Advanced Model Predictive Control

Recitation 4
Nonlinear robust MPC - Part II

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Nonlinear robust MPC

Nonlinear robust MPC [online]

$$
\begin{aligned}
\min_{V,z_0} \quad & \sum_{i=0}^{N-1} l(z_i, v_i) + l_{\mathrm{f}}(z_N) \\
\text{s.t.} \quad & z_{i+1} = f(z_i, v_i), \quad && i \in [0, N-1], \\
& (z_i, v_i) \in \overline{\mathcal{Z}}, && i \in [0, N-1], \\
& z_N \in \mathcal{X}_{\mathrm{f}}, \\
& (x(k), z_0) \in \Omega
\end{aligned}
$$

RPI set computation [offline]

$$
\begin{aligned}
\min_{E,Y} \quad & L(E, Y) \\
\text{s.t.} \quad & E \succ 0, \quad E \succeq I \\
& \begin{bmatrix} \rho^2 E & (A(x_1)E + BY)^\top \\ A(x_1)E + BY & E \end{bmatrix} \succeq 0
\end{aligned}
$$

RPI set computation for segway example

Computation of RPI set Ω [offline]

$$\begin{aligned} \min_{E,Y} \quad & L(E,Y) \\ \text{s.t.} \quad & E \succeq I, \\ & \begin{bmatrix} \rho^2 E & (A(x_1)E + BY)^\top \\ A(x_1)E + BY & E \end{bmatrix} \succeq 0, \quad \forall x_1 \in [-30^\circ, 30^\circ] \end{aligned}$$

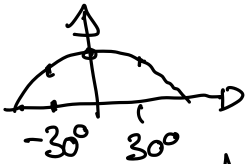
$$\begin{bmatrix} * & A_{1,2} & * \\ * & A_{1,2}^\top & * \end{bmatrix} \succeq 0 \qquad \alpha \begin{bmatrix} * & A_1 & * \\ * & A_1^\top & * \end{bmatrix} + (1-\alpha)\begin{bmatrix} * & A_2 & * \\ * & A_2^\top & * \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} * & A(x_1) & * \\ * & A(x_1)^\top & * \end{bmatrix} \succeq 0 \quad \forall A(x_1) \in Co(A_1, A_2)$$

Collaborative Exercise: We know from last recitation, that

$$A(x_1) = \begin{bmatrix} 1 & \delta t \\ -\delta t(k + \frac{g}{l}\underline{\cos(x_1)}) & 1 - \delta t \cdot c \end{bmatrix}.$$

How, do we make sure that the set Ω is RPI for all angles between -30° and $+30^\circ$?



$A_1 = A(x_1)|_{x_1 = 0^\circ}$

$A_2 = A(x_1)|_{x_1 = 30^\circ}$

$A(x_1) \in C_o(A_1, A_2)$

$A(x_1) = \alpha A_1 + (1-\alpha) A_2$

RPI set computation for segway example

Computation of RPI set Ω [offline]

$$\begin{aligned} \min_{E,Y} \quad & L(E,Y) \\ \text{s.t.} \quad & E \succeq I, \\ & \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0 \end{aligned}$$

Collaborative Exercise: Which cost function $L(E,Y)$ should we use for this problem?

$V(\Omega) \propto \log\det(E)$

$\log\det(E) \leq \mathrm{trace}(E - I)$
$= \mathrm{trace}(E) - \mathrm{trace}(I)$

Implementation

Offline Computations

Online Computations

Implementation

Offline Computations

Solve

$$
\begin{aligned}
\min_{E,\,Y} \quad & \mathrm{tr}\,(E) \\
\text{s.t.} \quad & E \succeq I, \\
& \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0
\end{aligned}
$$

where, $E = P^{-1}$ and $Y = KP^{-1}$.

Online Computations

Implementation

Offline Computations

Solve

$$
\begin{aligned}
\min_{E,Y} \quad & \mathrm{tr}\,(E) \\
\text{s.t.} \quad & E \succeq I, \\
& \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0
\end{aligned}
$$

where, $E = P^{-1}$ and $Y = KP^{-1}$.

Then, we can compute the tightenings as

$$
\tilde{b}_{x,j} = ||P^{-\frac{1}{2}}[A_x]_j^\top||_2 \delta
$$

$$
\tilde{b}_{u,j} = ||P^{-\frac{1}{2}}K^\top[A_x]_j^\top||_2 \delta
$$

where $\delta = \frac{\bar{\alpha}_w}{1-\rho} = \max_{w \in \mathcal{W}} ||w||_P \cdot \frac{1}{1-\rho}$.

Online Computations

$\|w\|_P = \sqrt{w^\top P w}$

Implementation

Offline Computations

Solve

$$
\begin{aligned}
\min_{E,\,Y} \quad & \operatorname{tr}(E) \\
\text{s.t.} \quad & E \succeq I, \\
& \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0
\end{aligned}
$$

where, $E = P^{-1}$ and $Y = KP^{-1}$.
Then, we can compute the tightenings as

$$
\tilde{b}_{x,j} = ||P^{-\frac{1}{2}}[A_x]_j^\top||_2\delta
$$

$$
\tilde{b}_{u,j} = ||P^{-\frac{1}{2}}K^\top[A_x]_j^\top||_2\delta
$$

where $\delta = \frac{\bar{\alpha}_w}{1-\rho} = \max_{w\in\mathcal{W}}||w||_P \cdot \frac{1}{1-\rho}$.

Online Computations

Solve

$$
\begin{aligned}
\min_{V,\,z_0} \quad & \sum_{i=0}^{N-1} z_i^T Q z_i + v_i^T R v_i \\
\text{s.t.} \quad & \forall i \in [0, N-1], \\
& z_{i+1} = f(z_i, v_i), \\
& [A_x]_j z_i \leq [b_x]_j - \tilde{b}_{x,j}, \quad j \in [1, n_x], \\
& [A_u]_j v_i \leq [b_u]_j - \tilde{b}_{u,j}, \quad j \in [1, n_u], \\
& z_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
& ||x(k) - z_0||_P^2 \leq \delta^2
\end{aligned}
$$

for the measured state $x(k)$.

Implementation

Homework (PS4):

1. Implement the computation of the RPI set and the tightenings in the `compute_tightening` method in the provided `Nonlinear_RMPC.m` file.
2. Compute the constraint tightenings for different choices of ρ and observe how the tightenings and Ω change. Choose a ρ for the remainder of the exercise.
3. Implement the nominal nonlinear MPC problem from the previous slide in the provided `Nonlinear_RMPC.m` file.

Offline Computations

Computation of RPI set Ω

$$
\begin{aligned}
\min_{\underline{E}, Y} \quad & \operatorname{tr}(E) \\
\text{s.t.} \quad & E \succeq I, \\
& \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0
\end{aligned}
$$

Computation of tightenings

$$\tilde{b}_{x,j} = \|[A_x]_j^\top\|_{P^{-1}} \cdot \max_{w \in \mathcal{W}} \|w\|_P \cdot \frac{1}{1-\rho}$$

$$\tilde{b}_{u,j} = \|K^\top [A_u]_j^\top\|_{\underbrace{P^{-1}}_{E}} \cdot \max_{w \in \mathcal{W}} \|w\|_{\underbrace{P}_{E^{-1}}} \cdot \frac{1}{1-\rho}$$

Collaborative Exercise: Can we do better than minimizing an upper bound of the volume of $\mathcal{E} = \{x \mid x^\top P x \le 1\}$?

$$\sum_{j=1}^{n_x} \tilde{b}_{x,j} + \sum_{j=1}^{n_u} \tilde{b}_{u,j} \leftarrow$$ use cost which minimizes the tightening

Offline Computations

Rewrite the state tightening:

$$\tilde{b}_{x,j} = \|[A_x]_j^\top\|_E \cdot \max_{w\in\mathcal{W}} \|w\|_{E^{-1}} \cdot \frac{1}{1-\rho}$$

$$= \sqrt{[A_x]_j E [A_x]_j^\top} \cdot \max_{w\in\mathcal{W}} \sqrt{w^\top E^{-1} w} \cdot \frac{1}{1-\rho}$$

$$\le \sqrt{\beta_j} \cdot \sqrt{\omega} \cdot \frac{1}{1-\rho}$$

Exercise: Rewrite the two inequalities

1. $[A_x]_j E [A_x]_j^\top \le \beta_j$
2. $\underline{\max_{w\in\mathcal{W}} \sqrt{w^\top E^{-1} w} \le \sqrt{\omega}}$

as LMIs in the variables E, β_j, ω.

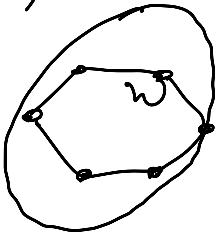
1.) $[A_x]_j E [A_x]_j^\top \le \beta_j$

4 $\quad E^{-1} E = I$

$$[A_x]_j E E^{-1} E [A_x]_j^\top \le \beta_j$$

$$\Leftrightarrow \begin{bmatrix} \beta_j & [A_x]_j E \\ E^\top [A_x]_j^\top & E \end{bmatrix} \succeq 0$$

2.) $w^\top E^{-1} w \le \omega \Leftrightarrow \begin{bmatrix} \omega & w^\top \\ w & E \end{bmatrix} \succeq 0$



$$\begin{bmatrix} \omega & v_w^\top \\ v_w & E \end{bmatrix} \succeq 0$$

$$\forall v_w \in \mathcal{V}(\mathcal{W})$$

Offline Computations

Rewrite the input tightening:

$$\tilde{b}_{u,j} = \|K^\top [A_u]_j^\top\|_E \cdot \max_{w\in\mathcal{W}} \|w\|_{E^{-1}} \cdot \frac{1}{1-\rho}$$
$$\leq \sqrt{[A_u]_j K E K^\top [A_u]_j^\top} \cdot \sqrt{\omega} \cdot \frac{1}{1-\rho}$$
$$\leq \sqrt{\gamma_j} \cdot \sqrt{\omega} \cdot \frac{1}{1-\rho}$$

Homework: Rewrite the inequality

$$[A_u]_j K E K^\top [A_u]_j^\top \leq \gamma_j$$

as an LMI in the variables E, Y, γ_j.

$$[A_u]_j K E K^\top [A_u]_j^\top \leq \gamma_j$$

$$E^{-1}E$$

$$\Leftrightarrow [A_u]_j \underbrace{K E}_{=Y} E^{-1} E^\top K^\top [A_u]_j^\top \leq \gamma_j$$

$$\Leftrightarrow \begin{bmatrix} \gamma_j & [A_u]_j Y \\ Y^\top [A_u]_j^\top & E \end{bmatrix} \succeq 0$$

Offline Computations

Computation of RPI set Ω

$$
\begin{aligned}
\min_{E \succeq I, Y, \beta_j, \gamma_j, \omega} \quad & \sum_{j=1}^{n_x} \tilde{b}_{x,j} + \sum_{j=1}^{n_u} \tilde{b}_{u,j} \\
\text{s.t.} \quad & \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \beta_j & [A_x]_j E \\ E^\top [A_x]_j^\top & E \end{bmatrix} \succeq 0, \ j \in [1, n_x], \\
& \begin{bmatrix} \gamma_j & [A_u]_j Y \\ Y^\top [A_u]_j^\top & E \end{bmatrix} \succeq 0, \ j \in [1, n_u], \\
& \begin{bmatrix} \omega & v_w^\top \\ v_w & E \end{bmatrix} \succeq 0, \ \forall v_w \in \mathcal{V}(\mathcal{W})
\end{aligned}
$$

Computation of tightenings

$$\tilde{b}_{x,j} = \frac{\sqrt{\beta_j \cdot \omega}}{1-\rho}$$

$$\tilde{b}_{u,j} = \frac{\sqrt{\gamma_j \cdot \omega}}{1-\rho}$$

$$\sqrt{\beta_j \cdot \omega} \leq \frac{\beta_j + \omega}{2}$$

AM-GM bound

Offline Computations

Rewrite the cost:

$$\sum_{j=1}^{n_x} \tilde{b}_{x,j} + \sum_{j=1}^{n_u} \tilde{b}_{u,j} = \sum_{j=1}^{n_x} \frac{\sqrt{\beta_j \cdot \omega}}{1-\rho} + \sum_{j=1}^{n_u} \frac{\sqrt{\gamma_j \cdot \omega}}{1-\rho} = \frac{1}{1-\rho} \left(\sum_{j=1}^{n_x} \sqrt{\beta_j \cdot \omega} + \sum_{j=1}^{n_u} \sqrt{\gamma_j \cdot \omega} \right)$$

Collaborative Exercise: Can we get rid of the bilinearities in the cost?

Yes, by using the AM-GM bound: $\sqrt{\beta_j \cdot \omega} \leq \frac{\beta_j + \omega}{2}$

$$\Rightarrow \sum_{j=1}^{n_x} \tilde{b}_{x,j} + \sum_{j=1}^{n_u} \tilde{b}_{u,j} \leq \frac{1}{1-\rho} \left(\sum_{j=1}^{n_x} \frac{\beta_j + \omega}{2} + \sum_{j=1}^{n_u} \frac{\gamma_j + \omega}{2} \right)$$

$$= \frac{1}{2(1-\rho)} \left((n_x + n_u)\omega + \sum_{j=1}^{n_x} \beta_j + \sum_{j=1}^{n_u} \gamma_j \right)$$

Offline Computations

Computation of RPI set Ω

$$
\begin{aligned}
\min_{E \succeq I, Y, \beta_j, \gamma_j, \omega} \quad & \frac{1}{2(1-\rho)}\left((n_x + n_u)\omega + \sum_{j=1}^{n_x} \beta_j + \sum_{j=1}^{n_u} \gamma_j\right) \\
\text{s.t.} \quad & \begin{bmatrix} \rho^2 E & (A_1 E + BY)^\top \\ A_1 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \rho^2 E & (A_2 E + BY)^\top \\ A_2 E + BY & E \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} \beta_j & [A_x]_j E \\ E^\top [A_x]_j^\top & E \end{bmatrix} \succeq 0, \ j \in [1, n_x], \\
& \begin{bmatrix} \gamma_j & [A_u]_j Y \\ Y^\top [A_u]_j^\top & E \end{bmatrix} \succeq 0, \ j \in [1, n_u], \\
& \begin{bmatrix} \omega & v_w^\top \\ v_w & E \end{bmatrix} \succeq 0, \ \forall v_w \in \mathcal{V}(\mathcal{W})
\end{aligned}
\tag{1}
$$

Implementation

Homework (PS4):

1. Implement optimization problem (1) in the `compute_min_tightening` method in the provided `Nonlinear_RMPC.m` file.
2. Compute the constraint tightenings for different choices of ρ and observe how the tightenings and Ω change. Choose a ρ and simulate the system with the nonlinear robust MPC controller.