

# Analytic Geometry

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## 1 Introduction

## 2 Norms

**Definition:** (Norm): A norm on a vector space  $V$  which assigns each vector  $\mathbf{x}$  its length  $\|\mathbf{x}\|$  is defined as

$$\begin{aligned}\|\cdot\| : V &\rightarrow \mathbb{R} \\ \mathbf{x} &\mapsto \|\mathbf{x}\|\end{aligned}\tag{1}$$

such that for all  $\lambda \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in V$  the following holds

- *Absolutely homogeneous*  $|\lambda|\|\mathbf{x}\| = \|\lambda\mathbf{x}\|$
- *Triangle equality*  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
- *Positive definite*  $\|\mathbf{x}\| \geq 0$  and  $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$

**Definition:** (Manhattan norm): Also called  $\ell_1$  norm, it's defined for  $\mathbf{x} \in \mathbb{R}^n$  as

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|\tag{2}$$

**Definition: (Euclidian norm):** Also called  $\ell_2$  norm, it's defined for  $\mathbf{x} \in \mathbb{R}^n$  as

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\mathbf{x}^\top \mathbf{x}} \quad (3)$$

### 3 Inner Products

**Definition: (Dot product):** A *dot product* or *scalar product* is a type of inner product. In  $\mathbb{R}^n$  it is defined as

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i \quad (4)$$

**Definition: (Bilinear mapping):** A *bilinear mapping* is a mapping  $\Omega$  with two arguments for which the following holds

$$\begin{aligned} \Omega(\lambda \mathbf{x} + \psi \mathbf{y}, \mathbf{z}) &= \lambda \Omega(\mathbf{x}, \mathbf{z}) + \psi \Omega(\mathbf{y}, \mathbf{z}) \\ \Omega(\mathbf{x}, \lambda \mathbf{y} + \psi \mathbf{z}) &= \lambda \Omega(\mathbf{x}, \mathbf{y}) + \psi \Omega(\mathbf{x}, \mathbf{z}) \end{aligned} \quad (5)$$

**Rule:** Given  $\Omega : V \times V \rightarrow \mathbb{R}$  then

- $\Omega$  is called *symmetric* if  $\Omega(\mathbf{x}, \mathbf{y}) = \Omega(\mathbf{y}, \mathbf{x})$
- $\Omega$  is called *positive definite* if

$$\forall \mathbf{x} \in V \setminus \{\mathbf{0}\} : \Omega(\mathbf{x}, \mathbf{x}) > 0, \quad \Omega(\mathbf{0}, \mathbf{0}) = 0 \quad (6)$$

- A positive definite, symmetric bilinear mapping  $\Omega$  is called an *inner product* and is written as  $\langle \mathbf{x}, \mathbf{y} \rangle$ .
- The pair  $(V, \langle \cdot, \cdot \rangle)$  is called *inner product space* or *Euclidian vector space* if  $\langle \cdot, \cdot \rangle$  is defined as the  $\ell_2$  norm.

**Definition:** (Symmetric, positive definite matrix): A symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  that satisfies

$$\forall \mathbf{x} \in V \setminus \{\mathbf{0}\} : \mathbf{x}^\top \mathbf{A} \mathbf{x} > 0 \quad (7)$$

If only  $\geq$  holds, it is *symmetric, positive semidefinite*.

**Rule:** Let  $V$  be a vector space with basis  $B$ . It holds that  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  is an inner product if and only if there exists a symmetric positive definite matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  with

$$\langle \mathbf{x}, \mathbf{y} \rangle = \hat{\mathbf{x}}^\top \mathbf{A} \hat{\mathbf{y}} \quad (8)$$

where  $\hat{\mathbf{x}}, \hat{\mathbf{y}}$  are the coordinate vectors of  $\mathbf{x}, \mathbf{y}$  with respect to  $B$ . Consequently, the following holds for  $\mathbf{A}$ :

- The null space of  $\mathbf{A}$  consist only of  $\mathbf{0}$ .
- The diagonal elements  $a_{ij}$  of  $\mathbf{A}$  are positive.

## 4 Lengths and Distances

**Rule:** Every inner product  $\langle \mathbf{x}, \mathbf{x} \rangle$  induces a norm  $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ .

**Definition:** (Cauchy-Schwartz inequality): For an inner product vector space  $(V, \langle \cdot, \cdot \rangle)$ , the induced norm  $\|\cdot\|$  satisfies the *Cauchy-Schwartz inequality* if

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\| \quad (9)$$

**Definition:** (Distance and Metric): The *distance* between two vectors  $\mathbf{x}, \mathbf{y}$  is defined as

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle} \quad (10)$$

If we use the dot product as the inner product then  $d$  is called the *Euclidian distance*. The mapping

$$d : V \times V \rightarrow \mathbb{R} \quad (\mathbf{x}, \mathbf{y}) \mapsto d(\mathbf{x}, \mathbf{y}) \quad (11)$$

is called a *metric* and must satisfies the following properties.

- *Positive definite*  $d(\mathbf{x}, \mathbf{y}) \geq 0$ ,  $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$
- *Symmetric*  $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
- *Triangle equality*  $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$

**Rule:** Given  $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ . The angle  $\omega$  between  $\mathbf{x}$  and  $\mathbf{y}$  is defined as

$$\cos \omega = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \quad (12)$$

and is contained inside  $[-1, 1]$ .

**Definition: (Orthogonality):** Two vectors  $\mathbf{x}, \mathbf{y}$  are *orthogonal* if  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ . We write  $\mathbf{x} \perp \mathbf{y}$ . Furthermore,  $\mathbf{x}, \mathbf{y}$  are *orthonormal* if  $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$ .

**Rule:**

- Whether two vectors are orthogonal depends on the norm being used.
- The  $\mathbf{0}$  vector is orthogonal to every vector in the vector space.
- Orthogonality is a generalization of the concept of perpendicularity.

**Definition: (Orthogonal matrix):** A matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is orthogonal if

$$\mathbf{A}\mathbf{A}^\top = \mathbf{I} = \mathbf{A}^\top \mathbf{A} \quad (13)$$

which implies that

$$\mathbf{A}^{-1} = \mathbf{A}^\top \quad (14)$$