

# Independent Trials and Trial Size

## Computational Physics Exercise 1

Gabriel Remiszewski  
Rojin Aksu

25<sup>th</sup> October 2024

In this paper a stochastic method for computing  $\pi$  is introduced and analyzed. The Idea is that the area of the unit circle is given by  $\pi \approx 3.1416$  and the area of a square with a width of 2 is 4. Using this it is possible to calculate  $\pi$  by generating  $P$  two-dimensional points (from a uniform distribution) inside the square and counting how many of them also are located inside the unit circle.

The points are labeled  $\vec{r}_p = (x_p, y_p)$  with  $\vec{r}_p \in [-1, 1] \times [-1, 1]$ . By using the Iverson Bracket

$$[X] = \begin{cases} 1 & \text{if } X \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

We can write the formula for  $\pi_x$ , where the  $x$  stands for the experiment, as

$$\pi_x = E[4[X^2 + Y^2 \leq 1]] = \frac{4}{P} \sum_{p=1}^P [x_p^2 + y_p^2 \leq 1]. \quad (1)$$

The experiment can be repeated  $X$  times, so that we get a final answer

$$\pi_f = \frac{1}{X} \sum_{x=1}^X \pi_x \quad \Delta\pi_f = \sqrt{\frac{1}{X-1} \sum_{x=1}^X (\pi_f - \pi_x)^2}. \quad (2)$$

## 1 One big experiment

Let  $P = 10000$  and  $X = 1$ . Since only one experiment is done, eq. (2) cannot be used to calculate the uncertainty. Because of that the uncertainty is defined as

$$\Delta\pi_x = \sqrt{\text{Var}(4[X^2 + Y^2 \leq 1])} \quad (3)$$

with  $\text{Var}(X)$  being the variance of the random variable  $X$ . Letting the experiment run results in a value of

$$\pi_x = 3.1 \pm 1.7, \quad (4)$$

which is considering the uncertainty of approximately 53 % close to the real value.

When doing just one experiment, the distribution of the generated points is of interest. For this the histogram of the radii  $R = \sqrt{X^2 + Y^2}$  and the squared radii are plotted in figs. 1 and 2. For the radii we can

see a linear rising distribution with a maximum at  $r = 1$ , after which the distribution drops. This drop is also visible for the squared radii, but in the region  $[0, 1]$  a constant distribution is visible. The drop after  $r = 1$  is easily explainable with the fact that points inside a  $2 \times 2$  square we generated and a circle with radius 1 is the biggest circle that fits inside. With this in mind the amount of points generated on the circumference of a circle with bigger radius has to get smaller. **TO BE CONTINUED**

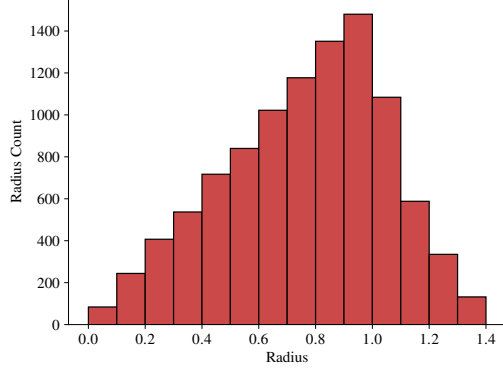


Figure 1: Histogram of radii from all the generated points of one big experiment with  $P = 10000$ .

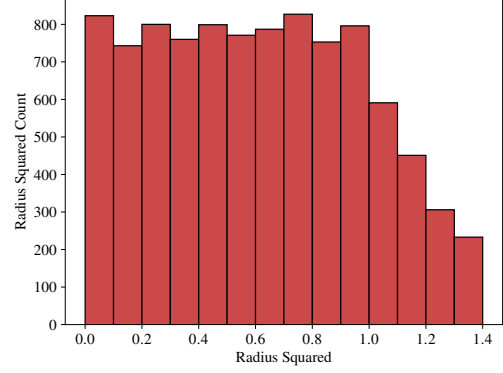


Figure 2: Histogram of the squared radii from all the generated points of one big experiment with  $P = 10000$ .

## 2 Rearranging Calculations

The number of calculations is now split into  $P = 100$  and  $X = 100$ . In this case we get a value of

$$\pi_f = 3.15 \pm 0.16. \quad (5)$$

This value has a much smaller error (5.1 %) compared to section 1. A histogram for the distribution of  $\pi_x$  is shown in fig. 4. In that case the  $\pi_x$  histogram resembles a gaussian distribution.

With  $P = 1$  and  $X = 10000$  we get a value of

$$\pi_f = 3.2 \pm 1.7, \quad (6)$$

which is identical to the calculation from section 1. In fact, they have to be the same because they are mathematically the same. In the special case where  $X = 1$ , the final  $\pi$  value is given by

$$\pi_f = \frac{4}{P} \sum_{p=1}^P [r_p^2 \leq 1] \quad \Delta\pi_f = \sqrt{\frac{1}{P-1} \sum_{p=1}^P (4[r_p^2 \leq 1] - \pi_f)^2}$$

with using the definition of the variance from above. For the special case of  $P = 1$  and the uncertainty from eq. (2) we get

$$\pi_f = \frac{4}{X} \sum_{x=1}^X [r_x^2 \leq 1] \quad \Delta\pi_f = \sqrt{\frac{1}{X-1} \sum_{x=1}^X (4[r_x^2 \leq 1] - \pi_f)^2}.$$

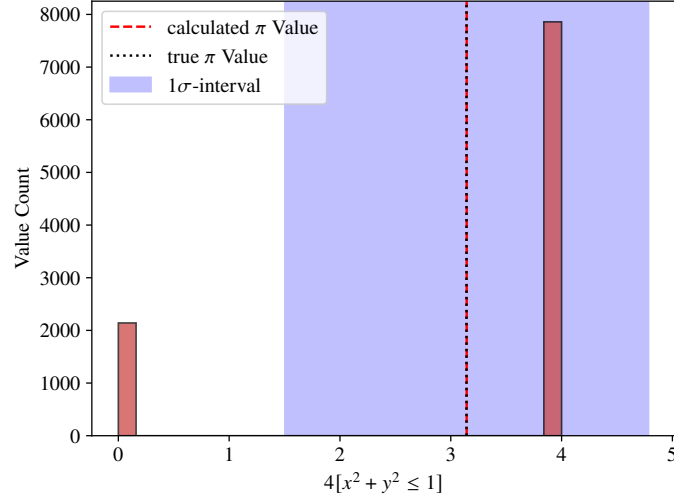


Figure 3: Histogram of the indicator variable  $[X^2 + Y^2 \leq 1]$  for an experiment with  $P = 10000$ .

For  $X = P$  it is clear that they are the same, so similar results are expected. Since we can compare the results for both extreme cases, one can make the assumption that the definition of eq. (3) for  $X = 1$  is as good as the definition in 2 for the general case.

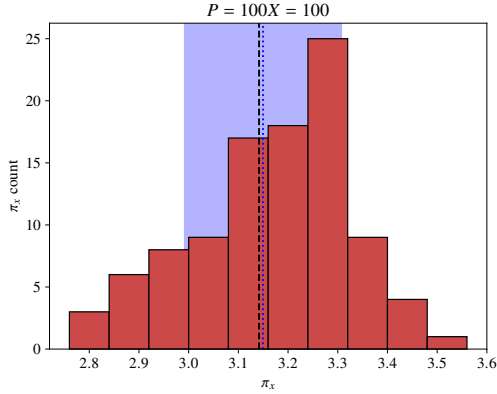


Figure 4: Histogram of calculated  $\pi$  values with  $P = 100$  and  $X = 100$ .

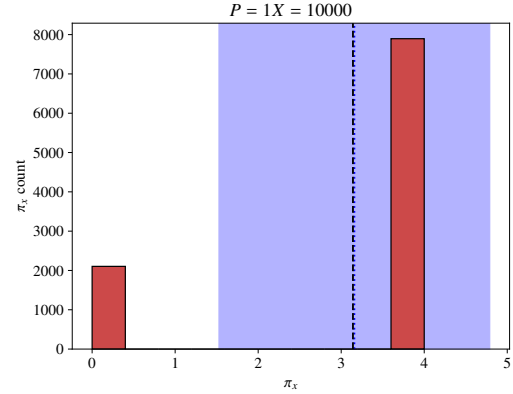


Figure 5: Histogram of calculated  $\pi$  values with  $P = 1$  and  $X = 10000$ .

Table 1: Calculated  $\pi$  values for different  $P$  and  $X$ .

$X \backslash P$	10	100	1000	10000
10	$3.160 \pm 0.332$	$3.136 \pm 0.161$	$3.125 \pm 0.042$	$3.147 \pm 0.015$
100	$3.124 \pm 0.465$	$3.134 \pm 0.141$	$3.133 \pm 0.044$	$3.141 \pm 0.017$
1000	$3.148 \pm 0.526$	$3.136 \pm 0.162$	$3.142 \pm 0.050$	$3.142 \pm 0.016$
10000	$3.144 \pm 0.523$	$3.140 \pm 0.166$	$3.141 \pm 0.052$	$3.141 \pm 0.016$

### 3 Uncertainty Dependency of $X$ and $P$

Now we focus on how the uncertainty changes as a function of  $X$  and  $P$ . The  $\pi$  values were calculated for 16 different combinations of point- and experiment count. The results are shown in table 1 with the corresponding histograms for the  $\pi_x$  distributions. It is easy to see that for a constant  $P$  the uncertainty does not change with higher  $X$  which is surprising, because it means that for a given number of generated points the  $\pi$  value cannot be made more precise with higher experiment count. The uncertainties for every  $P$  are shown in fig. 7. On the other hand the correlation between the uncertainty and  $P$  for constant  $X$  is visible. For the log-log plot in fig. 8 a linear function would be a good approximation and thus we can write

$$\log \text{err}(P) = -n \log P + b$$

with  $b, n > 0$ . This results in the relation

$$\text{err}(P) = \frac{e^b}{P^n}$$

for the uncertainty. This function will not be fitted to the data but from visual inspections it is clear that the same relation with same parameters  $b$  and  $n$  will hold for every  $X$ .

With this information we can conclude that the most efficient way to calculate  $\pi$  with a given  $PX$  is to maximize  $P$  and keep  $X$  small, although from section 1 it is known that  $X$  has to be at least bigger than 1.

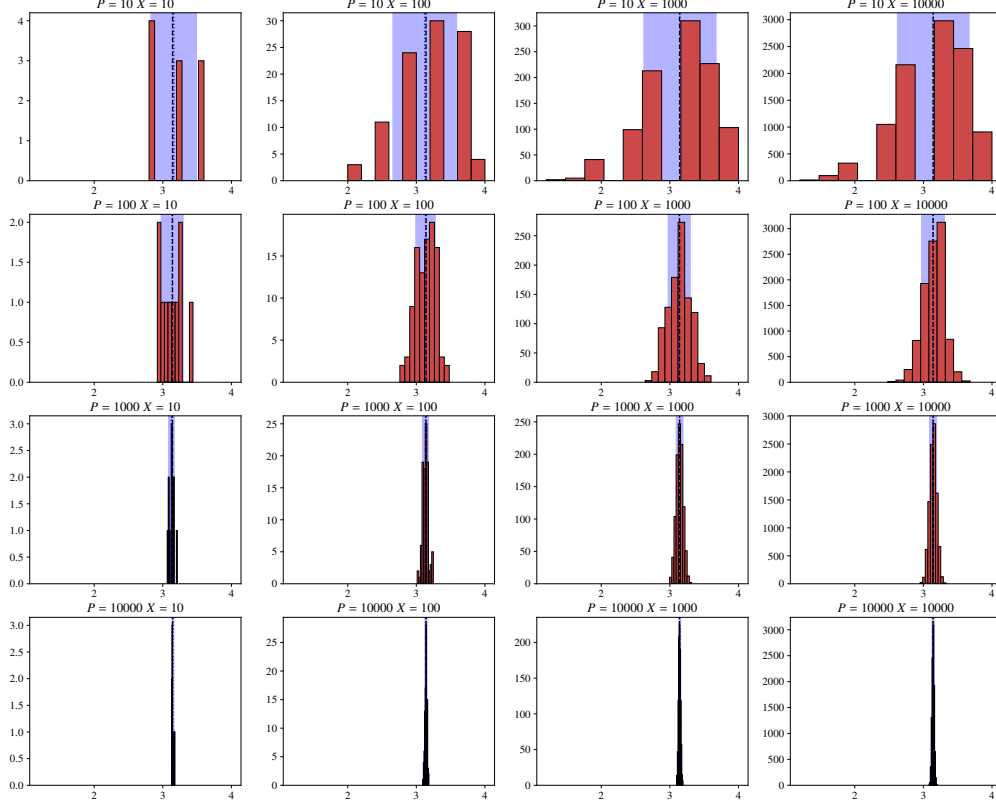


Figure 6:  $\pi_x$  distributions for different combinations of  $P$  and  $X$ .

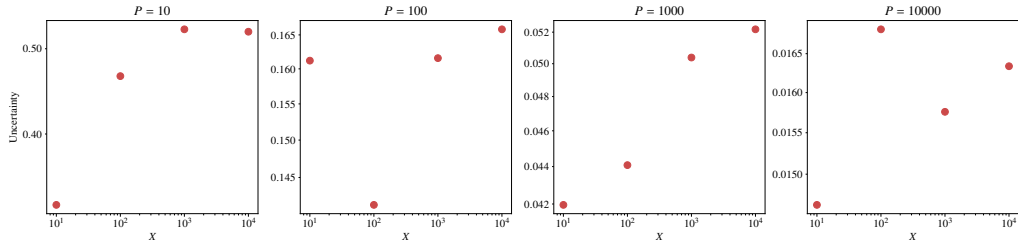


Figure 7: Uncertainty as a function of  $X$  for different  $P$  values.

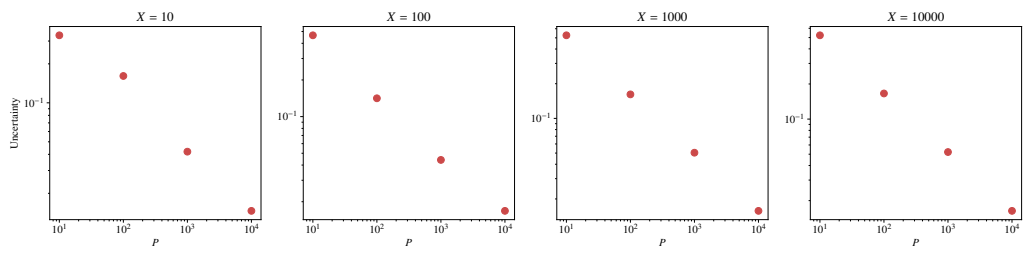


Figure 8: Uncertainty as a function of  $P$  for different  $X$  values.