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Solution 1

Introduction to LMIs and Optimization Toolboxes

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1 Exercise

Linear Matrix Inequalities

1. Consider the set of matrices

$$\mathcal{F} = \{F(x) \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^m \mid F(x) = F_0 + \sum_{i=1}^{m} F_i x_i, F(x) \succeq 0\},$$

   where $F_i \in \mathbb{R}^{n\times n}$. Show that this set is convex.

2. Consider the ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n | (x - x_c)^T P(x - x_c) \leq 1\}$, which P symmetric and $P \succeq 0$. Rewrite the inequality $(x - x_c)^T P(x - x_c) \leq 1$ as an LMI in the variable P^{-1} using Schur's complement. Can you also formulate the LMI in the variable P?

Semi-definite Programming

1. Write down an SDP which computes the largest ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n \mid x^T P x \leq 1\}$ fully contained in the polytope $\mathbb{X} = \{x \in \mathbb{R}^n | A_x x \leq b_x\}$, with $A_x \in \mathbb{R}^{n_x\times n}$, $b_x \in \mathbb{R}^{n_x}$.
   Hint: The volume of $\mathcal{E}$ is proportional to logdet(P^{-1}), where logdet is concave.
   Hint: An ellipsoid is contained within a halfspace $a^T x \leq b$ if $a^T P^{-1} a \leq b^2$.

2. Complete the provided MATLAB template to solve the SDP in 1. using YALMIP.

2 Solution

Linear Matrix Inequalities

1. Note that a set $\mathcal{A}$ is convex if it holds for all $x, y \in \mathcal{A}$ and $\lambda \in [0, 1]$, that $\lambda x + (1-\lambda)y \in \mathcal{A}$. Using the definition of $\mathcal{F}$, we have

$$\begin{aligned} F(\lambda x + (1-\lambda)y) &= F_0 + \sum_{i=1}^{m} (\lambda x_i + (1-\lambda)y_i)F_i \\ &= \lambda F_0 + (1-\lambda)F_0 + \lambda \sum_{i=1}^{m} x_i F_i + (1-\lambda) \sum_{i=1} y_i F_i \\ &= \lambda F(x) + (1-\lambda)F(y) \\ &\succeq 0 \end{aligned}$$

2. We have

$$\begin{aligned} &(x - x_c)^T P(x - x_c) \leq 1 \\ \Leftrightarrow &1 - (x - x_c)^T P(x - x_c) \geq 0 \\ \Leftrightarrow &\begin{bmatrix} 1 & (x - x_c)^T \\ (x - x_c) & P^{-1} \end{bmatrix} \succeq 0 \end{aligned}$$

By defining $E = P^{-1}$, we have

$$\Leftrightarrow \begin{bmatrix} 1 & (x - x_c)^T \\ (x - x_c) & E \end{bmatrix} \succeq 0.$$

Alternatively, we can write

$$\begin{aligned} &1 - (x - x_c)^T P(x - x_c) \geq 0 \\ \Leftrightarrow &1 - (x - x_c)^T P P^{-1} P(x - x_c) \geq 0 \\ \Leftrightarrow &\begin{bmatrix} 1 & (x - x_c)^T P \\ P(x - x_c) & P \end{bmatrix} \succeq 0 \end{aligned}$$

Semi-definite Programming

1. By using $E = P^{-1}$, we can write the SDP as

$$\begin{aligned} &\min_{E} - \operatorname{logdet} E \\ &\text{s.t. } \forall i = 0, \ldots, n_x \\ &\quad E \succeq 0 \\ &\quad [A_x]_i E [A_x]_i^T \leq [b_x]_i^2 \end{aligned}$$

2. The MATLAB Code for this question can be found on Moodle.