Advanced Model Predictive Control

Recitation 3
Nonlinear robust MPC - Part I

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Nonlinear robust MPC

$$
\begin{aligned}
V_N^\star(x(k)) = \min_{V,\, z_0} \quad & \sum_{i=0}^{N-1} l(z_i, v_i) + l_{\mathrm{f}}(z_N) \\
\text{s.t.} \quad & z_{i+1} = f(z_i, v_i), \quad && i \in [0, N-1], \\
& (z_i, v_i) \in \overline{\mathcal{Z}}, && i \in [0, N-1], \\
& z_N \in \mathcal{X}_{\mathrm{f}}, \\
& (x(k), z_0) \in \Omega
\end{aligned}
$$

Nonlinear system

$$x^+ = f(x, u)$$

↓ diff.

Differential dynamics

$$\delta x^+ = A(x,u)\delta x + B(x,u)\delta u$$

↓ LMIs

Differential Lyapunov function V_x

↓ Riemannian distance

Incremental Lyapunov function

$$V_\delta(x^+, z^+) \leq \rho V_\delta(x, z)$$

↓ RPI Condition

Constraint tightening

Segway

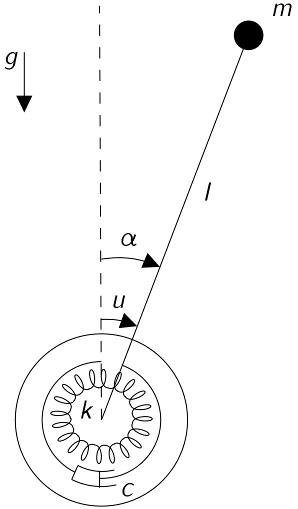
Dynamics:

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 + \delta t \cdot x_2 \\ x_2 + \delta t \left(-k x_1 - c x_2 + \frac{g}{l} \cdot \sin x_1 + u \right) \end{bmatrix} + w$$

Nonlinear system

$$x^+ = f(x, u)$$

[freepik.com]



Nonlinear robust MPC

Differential dynamics:

$$\delta_x(k+1) = A(x,u)\delta_x(k) + B(x,u)\delta_u(k),$$

with $A(x,u) = \left.\frac{\partial f}{\partial x}\right|_{(x,u)}$ and $B(x,u) = \left.\frac{\partial f}{\partial u}\right|_{(x,u)}$

Exercise: Derive the differential dynamics for the segway system.

Nonlinear system

$$x^+ = f(x,u)$$

$$\Big\downarrow \text{diff.}$$

Differential dynamics

$$\delta x^+ = A(x,u)\delta x + B(x,u)\delta u$$

$$A(x,u) = \left.\frac{\partial f}{\partial x}\right|_{(x,u)} = \begin{bmatrix} 1 & \delta t \\ -\delta t\, k + \delta t \frac{g}{\ell}\cos(x_1) & 1 - \delta t\, c \end{bmatrix}$$

$$B(x,u) = \left.\frac{\partial f}{\partial u}\right|_{(x,u)} = \begin{bmatrix} 0 \\ \delta t \end{bmatrix}$$

Nonlinear robust MPC

Use $\delta_u(k)=K(x)\delta_x(k)$, $A_K(x,u)=A(x,u)+B(x,u)K(x)$:

$$\delta_x(k+1) = A_K(x,u)\delta_x(k)$$

Conditions on diff. Lyapunov function $V_x(x,\delta_x) = ||\delta_x||_{M(x)}$:

$$c_1||\delta_x|| \leq V_x(x,\delta_x) \leq c_2||\delta_x||$$
$$V_x(f(x,u), A_K(x,u)\delta_x) \leq \rho V_x(x,\delta_x)$$

Conditions on contraction metric:

$$\underline{M} \preceq M(x) \preceq \overline{M}, \quad \underline{M}, \overline{M} \succ 0$$
$$A_K(x,u)^T M(x) A_K(x,u) \preceq \rho^2 M(x)$$

Nonlinear system
$$x^+ = f(x,u)$$

$\downarrow$ diff.

Differential dynamics
$$\delta x^+ = A(x,u)\delta x + B(x,u)\delta u$$

$\downarrow$ LMIs

Differential Lyapunov function V_x

$$(\delta_x^+)^T M(x)\, \delta_x^+ \leq \rho^2 \delta_x^T M(x)\, \delta_x$$

Nonlinear robust MPC

Use constant $P := M(x)$ and K:

$$P \succ 0$$

$$(A(x) + BK)^T P (A(x) + BK) \preceq \rho^2 P$$

Exercise: Use Schur's complement to rewrite the conditions as an LMI in $E := P^{-1}$ and $Y := KE$.

Nonlinear system
$$x^+ = f(x, u)$$

$\downarrow$ diff.

Differential dynamics
$$\delta x^+ = A(x, u)\delta x + B(x, u)\delta u$$

$\downarrow$ LMIs

Differential Lyapunov function V_x

$$\rho^2 P - (A(x)+BK)^T P (A(x)+BK) \succeq 0$$

$$\iff \begin{bmatrix} \rho^2 P & (A(x)+BK)^T \\ A(x)+BK & P^{-1} \end{bmatrix} \succeq 0$$

$$\iff \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \rho^2 P & (A(x)+BK)^T \\ A(x)+BK & E \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \succeq 0$$

$$\iff \begin{bmatrix} \rho^2 & (A(x)E+BKE)^T \\ A(x)+BK & E \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \succeq 0$$

$$\iff \begin{bmatrix} \rho^2 E & (A(x)E+BY)^T \\ A(x)E+BY & E \end{bmatrix} \succeq 0$$

Nonlinear robust MPC

We have differential Lyapunov function $V_x = ||\delta_x||_P$.
Riemannian distance and path integral controller:

$$V_\delta(x,z) = \inf_{\gamma \in \Gamma(x,z)} \left(\int_0^1 \sqrt{\left.\frac{\partial \gamma}{\partial s}\right|_s^\top M(\gamma(s)) \left.\frac{\partial \gamma}{\partial s}\right|_s} \right) ds,$$

$$\kappa(x,z,v) = v + \int_0^1 K(\gamma^\star(s)) \left.\frac{\partial \gamma^\star}{\partial s}\right|_s ds$$

With constant contraction metric and constant differential controller:

$$V_\delta(x,z) = ||x - z||_P$$
$$\kappa(x,z,v) = K(x-z) + v$$

Nonlinear system
$x^+ \;=\; f(x,u)$

$\downarrow$ diff.

Differential dynamics
$\delta x^+ \;=\; A(x,u)\delta x + B(x,u)\delta u$

$\downarrow$ LMIs

Differential Lyapunov function V_x

$\downarrow$ Riemannian distance

Incremental Lyapunov function
$V_\delta(x^+, z^+) \;\leq\; \rho V_\delta(x,z)$

Nonlinear robust MPC

RPI Condition for $\Omega = \{x, z | V_\delta(x, z) \leq \delta\}$:

$$(f(x, \kappa(x, z, v)) + w, f(z, v)) \in \Omega, \forall w \in \mathcal{W}, (x, z) \in \Omega$$

Exercise: Using the incremental Lyapunov function $V_\delta(x, z) = \|x - z\|_P$, show **i-ISS**.

$$V_\delta(x^+, z^+) = \|f(x, \kappa(x, z, v)) + w - f(z, v)\|_P$$

$$\leq \|f(x, \kappa(x, z, v)) - f(z, v)\|_P + \|w\|_P$$

$$\leq \underbrace{\rho \|x - z\|}_{\text{decrease}} + \underbrace{\|w\|_P}_{\alpha_w(\|w\|)}$$

Nonlinear system
$$x^+ = f(x, u)$$

$\downarrow$ diff.

Differential dynamics
$$\delta x^+ = A(x, u)\delta x + B(x, u)\delta u$$

$\downarrow$ LMIs

Differential Lyapunov function V_x

$\downarrow$ Riemannian distance

Incremental Lyapunov function
$$V_\delta(x^+, z^+) \leq \rho V_\delta(x, z)$$

$\downarrow$ RPI Condition

Constraint tightening

Nonlinear robust MPC

$\|x-z\|_P = \|e\|_P = \|P^{1/2} e\|_2 \leq \delta$

RPI Condition for $\Omega = \{x, z | V_\delta(x, z) \leq \delta\}$:

$$(f(x, \kappa(x, z, v)) + w, f(z, v)) \in \Omega, \forall w \in \mathcal{W}, (x, z) \in \Omega$$

Exercise: Using the incremental Lyapunov function $V_\delta(x, z) = \|x - z\|_P$, show **i-ISS**.

$$V_\delta(x^+, z^+) = \|f(x, \kappa(x, z, v)) + w - f(z, v)\|_P$$

$$\leq \|f(x, \kappa(x, z, v)) - f(z, v)\|_P + \|w\|_P$$

$$\leq \underbrace{\rho \|x - z\|_P}_{\text{decrease}} + \underbrace{\|w\|_P}_{\alpha_w(\|w\|)}$$

Using $\bar{\alpha}_w = \max_{w \in \mathcal{W}} \alpha_w(\|w\|)$, choose $\delta = \frac{\bar{\alpha}_w}{1-\rho}$ for RPI.

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↓ RPI Condition

Constraint tightening

Nonlinear robust MPC

State Constraints: $\mathcal{X} = \{x \mid A_x x \leq b_x\}$

$$[A_x]_i x = [A_x]_i(z+e) \leq [b_x]_i$$
$$\Rightarrow [A_x]_i z \leq [b_x]_i - \underbrace{[A_x]_i e}_{\text{tightening}}$$

Using $y := P^{\frac{1}{2}} e$, we have

$$\begin{aligned}
[A_x]_i e &= [A_x]_i P^{-\frac{1}{2}} y \\
&\leq \|P^{-\frac{1}{2}}[A_x]_i^T\|_2 \|y\|_2 \quad \text{(Cauchy-Schwarz)} \\
&\leq \|P^{-\frac{1}{2}}[A_x]_i^T\|_2 \delta \qquad (\Omega = \{y \mid \|y\|_2 \leq \delta\})
\end{aligned}$$

Homework: Derive the input constraint tightening.

Tightening: $[A_u]_i^T K(x-z)$

$\leq \|P^{-1/2} K^T [A_u]_i^T\| \delta$

$$\begin{aligned}
&\textbf{Nonlinear system} \\
&x^+ = f(x,u) \\
&\quad\downarrow \text{ diff.} \\
&\textbf{Differential dynamics} \\
&\delta x^+ = A(x,u)\delta x + B(x,u)\delta u \\
&\quad\downarrow \text{ LMIs} \\
&\textbf{Differential Lyapunov function } V_x \\
&\quad\downarrow \text{ Riemannian distance} \\
&\textbf{Incremental Lyapunov function} \\
&V_\delta(x^+, z^+) \leq \rho V_\delta(x,z) \\
&\quad\downarrow \text{ RPI Condition} \\
&\textbf{Constraint tightening}
\end{aligned}$$

Nonlinear robust MPC

$$
\begin{aligned}
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& (z_i, v_i) \in \overline{\mathcal{Z}}, && i \in [0, N-1], \\
& z_N \in \mathcal{X}_{\mathrm{f}}, \\
& (x(k), z_0) \in \Omega
\end{aligned}
$$

As terminal ingredients, use $\mathcal{X}_f = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$[1] and $l_f(z_N) = 0$.

[1]Terminal equality can be used for ISS if the system is weakly controllable, see MPC book p.116.

Nonlinear robust MPC

$$
\begin{aligned}
V_N^\star(x(k)) = \quad & \min_{V,\, z_0} \quad \sum_{i=0}^{N-1} z_i^T Q z_i + v_i^T R v_i \\
\text{s.t.} \quad & z_{i+1} = f(z_i, v_i), && i \in [0, N-1], \\
& [A_x]_j z_i \leq [b_x]_j - \|P^{-\frac{1}{2}} [A_x]_j^T\|_2 \frac{\bar{\alpha}_w}{1-\rho}, && i \in [0, N-1],\ j \in [1, n_x], \\
& [A_u]_j v_i \leq [b_u]_j - \|P^{-\frac{1}{2}} K^T [A_u]_j^T\|_2 \frac{\bar{\alpha}_w}{1-\rho}, && i \in [0, N-1],\ j \in [1, n_u], \\
& z_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
& \|x(k) - z_0\|_P \leq \delta
\end{aligned}
$$