Introductory Combinatorics - Permutations and Combinations

Diego ROJAS

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1 Four Basic Counting Principles

1.1 The Addition Principle

Suppose that a set S is partitioned into pairwise disjoint parts $S_1, S_2, \ldots, S_m$. The number of objects in S can be determined by finding the number of objects in each of the parts and adding the numbers obtained.

$$\begin{aligned} S &= S_1 \cup S_2 \cup \cdots \cup S_m \\ |S| &= |S_1| + |S_2| + \cdots + |S_m| \end{aligned} \tag{1}$$

The number of objects of a set S is denoted by $|S|$ and is called the size of S.

1.2 The Multiplication Principle

Let S be a set of ordered pairs (a, b) of objects, where the first object a comes from a set of size p, and for each choice of object a there are q choices for object b. Then the size of S is $p \times q$.
It's important for a and b to be independent choices.

1.3 The Substraction Principle

Let A be a set and let U be a larger set containing A. Let

$$\bar{A} = U \backslash A = \{x \in U : x \notin A\}$$

be the *complement* of A. Then the number of objects in A is given by:

$$|A| = |U| - |\bar{A}|$$

1.4 The Division Principle

Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by the rule:

$$k = \frac{|S|}{\text{number of objects in a part}}$$

2 Permutations of Sets

We denote $P(n, r)$ the number of *r-permutations* of a set of n elements.

Linear permutations: the number of *linear* r-permutations of a set of n elements where $n \geq r$ is given by:

$$P(n,r) = n \times (n-1) \times \cdots \times (n-r+1) = P(n,r) = \frac{n!}{(n-r)!} \quad (2)$$
$$P(n,n) = n!$$

Circular permutations: the number of *circular* r-permutations of a set of n elements where $n \geq r$ is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r \cdot (n-r)!}, \; P(n,n) = (n-1)! \quad (3)$$

3 Combinations (Subsets) of Sets

Combination: a *combination* or *subset* of a set S denotes an unordered selection of the elements of S. We denote by $\binom{n}{r}$ the number of r-subsets of a set S of size n.

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!} \quad (4)$$

It also stands that

$$\binom{n}{r} = \binom{n}{n-r} \quad (5)$$

Pascal's formula: for all integers n and k with $1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (6)$$

Theorem: for $n > 0$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \tag{7}$$

4 Permutations of Multisets

Infinite repetition: the number of r-permutations of k distinct objects, each available in unlimited supply (or if $r \leq$ supply), equals k^r.

Finite repetition: for a set S with k elements repeated $n_1, n_2, \cdots n_k$ times, the number of permutations is given by

$$\frac{n!}{n_1!n_2!\cdots n_k!} \tag{8}$$

Note there are no easy formula for the case $n < r$.

5 Combinations of Multisets

Infinite repetition: Let S be a multiset with objects of k types, each with an infinite repetition number (or at least r). Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1} \tag{9}$$

6 Finite Probability