bar{z}_0}}{z^2} \tag{100}$$

Now, substituting everything into equation 96, we get

$$\int_c \left( U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} ( \cancel{\frac{1}{z}} + \frac{z_0}{z^2} - \cancel{\frac{1}{z}} - \frac{\bar{z}_0}{z^2} - \cancel{\frac{1}{z}} - \frac{\frac{a^2}{\bar{z}_0}}{z^2} + \cancel{\frac{1}{z}} + \frac{\frac{a^2}{z_0}}{z^2} ) \right)^2 dz \tag{101}$$

$$= \int_c \left( U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} ( \frac{z_0}{z^2} - \frac{\bar{z}_0}{z^2} - \frac{\frac{a^2}{\bar{z}_0}}{z^2} + \frac{\frac{a^2}{z_0}}{z^2} ) \right)^2 dz \tag{102}$$

$$= \int_c U_\infty^2 (1 - \frac{a^2}{z^2})^2 + 2U_\infty (1 - \frac{a^2}{z^2}) \frac{i\Gamma}{2\pi} \left( \frac{z_0}{z^2} - \frac{\bar{z}_0}{z^2} - \frac{\frac{a^2}{\bar{z}_0}}{z^2} - \frac{\frac{a^2}{z_0}}{z^2} \right) -$$

$$\frac{\Gamma^2}{4\pi^2} \left( \frac{z_0}{z^2} - \frac{\bar{z}_0}{z^2} - \frac{\frac{a^2}{\bar{z}_0}}{z^2} + \frac{\frac{a^2}{z_0}}{z^2} \right)^2 dz \tag{103}$$

Looking for the coefficient of $\frac{1}{z}$ we can see that non of terms in equation 103 will lead to a term with $\frac{1}{z}$, thus in integral from 93, $a_i = 0$ and

$$\int_c (\frac{dw}{dz})^2 dz = 2\pi i 0 = 0 \tag{104}$$

Now, focusing on the integrals around the vortices, $C_1, C_2$, we can take the radius $r \to 0$ for the contours and neglect the self-induced velocities as shown in A.2. For $c_1$

$$\int_{c_1} (\frac{dw}{dz})^2 dz = \int_{c_1} \left( U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} ( \cancel{\frac{1}{z - z_0}} - \frac{1}{z - \bar{z}_0} - \frac{1}{z - \frac{a^2}{\bar{z}_0}} + \frac{1}{z - \frac{a^2}{z_0}} ) \right)^2 dz \tag{105}$$

And the same for $c_2$

$$\int_{c_2} (\frac{dw}{dz})^2 dz = \int_{c_2} \left( U_\infty (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi} ( \frac{1}{z - z_0} - \cancel{\frac{1}{z - \bar{z}_0}} - \frac{1}{z - \frac{a^2}{\bar{z}_0}} + \frac{1}{z - \frac{a^2}{z_0}} ) \right)^2 dz \tag{106}$$

We can easily see that both integrals won't lead to any terms such as $\frac{1}{z - z_0}$ or $\frac{1}{z - \bar{z}_0}$ respectively, due to the self-induced velocities being neglected. From that according to Cauchy's Residue Theorem as shown in 93, both integrals would be zero:

$$\int_{c_1} (\frac{dw}{dz})^2 dz = \int_{c_2} (\frac{dw}{dz})^2 dz = 0 \tag{107}$$

Now, going back to 95 after calculating all the needed integrals, we conclude that:

$$\int_{c_3} (\frac{dw}{dz})^2 dz = \int_c (\frac{dw}{dz})^2 dz - \int_{c_1} (\frac{dw}{dz})^2 dz - \int_{c_2} (\frac{dw}{dz})^2 dz = 0 \tag{108}$$

And now, going back to Blasius's theorem in 91, we get:

$$X - iY = \frac{1}{2} i\rho \int_C (\frac{dw}{dz})^2 dz = 0 \tag{109}$$

And we get no forces acting on the cylinder.

## C.2

In our case of potential flow, no force in the X direction (drag) was computed. As expected, we also computed no forces in the Y direction (lift) since the problem is symmetric around the X axis. Comparing the results to results from the literature, as shown below, we see that the values for $C_L$ and $C_D$ differ. A common trend of the coefficient nearing zero as the Re number increases can be seen. This trend follows the same results from B.1. As the Reynolds number increases, the flow approaches the ideal potential flow. Both $C_L$ and $C_D$ never reach zero identically due to the Re number never being $\infty$ as in the theoretical case we computed in C.1.

## C.3 Bonus

### C.3.1

Looking at the problem in which the lower vortex moves with a velocity of $0.5U_\infty$, we can take two different approaches to solving the problem:

1. The first is based on a purely physical observation of the problem: as the lower vortex moves further from the cylinder, its effect on the cylinder decreases. The speed at the lower part of the aft of the cylinder decreases, and with Bernoulli's principle, the pressure increases, and we get a net pressure difference and a force in the positive Y direction.

2. The second perspective is purely mathematical based on C.1. Looking at the integrals for $C_1$ and $C_2$ at 105, 106, we can see that the net force isn't a function of $z_0$ for every single point at the movement of the lower vortex, the integrals would've still given a net force of zero. Also not required, we can easily prove this mathematical concept using the force from a single vortex outside the cylinder (mirrored using the circular theorem) and the principle of superposition. For a single vortex at $z_0$, the complex velocity would be:

$$\frac{dw_0}{dz} = (1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi}(\frac{1}{z - z_0} + \frac{1}{z - \frac{a^2}{z_0}}) \tag{110}$$

   With a similar process as shown in A.1. Now doing the same process as shown in C.1, the integral around the vortex itself would be zero due to the neglected self-induced velocity. Looking at the far field, we can derive a Laurent series similarly to 97:

$$\int_{c_3} \left( U_\infty(1 - \frac{a^2}{z^2}) + \frac{i\Gamma}{2\pi}(\frac{1}{z} + \frac{z_0}{z} + \frac{1}{z} + \frac{\frac{a^2}{z_0}}{z^2}) \right)^2 dz \tag{111}$$

We can see that terms for $\frac{1}{z}$ will come out of the integral with a coefficient such as $C(\Gamma, U_\infty, a)\frac{1}{z}$, thus a force in the Y direction will emerge with **no relation to $z_0$, the location of the vortex**. Using the principle of superposition, any combination of two vortices with opposite strengths will sum up to a net zero force no matter their location in the space (either still or moving).

We got two contradictory results: the physical perspective gives a net force in the Y direction, but the purely mathematical perspective gives a net force of **zero**. So, looking purely at the integrals from C.1, the problem would still provide a net force of zero acting on the cylinder.

### C.3.2

Adding a vortex at the origin with a strength of $\Gamma$ will add a force in the Y direction: $F = \rho U_\infty \Gamma$. In potential flow problems, we can use the principle of superposition to add different problems together, now combining the problem from C.1 and the simple case of a rotating cylinder:

$$F = X - iY = F_1 + F_2 = 0 + 0i + \rho U_\infty \Gamma + 0i = \rho U_\infty \Gamma \tag{112}$$

A force only in the Y direction of $F = \rho U_\infty \Gamma$.