Linear Algebra

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May 21, 2022

1 Introduction

Linear algebra is a discipline of mathematics that deals with linear equations, vectors, matrices and their transformations.

2 Systems of linear equations

3 Matrices

4 Solving systems of linear equations

5 Vector spaces

Definition: **(Group):** Given a set $\mathcal{G}$ and an operation $\otimes$ defined on $\mathcal{G}$. $G := (\mathcal{G}, \otimes)$ is a group if the following holds:

- Closure of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
- Associativity: $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
- Neutral element: $\exists e \in \mathcal{G}\ \forall x \in \mathcal{G} : x \otimes e = x$

- Inverse element: $\forall x \in \mathcal{G}\ \exists y \in \mathcal{G} : x \otimes y = e$ where e is the neutral element.

Definition: **(Vector space):** A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations $+$ and $\cdot$ defined where the following holds:

- $(\mathcal{V}, +)$ is an abelian group.
- Distributivity:
  - $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}\ \forall \lambda \in \mathbb{R} : \lambda(\boldsymbol{x} + \boldsymbol{y}) = \lambda \boldsymbol{x} + \lambda \boldsymbol{y}$
  - $\forall \boldsymbol{x} \in \mathcal{V}\ \forall \lambda, \psi \in \mathbb{R} : (\lambda + \psi)\boldsymbol{x} = \lambda \boldsymbol{x} + \psi \boldsymbol{y}$
- Associativity (outer operation)
- Neutral element (outer operation)

Definition: **(Vector subspace):** Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{U} \subseteq \mathcal{V}, \mathcal{U} \neq \emptyset$. Then $U = (\mathcal{U}, +, \cdot)$ is a *vector subspace* (or *linear subspace*) if U is a vector space restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathcal{U} \times \mathbb{R}$. We write $U \subseteq V$.

Rule: Given $U \subseteq V$, the following properties of V are passed to U.

- Abelian group, distributivity, associativity and neutral element properties
- $\mathcal{U} \neq \emptyset$, in particular: $\boldsymbol{0} \in \mathcal{U}$
- Closure of U
  - $\forall \lambda \in \mathbb{R}\ \forall \boldsymbol{x} \in \mathcal{U} : \lambda \boldsymbol{x} \in \mathcal{U}$
  - $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{U} : \boldsymbol{x} + \boldsymbol{y} \in \mathcal{U}$

Rule: Every subspace $U \subseteq (\mathbb{R}^n, +, \cdot)$ is the solution space of a homogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ for $\boldsymbol{x} \in \mathbb{R}^n$

6 Linear independence

Definition: **(Linear combination):** Consider a vector space V and a finite number of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$. Then every $\boldsymbol{v} \in V$ of the form:

$$\boldsymbol{v} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V \tag{1}$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ is a linear combination of the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

Definition: **(Linear independence):** Given a vector space V and $k \in \mathbb{N}$. Vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$ are said to be linearly independent if there exists no non-trivial solution to $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$. Otherwise they're linearly dependent.

Rule:

- If at least one of the vector $\boldsymbol{x}_i$ is $\mathbf{0}$ then they are linearly dependent.
- The vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \ldots, k\}, k \geqslant 2$ are linearly dependent, if and only if, at least one of them is a linear combination of the others.
- To check whether $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$ are linearly independent we can use Gaussian elimation: write all vectors as columns of a matrix $\boldsymbol{A}$ and perform Gaussian elimination until the matrix is in row-echelon form.
  - Non-pivot columns can be expressed as linear combinations of vectors on their left.
  - Pivot columns are linearly independent from vectors on their left.

  If all columns are pivots, the vectors are linearly independent.

Rule: Given m linear combinations over k linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k \in V$.

$$\begin{aligned} \boldsymbol{x}_1 &= \sum_{i=1}^{k} \lambda_{i1} \boldsymbol{b}_i \\ &\vdots \\ \boldsymbol{x}_m &= \sum_{i=1}^{k} \lambda_{im} \boldsymbol{b}_i \end{aligned} \tag{2}$$

We can write, with $\boldsymbol{B} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_k]$, the following:

$$\boldsymbol{x}_j = \boldsymbol{B}\boldsymbol{\lambda}_j, \quad \boldsymbol{\lambda}_j = \begin{bmatrix} \lambda_1 j \\ \vdots \\ \lambda_k j \end{bmatrix}, \quad j = 1, \dots, m \tag{3}$$

We can test whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ are linearly independent using:

$$\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \boldsymbol{B}\boldsymbol{\lambda}_j = \boldsymbol{B} \sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j \tag{4}$$

Which means that $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ is linearly independent if the column vectors $\{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m\}$ are linearly independent.

Rule: In a vector space V, m linear combinations of $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$ are linearly independent if $m > k$.

7 Basis and rank

In a vector space V, we are interested in a set of vectors $\mathcal{A}$ that posess the property that any vector $\boldsymbol{v} \in V$ can be obtained through a linear combination of vectors in $\mathcal{A}$.

7.1 Generating Set and Basis

Definition: **(Generating Set and Span):** Given a set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\} \subseteq \mathcal{V}$. If every vector $\boldsymbol{v} \in \mathcal{V}$ can be expressed as a linear combination of vectors in $\mathcal{A}$, $\mathcal{A}$ is called a *generating set* of V. The set of all linear combinations of vectors in $\mathcal{A}$ is called the span of $\mathcal{A}$. We write $V = \text{span}[A]$ or $V = \text{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$.

Definition: **(Basis):** Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} \subseteq \mathcal{V}$. A generating set $\mathcal{A}$ of $\mathcal{V}$ is called *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \subsetneq \mathcal{A} \subseteq \mathcal{V}$ that spans $\mathcal{V}$. Every linearly independent generating set of V is minimal and is called a *basis* of V.

Rule: Given $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Then the following statements are equivalent:

- $\mathcal{B}$ is a basis of V
- $\mathcal{B}$ is a minimal generating set.
- $\mathcal{B}$ is a maximal linearly independent set of vectors in V such that adding any vector to the set $\mathcal{B}$ would make it linearly dependent.
- Every vector $\boldsymbol{x} \in V$ is a linear combination of vectors from $\mathcal{B}$, and every combination is unique, i.e., with

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i \tag{5}$$

Rule: Every vector space V posseses a basis B. There can be many bases for V, all have the same number of elements, the *basis vectors*.

Rule: A basis of a subspace $U = span[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k] \subseteq \mathbb{R}^n$ by:

- Write the spanning vectors as columns of a matrix $\boldsymbol{A}$
- Determine the row-echelon form of $\boldsymbol{A}$
- The spanning vectors associated with the pivots columns are a basis of U.

7.2 Rank

Definition: **(Rank):** The number of linearly independent columns of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ equals the number of linearly indepdendent rows and is called the *rank*. We write $\text{rk}(A)$

Rule:

- $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}^\top)$ .i.e column rank equals row rank
- The columns of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = \text{rk}(\boldsymbol{A})$. This subspace is called the *image* or *range*. A basis of U can be found by applying Gaussian elimination to $\boldsymbol{A}$. Same goes for the rows, given a subspace $W \in \mathbb{R}^n$.
- For all $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ it holds that $\boldsymbol{A}$ is regular (invertible) if and only $\text{rk}(\boldsymbol{A}) = n$.
- For all $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and all $\boldsymbol{b} \in \mathbb{R}^m$ it holds that the linear equation system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ can be solved only and if $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}|\boldsymbol{b})$
- A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is said to have *full-rank* if $\text{rk}(\boldsymbol{A}) = \min(m, n)$. Otherwise, it is *rank-deficient*.

8 Linear mappings

Definition: **(Linear mapping):** For vector spaces V, W, a mapping $\Phi : V \to W$