Analytic Geometry

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June 3, 2022

1 Introduction

2 Norms

Definition: **(Norm):** A norm on a vector space V which assigns each vector $\boldsymbol{x}$ its length $\|\boldsymbol{x}\|$ is defined as

$$\begin{aligned} \|\cdot\| : V &\to \mathbb{R} \\ \boldsymbol{x} &\mapsto \|\boldsymbol{x}\| \end{aligned} \tag{1}$$

such that for all $\lambda \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$ the following holds

- *Absolutely homogeneous* $|\lambda|\|\boldsymbol{x}\| = \|\lambda \boldsymbol{x}\|$
- *Triange equality* $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- *Positive definite* $\|\boldsymbol{x}\| \geqslant 0$ and $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$

Definition: **(Manhattan norm):** Also called ℓ_1 norm, it's defined for $\boldsymbol{x} \in \mathbb{R}^n$ as

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i| \tag{2}$$

Definition: (Euclidian norm): Also called ℓ_2 norm, it's defined for $\boldsymbol{x} \in \mathbb{R}^n$ as

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}} \tag{3}$$

3 Inner Products

Definition: (Dot product): A *dot product* or *scalar product* is a type of inner product. In $\mathbb{R}^n$ it is defined as

$$\boldsymbol{x}^\top \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \tag{4}$$

Definition: (Bilinear mapping): A *bilinear mapping* is a mapping Ω with two arguments for which the following holds

$$\begin{aligned} \Omega(\lambda \boldsymbol{x} + \psi \boldsymbol{y}, \boldsymbol{z}) &= \lambda \Omega(\boldsymbol{x}, \boldsymbol{z}) + \psi \Omega(\boldsymbol{y}, \boldsymbol{z}) \\ \Omega(\boldsymbol{x}, \lambda \boldsymbol{y} + \psi \boldsymbol{z}) &= \lambda \Omega(\boldsymbol{x}, \boldsymbol{y}) + \psi \Omega(\boldsymbol{x}, \boldsymbol{z}) \end{aligned} \tag{5}$$

Rule: Given $\Omega : V \times V \to \mathbb{R}$ then

- Ω is called *symmetric* if $\Omega(\boldsymbol{x}, \boldsymbol{y}) = \Omega(\boldsymbol{y}, \boldsymbol{x})$
- Ω is called *positive definite* if

$$\forall \boldsymbol{x} \in V \backslash \{\boldsymbol{0}\} : \Omega(\boldsymbol{x}, \boldsymbol{x}) > 0, \quad \Omega(\boldsymbol{0}, \boldsymbol{0}) = 0 \tag{6}$$

- A positive definite, symmetric bilinear mapping Ω is called an *inner product* and is written as $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$.
- The pair $(V, \langle \cdot, \cdot \rangle)$ is called *inner product space* or *Euclidian vector space* if $\langle \cdot, \cdot \rangle$ is defined as the ℓ_2 norm.

Definition: **(Symmetric, positive definite matrix):** A symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ that satisfies

$$\forall \boldsymbol{x} \in V \backslash \{\boldsymbol{0}\} : \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0 \tag{7}$$

If only $\geqslant$ holds, it is *symmetric, positive semidefinite.*

Rule: Let V be a vector space with basis B. It holds that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is an inner product if and only if there exists a symmatric positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \hat{\boldsymbol{x}}^\top \boldsymbol{A}\hat{\boldsymbol{y}} \tag{8}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ are the coordinate vectors of $\boldsymbol{x}, \boldsymbol{y}$ with respect to B. Consequently, the following holds for $\boldsymbol{A}$:

- The null space of $\boldsymbol{A}$ consist only of $\boldsymbol{0}$.
- The diagonal elements a_{ij} of $\boldsymbol{A}$ are positive.

4 Lengths and Distances

Rule: Every inner product $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ induces a norm $\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$.

Definition: **(Cauchy-Schwartz inequality):** For an inner product vector space $(V, \langle \cdot, \cdot \rangle)$, the induced norm $\| \cdot \|$ satisfies the *Cauchy-Schwartz inequality* if

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leqslant \|\boldsymbol{x}\|\|\boldsymbol{y}\| \tag{9}$$

Definition: **(Distance and Metric):** The *distance* between two vectors $\boldsymbol{x}, \boldsymbol{y}$ is defined as

$$d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle} \tag{10}$$

If we use the dot product as the inner product then d is called the *Euclidian distance*. The mapping

$$d : V \times V \to \mathbb{R} \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto d(\boldsymbol{x}, \boldsymbol{y}) \tag{11}$$

is called a *metric* and must satisfies the following properties.

- *Positive definite* $d(\boldsymbol{x}, \boldsymbol{y}) \geqslant 0,\ d(\boldsymbol{x}, \boldsymbol{y}) = 0 \iff \boldsymbol{x} = \boldsymbol{y}$
- *Symmetric* $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$
- *Triangle equality* $d(\boldsymbol{x}, \boldsymbol{z}) \leqslant d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$

5 Angles and Orthogonality

Rule: Given $\boldsymbol{x}, \boldsymbol{y} \neq \mathbf{0}$. The angle ω between $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\cos\omega = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\|\ \|\boldsymbol{y}\|} \tag{12}$$

and is contained inside $[-1, 1]$.

Definition: **(Orthogonality):** Two vectors $\boldsymbol{x}, \boldsymbol{y}$ are *orthogonal* if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$. We write $\boldsymbol{x} \perp \boldsymbol{y}$. Furthermore, $\boldsymbol{x}, \boldsymbol{y}$ are *orthonormal* if $\|\boldsymbol{x}\| = 1 = \|\boldsymbol{y}\|$.

Rule:

- Whether two vectors are orthogonal depends on the norm being used.
- The $\mathbf{0}$ vector is orthogonal to every vector in the vector space.
- Orthogonality is a generalization of the concept of perpendicularity.

Definition: **(Orthogonal matrix):** A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is orthogonal if

$$\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I} = \boldsymbol{A}^\top \boldsymbol{A} \tag{13}$$

which implies that

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^\top \tag{14}$$

6 Orthonormal Basis

Definition: **(Orthonormal basis):** A basis B of a vector space V is said to be *orthonormal* if

$$\begin{aligned} \langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle &= 0 \text{ for } i \neq j \\ \langle \boldsymbol{b}_i, \boldsymbol{b}_i \rangle &= 1 \end{aligned} \tag{15}$$

In other words, all column vectors of B are orthogonal to each other. This implies that every vector has length/norm 1, if this is not the case the basis is only said to be *orthogonal*.

Definition: **(Gram-Schmidt process):** Given a basis $\hat{\boldsymbol{B}}$, we can perform Gaussian eliminiation on the augmented matrix $[\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}^\top|\hat{\boldsymbol{B}}]$ to obtain an orthonormal basis.

7 Orthogonal Complement

Definition: **(Orthogonal complement):** Given an D dimensional vector space V and an M dimensional subspace $U \in V$. The *orthogonal complement* $U^\perp$ is a subspace of V with dimension $D - M$ that contains all vectors in V that are orthogonal to every vector in U with $U \cap U^\perp$ so that any vector $\boldsymbol{x}$ can be uniquely decomposed into

$$\boldsymbol{x} = \sum_{i=1}^{M} \lambda_i \boldsymbol{b}_i + \sum_{j=1}^{D-M} \psi_j \boldsymbol{b}_j^\perp \tag{16}$$

where $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_M)$ is the basis of U, $(\boldsymbol{b}_1^\perp, \dots, \boldsymbol{b}_{D-M}^\perp)$ is the basis of $U^\perp$.

Rule: Let U be a two dimensional subspace of a three dimensional vector space V. U is a plane of V. Then the vector $\boldsymbol{w}$ where $\|\boldsymbol{w}\| = 1$, which is orthogonal to the plane U is the basis vector of $U^\perp$.

8 Inner Products of Functions

Rule: An inner product of two functions $u : \mathbb{R} \to \mathbb{R}, v : \mathbb{R} \to \mathbb{R}$ can be defined as the definite integral

$$\langle u, v \rangle := \int_b^a u(x)v(x)dx \tag{17}$$

9 Orthogonal projections

Definition: **(Projection):** Given a vector space V and a subspace $U \subseteq V$, a *projection* is a linear mapping $\pi : V \to U$ such that $\pi^2 = \pi \circ \pi = \pi$. The transformation matrix of a projection is called a *projection matrix* denoted $\boldsymbol{P}_\pi$ such that $\boldsymbol{P}_\pi^2 = \boldsymbol{P}_\pi$.j

Rule: Given a vector space $V \in \mathbb{R}^n$, a subspace $U \in \mathbb{R}^n, U \subseteq V$ with basis $\boldsymbol{b}$. When we project $\boldsymbol{x} \in V$ onto U we seek the vector $\pi_U(\boldsymbol{x}) \in U$.

- The projection $\pi_U(\boldsymbol{x})$ is one for which $\|\boldsymbol{x} - \pi_U(\boldsymbol{x})\|$
- The segment $\boldsymbol{x} - \pi_U(\boldsymbol{x})$ is orthogonal to U and its basis.
- $\langle \pi_U(\boldsymbol{x}) - \boldsymbol{x}, b \rangle = 0$.
- $\pi_U(\boldsymbol{x}) = \lambda \boldsymbol{b}$.

Furthermore, the following equations hold.

$$\lambda = \frac{\langle \boldsymbol{x}, \boldsymbol{b} \rangle}{\langle \boldsymbol{b}, \boldsymbol{b} \rangle} \tag{18}$$

where λ is the coordinate of $\pi_U(\boldsymbol{x})$ with respect to $\boldsymbol{b}$.

$$\pi_U(\boldsymbol{x}) = \boldsymbol{b}\lambda = \frac{\boldsymbol{b}^\top \boldsymbol{x}}{\langle \boldsymbol{b}, \boldsymbol{b} \rangle}\boldsymbol{b} = \frac{\boldsymbol{b}\boldsymbol{b}^\top}{\langle \boldsymbol{b}, \boldsymbol{b} \rangle}\boldsymbol{x} \tag{19}$$

$$\boldsymbol{P}_\pi = \frac{\boldsymbol{b}\boldsymbol{b}^\top}{\langle \boldsymbol{b}, \boldsymbol{b} \rangle} \tag{20}$$