EX2

roi hezkiyahu

30 11 2022

Q1

Question 1.

Let $Y_1, \ldots, Y_n$ be a sample from an exponential distribution $\exp(\theta)$ with $\mu = EY = 1/\theta$.

1. Find UMVUEs for θ and μ, and check whether they achieve Cramer-Rao lower bound. (hint: recall that $Y_1 + \cdots + Y_n \sim \frac{1}{2\theta}\chi^2_{2n} = Gamma(\alpha = n, \beta = \theta)$)

2. The survival function of the exponential distribution $S(t)$ is defined as $S(t) = P(Y \geq t) = e^{-\theta t}$. Verify that for a given t, a trivial unbiased estimator for $S(t)$ is 1, if $Y_1 \geq t$ and 0 otherwise. Using it and applying the Rao-Blackwellization, find the UMVUE for $S(t)$. (hint: derive first the conditional density of Y_1 given $Y_1 + \ldots + Y_n$ via the joint density of Y_1 and $Y_1 + \ldots + Y_n$)

a

$$Y_1, \ldots Y_n \sim exp(\theta)$$
$$\hat{\mu} = \bar{y}$$
$$E(\hat{\mu}) = E(y_1) = \frac{1}{\theta} \Rightarrow \hat{\mu} \text{ is an unbiased estimator and also MLE so UMVUE}$$
$$\text{lets show it achives the CRB:}$$
$$\text{define: } g(\theta) = \frac{1}{\theta}$$
$$CRB(\mu) = \frac{g'(\theta)^2}{I(\theta)} = \frac{\frac{1}{\theta^4}}{-E(l''(\theta, y))} = \frac{\frac{1}{\theta^4}}{E(\frac{-n}{\theta^2})} = \frac{n}{\theta^2}$$
$$V(\bar{y}) = \frac{1}{4\theta^2}4n = \frac{n}{\theta^2} = CRB(\mu)$$
$$\text{from MLE properties we can derive: } \hat{\theta} = \frac{1}{\bar{y}} := \frac{n}{x}; \quad x = n\bar{y}$$
$$E(\hat{\theta}) = \int_0^\infty \frac{n}{x}\frac{\theta^n}{\Gamma(n)}e^{-\theta x}x^{n-1}dx = \frac{n\theta\Gamma(n-1)}{\Gamma(n)}\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)}e^{-\theta x}x^{n-2}dx = \frac{n\theta\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{n\theta}{(n-1)} \Rightarrow$$
$$\hat{\theta} \; is \; biased; \frac{(n-1)\hat{\theta}}{n} = \frac{n-1}{x} \; is \; unbiased$$
$$\text{lets see if it achives the CRB:}$$
$$V(\hat{\theta}) = V(\frac{n-1}{x}) = E((\frac{n-1}{x})^2) - E(\frac{n-1}{x})^2 = E((\frac{n-1}{x})^2) - \theta^2$$
$$E((\frac{n-1}{x})^2) = \int_0^\infty \frac{(n-1)^2}{x^2}\frac{\theta^n}{\Gamma(n)}e^{-\theta x}x^{n-1}dx = \frac{(n-1)^2\theta^n}{\Gamma(n)}\int_0^\infty e^{-\theta x}x^{n-3}dx =$$
$$= \frac{(n-1)^2\theta^2\Gamma(n-2)}{\Gamma(n)}\int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)}e^{-\theta x}x^{n-3}dx = \frac{(n-1)^2\theta^2\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)} =$$
$$= \frac{(n-1)\theta^2}{(n-2)} \Rightarrow V(\hat{\theta}) = \frac{(n-1)\theta^2}{(n-2)} - \theta^2 = \frac{\theta^2}{n-2} > \frac{\theta^2}{n} = \frac{1}{I(\theta)} = CRB(\theta)$$
$$\text{thus the MLE for } \theta \text{ does not achive the CRB}$$

b

$$define \; T_s(y) \in \{0, 1\}, T_s = 1 \iff Y_1 > t$$
$$E(T_s(y)) = P(Y_1 > t) = e^{-\theta y} \Rightarrow bias(T_s(y)) = 0$$
$$T_1 = E(T_s(y)|\sum_{i=1}^n y_i = x) = P(y_1 > t|\sum_{i=1}^n y_i = x) = \frac{P(y_1 > t, \sum_{i=1}^n y_i = x)}{P(\sum_{i=1}^n y_i = x)} = \frac{\theta e^{-t\theta}\frac{\theta^{n-1}}{\Gamma(n-1)}e^{-\theta(x-t)}(x-t)^{n-2}}{\frac{\theta^n}{\Gamma(n)}e^{-\theta x}x^{n-1}} = \frac{\frac{\theta^n}{\Gamma(n-1)}(x-t)^{n-2}}{\frac{\theta^n}{\Gamma(n)}x^{n-1}}$$
$$= \frac{\frac{\Gamma(n)}{\Gamma(n-1)}(x-t)^{n-2}}{x^{n-1}} = \frac{(n-1)(x-t)^{n-2}}{x^{n-1}}$$
$$T_1 \text{ is unbiased and } \sum_{i=1}^n y_i = x \text{ is complete thus from lemann scheffe theorom we get: } T_1 is \; UMVUE$$

Q2

Question 2.

Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with a finite mean μ and varaince σ^2. Find the asymptotic distributions of $\bar{Y}^2$ when $\mu \neq 0$ and $\mu = 0$, and of $e^{-\bar{Y}}$.

$$y_1, \ldots, y_m \sim f_{\mu,\sigma^2} \Rightarrow \bar{Y} \dot{\sim} N(\mu, \frac{\sigma^2}{n})$$

$$\text{in the case: } \mu = 0 : \bar{Y} \dot{\sim} N(0, \sigma^2) \Rightarrow \frac{\bar{Y}\sqrt{n}}{\sigma} \dot{\sim} N(0,1) \Rightarrow (\frac{\bar{Y}\sqrt{n}}{\sigma})^2 \dot{\sim} \chi^2_n \Rightarrow \bar{Y}^2 \dot{\sim} \frac{\sigma^2}{n}\chi^2_n$$

$$\text{in the case: } \mu \neq 0 :$$

$$\text{define } g(x) = x^2 \text{ and recall } \bar{Y} \text{ is a CAN estimator:}$$

$$\sqrt{n}(\bar{Y}^2 - \mu^2) \dot{\sim} N(0, 4\mu^2\sigma^2) \Rightarrow \bar{Y}^2 \dot{\sim} N(\mu^2, 4\mu^2\sigma^2 n^{-1})$$

$$e^{-\bar{Y}} \dot{\sim} N(e^{-\mu}, e^{-2\mu}\sigma^2 n^{-1})$$

Q3

Question 3.

Let $Y_1, \ldots, Y_n$ be a sample from a shifted exponential distribution with the rate one, that is, $f(y) = e^{-(y-\theta)},\ y \geq \theta$.

1. Find the MLE $\hat{\theta}$ for θ.
2. Is $\hat{\theta}$ consistent in MSE? Is it consistent (in probability)?
   (*Hint*: show that $\hat{\theta}$ has also a shifted exponential distribution with the same shift θ but with the rate n, that is, its density $g(u) = ne^{-n(u-\theta)},\ u \geq \theta$)
3. Show that $n(\hat{\theta} - \theta) \sim exp(1)$.
4. Is $\hat{\theta}$ a CAN estimator? If not, why does the asymptotic normality of MLE not hold in this case?

a

$$L(\theta, y) = e^{-\sum_i (y-\theta)} \Rightarrow \text{L is an ascending function of } \theta \text{ thus we want to get the maximal } \theta \text{ possible :}$$

$$\hat{\theta} = y_{(1)}$$

b

$$\text{as the hint suggets lets derive the density of } \hat{\theta}$$

$$P(\hat{\theta} \leq t) = P(y_{(1)} \leq t) = 1 - P(y_{(1)} > t) = 1 - \Pi_i^n P(y_i > t) = 1 - (1 - (1 - e^{t-\theta}))^n = 1 - e^{-n(t-\theta)} = F(shiftedexp(n, \theta))$$

$$V(\hat{\theta}) = \frac{1}{n^2} \text{(location doesnt change variance)}$$

$$E(\hat{\theta}) = \frac{1}{n} + \theta \overset{n\to\infty}{\to} \theta \Rightarrow consistent\ in\ MSE \Rightarrow consistent\ in\ probability$$

c

$$P(n(\hat{\theta} - \theta) < t) = P(\hat{\theta} < t/n + \theta) = 1 - e^{n(t/n - \theta + \theta)} = 1 - e^{-t} = F(exp(1))$$

d

$$\hat{\theta} \text{ is not a CAN estimator:}$$

$$P(\sqrt{n}(\hat{\theta} - \theta) < t) = P(\hat{\theta} < t/\sqrt{n} + \theta) = 1 - e^{n(t/\sqrt{n} - \theta + \theta)} = 1 - e^{-\sqrt{n}t} = F(exp(\sqrt{n}))$$

$$\text{when } n \to \infty, \quad \hat{\theta} \to 0$$

$$\text{in this case the MLE is not a CAN estimate because } f_\theta(y) \text{ is not a continous function it has an asymptote at } \theta$$

Q4

Question 4.

1. Let $Y_1, \ldots, Y_n \sim f_\theta(y)$. Show that the M-estimator corresponding to $\rho(y, \theta) = \alpha(y - \theta)_+ + (1 - \alpha)(\theta - y)_+$ for a given $0 < \alpha < 1$ is the sample $\alpha \cdot 100\%$-quantile $Y_{(\alpha)}$.
2. Show that under the reguralitty conditions, $Y_{(\alpha)}$ is a consistent estimator of a $\alpha \cdot 100\%$-quantile of the distribution $f_\theta(y)$.
3. Assuming the required regularity conditions, find the asymptotic distribution of $Y_{(\alpha)}$.

a

$$\text{we are looking for a } \hat{\theta} = argmin_\theta \sum_{i=1}^{n} \rho(y_i, \theta)$$

$$\text{denote } n_- = \sum_{i=1}^{n} I(y_i < \theta), \quad n_+ = n - n_-$$

$$\sum_{i=1}^{n} \rho(y, \theta) = n_+ \alpha + (1 - \alpha) n_-$$

$$\psi(y, \theta) = \begin{cases} -1, & y > \theta \\ 0, & y = \theta \\ 1, & y < \theta \end{cases}$$

$$\frac{\partial \sum_{i=1}^{n} \rho(y, \theta)}{\partial \theta} = -n_+ \alpha + n_- (1 - \alpha)$$

$$\text{if } \theta = \hat{\theta_{(\alpha)}} = y_{(\alpha)}:$$

$$n_+ = n(1 - \alpha), \quad n_- = n\alpha \Rightarrow \frac{\partial \sum_{i=1}^{n} \rho(y, \theta)}{\partial \theta} = -n(1 - \alpha)\alpha + (1 - \alpha)n\alpha = 0 \Rightarrow \hat{\theta_{(\alpha)}} = argmin_\theta \sum_{i=1}^{n} \rho(y_i, \theta)$$

notice that this is indeed a minimum point as the function is not bound from above w.l.o.g difine $\alpha > 0.5$
we can alwyas decreace θ and get a higher value for the function ρ

b

because M-stimator are unbiased all we need to show is that $E(\frac{n_-}{n}) = \alpha$ this is stright forward from the quantile defenition

c

$$\sqrt{n}(y_\alpha - Y_\alpha) \dot{\sim} N(0, V(Y_\alpha))$$

$$\text{we know from class that: } Y_\alpha = \frac{E(\psi^2)}{E^2(\psi')}$$

$$\text{but in our case: } \psi' = 0$$

thus the variance is not finite

Q5

Question 5.

The number of power failures in an electrical network per day is Poisson distributed with an unknown mean λ. During the last month (30 days), 5 power failures have been registered. Let p be the probability that there is no power failures during a day.

1. Find the MLE for p.
2. Derive 95% asymptotic confidence intervals for p using asymptotic normality of MLE and using the variance stabilizing transformation for Poisson data.

a

denote the number of power failures per day as :pfd, and power failures per month as: pfm

$$pfd \sim Pois(\lambda) \Rightarrow pfm \sim Pois(30\lambda)(\text{ as a sum of poisson variables})$$

$$L(\lambda, pfm) = (30\lambda)^5 e^{-30\lambda}/5! \Rightarrow l = 5ln(30\lambda) - 30\lambda - ln(5!)$$

$$\frac{\partial l}{\partial \lambda} = \frac{5}{\lambda} - 30 = 0 \iff \lambda = \frac{5}{30} = \frac{1}{6} := \hat{\lambda}_{MLE}$$

$$\hat{p}_{MLE} = e^{-\hat{\lambda}_{MLE}} \approx 0.846$$

b

$$-E(l'') = \frac{30\lambda}{\lambda^2} = \frac{30}{\lambda}$$
$$\hat{\lambda}_{MLE} \dot{\sim} N(30\lambda, \frac{1}{-E(l'')}) = N(30\lambda, \frac{\lambda}{30})$$
$$g(\lambda) := e^{-\lambda}$$
$$\text{from the variance stability transformation we get:}$$
$$\hat{p}_{MLE} = g(\hat{\lambda}_{MLE}) \dot{\sim} N(e^{-30\lambda}, \frac{\lambda^2 e^{-2\lambda}\lambda}{30}) = N(e^{-30\lambda}, \frac{\lambda^3 e^{-2\lambda}}{30})$$

Q6

Question 6.

Let $Y_1, \ldots, Y_m \sim B(1, p)$.

1. Derive the asymptotic α-level Wald, Rao/score and Wilks tests for testing $H_0 : p = p_0$ vs. $H_1 : p \neq p_0$.
2. Show that all the three test-statistics are asymptotically close under the null.

a

$$\hat{p} = \frac{\sum_{i=1}^{n} yi}{n}$$
$$\hat{p} \dot{\sim} N(p, \frac{1}{nI^*(p)})$$
$$I^*(p) = -E(l'') = -E[(y_i ln(p) + (1-y_i)ln(1-p))''] = -E[(\frac{y_i}{p^2} + \frac{1-y_i}{(1-p)^2})] = \frac{1}{1-p} \Rightarrow \hat{p} \dot{\sim} N(p, \frac{p(1-p)}{n})$$
$$Wald\ test : (\hat{p}-p_0)^2 I(\hat{p}) = (\hat{p}-p_0)^2 \frac{n}{p_0(1-p_0)} = \frac{n(\bar{y}-p_0)^2}{p_0(1-p_0)} > \chi^2_{1,1-\alpha}$$
$$Wilks\ test : 2l(\hat{p}) - 2l(p_0) = 2(n\bar{y}\ ln(\frac{\bar{y}}{p_0}) + n(1-\bar{y})ln(\frac{1-\bar{y}}{1-p_0})) > \chi^2_{1,1-\alpha}$$
$$Rao\ test : \frac{(l'(p_0))^2}{I(p_0)} = (\frac{n\bar{y}}{p_0} - \frac{n(1-\bar{y})}{1-p_0})^2 \frac{p_0(1-p_0)}{n} = \frac{n^2(\bar{y}-p_0)^2}{p_0^2(1-p_0)^2}\frac{p_0(1-p_0)}{n} = \frac{n(\bar{y}-p_0)^2}{p_0(1-p_0)} > \chi^2_{1,1-\alpha}$$

b

$$\text{we can see that Rao and Wald are equivelent, let's use tylor expention for the wilks test:}$$
$$Wilks = 2l(\hat{p}) - 2l(p_0) \approx l''(\hat{p})(\hat{p}-p_0)^2 = (\frac{n\bar{y}}{\bar{y}^2} + \frac{n(1-\bar{y})}{(1-\bar{y})^2})(\hat{p}-p_0)^2 = (\frac{n}{\bar{y}} + \frac{n}{(1-\bar{y})})(\hat{p}-p_0)^2 =$$
$$= (\frac{n}{\bar{y}(1-\bar{y})})(\hat{p}-p_0)^2 \xrightarrow[CLT]{n\to\infty} (\frac{n}{p(1-p)})(\hat{p}-p_0)^2$$
$$\text{under the null } p = p_0 \text{ thus wilks is equivelent to wald and rao}$$

Q7

Question 7.

A director of a large bank has a monthly information from its L branches about the numbers of new clients joined the bank for each of the last n months. Assume that the the number of new clients joined a j-th branch each month is $Pois(\lambda)$, there is no correlation neither between diffirent branches nor between different months, i.e. $Y_{ij} \sim Pois(\lambda_j),\ i = 1, \ldots, n;\ j = 1, \ldots, L$ and all Y_{ij}'s are independent.

1. Find the MLE $\hat{\lambda}$ for the vector $\lambda = (\lambda_1, \ldots, \lambda_L)^t$ and the asymptotic distribution of $\sqrt{n}(\hat{\lambda} - \lambda)$.
2. The director is particularly interested in the proportion of new clients joined a specific branch (say, the first), i.e. in $p = \lambda_1 / \sum_{j=1}^{L} \lambda_j$. Find the MLE and an asymptotic 100(1-α)% confidence interval for p.
3. Derive the α-level GLRT for testing the hypothesis that all the branches are equally successful in "hunting" after new clients.
4. Perform the test for a bank that has three branches and the average numbers of new clients joined the branches over the last 15 months were 10, 12 and 15 respectively (explain approximations you have used if any).

a

$$L = \Pi_{j=1}^{L}\Pi_{i=1}^{n}\lambda_j^{y_{ij}}e^{-\lambda_j}(y_{ij}!)^{-1} = \Pi_{j=1}^{L}\lambda_j^{\sum_i y_{ij}}e^{-n\lambda_j}(\Pi_i y_{ij}!)^{-1} \Rightarrow$$
$$l = \sum_j[(\sum_i y_{ij})ln(\lambda_j) - n\lambda_j + f(y)]$$
take the deravetive with respect to λ_j :
$$\frac{\partial l}{\partial \lambda_j} = \frac{(\sum_i y_{ij})}{\lambda_j} - n = 0 \iff \lambda_j = \bar{y}_j$$
$$\hat{\lambda}_{MLE} = (\bar{y}_1, \dots, \bar{y}_L)$$
$$E(\hat{\lambda}_{MLE}) = \lambda, \quad V(\lambda) = I\lambda$$
thus from CLT we get:
$$\sqrt(n)(\hat{\lambda}_{MLE} - \lambda) \dot{\sim} N(0, V(\lambda))$$

b

$$\text{define } g(\lambda) = (\lambda_i / \sum_j \lambda_j)_{i=1}^{n}$$
$$p = g(\lambda)$$
thus from the delta therom:
$$\hat{p} = (\hat{\lambda}_i / \sum_j \hat{\lambda}_j)_{i=1}^{n} \dot{\sim} N(p, \nabla g^T V(\lambda)\nabla g)$$
$$\nabla g(\lambda) = (\frac{\sum_j \lambda_j - \lambda_i}{(\sum_j \lambda_j)^2})_{i=1}^{n}$$
$$[\nabla g^T V(\lambda)\nabla g]_{ik} = y = \begin{cases} (\frac{\sum_j \lambda_j - \lambda_i}{(\sum_j \lambda_j)^2})^2\lambda_i, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$
100(1-a)% CI is:
$$\hat{p} \pm Z_{1-\alpha/2} * \sqrt{\nabla g(\hat{\lambda})^T V(\hat{\lambda})\nabla g(\hat{\lambda})n^{-1}}$$

c

we can use the wilks test:
$$H_0 : \lambda_1 = \ldots = \lambda_L$$
$$H_1 : \text{otherwise}$$
$$\text{wilks test: } 2l(\hat{\lambda}) - 2l(\lambda_0) \sim \chi_L^2$$
$$l(\lambda_0) = \sum_i\sum_j y_{ij}ln(\lambda_0) - nL\lambda_0 - f(y) = ln(\bar{y})\sum_i\sum_j y_{ij} - nL\bar{y} = ln(\bar{y})\sum_i\sum_j y_{ij} - n\sum_j \bar{y}_j$$
$$l(\hat{\lambda}) = \sum_j[\sum_i y_{ij}ln(\bar{y}_j)] - n\bar{y}_j$$
plug it back in the wilks test and we will reject H_0 if:
$$2(\sum_j[\sum_i y_{ij}ln(\bar{y}_j)] - ln(\bar{y})\sum_i\sum_j y_{ij}) > \chi_{L,1-\alpha}^2$$

d

difine s as the wilks statistic:
$$s = 2(150ln(10) + 180ln(12) + 225ln(15) - ln(37/3)555) = 15.30527 > 7.814 = \chi_{3,0.95}^2$$
thus we reject the null at 95% confidence