A Castelnuovo Bound for Projective Varieties Admitting a Stable Linear Projection onto a Hypersuface

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Abstract

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We find a bound on the length of a fiber of a stable morphism $f: X^n \to Y^{n+1}$. For a certain class of smooth complex projective varieties, including all those up to dimension 14, we use this result to derive a bound on regularity of an ideal sheaf that is sharp in the degree.

We also give a bound on the regularity of the ideal sheaf of a Cohen-Macaulay projective variety X, that is polynomial of total degree 2 in the degree of X the dimension of X, and in the regularity of the dualizing sheaf on X.

Contents

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I. Introduction

Consider a nondegenerate smooth irreducible complex projective variety X in $\mathbf{P}^r$ of degree d and dimension n. It is of interest to find an explicit bound, in terms of d, n and r, on the degrees of hypersurfaces that cut out a complete linear system on X. An optimal bound is suspected to be:

$$H^1(\mathbf{P}^r, \mathcal{J}_{x|\mathbf{P}^r}(k)) = 0 \text{ for } k \geq d+n-r. \tag{$*$}$$

This was first proved in case of space curves by Castelnuovo [C] in 1893. The curve case was completed in 1983 by Gruson, Lazarsfeld and Peskine [GLP], who showed that $(*)$ holds for reduced irreducible curves over an algebraically closed field of arbitrary characteristic. In 1984 Mumford (c.f. [BM]) showed that $H^1(\mathbf{P}^r, \mathcal{J}_{X|\mathbf{P}^r}(k)) = 0$ for $k \geq (n+1)(d-2)+1$. Subsequently, Pinkham [P] gave a sharper bound in the surface case by showing that hypersurfaces of degree $\geq d-2$ and $\geq d-1$, respectively, cut out a complete linear system on a smooth irreducible surface $X \subset \mathbf{P}^r$ when $r \geq 5$ and $r=4$, respectively. Lazarsfeld [L] completed Pinkham's result by establishing that for surfaces the bound $(*)$ holds and is optimal for $r \geq 5$. Gruson [G] asserted an extension of Pinkham's theorem to the case of threefolds nondegenerate in $\mathbf{P}^7$.

These results can be expressed in terms of the Castelnuovo-Mumford regularity of the ideal sheaf $\mathcal{J}_{X|\mathbf{P}^r}$. Recall [M, Lecture 14] that a coherent sheaf F on a projective space $\mathbf{P}$ is m-regular if $H^i(\mathbf{P}, F(k)) = 0$ for $i > 0, k \geq m-i$. Equivalently (c.f. [EG]), F is m-regular if there exists an exact sequence

$$\cdots \to F_i \to \cdots \to F_0 \to F \to 0\ ,$$

where $F_i = \oplus \mathcal{O}_{\mathbf{P}}(s_i^j)$, $s_i^j \geq -m-i$. The regularity of F, $reg(F)$, is the minimal integer m such that F is m-regular. Bayer and Stillman [BS] showed that an estimate on the regularity of an ideal sheaf gives a bound on the complexity of algorithms for computing syzygies.

It is the objective of this work to contribute to the study of the regularity of an ideal sheaf by showing that for each positive integer n there exists a constant $c(n)$ such that for any smooth irreducible nondegenerate projective variety $X \subset \mathbf{CP}^r$ of degree d and dimension n which admits a stable linear projection onto a hypersurface in $\mathbf{CP}^{n+1}$,

$$\mathrm{reg}(\mathcal{J}_{X|\mathbf{P}^r}) \leq d+n+1-r+c(n),$$

so that hypersurfaces of degree $d+n-r+c(n)$ or higher cut out a complete linear system on X. Even though $c(n)$ grows exponentially, for values of $n = 1,2,3$:

$$c(1) = 0, c(2) = 0, c(3) = 1.$$

Therefore, since for $n \leq 14$ a generic linear projective of X into $\mathbf{CP}^{n+1}$ is stable (c.f. [M3, p. 244]), our result extends Lazarsfeld's theorem for surfaces and Gruson's assertion for threefolds.

The proof of the result is based on a simplified version of Lazarsfeld's argument in [L]. That argument starts with the construction of a certain vector bundle on $\mathbf{CP}^{n+1}$. That construction was introduced by Halphen [H] and revived by Gruson and Peskine in the series of papers on space curves (c.f. [GP1],[GP2]). It turns out that for Halphen's vector bundle V, $\mathrm{reg}(V) \geq reg(\mathcal{J}_{X|\mathbf{CP}^r})$, so that the problem of finding an upper bound for the regularity of $\mathcal{J}_{X|\mathbf{CP}^r}$ is reduced to that of finding an upper bound for the regularity of V.

The classical case of space curves is considered in §II. In this case, Halphen's bundle V is a rank 2 vector bundle, so that there is a certain symmetry between cohomologies of twists of V. It allows us not only to give a bound on $reg(V)$ but also, under a mild assumption on the embedding of X in $\mathbf{P}^3$, to find the regularity of V. In general, the construction of Halphen's bundle depends on the ability to separate points in the scheme-theoretic fiber of a linear projection of X into a projective space $\mathbf{CP}^{n+1}$ by polynomials of a fixed degree. In §III we show that each fiber of a generic linear projection of X into $\mathbf{CP}^{n+1}$ consists of no more than $n+1$ distinct points. In §IV we show that each fiber of a stable projection of X into $\mathbf{CP}^{n+1}$ consists of no more than $n+1$ points counted together with multiplicities, so that polynomials of degree n separate points in the scheme-theoretic fiber. The proof of the main theorem is given in §V. In §VII we give a bound on regularity of the ideal sheaf of a Cohen-Macaulay projective variety X, which is polynomial on the degree of X, dimension of X, and regularity of the dualizing sheaf on X. In order to derive this bound, we use the Eagon-Northcott complex whose construction is given in §VI.

II. The Classical Case

In order to illustrate the argument used in the proof of the main theorem, consider the classical case of a smooth irreducible curve C of degree d in a projective space $\mathbf{P}^3$ over an algebraically closed field.

Let $f : C \to \mathbf{P}^2$ be a generic linear projection of C into a plane $\mathbf{P}^2 \subset \mathbf{P}^3$ given by the equation $z = 0$ where $z \in H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. Following Halphen [H], consider a map of sheaves on $\mathbf{P}^2$

$$\omega : \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1) \to f_* \mathcal{O}_C ,$$

where the map $\omega \mid_{\mathcal{O}_{\mathbf{P}^2}(-1)}: \mathcal{O}_{\mathbf{P}^2}(-1) \to f_*\mathcal{O}_C$ is induced by multiplication by z. Since $f(C)$ is a nodal curve, linear polynomials separate points in each fiber of ω, so that Halphen's map ω is surjective. Thus, one gets the following exact sequence of sheaves on $\mathbf{P}^2$:

$$O \to V \to \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1) \xrightarrow{\omega} f_*\mathcal{O}_C \to 0.$$

The image of the map

$$H^\circ(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k) \oplus \mathcal{O}_{\mathbf{P}^2}(k-1)) \to H^\circ(\mathbf{P}^2, f_*\mathcal{O}_C(k)) \simeq H^\circ(C, \mathcal{O}_C(k))$$

is the restriction on C of the subspace of $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k))$ spanned by linear on z homogeneous polynomials of degree k, so that vanishing of $H^1(\mathbf{P}^2, V(k))$ implies surjectivity of the map $H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k)) \to H^\circ(C, \mathcal{O}_C(k))$.

Since $f_*\mathcal{O}_C$ is a Cohen-Macaulay sheaf of dimension 1 (c.f. [S, Proposition IV. 11]), the Auslander-Buchsbaum formula

$$proj.dim_A M + depth_A M = depth A$$

(c.f. [S, Proposition IV. 21]) implies that V is a projective and, therefore, locally free sheaf of rank 2 with the first Chern class $c_1(V) = -d-1$. Then the perfect pairing $V \otimes V \to \wedge^2 V \simeq \mathcal{O}_{P^2}(-d-1)$ induces an isomorphism $V \simeq V^\vee(-d-1)$, where $V^\vee$ is the dual of V. Therefore, the Serre duality theorem implies that, for any integer m,

$$H^1(\mathbf{P}^2, V(m)) \simeq H^1(\mathbf{P}^2, V^\vee(m-d-1)) \simeq H^1(\mathbf{P}^2, V(d-2-m)),$$

and

$$H^\circ(\mathbf{P}^2, V(m)) \simeq H^\circ(\mathbf{P}^2, V^\vee(m-d-1)) \simeq H^2(\mathbf{P}^2, V(d-2-m)).$$

We may assume that C is not a plane curve. Then the map

$$\alpha_m : H^\circ(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(m) \oplus \mathcal{O}_{\mathbf{P}^2}(m-1)) \to H^\circ(\mathbf{P}^2, f_*\mathcal{O}_C(m)) \simeq H^\circ(C, \mathcal{O}_C(m))$$

is surjective for $m \leq 0$ and injective for $m \leq 1$.

Hence, for $s \geq 0$,

$$H^2(\mathbf{P}^2, V(d-3+s)) \simeq \mathit{ker}\, \alpha_{1-s} = 0 = \mathit{coker}\, \alpha_{-s} \simeq H^1(\mathbf{P}^2, V(d-2+s)).$$

It implies that V is $(d-1)$-regular. (Actually, the regularity of V is exactly $d-1$ under the mild assumption that C is not linearly normal in $\mathbf{P}^3$). Therefore, $\mathcal{J}_{C/\mathbf{P}^3}$, the ideal sheaf of C in $\mathbf{P}^3$, is $(d-1)$-regular, and the map $H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k)) \to H^\circ(C, \mathcal{O}_C(k))$ is surjective for $k \geq d-2$.

III. A Bound on the Number of Distinct Points in the Fiber of a Generic Projection.

In this paragraph we will give a self-contained exposition of the theorem of Mather on transversality of a generic linear projection to a modular stratum of the multi-jet bundle ${}_sJ^k(X, \mathbf{CP}^{n+1})$ in the case $k = 0$.

Theorem. *Let $X \subset P_1$ be an embedding of a smooth irreducible complex n-dimensional quasi-projective variety X into a complex projective space $P_1 = \mathbf{CP}^N$. Let P_2 be an $(n+1)$-dimensional linear subspace of P_1. Then, for any positive integer s, there exists a nonempty open subset Λ of the grassmanian $G(\mathbf{CP}^{N-n-2}, P_1)$ of $(N-n-2)$-dimensional subspaces of P_1 such that any $\lambda \in \Lambda$ gives a linear projection with center at $\lambda, p_\lambda : X \to P_2$, and the induced map*

$$p_\lambda^{(s)} = p_\lambda^s \mid_{X^{(s)}} : X^{(s)} = \{(x_1, \ldots, x_s) \in X^s : x_i \neq x_j \;\text{ for }\; i \neq j\} \to P_2^s$$

is transverse to the diagonal $\Delta^s = \Delta^s(P_2) = \{(y, \ldots, y) : y \in P_2\} \subset P_2^s$.

The dimension count

$$\dim X^{(n+2)} + \dim \Delta^{n+2} = n^2 + 3n + 1 < \dim\ P_2^{n+2} = n^2 + 3n + 2$$

shows that the theorem implies the following:

Corollary. *For generic* $\lambda \in G(\mathbf{CP}^{N-n-2}, P_1)$, *each fiber of the linear projection* $p_\lambda : X \to P_2$ *consists of no more than* $n+1$ *distinct points.*

Remark: *We note that this bound on the number of distinct points in a fiber of a generic linear projection of n-dimensional irreducible nondegenerate subvariety* $X \subset \mathbf{CP}^{2n+1}$ *is claimed in the paper [Ll]. However, the proof given there is faulty. Lluis implicitly assumes that the family of n-dimensional linear subspaces of* $\mathbf{CP}^{2n+1}$ *intersecting* X *at* $n+1$ *or more points is an irreducible variety whose generic point is a linear span of* $n+1$ *points of* X *in general position. Then a dimension count shows that a generic* $(n-1)$*-dimensional linear subspace of* $\mathbf{CP}^{2n+1}$ *is not contained in any n-dimensional linear subspace of* $\mathbf{CP}^{2n+1}$ *intersecting* X *at more than* $n+1$ *points, i.e., each fiber of a generic linear projection* $p : X \to \mathbf{CP}^{n+1}$ *consists of no more than* $n+1$ *distinct points. Lazarsfeld gave the following counterexample to Lluis's implicit assumption:*

> *if* $C \overset{i}{\hookrightarrow} \mathbf{CP}^2$ *is a plane curve of degree* $d \geq 3$ *then the surface* $X = \mathbf{CP}^1 \times C$ *admits a nondegenerate embedding into* $\mathbf{CP}^5$

$$X = \mathbf{CP}^1 \times C \xrightarrow{1 \times i} \mathbf{CP}^1 \times \mathbf{CP}^2 \xrightarrow{Segre} \mathbf{CP}^5$$

such that the family of trisecant planes has at least 2 components. Generic element of the first component is a linear span of 3 points of X in general position. Generic element of the second component is a plane passing through a line in the plane Segre $(t\times\mathbf{CP}^2)$, $t\in\mathbf{CP}^1$.

In order to prove the theorem we need a sufficient condition for the claimed transversality. For any morphism between nonsingular complex varieties $f: X^1 \to X^2$, introduce the following measure of deviation of f from being tranversel to a smooth subvariety $V \subset X^2$ at a point $x \in X^1$:

$$\delta(f,V,x) = \begin{cases} 0, & \text{if} \quad f(x) \notin V \\ \dim X^2 - \dim((T_{f(x)}V) + df(T_x X^1)), & \text{if} \quad f(x) \in V. \end{cases}$$

The topology we consider in the following lemma is the Zariski topology.

Lemma. *Let A, Y, Z be nonsingular irreducible complex varieties. Let X be an open subset of $A \times Y$. Let $F: X \to Z$ be a morphism defining a family of morphisms $F_a : Y_a \to Z$, where $Y_a = \{y \in Y : (a,y) \in X\}$ and $F_a(y) = F(a,y)$ for $y \in Y_a$. Let W be a nonsingular irreducible subvariety of Z. Suppose that for all $(a,y) \in X$, either $\delta(F_a, W, y) = 0$ or $\delta(F, W, (a,y)) < \delta(F_a, W, y)$. Then for generic $a \in A$, F_a is transverse to W.*

Proof: Let $\delta_F = max\{\delta(F,W,x) : x \in X\}$. The lemma is proved by induction on δ_F. If $\delta_F = 0$ then F is transverse to W, therefore, $W' = F^{-1}W$ is nonsingular. By the Sard's lemma for algebraic varieties (c.f. [Mu, p. 42]), the set of regular values of the restriction on W' of the projection $\pi_A : X \to A$ contains a nonempty open subset $V \subset A$. For any $a \in V$ and any $y \in F_a^{-1}(W)$ we have:

$$dF_a(T_yY_a) + T_{F_a(y)}W = dF(T_{(a,y)}(a \times Y)) + dF(T_{(a,y)}W') + T_{F(a,y)}W$$
$$= dF(T_{(a,y)}(A \times Y)) + T_{F(a,y)}W = T_{F_a(y)}Z.$$

Thus, F_a is transverse to W for all $a \in V$.

Consider now the case $\delta_F > 0$. Denote $\sum = \{\sigma \in X : \delta(F, W, \sigma) = \delta_F\}$. For every $\sigma \in \Sigma$ choose $X_\sigma \subset X, Z_\sigma \subset Z, W_\sigma \subset Z_\sigma$ such that

- X_σ is an open neighborhood of σ in X;
- Z_σ is an open neighborhood of $F(X_\sigma)$ in Z;
- W_σ is a nonsingular irreducible subvariety of Z_σ;
- $W_\sigma \supset W \cap Z_\sigma$;
- dim $W_\sigma =$ dim $W + \delta_f$;
- $F_\sigma = F\mid_{X_\sigma}: X_\sigma \to Z_\sigma$ is transverse to W_σ.

In order to do so, locally extend W along a subspace of $T_{F(\sigma)}Z$ that is complementary to $dF(T_\sigma X) + T_{F(\sigma)}W$.

For $\sigma \in \Sigma, a \in A$, denote $Y_{\sigma,a} = \{y \in Y : (a, y) \in X_\sigma\}$. If $a \in \pi_A(\Sigma \cap X_\sigma)$ then $F_{\sigma,a} : Y_{\sigma,a} \to Z_\sigma$ is not transverse to W_σ because for $y \in Y_{\sigma,a}$ such that $(a, y) \in \Sigma \cap X_\sigma$ we have:

$$\delta(F_{\sigma,a}, W_\sigma, y) \geq \delta(F_{\sigma,a}, W \cap Z_\sigma, y) - (dim\, W_\sigma - dim W)$$
$$> \delta(F_\sigma, W, (a, y)) - \delta_F = 0.$$

Therefore, since F_σ is transverse to W_σ, the complement of $\pi_A(\Sigma \cap X_\sigma)$ contains an open dense subset of A. Since X is covered by finite number of X_σ's, the complement of $\pi_A(\Sigma)$ contains an open dense subset of A. Induction on δ_F completes the proof of the lemma. $\square$

Proof of the Theorem:

Denote $A = \{\lambda \in G(\mathbf{CP}^{N-n-2}, P_1) : \lambda \cap P_2 = \phi, \lambda \cap X = \phi\}$, an open subset of the grassmanian of $(N-n-2)$-dimensional linear subspaces of P_1 consisting of centers of linear projections from X into P_2. In order to prove the theorem, we will show that the map $p^{(s)} : A \times X^{(s)} \to P_2^s$, defined as $p^{(s)}(\lambda, (x_1, \ldots, x_s)) = (p_\lambda(x_1), \ldots, p_\lambda(x_s))$, satisfies the hypotheses of the lemma: for any $\lambda \in A, \mathbf{x} \in X^{(s)}$, either $\delta(p_\lambda^{(s)}, \Delta^s, \mathbf{x}) = 0$ or $\delta(p^{(s)}, \Delta^s, (\lambda, \mathbf{x})) < \delta(p_\lambda^{(s)}, \Delta^s, \mathbf{x})$.

Suppose that $\lambda \in A, \mathbf{x} = (x_1, \ldots, x_s) \in X^s, y \in P_2$ are such that $p_\lambda^{(s)}(\mathbf{x}) = \mathbf{y} = \underbrace{(y, \ldots, y)}_{s}$ and $\delta(p^{(s)}, \Delta^s, (\lambda, \mathbf{x})) = \delta(p_\lambda^{(s)}, \Delta^s, \mathbf{x})$. Therefore,

$$(dp^{(s)}(., \mathbf{x}))T_\lambda A \subset T = dp_\lambda^{(s)}(T_\mathbf{x} X^{(s)}) + T_\mathbf{y} \Delta^s. \tag{1}$$

The objective now is to show that the inclusion (1) implies that $T = T_\mathbf{y}(P_2^{(s)})$ so that $\delta(p_\lambda^{(s)}, \Delta^s, \mathbf{x}) = 0$.

For a projective space P, let $V(P)$ denote an affine cone over P, i.e., if $P = P(W)$ is the space of lines in a linear space W, then $V(P) = W$.

For linear subspaces μ, ν of a projective space P such that $\mu \cap \nu = \phi, \dim \mu + \dim \nu = \dim P - 1$, consider $\Gamma^{\mu,\nu} : Hom_\mathbf{C}(V(\mu), V(\nu)) \to G(\mathbf{CP}^{\dim \mu}, P)$, an isomorphism of the vector space $Hom_\mathbf{C}(V(\mu), V(\nu))$ onto an open neighborhood of μ in the grassmanian $G(\mathbf{CP}^{dim \mu}, P)$, which assoociates to each linear map $\ell : V(\mu) \to V(\nu)$ its projectivized graph

$$\Gamma^{\mu,\nu}(\ell) = P(\text{graph of } \ell) = P(\{v + \ell(v) : v \in V(\mu)\}).$$

The map $\Gamma^{\mu,\nu}$ induces a linear isomorphism

$$\gamma^{\mu,\nu} = (d\Gamma^{\mu,\nu})_0 : Hom_\mathbf{C}(V(\mu), V(\nu)) \to T_\mu G(\mathbf{CP}^{dim \mu}, P).$$

Let P_3 be a hyperplane in P_2 not containing y. Denote $\Lambda = V(\lambda)$. Denote $V_i = V(P_i)$ for i = 1, 2, 3. Choose $0 \neq u \in V(y)$. For $i = 1, \dots, s$ there is uniquely defined $w_i \in \Lambda$, such that $w_i + u \in V(x_i)$. Suppose $\alpha \in Hom_{\mathbf{C}}(\Lambda, V_3) \subset Hom_{\mathbf{C}}(\Lambda, V_2)$. Then $\Gamma^{\lambda, P_2}(\alpha)$ does not contain $x_1, \dots, x_s$. For $i = 1, \dots, s$

$$V(x_i) \ni w_i + u = (w_i + \alpha(w_i)) + (u - \alpha(w_i)),$$

where $w_i + \alpha(w_i) \in V(\Gamma^{\lambda, P_2}(\alpha)), u - \alpha(w_i) \in V_2$. Therefore,

$$(ev_u \circ (\gamma^{y, P_3})^{-1} \circ dp(., x_i) \circ \gamma^{\lambda, P_2})(\alpha) = -\alpha(w_i), \tag{2}$$

where $p(\mu, x) = p_\mu(x)$, and $ev_u : Hom_{\mathbf{C}}(V(y), V_3) \to V_3$ is a linear isomorphism, which evaluates a linear map $\ell : V(y) \to V_3$ at u.

Introduce the notation $S = \oplus_{k=0}^{\infty} S^k$ for the graded ring of complex polynomial function on Λ. Consider a homomorphism of C-algebras $\sigma : S \to \mathrm{End}_{\mathbf{C}}(T_{\mathbf{y}} P_2^s)$ given by the action

$$f \circ \mathbf{v} = (f(w_1)v_1, \dots, f(w_s)v_s)$$

of a polynomial of $f \in S$ on an element $\mathbf{v} = \underbrace{(v_1, \dots, v_s)}_{s} \in T_{\mathbf{y}} P_2^s = \Pi_{i=1}^s T_y P_2$.

The subspace

$$T_1 = dp_\lambda^{(s)}(T_{\mathbf{x}} X^{(s)}) = \Pi_{i=1}^s dp_\lambda(T_{x_i} X) \subset T_{\mathbf{y}} P_2^{(s)} = \Pi_{i=1}^s T_y P_2$$

is invariant under the action of $\sigma(S)$.

For any $\ell \in S^1 = Hom_{\mathbf{C}}(\Lambda, \mathbf{C})$ and any $\mathbf{v} = (v, \dots, v) \in T_2 = T_{\mathbf{y}} \Delta^s$, define $L \in Hom_{\mathbf{C}}(\Lambda, V_3) \subset Hom_{\mathbf{C}}(\Lambda, V_2)$ by the formula $L(w) = -\ell(w) \cdot (ev_u \circ (\gamma^{y, P_3})^{-1})(v)$. Then the formula (2) shows that $\ell \circ \mathbf{v} = dp^{(s)}(\cdot, \mathbf{x})(N)$, where

$$N = \gamma^{\lambda, P_2}(L) \in T_\lambda G(\mathbf{C}\mathbf{P}^{N-n-2}, P_1) = T_\lambda A.$$

Therefore, the inclusion (1) implies that $\ell \circ \mathbf{v} \in T$. Thus, the subspace $T = T_1 + T_2 \subset T_{\mathbf{y}}(P_2^s)$ is invariant under the action of $\sigma(S^1)$ on $T_y(P_2^s)$.

Since S^1 generates the algebra S over $S^\circ = \mathbf{C}$, T is invariant under the action of $\sigma(S)$. Since $w_1, \ldots, w_s$ are distinct vectors in Λ, there exist polynomials $\sigma^i \in S = \mathbf{C}[\lambda]$ such that

$$\delta^i(w_j) = \delta^i_j = \begin{cases} i & , \; j = 1 \\ 0 & , \; j \neq i \end{cases} .$$

Then for $v \in T_y P_2$,

$$\delta^i \circ \mathbf{v} = (\underbrace{0, \ldots, 0}_{i-1}, v, \underbrace{0, \ldots, 0}_{s-i}),$$

where

$$\mathbf{v} = (\underbrace{v, \ldots, v}_{s}) \in T_{\mathbf{y}}\Delta^s \subset T.$$

Therefore, $T = \sigma(S) \circ T = T_{\mathbf{y}}(P_2^s)$ so that $\delta(p_\lambda^{(s)}, \Delta^s, x) = 0$. Hence, the assumptions of the lemma are satisfied, and the proof of the theorem is completed. $\square$

IV. A Bound on the Length of a Fiber of a Stable Projection onto a Hypersurface.

As we have already proved, for a generic linear projection of n-dimensional smooth projective variety $X \subset \mathbf{CP}^N$ onto a hypersurface in an $(n+1)$-dimensional linear subspace $P \subset \mathbf{CP}^N$, each fiber consists of no more than $n+1$ distinct points. Using Mather's theory of stable singularities, we will show in this paragraph that the same bound holds for the length of a fiber of a stable map of X onto P. In particular, if the pair $(dimX, dimP)$

is in the range of the *nice dimensions* (i.e., when $n \leq 14$), the length of a fiber is bounded by $n + 1$, since, in this case, a generic linear projection is stable (c.f. [M3]).

Conjecture. *The length of a fiber of a generic linear projection of X into P is bounded by $n + 1$.*

Remark. *It was pointed out by Piene and Ronga [PR, Remark 2.4] that for $n \leq 5$ the bound $n+1$ on the length of a fiber of a generic linear projection $f : X \to P$ follows from Mather's theorem [M3, Theorem 2] on transversality of a generic linear projection with respect to each stratum of the Thom-Boardman stratification of the multi-jet bundle ${}_sJ^k(X, P)$.*

Remark. *The result of Damon and Galligo [DG] on a topological interpretation of the local multiplicity of certain stable maps, combined with the established bound on the number of points in a fiber of a generic projection, implies a bound on the length of a fiber of some stable projections, including all those in the nice dimensions. That bound is quadratic in dim X.*

A differentiable map $f : X_1 \to X_2$ is called stable if it is "genuinely generic" in the sense that all nearby differentiable maps coincide with f up to diffeomorphisms of X_1 and X_2. Mather [M2] proved that a proper differentiable map $f : X_1 \to X_2$ (e.g., when X_1 is compact) is stable if and only if it is locally infinitesimally stable. It is the notion of local infinitesimal stability that has a direct analogue in algebraic geometry.

For a holomorphic map-germ $f : (N, S) \to (P, y)$ where S is a finite subset of N and $y \in P$, let $A = \mathcal{O}_{P,y}(TP)$ denote the $\mathcal{O}_{P,y}$-module of germs at y of holomorphic vector fields on P, let $B = \mathcal{O}_{N,S}(TN)$ denote the $\mathcal{O}_{N,S}$-module of germs at S of holomorphic vector fields on N, let $\theta(f) = \mathcal{O}_{N,S}(f^*(TP))$ denote the $\mathcal{O}_{N,S}$-module of germs at S of holomorphic vector fields along f. Let $wf : A \to \theta(f)$ and $tf : B \to \theta(f)$ be given as $wf(\eta) = \eta \circ f$ and $tf(\xi) = df \circ \xi$.

Definition. *A holomorphic map-germ $f : (N,S) \to (P,y)$ is called infinitesimally stable if $wf(A) + tf(B) = \theta(f)$.*

Definition. *A holomorphic map $f : N \to P$ is called locally infinitesimally stable (or just stable) if, for any $y \in P$ and any finite subset S of $f^{-1}(y)$, the induced by f holomorphic map-germ $f_S : (N,S) \to (P,y)$ is infinitesimally stable.*

If $f : (N,S) \to (P,y)$ is an infinitesimally stable map-germ, then wf induces a surjective $\mathbf{C}$-linear map

$$\begin{aligned} \overline{wf} : A/m_y A = T_y P &\to \theta(f)/(f^*(m_y)\theta(f) + tf(B)) \\ &= \oplus_{x \in S}\theta(f_x)/(f_x^*(m_y)\theta(f_x) + tf_x(B_x)), \end{aligned} \tag{3}$$

where for $x \in S$, $f_x : (N,x) \to (P,y)$ is a germ induced by f.

For a holomorphic map-term $f : (N,x) \to (P,y)$ denote

$$\mu(f) = dim_{\mathbf{C}}(\theta(f)/(f^*(m_y)\theta(f) + tf(B)),$$

and let

$$\delta(f) = dim_{\mathbf{C}}(\mathcal{O}_{N,x}/f^*(m_y)\mathcal{O}_{N,x})$$

denote a multiplicity of f at $x \in N$. Neither $\delta(f)$ nor $\mu(f)$ are necessarily finite.

Proposition. *For any holomorphic map-germ $f : (N, x) \to (P, y), \mu(f) \geq (dim\, P - dim\, N) \cdot \delta(f)$. Furthermore, $\mu(f) = (dim\, P - dim\, N) \cdot \delta(f) < \infty$ if and only if $\delta(f) = 1$.*

Proof: For an integer k, introduce finite numbers δ_k, μ_k defined as

$$\delta_k = dim_{\mathbf{C}} \mathcal{O}_{N,x}/(f^*(m_y)\mathcal{O}_{N,x} + m_x^k),$$

$$\mu_k = dim_{\mathbf{C}}(\theta(f)/(f^*(m_y)\theta(f) + tf(B) + m_x^k\theta(f)).$$

Then μ_k is the dimension of the cokernel of the C-linear map $\bar{t}_k f : B/((f^*(m_y) + m_x^k)B \to \theta(f)/(f^*(m_y) + m_x^k)\theta(f)$. Therefore,

$$\begin{aligned}
\mu_k &= dim_{\mathbf{C}}(coker\, \bar{t}_k f) \\
&\geq dim_{\mathbf{C}}(\theta(f)/(f^*(m_y) + m_x^k)\theta(f)) - dim_{\mathbf{C}}(B/(f^*(m_y) + m_x^k)B) \\
&= dim_{\mathbf{C}}(\oplus_{i=1}^{dim\, P} \mathcal{O}_N, x/(m_x^k + f^*(m_y)\mathcal{O}_{N,x})) \\
&- dim_{\mathbf{C}}(\oplus_{i=1}^{dim\, N} \mathcal{O}_{N,x}(/m_x^k + f^*(m_y)\mathcal{O}_{N,x})) \\
&= (dim\, P - dim\, N)\delta_k(f).
\end{aligned}$$

Thus, $\mu(f) = lim_{k\to\infty} \mu_k \geq lim_{k\to\infty}(dim\, P - dim\, N) \cdot \delta_k(f) = (dim\, P - dim\, N) \cdot \delta(f)$. Furthermore, if $\mu(f) = \delta(f) \cdot (dim\, P - dim\, N) < \infty$, then the map $\bar{t}f : B/f^*(m_y) \to \theta(f)/f^*(m_y)\theta(f)$ is injective. Therefore, kernel of the map $tf : B \to \theta(f)$ is contained in $f^*(m_y)B \subset m_x B$. By the Nullstellensatz, $B = m_x B \oplus B/m_x B = m_x B \oplus T_x N, \theta(f) = m_x\theta(f) \oplus \theta(f)/m_x\theta(f) = m_x\theta(f) \oplus T_y P$, so that the restriction of tf on $T_x N$ factors through the map $df : T_x N \to T_y P$. Therefore, $ker\, df \subset T_x N \cap m_x B = \{0\}$. Thus, the map $df : T_x N \to T_y P$ is injective and $\delta(f) = 1$. On the other hand, if $\delta(f) = 1$, then the map $\bar{t}f : B/f^*(m_y)B \to \theta/f^*(m_y)\theta(f)$ coincides with the injective map $df : T_xN \to T_yP$ so that $\mu(f) = dim_{\mathbf{C}}\, coker(\bar{t}f) = dim\, P - dim\, N$. □

Theorem. *If $f : N \to P$ is a stable morphism from a nonsingular n-dimensional complex variety N into a nonsingular $(n+1)$-dimensional complex variety P, then each fiber of f consists of no more than $n+1$ points counted together with multiplicites. Furthermore, this bound is attained only for finite number of points in P.*

Proof: It follows form the proposition and surjectivity of the map (3) that for any $y \in P$:

$$\sum_{x \in f^{-1}(y)} \delta(f_x) \leq \sum_{x \in f^{-1}(y)} \mu(f_x) \leq dim_{\mathbf{C}}\, T_yP = n+1.$$

Moreover, the equality $\sum_{x \in f^{-1}(y)} \delta(f_x) = n+1$ implies that for any $x \in f^{-1}(y), \delta(f_x) = \mu(f_x) = 1$, i.e, the fiber over y consists of $n+1$ distinct points of multiplicty 1. On the other hand, stability of f implies that f is generic in the sense of §2 – for any finite subset $S \subset f^{-1}(p)$, the map-germ $f_S : (N, S) \to (P, p)$ is infinitesimally stable, therefore, wf_S induces a surjective **C**-linear map

$$\begin{aligned} \tilde{w}f_S : A/m_y A = T_pP &\to \theta(f_S)/(m_S\theta(f_S) + tf_S(B_S)) \\ &= \oplus_{x \in S}\, \theta(f_x)/(m_x\theta(f_x) + tf_x(B_x)) \\ &= \oplus_{x \in S} T_pP/df(T_xN). \end{aligned}$$

Therefore,

$$T_{\mathrm{p}}\Delta^s(P) + \oplus_{x \in S}\, df(T_x) = T_{\mathrm{p}}(P^N),$$

where $s = \#(S), \mathbf{p} = (\underbrace{p, \ldots, p}_{s})$. Hence, for any positive integer s, the map $f^{(s)} : N^{(s)} \to P^{(s)}$ is transversal to the diagonal $\Delta^s(P)$. Thus, the preimage of the diagonal $\Delta^{n+1}(P)$ under the morphism $f^{(n+1)} : N^{(n+1)} \to P^{n+1}$ is a subvariety of $N^{(n+1)}$ of dimension equal

$$dim\, N^{(n+1)} + dim\, \Delta^{n+1}(P) - dim\, P^{n+1} = 0,$$

hence, a finite subset of $N^{(n+1)}$. Therefore, there are only finite number of points in P with $n+1$ points in the scheme-theoretic fiber. $\square$

V. The Main Theorem

For a positive integer n, denote

$$c(n) = \begin{cases} 0, & \text{if } \ n = 1, 2 \\ n - 2 + \sum_{k=3}^{n-1} \binom{n-1+k}{k} (k-2), & \text{if } \ n \geq 3 \end{cases}.$$

Theorem. *Let $X \subset \mathbf{CP}^r$ be a smooth complex n-dimensional nondegenerate irreducible projective variety of degree d. Choose a linear projection $p : \mathbf{C}^r - - \to \mathbf{CP}^{2n+1}$ that induces an embedding of X. Suppose that there exists a linear projection $\mathbf{C}^{2n+1} - - \to \mathbf{CP}^{n+1}$ whose restriction on $p(X)$ is stable. Then the ideal sheaf $\mathcal{J}_{X/\mathbf{CP}^r}$ is $(d+n-r+1+c(n))$-regular.*

Proof. The proof of the theorem is based on a simplified version of Lazarsfeld's argument in [L].

Let L denote the center of the projection

$$\begin{tikzcd} p_L : \mathbf{CP}^r \arrow[r, dashed] & \mathbf{CP}^{n+1}, \\ X \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

an $(r-n-2)$-dimensional subspace of $\mathbf{CP}^r$. Let ℓ, ℓ' be disjoint linear subspaces of L such that $p_L = p_\lambda \circ p_\ell$, where $p_\ell : \mathbf{CP}^r - - \to \mathbf{CP}^{2n+1}$ induces an embedding of X, $\lambda = p_\ell(\ell'), p = p_\lambda |_{p_\ell(X)}$: $p_\ell(X) \to \mathbf{CP}^{n+1}$ is stable.

Choose a $\mathbf{C}$-algebra homomorphism

$$\alpha : \mathbf{C}[L] = \oplus_{k=0}^{\infty} H^\circ(L, \mathcal{O}_L(k)) \to \mathbf{C}[\mathbf{CP}^r] = \oplus_{k=0}^{\infty} H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^r}(k)),$$

which is a lifting of the restriction map $\mathbf{C}[\mathbf{CP}^r] \to \mathbf{C}[\lambda]$. Also choose a $\mathbf{C}$-algebra homomorphism $\beta : \mathbf{C}[\lambda] \to \mathbf{C}[L]$, which is a lifting of the surjective homomorphism

$$\mathbf{C}[L] \to \mathbf{C}[\ell'] \simeq \mathbf{C}[\lambda].$$

Restricting on X an element

$$v \in H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^r}(k))$$

determines a section in

$$H^\circ(X, \mathcal{O}_X(k)) \simeq H^\circ(\mathbf{CP}^{n+1}, f_* \mathcal{O}_X(k)) \simeq Hom(\mathcal{O}_{\mathbf{CP}^{n+1}}(-k), f_* \mathcal{O}_X)$$

giving a map $w_v : \mathcal{O}_{\mathbf{CP}^{n+1}}(-k) \to f_* \mathcal{O}_X$, which depends linearly on v. Thus, a sequence of linear subspaces

$$\mathcal{V} = (V_0 \subset H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^r}), \ldots, V_k \subset H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^r}(k)))$$

determines a map $w_{\mathcal{V}} : \oplus_{i=0}^{k} \mathcal{O}_{\mathbf{CP}^{n+1}}(-i) \otimes V_i \to f_* \mathcal{O}_X$. The image of the map

$$H^\circ(\mathbf{CP}^{n+1}, (\oplus_{i=0}^{k} \mathcal{O}_{\mathbf{CP}^{n}+1}(-i) \otimes \alpha(H^\circ(L, \mathcal{O}_L(i)))(m))$$
$$\to H^\circ(\mathbf{CP}^{n+1}, f_* \mathcal{O}_x(m)) \simeq H^\circ(X, \mathcal{O}_X(m))$$

is contained in the image of the restriction map

$$H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^r}(m) \to H^\circ(X, \mathcal{O}_X(m)).$$

Therefore, if the sequence

$$\mathcal{V} = (V_0 \subset \alpha(H^\circ(L, \mathcal{O}_L)), \ldots, V_k \subset \alpha(H^\circ(L, \mathcal{O}_L(k))))$$

is such that the map

$$w_{\mathcal{V}} : \oplus_{i=0}^{k} \mathcal{O}_{\mathbf{CP}}^{n+1}(-i) \otimes V_i \to f_* \mathcal{O}_X$$

is surjective, then $reg(ker\, w_{\mathcal{V}}) \geq reg(\mathcal{J}_{X/\mathbf{CP}^r})$.

By the theorem of §IV, the length of each fiber of a stable map $p : p_\ell(X) \to \mathbf{CP}^{n+1}$ is bounded by $n+1$. Furthermore, there is only a finite number of fibers of p of length $n+1$. Therefore, there exists a 1-dimensional subspace U of $H^\circ(\lambda, \mathcal{O}_\lambda(n))$ such that the sequence $\mathcal{V} = (V_0 = \alpha(W_0), \ldots, V_n = \alpha(W_n))$, where

$$W_i = \begin{cases} H^\circ(L, \mathcal{O}_L(i)) & \text{if } \ i = 0, 1, 2 \\ \beta(H^\circ(\lambda, \mathcal{O}_\lambda(i))) & \text{if } \ 3 \leq i \leq n-1 \\ \beta(U) & \text{if } \ i = n \geq 3 \end{cases},$$

determines a surjective map

$$w_{\mathcal{V}} : A = \oplus_{i=0}^{n} \mathcal{O}_{\mathbf{CP}^{n+1}}(-i) \otimes V_i \to f_* \mathcal{O}_X.$$

It induces the following exact sequence of sheaves on $\mathbf{CP}^{n+1}$:

$$0 \to B \to A \to f_* \mathcal{O}_X \to 0. \qquad (*)$$

Since $f_*\mathcal{O}_X$ is a Cohen-Macaulay sheaf of dimension n (c.f. [S, Proposition IV. 11]), the Auslander-Buchsbaum formula

$$proj.\ dim_A M + depth_A M = depth A$$

(c.f. [S, Proposition IV. 21]) implies that B is a locally free sheaf of rank $b = rk(B) = rk(A) = \sum_{i=0}^{n} h_i$, where we denote $h_i = dim_{\mathbf{C}} V_i$. The first Chern class of B is

$$c_1 = c_1(B) = c_1(A) - c_1(f_*\mathcal{O}_X) = -d - \sum_{i=0}^{n} ih_i.$$

The multiplication map

$$\wedge^{b-1} B \otimes B \rightarrow \wedge^b B = \mathcal{O}_{\mathbf{CP}^{n+1}}(c_1)$$

is a perfect pairing, i.e., it induces an isomorphism

$$B \simeq \wedge^{b-1} B^{\vee} \otimes \mathcal{O}_{\mathbf{CP}^{n+1}}(c_1),$$

where $B^{\vee}$ is the dual of B.

Lemma 1. *(analogues to [L, Lemma 2.1]). The sheaf $B^{\vee}$ is (-2)-regular.*

Lemma 2. *(Lazarsfeld [L, Lemma 2.7]). If $\mathcal{F}$ is a locally free sheaf on a projective space* $\mathbf{P}$ *over an algebraically closed field of arbitrary characteristic then, for any $q \geq 0, reg(\mathcal{F})^{\otimes q} \leq q \cdot reg(\mathcal{F})$.*

The lemmas will be proved later.

Since $\mathbf{C}$ has characteristic 0, the sheaf $\wedge^q B^{\vee}$ is a direct summand of the sheaf $(B^{\vee})^{\otimes q}$, and the lemmas imply that, for any $q \geq 0, reg(\wedge^q B^{\vee}) \leq$

$reg((B^\vee)^{\otimes q}) \leq q \cdot reg(B^\vee) \leq -2q$. Therefore, $reg(J_{X/\mathbf{CP}^r}) \leq reg(B) = -c_1 + reg(\wedge^{b-1}B^\vee) \leq -c_1 - 2(b-1) = d + n - r + 1 + c(n)$.

Proof of the Lemma 1: We have to check that

$$H^i(\mathbf{CP}^{n+1}, B^\vee(m)) = 0 \quad \text{if} \quad 1 \leq i \leq n+1 \ \text{ and } \ m \geq -i-2.$$

By the Serre duality theorem,

$$h^i_m = dim_{\mathbf{C}}(H^i(\mathbf{CP}^{n+1}, B^\vee(m)) = dim_{\mathbf{C}} H^{n+1-i}(\mathbf{CP}^{n+1}, B(-m-n-2)).$$

Consider 3 cases:

(i) $\quad 1 \leq i \leq n-1, m \geq -i-2.$

In this case, it follows from the exact sequence $(*)$ that

$$\begin{aligned} h^i_m &= dim_{\mathbf{C}} H^{n-i}(\mathbf{CP}^{n+1}, f_*\mathcal{O}_X(-m-n-2)) \\ &= dim_{\mathbf{C}} H^{n-i}(X, \mathcal{O}_X(-m-n-2)). \end{aligned}$$

Therefore, by the Serre duality theorem,

$$h^i_m = dim_{\mathbf{C}} H^i(X, K_X(m+n+2)).$$

Thus, since $i > 0$ and $m+n+2 \geq -i-2+n+2 > 0$, the Kodaira vanishing theorem implies that $h^i_m = 0$.

(ii) $\quad i = n, m \geq -i-2.$

In this case, $k = -m-n-2 \leq 0$. Therefore,

$$h^i_m = dim_{\mathbf{C}} H^1(\mathbf{CP}^{n+1}, B(k)) = dim_{\mathbf{C}}\, coker(\Omega_k),$$

where Ω_k is a $\mathbf{C}$-linear map

$$\Omega_k : H^\circ(\mathbf{CP}^{n+1}, A(k)) = H^\circ(\mathbf{CP}^{n+1}, \mathcal{O}_{\mathbf{CP}^{n+1}}(k))$$
$$\to H^\circ(\mathbf{CP}^{n+1}, f_*\mathcal{O}_X(k)) \simeq H^\circ(X, \mathcal{O}_X(k)).$$

Since Ω_k is surjective, $h^i_m = 0$.

(iii) $\quad i = n+1, m \geq -i-2.$

In this case, $k = -m-n-2 \leq 1$. Therefore,

$$h^i_m = dim_{\mathbf{C}} H^\circ(\mathbf{CP}^{n+1}, B(k)) = dim_{\mathbf{C}} ker(\Omega_k),$$

where Ω_k is a restriction map

$$\Omega_k : H^\circ(\mathbf{CP}^{n+1}, A(k)) \simeq H^\circ(\mathbf{CP}^r, \mathcal{O}_{\mathbf{CP}^{n+1}}(k)) \to H^\circ(\mathbf{CP}^{n+1}, f_*\mathcal{O}_X(k))$$
$$\simeq H^\circ(X, \mathcal{O}_X(k)).$$

Since X is nondegenerate in $\mathbf{CP}^{n+1}$, the map Ω_k is injective and $h^i_m = 0$. That completes the proof of the lemma 1. $\square$

Proof of the Lemma 2: There exists an exact sequence

$$\cdots \to \mathcal{F}_i \to \cdots \mathcal{F}_0 \to \mathcal{F} \to 0,$$

where $\mathcal{F}_i = \oplus \mathcal{O}_{\mathbf{P}}(-i-r), r = reg(\mathcal{F})$. Tensoring this exact sequence by a locally free sheaf $\mathcal{F}^{\otimes(q-1)}$ gives an exact sequence

$$\cdots \to \oplus \mathcal{F}^{\otimes(q-1)}(-i-r) \to \cdots \to \oplus \mathcal{F}^{\otimes(q-1)}(-r) \to \mathcal{F}^{\otimes q} \to 0.$$

Therefore, induction on q yields that the sheaf $\mathcal{F}^{\otimes q}$ is qr-regular. That completes the proof of the lemma 2 and the proof of the theorem. $\square$

VI. The Eagon-Northcott complex.

We follow the construction of the Eagon-Northcott complex outlined in [Sz,§2.1].

Let $0 \to \mathcal{E} \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ be an exact sequence of vector bundles of ranks e, f, g, respectively, over a noëtherian scheme X.

Let $P = \mathbf{Proj}(\mathcal{G})$. The relative projective space P is endowed with the invertible sheaf $\mathcal{O}_P(1)$ and the projection $\pi : P \to X$.

Let $\lambda : (\pi^*\mathcal{F})(-1) \to \mathcal{O}_P$ denote the surjective homomorphism obtained by twisting by $\mathcal{O}_P(-1)$ the composition of $\pi^*\phi : \pi^*\mathcal{F} \to \pi^*\mathcal{G}$ with the canonical homomorphism $\pi^*\mathcal{G} \to \mathcal{O}_P(1)$. The homomorphism λ uniquely extends to a degree decreasing differentiation R_λ of the exterior algebra $Q = \oplus_{i=0}^{f} \wedge^i ((\pi^*\mathcal{F})(-1))$, thus giving rise to the following Koszul complex:

$$O \to \wedge^f ((\pi^*\mathcal{F})(-1)) \to \cdots \to (\pi^*\mathcal{F})(-1) \to \mathcal{O}_P \to 0. \tag{4}$$

Let $p \in P$. Then an element $v \in (\pi^*\mathcal{F})(-1))_p$ defines the degree increasing differentiation $L_v = \wedge v : Q_p \to Q_p$, such that $L_v \cdot R_{\lambda,p} + R_{\lambda,p} \cdot L_v = \lambda_p(v) Id_{Q_p}$. Therefore, for appropriate v, L_v gives a homotopy of the complex (4) localized at p. This shows that the Koszul complex (4) is acyclic, i.e., it is an exact sequence. Twisting (4) by $\mathcal{O}_P$ gives the following exact sequence:

$$\begin{aligned} 0 \to \mathcal{O}_P(1-f) \otimes \pi^*(\wedge^f \mathcal{F}) \to \cdots \to \mathcal{O}_P(-1) \otimes \pi^*(\wedge^2 \mathcal{F}) \\ \to \pi^*\mathcal{F} \to \mathcal{O}_P(1) \to 0. \end{aligned} \tag{5}$$

The higher direct images of vector bundles in (5) give a converging to 0 spectral sequence of sheaves on X:

$$\begin{aligned} E^{p,q} &= R^p\pi_*(\mathcal{O}_P(1-q) \otimes \pi^*(\wedge^q \mathcal{F})) \\ &= \wedge^q \mathcal{F} \otimes R^p\pi_*(\mathcal{O}_P(1-q)) \Rightarrow 0. \end{aligned} \tag{6}$$

The only nonzero elements of this sequence are:

$$E^{0,0} = \mathcal{G},$$

$$E^{0,1} = \mathcal{F},$$

$$E^{g-1,q} = \wedge^q \mathcal{F} \otimes S^{q-1-g}(\mathcal{G}^\vee) \otimes (\wedge^g \mathcal{G})^\vee, \text{ if } f \geq q \geq g+1$$

(see [Ha,p. 253] for the case $g \geq 2$; in case $g = 1$ the sequence (6) coincides with (5)). The total complex $\cdots \to \underset{q-p=i}{\oplus} E^{p,q} \to \cdots$ associated to the spectral sequence (6) is, therefore, the following exact sequence:

$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[dr] & & 0 & & \\
 & & & & \mathcal{E} \arrow[ur] \arrow[dr] & & & \\
O \arrow[r] & \wedge^f \mathcal{F} \otimes (S^{f-1-g}\mathcal{G})^\vee \otimes (\wedge^g \mathcal{G})^\wedge \arrow[r] & \cdots & \wedge^{g+1} \mathcal{F} \otimes (\wedge^g \mathcal{G})^\vee \arrow[ur] \arrow[rr] & & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \arrow[r] & 0.
\end{tikzcd}$$

This exact sequence induces the Eagon-Northcott complex - the following exact sequence of vector bundles on X:

$$0 \to \wedge^f \mathcal{F} \otimes (S^{f-1-g}\mathcal{G})^\vee \otimes (\wedge^g \mathcal{G})^\vee \to \cdots \to \wedge^{g+i+1}\mathcal{F} \otimes (S^i \mathcal{G})^\vee \otimes (\wedge^g \mathcal{G})^\vee$$
$$\to \cdots \to \wedge^{g+1}\mathcal{F} \otimes (\wedge^g \mathcal{G})^\vee \to \mathcal{E} \to 0.$$

VII. Regularity of the ideal sheaf of a Cohen-Macaulay projective variety

Theorem. *Let $X \subset \mathbf{P}_k^N$ be an equidimensional Cohen-Macaulay variety of dimension n and degree d over an algebraically closed field k of characteristic greater or equal n. Let K_X denote the dualizing sheaf on X. Define $r = reg(K_X) - n - 1$. Then, $reg(\mathcal{J}_{X|\mathbf{P}_k^N}) \leq (n-1)(r-1) + d(d+n)$.*

Remark *If X is smooth projective variety over a field of characteristic 0 then the Kodaira vanishing theorem implies that $r \leq 0$.*

Proof of the theorem: Consider a linear projection $\pi : X \to \mathbf{P}^n_k$ with center λ, an $(N-n-1)$-dimensional linear subspace of $\mathbf{P}^N_k$.

Restricting on X an element

$$v \in H^\circ(\mathbf{P}^N_k, \mathcal{O}_{\mathbf{P}^N_k}(m))$$

determines a section in

$$H^\circ(X, \mathcal{O}_X(m)) \simeq H^\circ(\mathbf{P}^n_k, \pi_* \mathcal{O}_X(m)) \simeq \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^n_k}(-m), \pi_* \mathcal{O}_X)$$

giving a map $R_v : \mathcal{O}_{\mathbf{P}^n_k}(-m) \to \pi_* \mathcal{O}_X$ which depends linearly on v. Choose a k-algebra homomorphism

$$\alpha : k[\lambda] = \oplus_{m=0}^{\infty} H^\circ(\lambda, \mathcal{O}_\lambda(m)) \to k[\mathbf{P}^N_k] = \oplus_{m=0}^{\infty} H^\circ(\mathbf{P}^N_k, \mathcal{O}_{\mathbf{P}^N_k}(m)),$$

a lifting of the restriction homomorphism $k[\mathbf{P}^N_k] \to k[\lambda]$. Consider a map

$$\begin{aligned} R : \oplus_{m=0}^{d} \mathcal{O}_{\mathbf{P}^n_k}(-m) \otimes H^\circ(\lambda, \mathcal{O}_\lambda(m)) &\to \pi_* \mathcal{O}_X \\ : f \otimes v &\mapsto R_{\alpha(v)}(f). \end{aligned}$$

Since the map $\pi : X \to \mathbf{P}^n_k$ is a $d : 1$ covering, polynomials of degree d separate the points in each scheme-theoretic fiber of π. Therefore, the map R is surjective. Hence, there is an exact sequence on $\mathbf{P}^n_k$:

$$0 \to \mathcal{E} \to \mathcal{F} = \oplus \mathcal{O}_{\mathbf{P}^n_k}(-d) \to \pi_* \mathcal{O}_X \to 0, \tag{7}$$

such that, for any integer t, the image of the map

$$H^\circ(\mathbf{P}^n_k, \mathcal{F}(t)) \to H^\circ(\mathbf{P}^n_k, \pi_* \mathcal{O}_X(t)) \simeq H^\circ(X, \mathcal{O}_X(t))$$

is contained in the image of the map

$$H^\circ(\mathbf{P}^N_k, \mathcal{O}_{\mathbf{P}^N_k}(t)) \to H^\circ(X, \mathcal{O}_X(t)).$$

It follows that $reg(\mathcal{J}_{X/\mathbf{P}_k^N}) \leq reg(\mathcal{E})$.

$\pi_* \mathcal{O}_X$ is a Cohen-Macaulay sheaf on $\mathbf{P}_K^n$ of dimension n (c.f. [S, Proposition IV. 11]), hence, a locally free sheaf of rank d (c.f. [S, Proposition IV. 21]). Therefore, the sequence (7) is an exact sequence of vector bundles on $\mathbf{P}_k^n$, which gives rise to the corresponding Eagon-Northcott complex – the following exact sequence of vector bundles on X:

$$\cdots \to \mathcal{E}_i \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{E} \to 0, \tag{8}$$

where

$$\mathcal{E}_i = \wedge^{d+i+1} \mathcal{F} \otimes (S^i \pi_* \mathcal{O}_X)^\vee \otimes (\wedge^d \pi_* \mathcal{O}_X)^\vee$$
$$= \oplus (S^i \pi_* \mathcal{O}_X)^\vee (-d(d+i+1) - c_1(\pi_* \mathcal{O}_X)).$$

Lemma 1. $c_1(\pi_* \mathcal{O}_X) \leq 0$.

Proof. Taking successive generic hyperplane sections and using the fact that the first Chern class $c_1(\pi_* \mathcal{O}_X)$ is preserved, we may assume that $dim X = 1$. Then, according to the Grothendieck theorem [Gr, Théorème 2.1], the sheaf $\pi_* \mathcal{O}_X$, as any vector bundle on $\mathbf{P}_k^1$, splits into a direct sum of line bundles: $\pi_* \mathcal{O}_X = \oplus_{i=1}^d \mathcal{O}(\alpha_i)$. Since $H^\circ(\mathbf{P}_k^1, \pi_* \mathcal{O}_X(-1)) = 0, \alpha_i \leq 0$ for any $i = 1, \ldots, d$. Therefore, $c_1(\pi_* \mathcal{O}_X) = \sum_{i=1}^d \alpha_i \leq 0$. □

Lemma 2. $reg(\pi_* \mathcal{O}_X^\vee) \leq r$.

Proof:

By definition of regularity, we must show that if $i \geq 1, s \geq 0$ then

$$H^i(\mathbf{P}_k^n, (\pi_* \mathcal{O}_X)^\vee (r + s - i)) = 0.$$

Applying twice the Serre duality theorem, we get:

$$H^i(\mathbf{P}^n_k, (\pi_* \mathcal{O}_X)^\vee (r+s-i)) \simeq H^{n-i}(\mathbf{P}^n_k, \pi_* \mathcal{O}_X(-reg(K_X) - s + i))$$
$$\simeq H^{n-i}(X, \mathcal{O}_X(-reg(K_X) - s + i)) \simeq H^i(X, K_X(reg(K_X) + s - i)) = 0.$$

□

Turning to the proof of the theorem, notice that if $i < n \leq char\, k$ then the sheaf $(S^i \pi_* \mathcal{O}_X)^\vee$ is a direct summand of the sheaf $(\pi_* \mathcal{O}_X^\vee)^{\otimes i}$, whose regularity is bounded, according to Lemma 2 of §V, by $i \cdot reg(\pi_* \mathcal{O}_X^\vee)$.

Chopping (8) into short exact sequences, we get that

$$reg(\mathcal{E}) \leq \max_{0 \leq i \leq n-1} (reg(\mathcal{E}_i) - i).$$

Then the lemmas imply that

$$\begin{aligned} reg(\mathcal{J}_{X/\mathbf{P}^N_k}) \leq reg(\mathcal{E}) &\leq \max_{0 \leq i \leq n-1} (reg(\mathcal{E}_i) - i) \\ &\leq \max_{0 \leq i \leq n-1} (i(reg(\pi_* \mathcal{O}_X^\vee) - 1) + d(d+i+1)) \\ &= (n-1)(reg(\pi_* \mathcal{O}_X^\vee) - 1) + d(d+n) \\ &\leq (n-1)(r-1) + d(d+n). \end{aligned}$$

That completes the proof of the theorem. □

Applying the argument used in the proof of the theorem to the case $n = 2$, we get the following:

Proposition. Let $X \subset \mathbf{P}^N_k$ be a Cohen-Macaulay surface of degree d over an algebraically closed field k of arbitrary characteristic. Then, $reg(\mathcal{J}_{X/\mathbf{P}^N_k}) \leq d^2 + 2d - 1$.

Proof: The sequence (8) yields the following exact sequence of vector bundles on $\mathbf{P}^2_k$:

$$\oplus(\pi_*\mathcal{O}_X)^\vee(-d(d+2)-c_1) \to \oplus\mathcal{O}_{\mathbf{P}^2_k}(-d(d+1)-c_1) \xrightarrow{\varphi} \mathcal{E} \to 0\ , \tag{9}$$

where $c_1 = c_1(\pi_*\mathcal{O}_X) \leq 0$.

Then, for any integer s, we get the following exact sequences of cohomology modules:

$$0 = H^1(\mathbf{P}^2_k, \oplus\mathcal{O}_{\mathbf{P}^2_k}(s-d(d+1)-c_1)) \to H^1(\mathbf{P}^2_k, \mathcal{E}(s)) \to H^2(\mathbf{P}^2_k, (ker\varphi)(s)), \tag{10}$$

$$H^2(\mathbf{P}^2_k, \oplus(\pi_*\mathcal{O}_X)^\vee(s-d(d+2)-(c_1)) \to H^2(\mathbf{P}^2_k, (ker\varphi)(s)) \to 0, \tag{11}$$

and

$$H^2(\mathbf{P}^2_k, \oplus\mathcal{O}_{\mathbf{P}^2_k}(s-d(d+1)-c_1)) \to H^2(\mathbf{P}^2_k, \mathcal{E}(s)) \to 0. \tag{12}$$

Since

$$H^2(\mathbf{P}^2_k, (\pi_*\mathcal{O}_X)^\vee(t)) \cong H^\circ(\mathbf{P}^2_k, (\pi_*\mathcal{O}_X)(-3-t)) \cong H(X, \mathcal{O}_X(-3-t)) = 0$$

if $t \geq -2$, we get from (10) and (11):

$$H^1(\mathbf{P}^2_k, \mathcal{E}(s)) = 0 \quad \text{if} \quad s \geq d(d+2)-2\ .$$

From (12):

$$H^2(\mathbf{P}^2_k, \mathcal{E}(s)) = 0 \qquad \text{if} \qquad s \geq d(d+1)-2\ .$$

Therefore,

$$reg(\mathcal{J}_{X/\mathbf{P}^N_k}) \leq reg(\mathcal{E}) \leq d(d+2)-1\ .$$

□

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