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A family of iterative methods for computing the approximate inverse of a square matrix and inner inverse of a non-square matrix

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ABSTRACT

Based on a quadratical convergence method, a family of iterative methods to compute the approximate inverse of square matrix are presented. The theoretical proofs and numerical experiments show that these iterative methods are very effective. And, more importantly, these methods can be used to compute the inner inverse and their convergence proofs are given by fundamental matrix tools.

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1. Introduction

It is well-known that matrix inverse and generalized inverse are important in applied fields of nature science, such as solution to various systems of linear equation, eigenvalue problems, the linear least square problems. Of course, many important methods to find inverse and generalized inverse of matrix have been developed . In these methods, direct methods usually need much cost in both time and space in order to achieve the desirable results, sometimes it may not be able to work at all. So iterative methods are often effective especially for large scale systems with sparse matrix.

In this paper, we consider a family of methods for computing approximate inverse of nonsingular matrix and generalized inverse of any $m \times n$ matrix.

In [1], the authors mentioned an iterative formula from [2], that is

$$V_{q+1} = V_q(2I - AV_q) \tag{1.1}$$

and showed

$$\|E_{q+1}\| \leqslant \|A\|\|E_q\|^2,$$

where $E_q = I - AV_q$, I is an identity matrix with the same dimension as matrix A, and $\|I - AV_0\| < 1$. In [3], the author gave some comments on convergence analysis in this iterative algorithm for the inverse of an nonsingular square matrix considered in paper [1]. Motivated by Saberi Najafi et al. [1], Wu [3] and SMS algorithm for computing Drazin inverse in [4] and outer inverse in [5], we can get a high order method to find the inverse of nonsingular square matrix and the inner inverse of any $m \times n$ matrix. Indeed, there are many current and related references, some of which are interesting and beneficial to our paper: computing Moore-Penrose inverses of matrices by Newton's iteration [6] or by interval iterative methods [7], computing the group inverses of singular Toeplitz matrices by Modified Newton's algorithm [8], computing generalized matrix inversion and rank by successive matrix powering method [9], approximating outer generalized inverse [10], Drazin inverse [11] and the generalized inverse $A^{(2)}_{T,S}$ of a matrix or an operator A [12,13] by constructive methods.

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2. A family of iterative methods for computing approximate inverse

Let A be nonsingular and V_0 as an initial approximation to A^{-1}. With (1.1), we get a sequence of approximation $\{V_q\}$ to A^{-1}. In [3], the author proved that V_q generated from the iterative formula (1.1) is quadratically convergent to A^{-1}.

In this section, we first introduce a third-order convergence iterative method for finding the inverse of a square matrix as follows:

$$V_{q+1} = V_q(3I - 3AV_q + (AV_q)^2). \tag{2.1}$$

Now we have

Theorem 2.1. *If A is nonsingular and initial approximation V_0 satisfies*

$$\|E_0\| = \|I - AV_0\| < 1,$$

then the iterative formula (2.1) is convergent and V_q converges cubically to A^{-1}.

Proof. From (2.1) and let

$$E_q = I - AV_q,$$

then

$$E_{q+1} = I - AV_{q+1} = I - AV_q(3I - 3AV_q + (AV_q)^2) = (I - AV_q)^3 = E_q^3. \tag{2.2}$$

Since

$$\|E_0\| = \|I - AV_0\| < 1.$$

So

$$\|E_q\| \leqslant \|E_{q-1}\|^2 \leqslant \cdots \|E_0\|^{3^q} \to 0,$$

when $q \to \infty$. Namely,

$$I - AV_q \to 0,$$

when $q \to \infty$, that is,

$$\lim_{q\to\infty} V_q = A^{-1}.$$

Let

$$e_q = A^{-1} - V_q,$$

then

$$Ae_q = I - AV_q = E_q.$$

From (2.2), we have

$$(Ae_q)(Ae_q)^2 = E_q^3 = E_{q+1}.$$

By $Ae_{q+1} = E_{q+1}$, we obtain

$$e_{q+1} = e_q(Ae_q)^2.$$

Therefore it follows immediately that

$$\|e_{q+1}\| \leqslant \|e_q(Ae_q)^2\| \leqslant \|A\|^2\|e_q\|^3.$$

Consequently, it is proved that the iterative formula (2.1) is convergent and V_q at least converges cubically to A^{-1}. □

Numerical experiments show the results of formula (1.1) are evidently improved by (2.1). See Examples 4.1 and 4.2.

In fact, under the same conditions of Theorem 2.1, we can present a family of formulae as follows:

$$V_{q+1} = V_q\left[kI - \frac{k(k-1)}{2}AV_q + \cdots + (-1)^{k-1}(AV_q)^{k-1}\right],\ k = 2, 3, \ldots. \tag{2.3}$$

We can obtain the following conclusion that is similar to Theorem 2.1.

Theorem 2.2. *If A is nonsingular and initial approximation V_0 satisfies*

$$\|E_0\| = \|I - AV_0\| < 1,$$

then V_q generated from iterative formula (2.3) is convergent to A^{-1} and the order of convergence of this formula is k.

Proof. From (2.1) and let

$$E_q = I - AV_q,$$

then

$$E_{q+1} = I - AV_{q+1} = I - AV_q\left[kI - \frac{k(k-1)}{2}AV_q + \cdots + (-1)^{k-1}(AV_q)^{k-1}\right] = (I - AV_q)^k = E_q^k. \tag{2.4}$$

The rest of this proof is similar to that of Theorem 2.1. □

Especially, when $k = 2$, the formula (2.3) is just the formula (1.1) from [2], and when $k = 3$, the formula (2.3) is just the formula (2.1).

In fact, the iterative methods to find the approximate inverse of nonsingular matrix are very powerful in practice but a difficult problem is to find a good initial approximate inverse besides the convergence of iterative sequence. In the following theorem, we will give an easy method to find an initial approximate inverse of any nonsingular matrix.

Theorem 2.3. *For any nonsingular matrix $A \in C^{n\times n}$ and any $0 < \alpha < \frac{2}{\sigma_1^2(A)}$, where $\sigma_1(A) = \|A\|_2$, and setting $V_0 = \alpha A^*$, where A^* is the conjugate transpose matrix of A, then V_q generated from the iterative formula (2.3) is convergent to A^{-1} with k-order convergence.*

Proof. Since A is nonsingular, there exists an unitary matrix Q such that

$$Q^*A^*AQ = diag(\sigma_1^2, \ldots, \sigma_n^2),$$

where $\sigma_1^2 \geqslant \cdots \geqslant \sigma_n^2 > 0$. Based on the condition $0 < \alpha < \frac{2}{\sigma_1^2(A)}$, it is easy to see that

$$-1 < 1 - \alpha\sigma_1^2 \leqslant \lambda_j(I - \alpha A^*A) \leqslant 1 - \alpha\sigma_n^2 < 1.$$

Let

$$d = \max\{|1 - \alpha\sigma_1^2|, 1 - \alpha\sigma_n^2\},$$

then $\rho(I - \alpha A^*A) = d < 1$, where $\rho(I - \alpha A^*A)$ is the spectral radius of matrix $I - \alpha A^*A$.

By $d = \rho(I - V_0A) < 1$ and Lemma 5.6.10 in [14], there exists a constant $\epsilon > 0$ and a matrix norm $\|\cdot\|_*$ such that

$$\|I - V_0A\|_* \leqslant \rho(I - V_0A) + \epsilon = d + \epsilon < 1.$$

Based on Theorem 2.2, we have gained our conclusion. □

3. Computing approximate inner (generalized) inverse of non-square matrix

Much more numerical experiments show the family of formulae (2.3) can be used to compute approximate inner (generalized) inverse of a matrix. The following is a detailed deduction of this statement.

Lemma 3.1 [15]. *Let $G \in C^{n\times n}$. Then $\lim_{r\to\infty} G^r$ exists if and only if the following three conditions hold at the same time:*

(1) $\rho(G) \leqslant 1$, *where $\rho(G)$ is the spectral radius of matrix G;*
(2) *if $\rho(G) = 1$, then these eigenvalues whose modulus are 1 must be 1;*
(3) *if $\rho(G) = 1$, then these eigenvalues whose modulus are 1 must be semi-simple (i.e., corresponding Jordan block is 1×1).*

Lemma 3.2. *Let $A \in C^{m\times n}$, $\{V_q\} \subseteq C^{n\times m}$. Then $\lim_{q\to\infty}(I - V_qA) = \lim_{q\to\infty}(I - V_0A)^{k^q}$ exists if and only if the following two conditions hold.*

(1) $|1 - \mu_j(V_0A)| \leqslant 1, j = 1, \ldots, n$;
(2) if $|1 - \mu_j(V_0A)| = 1$, *then there must be $\mu_j(V_0A) = 0$, and these eigenvalues are semi-simple, where I is identity matrix, V_q comes from (2.3) and $\mu_j(V_0A), j = 1, \ldots, n$, are eigenvalues of V_0A.*

Proof. Let $G = I - V_0A$ and $r = k^q$. From Lemma 3.1 and $\lambda_j(I - V_0A) = 1 - \mu_j(V_0A)$, $j = 1, \ldots, n$, the conclusion is clear. □

Note 3.1. The conditions of Lemma 3.2 are equivalent to that the eigenvalues of (V_0A) are in the circle on the complex plane, whose center is at $(1, 0)$ and whose radius is 1. Or the eigenvalues of (V_0A) are in the origin (these eigenvalues are semi-simple). See Fig. 3.1. In this situation,

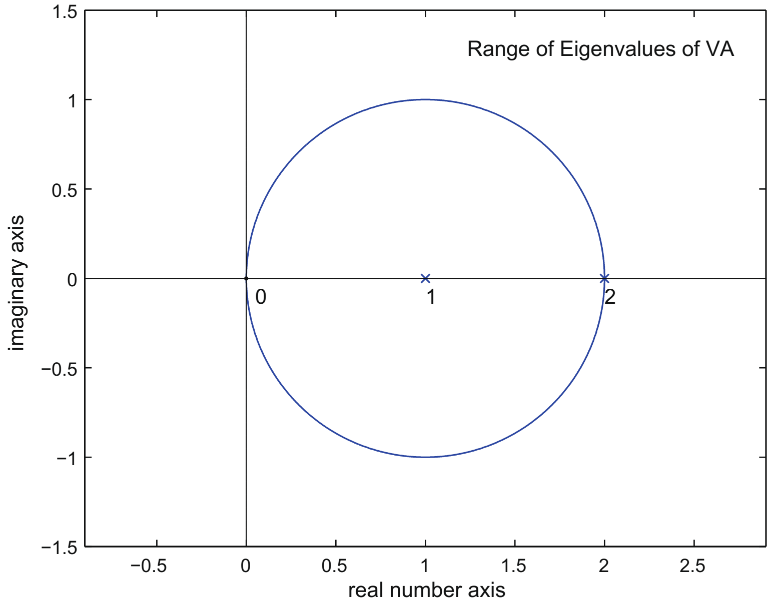

Fig. 3.1. The eigenvalues of V_0A are in the circle or at origin on the complex plane (Note 3.1).

$$\lim_{q\to\infty}(I - V_qA) = \lim_{q\to\infty}(I - V_0A)^{k^q} = PBP^{-1}, \tag{3.1}$$

where P is a similar matrix of Jordan canonical form of matrix $V_0A, r = rank(V_0A)$,

$$P^{-1}(V_0A)P = \begin{pmatrix} 0_{n-r} & 0 \\ 0 & J_r \end{pmatrix}$$

and

$$B = \begin{pmatrix} I_{n-r} & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition 3.1 [16]. Let $A \in C^{m\times n}$. If there exists a matrix $V \in C^{n\times m}$ satisfying $AVA = A$, then V is called the inner (generalized) inverse of A or $\{1\}$ inverse of A.

Note 3.2. Inner inverse is not unique. In general textbooks, the set of the inner inverse of the matrix A is denoted as A^-. Let $A \in C^{m\times n}$ and $rank(A) = r$. If there exist two invertible matrices $P \in C^{n\times n}$ and $R \in C^{m\times m}$ such that

$$RAP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

then $V \in A^-$ if and only if

$$V = P\begin{bmatrix} I_r & U \\ W & T \end{bmatrix}R,$$

where $U \in C^{r\times(m-r)}, W \in C^{(n-r)\times r}, T \in C^{(n-r)\times(m-r)}$ are arbitrary.

Lemma 3.3 [16]. *If $V \in A^-$, then $x = Vb$ is a special solution of compatible systems of linear equations $Ax = b$.*

Consider the sequence $\{V_q\}$ generated from (2.3). We can choose $V_0 = \alpha A, 0 < \alpha < \frac{2}{\sigma_1^2(A)}$, where $\sigma_1(A)$ is the maximum singular value of matrix A. In the following, we will prove that this sequence satisfies $\lim_{q\to\infty} AV_qA = A$ for any $A \in C^{m\times n}$ and any k. That is, V_k is an approximate inner inverse of matrix A.

Lemma 3.4. *For any $A \in C^{m\times n}$ and any $X \in C^{n\times l}, AX = 0$ if and only if $A^*AX = 0$.*

Proof. If $AX = 0$, then $A^*AX = 0$. Inversely, if $A^*AX = 0$, then $A^*AX_j = 0, j = 1,\ldots,l$, where $X = [X_1,\ldots,X_l]$. Therefore, $X_j^*A^*AX_j = 0, j = 1,\ldots,l$, that is, $(AX_j)^*(AX_j) = 0$, or $AX_j = 0, j = 1,\ldots,l$. So we get $AX = 0$. $\square$

Now we prove the main theorem of this section.

Theorem 3.1. *For any $A \in C^{m\times n}$ and any $0 < \alpha < \frac{2}{\sigma_1^2(A)}$, let $V_0 = \alpha A^*$, then the sequence (2.3) satisfies*

$$\lim_{q\to\infty} AV_qA = A \tag{3.2}$$

and the order of convergence is k.

Proof. Since $0 < \alpha < \frac{2}{\sigma_1^2(A)}$, the eigenvalues of matrix αA^*A satisfies

$$0 \leqslant \mu_j(\alpha A^*A) < 2, \quad j = 1,\ldots,n,$$

and $\mu_j(\alpha A^*A)$ are all semi-simple. Obviously, there exists an unitary matrix Q such that

$$Q^*A^*AQ = \begin{pmatrix} 0_{n-r} & 0 \\ 0 & \Lambda_r \end{pmatrix}, \tag{3.3}$$

where $\Lambda_r = diag(\sigma_1^2,\ldots,\sigma_r^2)$, and $\sigma_j^2 > 0, j = 1,\ldots,r$ are singular values of matrix A. Then the sequence (2.3) satisfies the conditions of Lemma 3.2. So we know that

$$\lim_{q\to\infty}(I - V_qA) = \lim_{q\to\infty}(I - \alpha A^*A)^{k^q} = QBQ^* = Q\begin{pmatrix} I_{n-r} & 0 \\ 0 & 0, \end{pmatrix}Q^*$$

exists. Therefore, we have

$$\lim_{q\to\infty}(AV_qA - A) = -\lim_{q\to\infty} A(I - V_qA) = -AQ\begin{pmatrix} I_{n-r} & 0 \\ 0 & 0. \end{pmatrix}Q^*. \tag{3.4}$$

By (3.3),

$$A^*AQ\begin{pmatrix} I_{n-r} & 0 \\ 0 & 0 \end{pmatrix} = Q\begin{pmatrix} 0_{n-r} & 0 \\ 0 & \Lambda_r \end{pmatrix}\begin{pmatrix} I_{n-r} & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{3.5}$$

Again from Lemma 3.4,

$$AQ\begin{pmatrix} I_{n-r} & 0 \\ 0 & 0. \end{pmatrix} = 0. \tag{3.6}$$

Therefore, from (3.4) and (3.6), we obtain $\lim\limits_{q\to\infty}(AV_qA) = A$.

In order to prove that AV_qA has k-order convergence, from above converge proof, we only need to prove that $(I - V_qA)$ converges to Q^*BQ with k-order, or to prove that $(I_r - \alpha\Lambda_r)^{k^q}$ converges to zero matrix with k-order. By $0 < \alpha\sigma_j^2 < 2, -1 < d_j = 1 - \alpha\sigma_j^2 < 1, j = 1,\ldots,r$, and $(d_j)^{k^q} \to 0$ when0 $q \to \infty$, it is self-evident. □

Note 3.3. From (3.2), we can say that V_k is the approximation inner inverse of A. But we cannot prove that V_k converges (in norm) to a fixed inner inverse of A. (In fact, V_k is weakly convergent to an inner inverse of A.) A lot of numerical experiments can show this fact.

4. Numerical experiments

We first present two examples to illustrate the efficiency of the new iterative method with the new initial approximation method to compute the approximate inverse of a nonsingular square matrix. We compare the results of formulae (2.3) when $k = 2$ and $k = 3$. In the following examples, we all set that termination parameter $\varepsilon = 10^{-8}$ and maximum iterative number of step $n = 1000$.

Example 4.1. Let $A = rand(100,100), \alpha = \frac{1}{\|A\|_2^2}$ and $V_0 = \alpha A^*$. The stop criterion is $\|I - VA\|_2 < \varepsilon$. We have tested 50 times with MATLAB 7 and carried out with double precision arithmetic on the computer pentium 4. The time consuming (s) and the iterative times are compared in Figs. 4.1 and 4.2, respectively. x-axis represents the times of choice random matrices, y-axis in Fig. 4.1 represents the computing time and y-axis in Fig. 4.2 represents the iterative times. The dashed line corresponds to the iteration by (1.1) and solid line to the iteration (2.1). We see that (2.1) has a more considerable improvement than (1.1) in both the computing time and the iterative times.

Example 4.2. Let $A = randn(300,300), \ \alpha = \frac{1}{\|A\|_2^2}$ and $V_0 = \alpha A^*$. The stop criterion is $\|I - VA\|_2 < \varepsilon$. We have tested 20 times with MATLAB 7 and carried out with double precision arithmetic on the computer pentium 4. The time consuming (s) and the iterative times are compared in Figs. 4.3 and 4.4, respectively. From Figs. 4.3 and 4.4, we can get the same conclusion as that in Example 4.1.

Example 4.3. We will compare iterative methods (1.1) and (2.1) for computing the inner inverse of non-square matrix A. We also consider the following compatible systems of linear algebraic equations

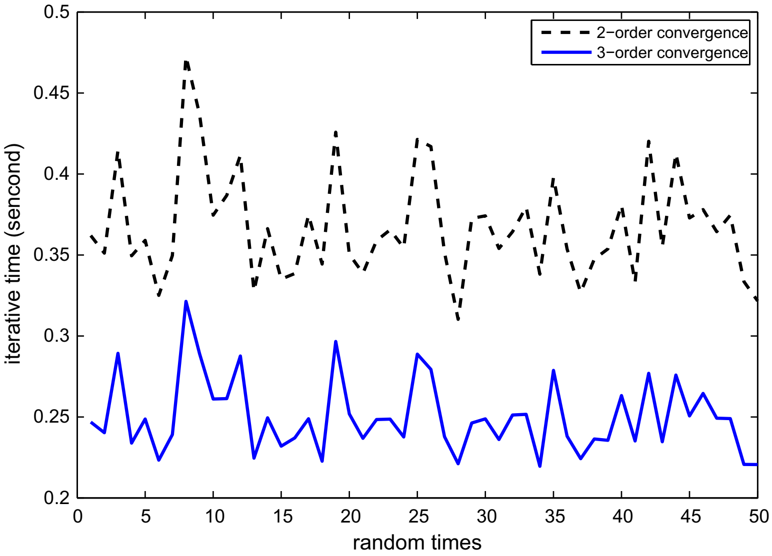

Fig. 4.1. Comparison of consuming time (Example 4.1).

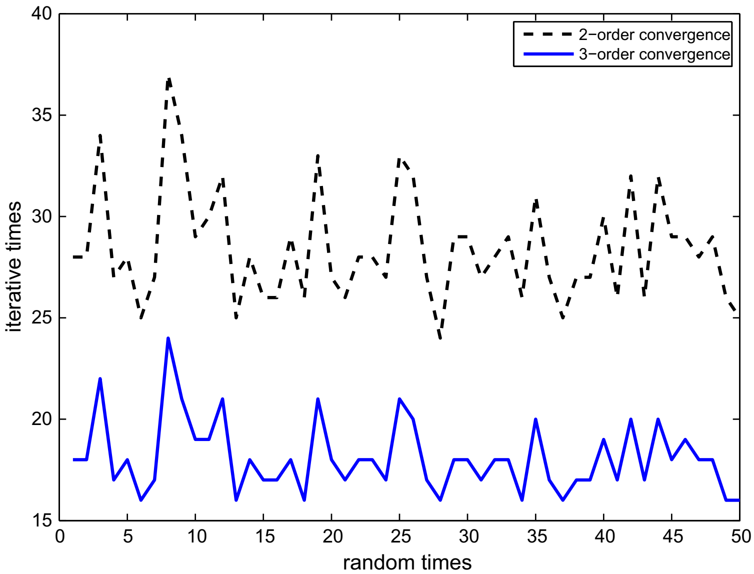

Fig. 4.2. Comparison of iterative times (Example 4.1).

$$Ax = b,\ A = (a_{ij}), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{4.1}$$

where

$$b_i = \sum_{l=1}^{m} a_{il}, \quad i = 1, 2, \ldots, n. \tag{4.2}$$

It is easy to see that the accuracy solution of the system (4.1) is given by

$$x = x^* = (1, 1, \ldots, 1)^T.$$

Let $A = \begin{pmatrix} randn(200, 210) \\ ones(40, 210) \end{pmatrix}$, $\alpha = \frac{1}{\|A\|_2^2}$ and $V_0 = \alpha A^*$. The stop criterion is $\|A - AVA\|_2 < \varepsilon$. We have tested 50 times with MATLAB 7 and carried out with double precision arithmetic on the computer pentium 4. The time consuming (s), the iterative times and the absolute error between the approximate solution V_qB and the accuracy solution x^* of the compatible systems are compared in Figs. 4.5, 4.6, and 4.7, respectively. x-axis represents the times of choice random matrices, y-axis in

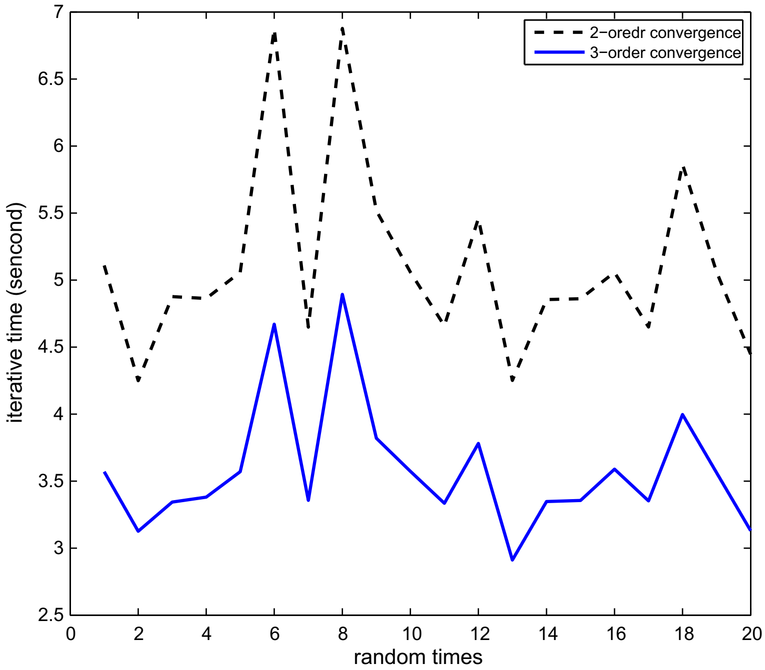

Fig. 4.3. Comparison of consuming time (Example 4.2).

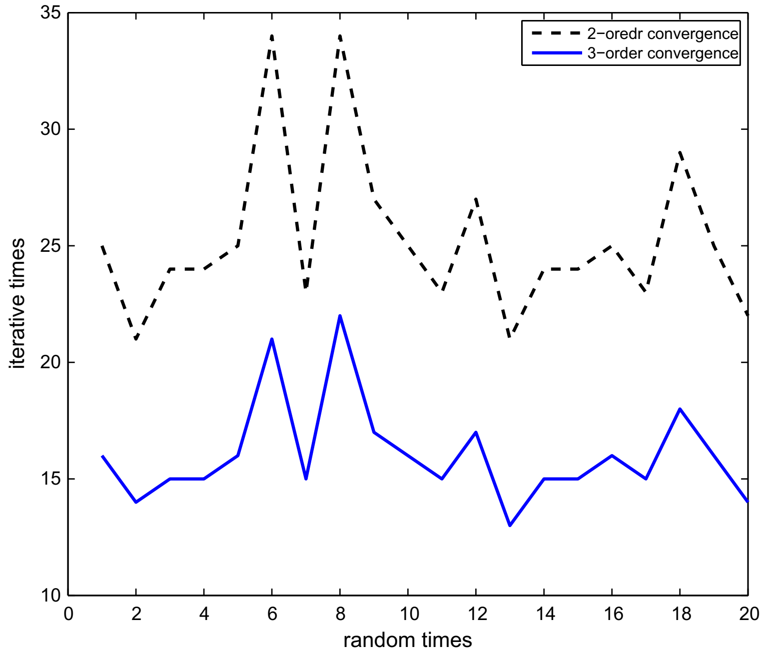

Fig. 4.4. Comparison of iterative times (Example 4.2).

Fig. 4.5 represents the computing time, y-axis in Fig. 4.6 represents the iterative times and y-axis in Fig. 4.7 represents the absolute error between the approximate solution V_qB and the accuracy solution x^* of the compatible systems. The dashed line corresponds to the iteration by (1.1) and the solid line to the iteration (2.1). We see that the iteration method (2.1) has a more considerable improvement than (1.1) in all the computing time and the iterative times.

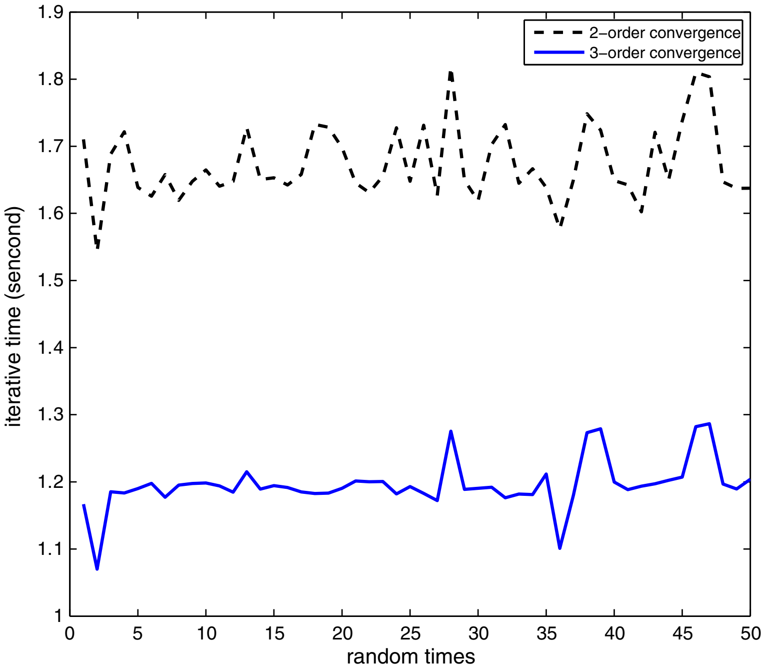

Fig. 4.5. Comparison of consuming time (Example 4.3).

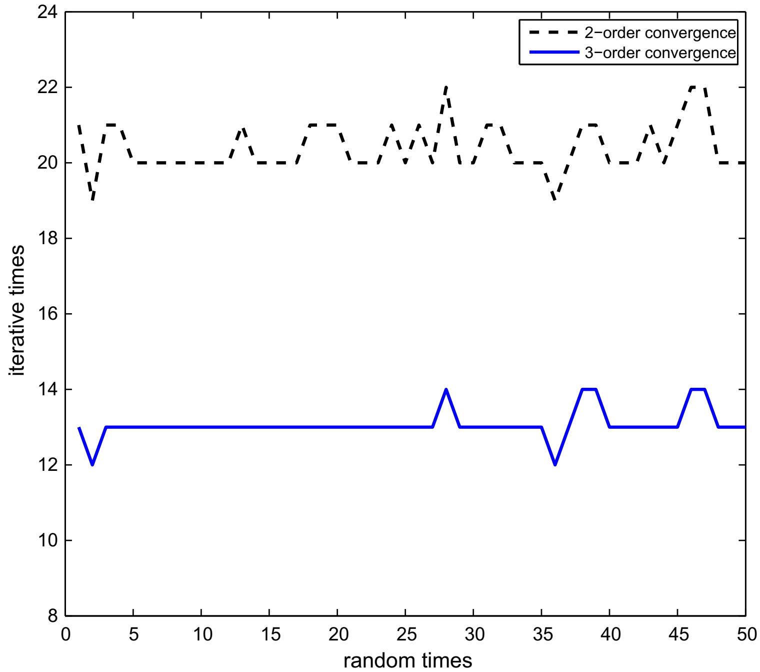

Fig. 4.6. Comparison of iterative times (Example 4.3).

Example 4.4. As an important application, we consider the typical ill-conditioned systems of linear algebraic equations with coefficient matrix (Hilbert) $A = (h_{ij})_{n\times n}$, where $h_{ij} = \frac{1}{i+j-1}$, $i,j = 1,2,\ldots,n$.

$$Ax = b, \tag{4.3}$$

where

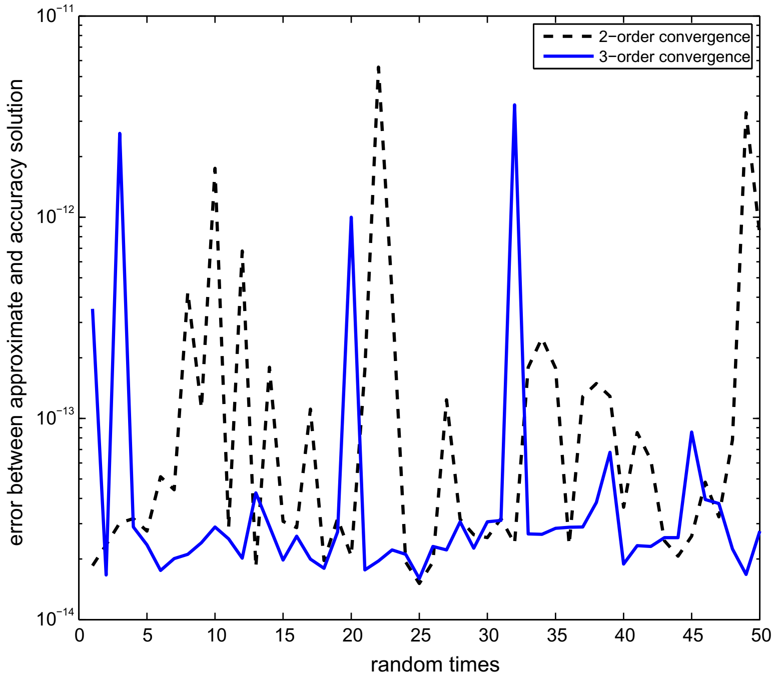

Fig. 4.7. Absolute error (Example 4.3).

Table 4.1
Range of Eigenvalues of VA.

n/cond(A)	Consuming time (s)		Computing times		Absolute error of solution	
	By (1.1)	By (2.1)	By (1.1)	By (2.1)	By (1.1)	By (2.1)
$10/1.6\times10^{13}$	0.015	0.011	49	31	7.14×10^{-5}	7.20×10^{-5}
$50/9.4\times10^{18}$	0.112	0.079	50	32	8.42×10^{-4}	7.35×10^{-4}
$100/1.5\times10^{20}$	0.623	0.441	53	34	0.0015	0.0018
$200/1.1\times10^{20}$	3.29	2.48	52	33	0.0034	0.0033
$300/8.5\times10^{19}$	11.88	8.29	53	34	0.0047	0.0035
$500/1.9\times10^{20}$	45.38	31.49	53	34	0.0081	0.0073

$$b_i = \sum_{l=1}^{n} 0.01 \times l \times h_{il}, \quad i = 1, 2, \ldots, n. \tag{4.4}$$

It is easy to see that the accuracy solution of the system (4.3) is given by

$$x = x^* = (0.01, 0.02, \ldots, 0.01 \times n)^T.$$

Letting $n = 10, 50, 100, 200, 300$, and 500, the corresponding spectral condition numbers from MATLAB function $cond(A)$ and the absolute errors between the approximate solution $V_q b$ and the accuracy solution x^* of the compatible systems are shown in Table 4.1.

We compute the inverse of Hilbert matrices by the iterative methods with $\alpha = \frac{1}{\|A\|_2^2}$ and $V_0 = \alpha A^*$. The stop criteria is $\|A - AVA\|_2 < \varepsilon$. We have tested with MATLAB 7 and carried out with double precision arithmetic on the computer pentium 4. The time consuming (s), the iterative times and the absolute error between the approximate solution and the accuracy solution are compared in Table 4.1.

5. Conclusions

In this paper, we have developed a family of k-order iterative convergence formulae (2.3), which has generalized the quadratically convergent method in [1–3]. Especially, a three-order method is tested. The numerical experiments show the efficiency and improvement to second-order methods. Furthermore, these formulae can be used to compute the inner (generalized) inverse of any matrix. At the same time, we also have given an easy method to furnish an initial approximate inverse which is very effective.

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