Linear Algebra

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1 Introduction

Linear algebra is a discipline of mathematics that deals with linear equations, vectors, matrices and their transformations.

2 Systems of linear equations

3 Matrices

4 Solving systems of linear equations

5 Vector spaces

6 Linear independence

Definition: **(Linear combination):** Consider a vector space V and a finite number of vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in V$. Then every $\boldsymbol{v} \in V$ of the form:

$$\boldsymbol{v} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V \tag{1}$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ is a linear combination of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

Definition: **(Linear independence):** Given a vector space V and $k \in \mathbb{N}$. Vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$ are said to be linearly independent if there exists no non-trivial solution to $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$. Otherwise they're linearly dependent.

Rule:

- If at least one of the vector $\boldsymbol{x}_i$ is $\mathbf{0}$ then they are linearly dependent.
- The vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \ldots, k\} \geqslant 2$ are linearly dependent, if and only if, at least one of them is a linear combination of the others.
- To check whether $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$ are linearly independent we can use Gaussian elimation: write all vectors as columns of a matrix $\boldsymbol{A}$ and perform Gaussian elimination until the matrix is in row-echelon form.
  - Non-pivot columns can be expressed as linear combinations of vectors on their left.
  - Pivot columns are linearly independent from vectors on their left.

  If all columns are pivots, the vectors are linearly independent.

Rule: Given m linear combinations over k linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k \in V$.

$$\begin{aligned} \boldsymbol{x}_1 &= \sum_{i=1}^{k} \lambda_{i1} \boldsymbol{b}_i \\ &\vdots \\ \boldsymbol{x}_m &= \sum_{i=1}^{k} \lambda_{im} \boldsymbol{b}_i \end{aligned} \tag{2}$$

We can write, with $\boldsymbol{B} = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k]$, the following:

$$\boldsymbol{x}_j = \boldsymbol{B}\boldsymbol{\lambda}_j, \quad \boldsymbol{\lambda}_j = \begin{bmatrix} \lambda_1 j \\ \vdots \\ \lambda_k j \end{bmatrix}, \quad j = 1, \ldots, m \tag{3}$$

We can test whether $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are linearly independent using:

$$\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \boldsymbol{B}\boldsymbol{\lambda}_j = \boldsymbol{B} \sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j \tag{4}$$

Which means that $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ is linearly independent if the column vectors $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m\}$ are linearly independent.

Rule: In a vector space V, m linear combinations of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ are linearly independent if $m > k$.

7 Basis and rank

In a vector space V, we are interested in a set of vectors $\mathcal{A}$ that posess the property that any vector $\boldsymbol{v} \in V$ can be obtained through a linear combination of vectors in $\mathcal{A}$.

7.1 Generating Set and Basis

Definition: **(Generating Set and Span):** Given a set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq V$. If every vector $\boldsymbol{v} \in V$ can be expressed as a linear combination of vectors in $\mathcal{A}$, $\mathcal{A}$ is called a *generating set* of V. The set of all linear combinations of vectors in $\mathcal{A}$ is called the span of $\mathcal{A}$.

Definition: **(Basis):** Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} \subseteq \mathcal{V}$. A generating set $\mathcal{A}$ of $\mathcal{V}$ is called *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \subsetneq \mathcal{A} \subseteq \mathcal{V}$.