Computational Physics (Physics 760) Exercise #2

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2 Simple Sampling Experiments

2.1 (6pt) Sequences of Fair Coin Flips Can Be Understood as Sampling the Uniform Distribution on $[0, 1]$

Suppose we flip a fair coin p times and record f_1 through f_p. If the coin lands heads on the jth flip, $f_j = 1$ and if it lands tails, $f_j = 0$.

If you like, the outcome space of the p flips is $\vec{f} \in \Omega = \{\text{heads}, \text{tails}\}^{\otimes p}$. The coin flips are independent, so we can multiply each coin's probability. Each subspace (each coin) is identically distributed with $P(\text{heads}) = 1 - P(\text{tails}) = 0.5$. So, the components of f in a single sequence is iid.

Let us reinterpret each sequence of p flips as a binary number with fixed precision of p digits,

$$n = 0.f_1 f_2 f_3 f_4 \cdots f_p = \sum_{j=1}^{p} \frac{f_j}{2^j} \tag{1}$$

where we're concatenating the 0s and 1s into a bit string, not multiplying. In the language of observables or random variables, you could say

$$real(\vec{f}) = \sum_{j=1}^{p} \frac{f_j}{2^j}. \tag{2}$$

is a map $real:\ \Omega \to [0, 1] \subset \mathbb{R}$. We will demonstrate numerically that the induced probability distribution on the *real* observable is flat.

- (1pt) What is the minimum possible value for n (in base-10, or some simple expression, not binary!) if $p = 16$?
- (1pt) What is the maximum possible value for n (in base-10, or some simple expression, not binary!) if $p = 16$?
- (1pt) What is the maximum possible if we send $p \to \infty$?

For $S \in \{10^2, 10^4, 10^6\}$ sequences

- (3pt) Generate S sequences of $p = 32$ fair coin flips. Reinterpret them as S real numbers. Make a histogram. (This part used to ask for $p = 16$ but 2^{16} is small enough to quickly try every possibility. Don't do that; use a randomized sampling algorithm.)

2.2 (5pt) Independent Coins with Different Distributions

So far we have flipped a fair coin in sequences of p iid flips. It doesn't matter whether you were imagining flipping the same coin p times or flipping p different coins that all happen to be fair. In this exercise we'll see that if the different flips in a sequence have different distributions we get a different distribution for the *real* observable.

Let's flip coins with different biases, so that the p flips are indepdent but not identically distributed. Suppose the jth coin has

$$P_j(\text{tails}) = 1 - P_j(\text{heads}) = \frac{1}{1 + e^{-\lambda/2^{j+1}}}. \tag{3}$$

Because the coins have biases the induced probability distribution on the *real* observable is not flat.

- (3pt) For $\lambda \in 0, 0.5, 1, 2$ sample 10^6 sequences of $p = 32$ different coins; the jth coin in each sequence should be biased as in equation (3). Evaluate the *real* observable on each sequence, interpreting these flips as 10^6 *real* observables in $[0, 1]$. Make a histogram.

We won't prove it, but the *real* observables x are distributed exponentially on the interval $[0, 1]$. The probability density is, up to normalization,

$$P(x) \propto e^{-\lambda x} \tag{4}$$

- (1pt) Find the normalization constant of P.
- (1pt) Plot the normalized $P(x)$ on top of the appropriate histogram.

2.3 (2pt) Transforming CDFs

The CDF of the normal distribution with mean μ and standard deviation σ is

$$N_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}. \tag{5}$$

The inverse CDF with $\mu = 0$ and $\sigma = 1$ is

$$CDF_{\mu=0,\sigma=1}(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right) \tag{6}$$

where erf is the *error function*.

- (2pt) Derive the CDF for arbitrary μ and σ. You can look up the answer if you need to, but the point of the exercise is to prove it using the definition of the CDF, the particular value of the CDF for $\mu = 0$ and $\sigma = 1$, and rules of calculus. Try without looking!

2.4 (7pt) Inverse Transform Sampling

The CDF of some distribution we are interested in sampling from is given by

$$CDF_{\mu,s}(x) = \left(\frac{1 + \tanh\frac{x-\mu}{2s}}{2}\right)^2 \tag{7}$$

which is the square of the CDF of the logistic distribution. I just made this CDF up, so who knows what the PDF is?! I don't! (Yet; you'll compute it below.) Nevertheless, let's generate samples according to its PDF.

Let $\mu = 0$, and $s = 3$.

- (3pt) Draw 10^5 samples of u from the uniform distribution on $[0, 1]$, which is the range of the CDF. For each sample compute $x = CDF^{-1}_{\mu,s}(u)$.
- (1pt) To make sure you've succeeded in applying the inverse, make a scatter plot of (x, u) on top of a plot of the CDF.
- (1pt) Make a histogram of x. Indicate the mean of the samples.
- (2pt) Compute the PDF analytically and draw it on top of your histogram. Be honest and do this part last so that you are really checking your sampling!