Borcherds products and Lie superalgebras

Kaiwen Sun (USTC)

Workshop "Representation Theory on Ice", Luleå

February 4, 2026

Joint works with Haowu Wang and Brandon Williams

Monstrous moonshine

- **Observation (McKay–Thompson, 1978):** The Fourier coefficients of the Klein j–function

$$j(\tau) = q^{-1} + 744 + 196884\, q + 21493760\, q^2 + \cdots$$

  decompose into sums of dimensions of irreducible representations of the *Monster group* $\mathbb{M}$:

$$1,\ 196883,\ 21296876, \ldots$$

- **Moonshine Conjecture (Conway–Norton):** For each $g \in \mathbb{M}$, there exists a genus–zero modular function (McKay–Thompson series) whose q–expansion encodes the character of $g \in \mathbb{M}$.
- **Vertex Operator Algebra (Frenkel–Lepowsky–Meurman):** Construction of the Moonshine module $V^\natural$, a holomorphic VOA with

$$\mathrm{Aut}(V^\natural) = \mathbb{M}, \qquad c = 24.$$

- **Borcherds' Theorem (1992):** Proof of the Moonshine conjecture.

$c = 24$ holomorphic VOA

Classification of $c = 24$ holomorphic VOA

In 1993, Schellenkens conjectured there are in total 71 of them:

- Monster VOA/Moonshine module $V^\natural$
- Leech lattice VOA
- 69 cases with affine Kac-Moody algebras as V_1

Proved by many works of (Höhn, Scheithauer, Möller, van Ekeren, Dong, Mason, Lam, Shimakura, Lin, Kawasetsu ...)

$c = 12$ holomorphic VOSA

Classification of $c = 12$ holomorphic VOSA

In 2018, Creutzig–Duncan–Riedler proved the classification as

- supersymmetric E_8 lattice VOA (Scheithauer 00)
- Conway VOSA (Duncan 15)
- F_{24}: 24 fermions

In 2020, Harrison-Paquette-Persson-Volpato find F_{24} allows 8 affine Kac-Moody algebras as $V_{1/2}$:

$$A_{1,2}^8,\ A_{2,3}^3,\ A_{4,5},\ A_{3,4}A_{1,2}^3,\ B_{2,3}G_{2,4},\ B_{2,3}A_{2,3}A_{1,2}^2,\ B_{3,5}A_{1,2},\ C_{3,4}A_{1,2}.$$

$c = 24$ $\mathbb{Z}$-graded holomorphic VOSA

In 2023 in Heidelberg I heard the following open question from Creutzig

Open question

Does there exist a $c = 24$ $\mathbb{Z}$-graded holomorphic VOSA?

$c = 24$ $\mathbb{Z}$-graded holomorphic VOSA

In 2023 in Heidelberg I heard the following open question from Creutzig

Open question

Does there exist a $c = 24$ $\mathbb{Z}$-graded holomorphic VOSA?

So far there is only one result on this question.

Theorem (van Ekeren-Morales 23)

If it exists, the V_1 space must be one of 1227 Lie superalgebras.

$c = 24$ $\mathbb{Z}$-graded holomorphic VOSA

In 2023 in Heidelberg I heard the following open question from Creutzig

Open question

Does there exist a $c = 24$ $\mathbb{Z}$-graded holomorphic VOSA?

So far there is only one result on this question.

Theorem (van Ekeren-Morales 23)

If it exists, the V_1 space must be one of 1227 Lie superalgebras.

However, till now no example is constructed.

Classification of hyperbolization of affine Lie (super)algebra

These VOAs and VOSAs are closely related to an open problem in number theory that is to classify the Borcherds product of singular weight. From algbraic viewpoint, this is to classify the hyperbolization of affine Lie (super)algebras.

Classification (KS-Wang-Williams 23, 26)

	affine Lie algebra	affine Lie superalgebra
symmetric	$8+4$	4
anti-symmetric	69	$0?$

Conjecture

There exists no $c = 24$ $\mathbb{Z}$-graded holomorphic VOSA.

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Recall Jacobi symbol

$$\left(\begin{array}{c}-4\\ m\end{array}\right)=\begin{cases}\pm 1, & m\equiv \pm 1 \mod 4,\\ 0, & m\equiv 0 \mod 2,\end{cases}$$

and Jacobi theta function

$$\theta_1(\tau,z)=\sum_{m\in\mathbb{Z}}\left(\begin{array}{c}-4\\ m\end{array}\right)q^{m^2/8}r^{m/2}.$$

(Gritsenko-Nikulin 98) defined the Siegel theta constant

$$\Delta_{1/2}(Z)=\frac{1}{2}\sum_{m,n\in\mathbb{Z}}\left(\begin{array}{c}-4\\ m\end{array}\right)\left(\begin{array}{c}-4\\ n\end{array}\right)q^{m^2/8}r^{mn/2}s^{n^2/8},\quad Z=\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix},$$

and proved this is a weight-$1/2$ automorphic form for paramodular group Γ_4^+.

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Moreover, (Gritsenko-Nikulin 98) show that $\Delta_{1/2}(Z)$ is a reflective Borcherds product of singular weight

$$\Delta_{1/2}(Z) = \mathbf{B}(\phi) := q^A r^B s^C \prod_{\substack{n,l,m\in\mathbb{Z}\\ (n,l,m)>0}} (1 - q^n r^l s^m)^{f(nm,l)},$$

where $f(-,-)$ are the Fourier coefficients of weight-0 Jacobi form ϕ:

$$\phi(\tau, z) := \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \sum_{n,l\in\mathbb{Z}} f(n,l) q^n r^l,$$

and

$$A = \frac{1}{24}\sum_{l} f(0,l) = \frac{1}{8}, \quad B = \frac{1}{2}\sum_{l>0} l f(0,l) = \frac{1}{2}, \quad C = \frac{1}{4}\sum_{l} l^2 f(0,l) = \frac{1}{8}.$$

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

$\Delta_{1/2}(Z)$ is also a (trivial) Gritsenko additive lift of the A_1 theta block $\theta_{A_1} = \theta_1$:

$$\Delta_{1/2}(Z) = \mathbf{G}(\theta_1)(\tau, z, \omega) = \sum_{m=1}^{\infty} \left(\begin{array}{c} -4 \\ m \end{array} \right) \theta_1(\tau, mz) e^{2\pi i m^2 \omega/8}.$$

In fact, $-\theta_1(\tau, 3z)$ can be defined as a Hecke image of the theta block $\theta_1(\tau, z)$ such that

$$\phi = -\frac{\theta_{A_1}|T}{\theta_{A_1}}.$$

Gritsenko-Nikulin's $D_{1/2}(Z)$

(Gritsenko-Nikulin 98) also observed a quintiple product formula

$$\vartheta_{3/2}(\tau,z) := \eta(\tau)\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)} = \sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right) q^{\frac{n^2}{24}}\zeta^{\frac{n}{2}}$$

$$= q^{\frac{1}{24}}(\zeta^{\frac{1}{2}}+\zeta^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1+q^n\zeta)(1+q^n\zeta^{-1})(1-q^{2n-1}\zeta^2)(1-q^{2n-1}\zeta^{-2})(1-q^n).$$

Using it they constructed an exotic modular form as the additive lift

$$D_{1/2}(Z) := \mathbf{G}(\vartheta_{3/2})(\omega,z,\tau) = \sum_{m=1}^{\infty}\left(\frac{12}{m}\right)\vartheta_{3/2}(\tau,mz)\cdot e^{2\pi i m^2\omega/24},$$

This is a modular form of singular weight on $2U\oplus A_1(36)$ which can be realized as a Siegel paramodular form of Γ_{36}^*. They also proved this is a Borcherds product $\mathbf{B}(\phi_{0,36})$ by lifting

$$\phi_{0,36}(\tau,z) := \frac{\vartheta_{3/2}(\tau,5z)}{\vartheta_{3/2}(\tau,z)} = \frac{\theta_1(\tau,10z)\theta_1(\tau,z)}{\theta_1(\tau,5z)\theta_1(\tau,2z)} = (\zeta^2+\zeta^{-2})-(\zeta+\zeta^{-1})+1+O(q).$$

Oberservation

Number theory and algebra

- $\Delta_{1/2}$ is the denominator of a Borcherds-Kac-Moody algebra as hyperbolization of affine Lie algebra $A_{1,16}$.
- $D_{1/2}$ is the denominator of a Borcherds-Kac-Moody algebra as hyperbolization of affine Lie superalgebra $osp(1|2)_{36}$.

Note $(\zeta^2 + \zeta^{-2}) - (\zeta + \zeta^{-1}) + 1$ is the supercharacter of $osp(1|2)$.

Main question

Elliptic Semi-simple Lie algebras – Classical symmetries

Parabolic Affine Kac-Moody algebras – 2d Wess-Zumino-Witten CFTs

Hyperbolic Borcherds-Kac-Moody algebras – Algebra of BPS states (Harvey-Moore 95,96)

Question (Feingold-Frenkel 83, Gritsenko 12, Gritsenko-Wang 19,20...)

What kinds of affine Kac-Moody algebras allow hyperbolization?

This is to classify the reflective Borcherds products $F(\omega, \mathfrak{z}, \tau)$ of singular weight $\frac{1}{2}\mathrm{rk}(L)$ on $2U \oplus L$, where U is hyperbolic plane and L is a positive definite even lattice.

Main results

We give a complete classification of hyperbolization of affine Kac-Moody algebras

Main theorem (KS-Wang-Williams 23)

There are precisely 81 such affine Kac-Moody algebras:

1. 69 cases associated to Schellenkens' list of $c = 24$ holomorphic CFT/VOAs
2. 8 cases associated to the $c = 12$ holomorphic SCFT/self-dual SVOAs
3. 4 exotic cases $A_{1,16}, A_{2,9}, A_{1,8}^2, A_{1,4}^4$ associated to exceptional modular invariants from nontrivial automorphism of fusion algebras

Remark

Depending on $F(\omega, \mathfrak{z}, \tau)$ anti-invariant/invariant under the involution $(\omega, \mathfrak{z}, \tau) \mapsto (\tau, \mathfrak{z}, \omega)$, the class 1 is called antisymmetric and was also recently studied by (Driscoll-Spittler, Scheithauer, Wilhelm, 2312.02768) from different viewpoint. The class 2 and 3 are called symmetric.

Main idea

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\text{Affine Lie algebra} \arrow[r] \arrow[d, "\text{Denominator}"] \arrow[dr, "\text{Character}"] \arrow[rr, bend left, "\textcolor{red}{\text{Hyperbolization}}"] & \text{Holomorphic VOA} \arrow[r, "\text{BRST}"] \arrow[d, "\text{Character}"] & \text{BKM algbera} \arrow[d, "\text{Denominator}"] \\
\text{Theta block} \arrow[r, "\text{Hecke}"] & \textcolor{red}{\text{Weight } 0 \text{ Jacobi form}} \arrow[r, "\text{Lift}"] & \text{Borcherds product} \arrow[ll, bend left, "\text{Leading Fourier-Jacobi coefficient}"]
\end{tikzcd}
$$

Hyperbolization of affine Kac-Moody algebras

Definition

Let $\mathfrak{g}$ be an affine Kac–Moody algebra. A BKM algebra $\mathbf{g}$ is called a hyperbolization of $\mathfrak{g}$ if there exists an even positive definite lattice L such that the root lattice of $\mathbf{g}$ is $U \oplus L$ and the denominator of $\mathbf{g}$ defines a Borcherds product F of weight $\mathrm{rk}(L)/2$ on $2U \oplus L$ whose leading Fourier–Jacobi coefficient coincides with the denominator $\theta_{\mathfrak{g}}$ of $\mathfrak{g}$. We also call F the hyperbolization of $\theta_{\mathfrak{g}}$.

Hyperbolization of affine Kac-Moody algebras

Theorem (KS-Wang-Williams 23)

Suppose that there is a reflective Borcherds product F of singular weight on $2U \oplus L$ whose Jacobi form input has non-negative q^0-term. Then (a) or (b) below holds.

(a) *L is the Leech lattice and F is the denominator Φ_{12} of the fake monster algbera.*

(b) *There exists a semi-simple Lie algebra $\mathfrak{g} = \oplus_{j=1}^{s} \mathfrak{g}_{j,k_j}$ of rank $\mathrm{rk}(L)$ satisfying restrictions (1) or (1'). Let $\lambda \in U$ with $\lambda^2 = 2$.*

  (1) *If F vanishes on $\lambda^\perp$, then*

$$C := \frac{\dim \mathfrak{g}}{24} - 1 = \frac{h_j^\vee}{k_j}, \quad \text{for } 1 \le j \le s. \tag{1}$$

  (1') *If F does not vanish on $\lambda^\perp$, then*

$$C := \frac{\dim \mathfrak{g}}{24} = \frac{h_j^\vee}{k_j} \quad \text{and} \quad k_j > 1, \quad \text{for } 1 \le j \le s. \tag{2}$$

Necessary conditions for hyperbolization of $\bigoplus \mathfrak{g}_{i,k_i}$

Antisymmetric Constraints for $\bigoplus \mathfrak{g}_{i,k_i}$: Schellekens' first trace identity!

$$\frac{1}{24}\sum_i \dim(\mathfrak{g}_i) - 1 = C = \frac{h_i^\vee}{k_i}.$$

The VOA central charge $c = 24$. The weight 0 Jacobi form

$$\phi_{\mathrm{Borch}}|_{m_{\mathfrak{g}}\to 0} = J(\tau) + N = q^{-1} + N + 196884q + \dots.$$

Symmetric Constraints for $\bigoplus \mathfrak{g}_{i,k_i}$:

$$\frac{1}{24}\sum_i \dim(\mathfrak{g}_i) = C = \frac{h_i^\vee}{k_i}, \qquad k_i > 1.$$

The VOA central charge $c = 24C/(C+1)$ and

$$\phi_{\mathrm{Borch}}|_{m_{\mathfrak{g}}\to 0} = \mathrm{const}.$$

Antisymmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 69 out of 221

C	$\mathfrak{g}$
1/24	$A_{1,48}^{3}A_{2,72}^{2}$
1/24	$A_{1,48}^{5}B_{2,72}$
1/24	$A_{1,48}A_{2,72}G_{2,96}$
1/24	$A_{3,96}B_{2,72}$
1/12	$A_{1,24}^{6}A_{2,36}$
1/12	$A_{1,24}^{4}G_{2,48}$
1/12	$A_{1,24}^{2}B_{2,36}^{2}$
1/12	$A_{2,36}^{2}B_{2,36}$
1/12	$A_{1,24}A_{2,36}A_{3,48}$
1/8	$A_{1,16}B_{2,24}G_{2,32}$
1/8	$A_{1,16}A_{2,24}^{3}$
1/8	$A_{1,16}^{2}C_{3,32}$
1/8	$A_{1,16}^{3}A_{2,24}B_{2,24}$
1/8	$A_{1,16}^{9}$
1/8	$A_{1,16}^{4}A_{3,32}$
1/8	$A_{1,16}^{2}B_{3,40}$
1/8	$A_{1,16}A_{4,40}$
1/6	$A_{1,12}^{2}A_{2,18}G_{2,24}$
1/6	$A_{1,12}^{6}B_{2,18}$
1/6	$A_{1,12}^{4}A_{2,18}^{2}$
1/6	$D_{4,36}$
1/6	$G_{2,24}^{2}$
1/6	$A_{1,12}A_{3,24}B_{2,18}$
1/6	$A_{2,18}B_{2,18}^{2}$
1/4	$A_{1,8}^{3}C_{3,16}$
1/4	$A_{1,8}^{10}$
1/4	$A_{1,8}^{2}A_{2,12}^{3}$
1/4	$A_{1,8}^{5}A_{3,16}$
1/4	$A_{1,8}^{2}B_{2,12}G_{2,16}$
1/4	$A_{1,8}^{3}B_{3,20}$
1/4	$A_{3,16}^{2}$
1/4	$A_{1,8}^{2}A_{4,20}$
1/4	$A_{1,8}^{4}A_{2,12}B_{2,12}$
1/4	$A_{2,12}^{2}G_{2,16}$
1/4	$B_{2,12}^{3}$
1/3	$A_{2,9}B_{2,9}G_{2,12}$
1/3	$A_{1,6}^{8}A_{2,9}$
1/3	$A_{1,6}A_{2,9}C_{3,12}$
1/3	$A_{2,9}A_{4,15}$
1/3	$A_{1,6}A_{2,9}B_{3,15}$
1/3	$A_{1,6}^{6}G_{2,12}$
1/3	$A_{1,6}^{2}A_{2,9}^{2}B_{2,9}$
1/3	$A_{1,6}^{4}B_{2,9}^{2}$
1/3	$A_{1,6}^{3}A_{2,9}A_{3,12}$
1/3	$A_{2,9}^{4}$
1/3	$A_{1,6}A_{3,12}G_{2,12}$
1/2	$A_{2,6}G_{2,8}^{2}$
1/2	$A_{1,4}^{2}A_{3,8}^{3}$
1/2	$A_{1,4}^{4}A_{2,6}^{3}$
1/2	$A_{1,4}^{4}A_{4,10}$
1/2	$A_{1,4}A_{2,6}A_{3,8}B_{2,6}$
1/2	$A_{1,4}^{7}A_{3,8}$
1/2	$C_{4,10}$
1/2	$A_{1,4}^{5}B_{3,10}$
1/2	$A_{2,6}^{2}B_{2,6}^{2}$
1/2	$A_{3,8}B_{3,10}$
1/2	$A_{1,4}^{6}A_{2,6}B_{2,6}$
1/2	$A_{1,4}^{2}A_{2,6}^{2}G_{2,8}$
1/2	$B_{4,14}$
1/2	$A_{1,4}^{12}$
1/2	$A_{1,4}^{4}B_{2,6}G_{2,8}$
1/2	$A_{2,6}D_{4,12}$
1/2	$A_{1,4}^{5}C_{3,8}$
1/2	$A_{1,4}^{2}B_{2,6}^{3}$
1/2	$A_{3,8}C_{3,8}$
2/3	$A_{1,3}^{4}G_{2,6}^{2}$
2/3	$A_{1,3}^{4}D_{4,9}$
3/4	$A_{2,4}^{4}B_{2,4}$
1	$A_{1,2}A_{2,3}^{3}B_{3,5}$
1	$A_{1,2}^{3}A_{2,3}B_{2,3}C_{3,4}$
1	$A_{1,2}A_{3,4}B_{2,3}^{3}$
1	$A_{1,2}A_{4,5}C_{3,4}$
1	$A_{6,7}$
1	$A_{1,2}^{4}A_{2,3}G_{2,4}^{2}$
1	$A_{1,2}^{4}B_{4,7}$
1	$A_{1,2}^{16}$
1	$A_{1,2}A_{5,6}B_{2,3}$
1	$A_{1,2}D_{5,8}$
1	$A_{1,2}^{6}A_{2,3}^{2}G_{2,4}$
1	$A_{1,2}^{2}A_{2,3}B_{2,3}^{2}G_{2,4}$
1	$A_{1,2}^{9}B_{3,5}$
1	$B_{2,3}^{2}D_{4,6}$
1	$A_{1,2}B_{2,3}B_{3,5}G_{2,4}$
1	$A_{2,3}^{3}A_{4,5}$
1	$A_{1,2}A_{4,5}B_{3,5}$
1	$A_{1,2}^{2}A_{2,3}A_{4,5}B_{2,3}$
1	$A_{1,2}^{2}B_{3,5}^{2}$
1	$A_{1,2}B_{2,3}C_{3,4}G_{2,4}$
1	$A_{1,2}^{3}A_{2,3}^{3}A_{3,4}$
1	$A_{1,2}A_{3,4}^{3}$
1	$A_{1,2}^{8}A_{2,3}^{3}$
1	$A_{2,3}^{6}$
1	$A_{1,2}^{11}A_{3,4}$
1	$A_{1,2}^{5}A_{2,3}A_{3,4}B_{2,3}$
1	$A_{1,2}^{8}B_{2,3}G_{2,4}$
1	$A_{1,2}A_{2,3}^{3}C_{3,4}$
1	$A_{2,3}^{3}B_{2,3}G_{2,4}$
1	$A_{1,2}^{4}A_{2,3}D_{4,6}$
1	$A_{1,2}^{4}A_{3,4}C_{3,4}$
1	$A_{1,2}^{3}A_{3,4}A_{4,5}$
1	$A_{1,2}^{2}G_{2,4}^{3}$
1	$A_{1,2}^{2}C_{3,4}^{2}$
1	$A_{1,2}^{6}A_{3,4}^{2}$
1	$A_{2,3}A_{3,4}^{2}B_{2,3}$
1	$A_{1,2}^{2}A_{2,3}^{4}B_{2,3}$
1	$A_{1,2}^{2}B_{3,5}C_{3,4}$
1	$A_{1,2}^{8}A_{4,5}$
1	$A_{4,5}^{2}$
1	$A_{1,2}^{4}A_{2,3}^{2}B_{2,3}^{2}$
1	$A_{2,3}B_{2,3}^{4}$
1	$B_{2,3}^{2}G_{2,4}^{2}$
1	$A_{1,2}^{9}C_{3,4}$
1	$A_{1,2}^{4}A_{3,4}B_{3,5}$
1	$A_{4,5}B_{2,3}G_{2,4}$
1	$A_{1,2}A_{2,3}^{2}A_{3,4}G_{2,4}$
1	$A_{1,2}^{4}C_{4,5}$
1	$A_{1,2}^{3}A_{2,3}B_{2,3}B_{3,5}$
1	$A_{1,2}^{10}A_{2,3}B_{2,3}$
1	$A_{1,2}^{3}A_{3,4}B_{2,3}G_{2,4}$
1	$A_{1,2}^{2}D_{4,6}G_{2,4}$
1	$A_{1,2}^{6}B_{2,3}^{3}$
4/3	$C_{3,3}^{2}G_{2,3}$
4/3	$G_{2,3}^{4}$

TABLE 3. The 221 solutions of equation (4.1) in the order of increasing C. Those allowing hyperbolization are colored blue (continued on next page).

C	$\mathfrak{g}$
3/2	$A_{2,2}^{5}B_{2,2}^{2}$
3/2	$A_{2,2}^{4}D_{4,4}$
3/2	$B_{2,2}^{6}$
3/2	$A_{2,2}F_{4,6}$
2	$A_{1,1}^{2}C_{3,2}D_{5,4}$
2	$A_{1,1}^{10}C_{3,2}^{2}$
2	$A_{1,1}^{5}A_{3,2}C_{3,2}^{2}$
2	$A_{1,1}^{9}D_{5,4}$
2	$A_{3,2}^{2}G_{2,2}^{3}$
2	$A_{1,1}^{14}A_{3,2}^{2}$
2	$A_{1,1}^{10}G_{2,2}^{3}$
2	$A_{1,1}^{19}A_{3,2}$
2	$A_{1,1}^{5}A_{3,2}G_{2,2}^{3}$
2	$A_{1,1}^{3}A_{5,3}D_{4,3}$
2	$A_{1,1}^{9}A_{3,2}^{3}$
2	$A_{1,1}^{24}$
2	$A_{1,1}^{3}C_{3,2}D_{4,3}G_{2,2}$
2	$A_{1,1}^{7}A_{3,2}^{2}C_{3,2}$
2	$A_{1,1}^{3}A_{7,4}$
2	$A_{1,1}^{2}D_{6,5}$
2	$A_{1,1}^{5}A_{3,2}D_{4,3}G_{2,2}$
2	$A_{1,1}^{3}C_{3,2}G_{2,2}^{3}$
2	$A_{1,1}^{3}C_{3,2}^{3}$
2	$A_{3,2}^{2}D_{4,3}G_{2,2}$
2	$A_{1,1}^{4}A_{3,2}^{4}$
2	$A_{1,1}^{17}C_{3,2}$
2	$A_{1,1}C_{5,3}G_{2,2}$
2	$A_{3,2}^{2}C_{3,2}^{2}$
2	$A_{1,1}^{2}A_{3,2}^{3}C_{3,2}$
2	$A_{1,1}^{3}A_{5,3}G_{2,2}^{2}$
2	$A_{1,1}^{12}A_{3,2}C_{3,2}$
2	$A_{1,1}^{10}D_{4,3}G_{2,2}$
2	$A_{1,1}^{4}A_{3,2}D_{5,4}$
5/2	$A_{4,2}^{2}C_{4,2}$
5/2	$B_{3,2}^{4}$
3	$A_{2,1}^{2}A_{8,3}$
3	$A_{2,1}B_{2,1}E_{6,4}$
3	$A_{2,1}^{5}D_{4,2}^{2}$
3	$A_{2,1}^{2}B_{2,1}^{8}$
3	$A_{2,1}^{7}B_{2,1}^{4}$
3	$A_{2,1}^{6}B_{2,1}^{2}D_{4,2}$
3	$A_{2,1}^{12}$
3	$A_{2,1}^{2}D_{4,2}F_{4,3}$
3	$A_{2,1}^{3}B_{2,1}^{2}F_{4,3}$
3	$B_{2,1}^{4}D_{4,2}^{2}$
3	$A_{2,1}^{2}A_{5,2}^{2}B_{2,1}$
3	$A_{2,1}B_{2,1}^{6}D_{4,2}$
7/2	$B_{4,2}^{3}$
4	$A_{3,1}C_{7,2}$
4	$A_{3,1}^{5}D_{5,2}$
4	$A_{3,1}C_{3,1}^{5}$
4	$A_{3,1}^{2}D_{5,2}^{2}$
4	$A_{3,1}A_{7,2}C_{3,1}^{2}$
4	$C_{3,1}^{2}E_{6,3}$
4	$A_{3,1}^{8}$
4	$A_{3,1}C_{3,1}G_{2,1}^{6}$
4	$A_{3,1}A_{7,2}G_{2,1}^{3}$
4	$A_{3,1}D_{7,3}G_{2,1}$
4	$A_{3,1}C_{3,1}^{3}G_{2,1}^{3}$
4	$E_{6,3}G_{2,1}^{3}$
9/2	$A_{8,2}F_{4,2}$
5	$A_{4,1}B_{3,1}^{4}C_{4,1}$
5	$A_{4,1}^{6}$
5	$A_{4,1}^{3}C_{4,1}^{2}$
5	$A_{4,1}A_{9,2}B_{3,1}$
5	$C_{4,1}^{4}$
5	$B_{3,1}^{2}C_{4,1}D_{6,2}$
11/2	$B_{6,2}^{2}$
6	$D_{4,1}^{6}$
6	$A_{5,1}C_{5,1}E_{6,2}$
6	$A_{5,1}E_{7,3}$
6	$A_{5,1}^{4}D_{4,1}$
7	$B_{4,1}^{2}D_{8,2}$
7	$B_{4,1}C_{6,1}^{2}$
7	$A_{6,1}^{4}$
7	$A_{6,1}B_{4,1}^{4}$
8	$A_{7,1}D_{9,2}$
8	$A_{7,1}^{2}D_{5,1}^{2}$
9	$C_{8,1}F_{4,1}^{2}$
9	$B_{5,1}E_{7,2}F_{4,1}$
9	$A_{8,1}^{3}$
10	$D_{6,1}^{4}$
10	$A_{9,1}^{2}D_{6,1}$
11	$B_{6,1}C_{10,1}$
23/2	$B_{12,2}$
12	$E_{6,1}^{4}$
12	$A_{11,1}D_{7,1}E_{6,1}$
13	$A_{12,1}^{2}$
14	$D_{8,1}^{3}$
15	$B_{8,1}E_{8,2}$
16	$A_{15,1}D_{9,1}$
18	$D_{10,1}E_{7,1}^{2}$
18	$A_{17,1}E_{7,1}$
22	$D_{12,1}^{2}$
25	$A_{24,1}$
30	$E_{8,1}^{3}$
30	$D_{16,1}E_{8,1}$
46	$D_{24,1}$

TABLE 3. (continued).

Schellenkens' list (Schellenkens 93)

There are in total 71 holomorphic VOAs of $c = 24$.

1. Monster VOA, $N = 0$
2. Leech lattice VOA, $N = 24$
3. 69 cases with affine Kac-Moody structures, $N \geq 36$

- $N = 0$ case $\to$ Monster Lie algebra
- $N = 24$ case $\to$ fake Monster Lie algebra
- The 69 cases corresponding to 11 conjugacy classes of Conway group Co_0, 8 of which have been constructed to associate to BKM algebras (Höhn, Scheithauer, Gritsenko, Lam, Möller...)
- For existence part, we only need to deal with the rest 3 conjugacy classes.
- For the non-existence part, we use the forbidden component technique in the theory of Borcherds products.

A simple example: $B_{12,2}$, Schellekens' list No.57

The holomorphic CFT character can be expressed by affine characters as

$$\chi_V = \chi_{0,0}^{B_{12,2}} + \chi_{w_1+w_{12},2}^{B_{12,2}} + \chi_{w_{10},3}^{B_{12,2}} + \chi_{w_5,2}^{B_{12,2}}.$$

Decompose the reps into Weyl orbits with norm defined by $(,)_{B_{12}}/2$. Then $\chi_V = q^{-1} + (O_{w_2,1} + O_{w_1,\frac{1}{2}} + 12) + \sum_{i=1}^{\infty} c_i q^i$. We calculate

$$\begin{aligned} c_1 = & O_{2w_2,4} + O_{w_1+w_3,3} + O_{w_5,\frac{5}{2}} + O_{w_1+w_{12},\frac{5}{2}} + O_{w_1+w_2,\frac{5}{2}} + 4O_{w_4,2} \\ & + 12O_{2w_1,2} + 12O_{w_3,\frac{3}{2}} + 12O_{w_{12},\frac{3}{2}} + 44O_{w_2,1} + 90O_{w_1,\frac{1}{2}} + 300. \end{aligned}$$

Clearly all orbits in c_1 with norm > 2 have coefficients 1.

$$\begin{aligned} c_2 = & O_{w_{10},5} + O_{2w_0+w_1,5} + O_{w_2+w_4,5} + O_{w_9,\frac{9}{2}} + O_{3w_1,\frac{9}{2}} + O_{w_2+w_3,\frac{9}{2}} + O_{2w_1+w_{12},\frac{9}{2}} \\ & + O_{w_3+w_{12},\frac{9}{2}} + O_{w_1+w_6,\frac{9}{2}} + 4O_{w_8,4} + 12O_{2w_2,4} + 4O_{w_1+w_5,4} + 12O_{w_7,\frac{7}{2}} \\ & + 12O_{w_2+w_{12},\frac{7}{2}} + 12O_{w_1+w_4,\frac{7}{2}} + 32O_{w_6,3} + 44O_{w_1+w_3,3} + 90O_{w_5,\frac{5}{2}} \\ & + 90O_{w_1+w_{12},\frac{5}{2}} + 90O_{w_1+w_2,\frac{5}{2}} + 224O_{w_4,2} + 288O_{2w_1,2} \\ & + 520O_{w_{12},\frac{3}{2}} + 520O_{w_3,\frac{3}{2}} + 1242O_{w_2,1} + 2535O_{w_1,\frac{1}{2}} + 5792. \end{aligned}$$

All orbits with norm > 4 in c_2 have coefficients 1. This implies singular weight.

Antisymmetric case

After laborious works in lattice theory and Borcherds products for all 221 solutions, we prove surprisingly

The affine Lie algebras allowing antisymmetric hyperbolization are one to one corresponding to the 69 affine structures in Schellenkens' list and

$$\phi_{\mathrm{Borch}} = \chi_V = q^{-1} + \sum_{\alpha \in \Delta_{\mathfrak{g}}} e^{2\pi i \langle \alpha, \mathfrak{z} \rangle} + \mathrm{rk}(\mathfrak{g}) + O(q).$$

Theorem (KS-Wang-Williams 23)

Let V be a holomorphic VOA of central charge 24 with semi-simple $V_1 = \mathfrak{g}$. Let $L_{\mathfrak{g}}$ denote Höhn's orbit lattice of $\mathfrak{g}$ and χ_V denote the full character of V. Then the $\mathbf{B}(\chi_V)$ of χ_V is a reflective Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. Moreover, the leading Fourier–Jacobi coefficient of $\mathbf{B}(\chi_V)$ coincides with the denominator of the affine Kac–Moody algebra $\mathfrak{g}$.

Symmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 12 out of 17

C	$\mathfrak{g}$
$1/8$	$A_{1,16}$
$1/4$	$A_{1,8}^2$
$1/3$	$A_{2,9}$
$1/2$	$A_{1,4}^4$
$3/4$	$A_{2,4}B_{2,4}$
1	$A_{1,2}^8$

C	$\mathfrak{g}$
1	$A_{2,3}^3$
1	$A_{4,5}$
1	$A_{3,4}A_{1,2}^3$
1	$B_{2,3}G_{2,4}$
1	$B_{2,3}A_{2,3}A_{1,2}^2$
1	$B_{3,5}A_{1,2}$

C	$\mathfrak{g}$
1	$C_{3,4}A_{1,2}$
$3/2$	$A_{2,2}D_{4,4}$
$3/2$	$A_{2,2}^2B_{2,2}^2$
$5/2$	$A_{4,2}C_{4,2}$
$7/2$	$A_{6,2}B_{4,2}$

Main Theorem

The affine Lie algebras allowing symmetric hyperbolization are

- $C = 1$ Eight affine structures in F_{24} holomorphic VOSA of $c = 12$
- $C < 1$ Four exotic cases related to exceptional modular invariants

An interesting coincidence

8 special affine Lie algebras appeared in math and physics around the same time

Corollary (Corollary 7.6). *The following infinite series of pure theta blocks of q-order 1 satisfy the theta block conjecture.*

weight	*root system*	*theta block*
2	A_4	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_c\vartheta_d\vartheta_{a+b}\vartheta_{b+c}\vartheta_{c+d}\vartheta_{a+b+c}\vartheta_{b+c+d}\vartheta_{a+b+c+d}$
	$A_1 \oplus B_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{b+c}\vartheta_{b+2c+2d}\vartheta_{b+c+d}\vartheta_{b+c+2d}\vartheta_c\vartheta_{c+d}\vartheta_{c+2d}\vartheta_d$
	$A_1 \oplus C_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{2b+2c+d}\vartheta_{b+c}\vartheta_{b+2c+d}\vartheta_{b+c+d}\vartheta_c\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
	$B_2 \oplus G_2$	$\eta^{-6}\vartheta_a\vartheta_{a+b}\vartheta_{a+2b}\vartheta_b\vartheta_c\vartheta_{3c+d}\vartheta_{3c+2d}\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
3	$3A_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_1+b_1}\vartheta_{b_1}\vartheta_{a_2}\vartheta_{a_2+b_2}\vartheta_{b_2}\vartheta_{a_3}\vartheta_{a_3+b_3}\vartheta_{b_3}$
	$3A_1 \oplus A_3$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_4+a_5}\vartheta_{a_5+a_6}\vartheta_{a_4+a_5+a_6}$
	$2A_1 \oplus A_2 \oplus B_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_3+a_4}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_5+a_6}\vartheta_{a_5+2a_6}\vartheta_{a_6}$
4	$8A_1$	$\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_7}\vartheta_{a_8}$

Figure: Dittmann, Wang, *Theta blocks related to root systems*, 2006.12967.

3. F_{24}: This is a theory of 24 free chiral fermions. One can build an $\mathcal{N}=1$ superconformal structure by taking a linear combination of cubic Fermi terms, and the allowed choices are classified by semisimple Lie algebras of dimension 24. Each of these generates an affine Kac-Moody algebra, of which there are eight possibilities:

$$(\widehat{su}(2)_2)^{\oplus 8}\ ,\quad (\widehat{su}(3)_3)^{\oplus 3}\ ,\quad \widehat{su}(4)_4 \oplus (\widehat{su}(2)_2)^{\oplus 3}\ ,\quad \widehat{su}(5)_5\ ,\quad \widehat{so}(5)_3 \oplus \hat{g}_{2,4}\ ,$$
$$\widehat{so}(5)_3 \oplus \widehat{su}(3)_3 \oplus (\widehat{su}(2)_2)^{\oplus 2}\ ,\quad \widehat{so}(7)_5 \oplus \widehat{su}(2)_2\ ,\quad \widehat{sp}(6)_4 \oplus \widehat{su}(2)_2\ ,$$

Figure: Harrison, Paquette, Persson, Volpato, *Fun with F_{24}*, 2009.14710.

2d holomorphic VOSA with $c = 12$

The supercharacters of the eight F_{24} VOSAs are computed as

$$\chi_* = \eta^{-12}\theta_i^{r/2} \prod_{\alpha \in \Delta_+} \theta_i(z_\alpha), \quad i = 3, 4, 2 \text{ for } \mathrm{NS}, \widetilde{\mathrm{NS}}, \mathrm{R}.$$

Theorem (KS-Wang-Williams 23)

The input Jacobi form of Borcherds product is given by

$$\phi_{\mathfrak{g}} = \chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}}.$$

We also have a universal formula by theta block

$$\phi_{\mathfrak{g}} = -\frac{\theta_{\mathfrak{g}}|_{\mathrm{rk}(\mathfrak{g})/2} T_-^{(1)}(2)(\tau, \mathfrak{z})}{\theta_{\mathfrak{g}}(\tau, \mathfrak{z})}.$$

The 4 exotic VOAs

There exist four more known reflective Borcherds products of singular weights. In affine Lie algebra language, they are

$\mathfrak{g}$	$A_{1,16}$	$A_{1,8}^2$	$A_{1,4}^4$	$A_{2,9}$
C	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
c	$\frac{8}{3}$	$\frac{24}{5}$	8	6
rk	1	2	4	2
dim	3	6	12	8
ref	Gritsenko-Nikulin 98	Grit 19	Grit 18	Gritsenko-Skoruppa-Zagier 19

Question

Is there any VOA meaning for these four Borcherds products?

The 4 exotic VOAs

Answer

Yes! They are related to some very peculiar exceptional modular invariants that come from the nontrivial automorphism of the fusion algebra of the simple current extension.

Such peculiarity of $A_{1,16}$ and $A_{2,9}$ was first noticed by (Moore-Seiberg 89). later for $A_{1,8}^2$ by (Verstegen 90) and for $A_{1,4}^4$ by (Gannon 94).

The nontrivial automorphism of the fusion algebra happens rarely. It can be proved for $A_{1,k}$ this only happens at $k=16$, while for $A_{2,k}$ only at $k=9$.

We find the input Jacobi form can be expressed universally as

$$\phi_{\mathfrak{g}}(\tau,\mathfrak{z}) = -\frac{\left(\vartheta_{\mathfrak{g}}|_{\frac{1}{2}\,\mathrm{rk}(\mathfrak{g})}T_-^{(\frac{1}{C})}(1/C+1)\right)(\tau,\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})}.$$

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

- $\Delta_{1/2}(Z)$ is the denominator of a Borcherds-Kac-Moody algebra $\mathfrak{g}$ with infinite dimensional Cartan matrix $a_{mn} = 2 - 4(m-n)^2$.
- This $\mathfrak{g}$ can be regarded as the hyperbolization of affine Kac-Moody algebra $A_{1,16}$!
- Notice the weight-0 Jacobi form can be written as

$$\phi(\tau, z) = \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \chi_2(\tau, z) + \chi_{14}(\tau, z) - \chi_8(\tau, z).$$

- For $z \to 0$, this reduces to

$$3 = \chi_2(\tau) + \chi_{14}(\tau) - \chi_8(\tau),$$

which is a consequence of the Macdonald identity for $A_{1,2p^2-2}$ with $p = 3$:

$$p = \sum_{j=0}^{p-1} \chi_{2p^2-1-(4j+1)p}^{A_{1,2p^2-2}}(\tau).$$

- The same identity was used in (Moore-Seiberg 89) to construct the E_7 type modular invariant for $\hat{A}_1$.

Example: $A_{1,16}$

Affine A_1 has an ADE classification of modular invariants. The D_{10} modular invariant of $A_{1,16}$, – a simple current extended modular invariant:

$$\begin{aligned} Z_{D_{10}} &= |\chi_0+\chi_{16}|^2+|\chi_2+\chi_{14}|^2+|\chi_4+\chi_{12}|^2+|\chi_6+\chi_{10}|^2+2|\chi_8|^2 \\ &= |\phi_0|^2+|\phi_1|^2+|\phi_2|^2+|\phi_3|^2+|\phi_4|^2+|\phi_4'|^2. \end{aligned}$$

The S-matrix for the six extended fields $\phi_{0,1,2,3,4,4'}$ is

$$\frac{1}{3}\begin{pmatrix} 2\sin\left(\frac{\pi}{18}\right) & 1 & 2\cos\left(\frac{2\pi}{9}\right) & 2\cos\left(\frac{\pi}{9}\right) & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & -1 \\ 2\cos\left(\frac{2\pi}{9}\right) & 1 & -2\cos\left(\frac{\pi}{9}\right) & -2\sin\left(\frac{\pi}{18}\right) & 1 & 1 \\ 2\cos\left(\frac{\pi}{9}\right) & -1 & -2\sin\left(\frac{\pi}{18}\right) & 2\cos\left(\frac{2\pi}{9}\right) & -1 & -1 \\ 1 & -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & -1 & 2 \end{pmatrix}.$$

(Moore-Seiberg 89) observed a nontrivial automorphism $\omega : \phi_1 \leftrightarrow \phi_4$ for the fusion algebra! This only happens at level $k = 16$!

E_7 modular invariant

From such automorphism, (Moore-Seiberg 89) obtained the E_7 modular invariant:

$$
\begin{aligned}
Z_{E_7} &= Z_{D_{10}}\Big|_{\omega \text{ in holomorphic sector}} \\
&= |\phi_0|^2 + \phi_1\bar{\phi}_4 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4'|^2 + \phi_4\bar{\phi}_1 \\
&= |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 \\
&\quad + ((\chi_2 + \chi_{14})\bar{\chi}_8 + c.c.) \\
&= Z_{D_{10}} - |\phi_{A_1}|^2.
\end{aligned}
$$

The weight 0 Jacobi form

$$
\phi_{A_1} = \frac{\theta_1(3z)}{\theta_1(z)} = \chi^{A_{1,16}}_{2,\frac{1}{9}} + \chi^{A_{1,16}}_{14,\frac{28}{9}} - \chi^{A_{1,16}}_{8,\frac{10}{9}}
$$

is exactly the Jacobi form for Gritsenko-Nikulin's $\Delta_{1/2}(Z)$.

Generalize to affine Lie superalgebras

Gritsenko-Nikulin's $D_{1/2}(Z)$ is the denominator of an exotic BKM algebra with odd real roots. We find it can be regarded as the hyperbolization of affine Lie superalgebra $A^*_{1,36}$. Denote $osp(1|2)$ as A^*_1. We find the weight-0 Jacobi form of $D_{1/2}(Z)$ can be written as linear combination of affine characters of $osp(1|2)_{36}$.

$$\phi(z) = \frac{\theta_1(10z)\theta_1(z)}{\theta_1(2z)\theta_1(5z)} = \chi_2(z) - \chi_{12}(z) - \chi_{17}(z) + \chi_{27}(z) + \chi_{32}(z).$$

$osp(1|2)$ has $h^\vee = 3/2$ and dimension 1. The associated theta block is defined as $\vartheta_{A^*_1} = \vartheta_{3/2}$. Here $\phi(z)_{z\to 0} = 1$ is a Macdonald identity of $osp(1|2)$.

Question

Is this the unique example of reflective Borcherds product of singular weight with odd real roots?

Hyperbolization of affine Lie superalgebras

Answer

No! Recently we are able to construct two new examples.

Theorem (KS-Wang-Williams 24)

If $2U \oplus L$ has a symmetric reflective Borcherds product F of singular weight for which the q^0-term of the Jacobi form input involves negative Fourier coefficients, L is isomorphic to $A_1(36)$, $A_1(9) \oplus A_1(3)$, $2A_1(10)$ or $A_1(3) \oplus A_1 \oplus A_2(2)$. They are denominator of the hyperbolization of affine Lie superalgebras $A_{1,36}^*$, $A_{1,9}^* A_{1,12}$, $C_{2,10}^*$ and $A_{1,3}^* A_{1,4} A_{2,6}$.

Here we use $C_n^*, n \geq 2$ to denote Lie superalgebra $osp(1|2n)$.

This replies on the crucial fact on the modularity of affine (super)characters

Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra. If affine (super)characters of $(\mathfrak{g})_k$ are vector-valued modular forms on $\mathrm{SL}(2,\mathbb{Z})$, $\mathfrak{g}$ must be either a simple Lie algebra or a type $osp(1|2n)$ Lie superalgebra, and k must be a positive integer.

Hyperbolization of affine Lie superalgebras

The hyperbolization of symmetric type for affine Lie superalgebras $\oplus_{j=1}^{s}\mathfrak{g}_{j,k_j}$ requires the following necessary condition

$$\frac{1}{24}\sum_i \mathrm{sdim}(\mathfrak{g}_i) = C = \frac{h_i^\vee}{k_i}, \qquad k_i > 1.$$

This condition gives 25 possible solutions and we prove only 4 of them can be lifted to a reflective Borcherds product of singular weight.

The case $C^*_{2,10}$

Denote χ_{k_1,k_2} as the affine supercharacters for $osp(1|4)_k$. They can computed from Kac-Wakimoto character formulas. Here $k_i = 0, 1, \ldots, k$ and $k_1 + k_2 \leq k$.

Theorem (KS-Wang-Williams 24)

Let

$$\Psi = \chi_{2,0} - \chi_{4,2} - \chi_{1,4} + \chi_{8,1} + \chi_{0,8} + \chi_{5,5}.$$

Then $\mathbf{B}(\Psi)$ *is a symmetric reflective Borcherds product of singular weight.*

Hyperbolization of affine Lie superalgebras

The hyperbolization of antisymmetric type is connected to $\mathbb{Z}$-graded $c = 24$ holomorphic SVOA.

Open question

Does there exist any $\mathbb{Z}$-graded $c = 24$ holomorphic SVOA?

Necessary condition: first trace identity

$$\frac{1}{24}\sum_i \mathrm{sdim}(\mathfrak{g}_i) - 1 = C = \frac{h_i^\vee}{k_i}.$$

Theorem (van Ekeren-Morales 23)

There exist 1227 Lie superalgebras as possible V_1 space.

So far we restrict to around one hundred cases.

Conjecture

There exists no $\mathbb{Z}$-graded $c = 24$ holomorphic SVOA.

Summary on the classification of hyperbolization

	affine Lie algebra	affine Lie superalgebra
symmetric	$8+4$	4
anti-symmetric	69	$0?$

Table: Classification of hyperbolization of affine Lie (super)algebras.

Thank you!