Independent Trials and Trial Size

Computational Physics Exercise 1

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In this paper a stochastic method for computing π is introduced and analyzed. The Idea is that the area of the unit circle is given by $\pi \approx 3.1416$ and the area of a square with a width of 2 is 4. Using this it is possible to calculate π by generating P two-dimensional points (from a uniform distribution) inside the square and counting how many of them also are located inside the unit circle.

The points are labeled $\vec{r}_p = (x_p, y_p)$ with $\vec{r}_p \in [-1, 1] \times [-1, 1]$. By using the Iverson Bracket

$$[X] = \begin{cases} 1 & \text{if X is true} \\ 0 & \text{otherwise} \end{cases}$$

We can write the formula for π_x, where the x stands for the experiment, as

$$\pi_x = \mathrm{E}[4[X^2 + Y^2 \leq 1]] = \frac{4}{P} \sum_{p=1}^{P} [x_p^2 + y_p^2 \leq 1]. \tag{1}$$

The experiment can be repeated X times, so that we get a final answer

$$\pi_\mathrm{f} = \frac{1}{X} \sum_{x=1}^{X} \pi_x \qquad \Delta\pi_\mathrm{f} = \sqrt{\frac{1}{X-1} \sum_{x=1}^{X} (\pi_\mathrm{f} - \pi_x)^2}. \tag{2}$$

1 One big experiment

Let $P = 10\,000$ and $X = 1$. Since only one experiment is done, eq. (2) cannot be used to calculate the uncertainty. Because of that the uncertainty is defined as

$$\Delta\pi_x = \sqrt{\mathrm{Var}(4[X^2 + Y^2 \leq 1])} \tag{3}$$

with $\mathrm{Var}(X)$ being the variance of the random variable X. Letting the experiment run results in a value of

$$\pi_x = 3.1 \pm 1.7, \tag{4}$$

which is considering the uncertainty of approximately 53 % close to the real value.

When doing just one experiment, the distribution of the generated points is of interest. For this the histogram of the radii $R = \sqrt{X^2 + Y^2}$ and the squared radii are plotted in figs. 1 and 2. For the radii we can

see a linear rising distribution with a maximum at $r = 1$, after which the distribution drops. The linear rise is explainable with the following argumentation:
The points generated inside the square are distributed uniformly, but since the circumference of a circle is proportional to the radius, the point density $n(r)$ also has to be $n(r) \sim r$. The drop after $r = 1$ occurs because a circle with radius 1 is the biggest that fits inside a 2×2 square, so there will be fewer points that can be generated with that radius.

For the R^2 distribution a constant distribution up until $r = 1$ is visible. This can be easily calculated. For two random variables X, Y and a function $V : X \to Y$ and given distribution $n(x)$, the distribution of Y can be calculated with the delta distribution $\delta(x)$ by

$$p(y) = \int dx\, n(x)\delta(y - V(x)).$$

For the variable $\rho = r^2$ we get

$$\begin{aligned} p(\rho) &= \text{const} \cdot \int_0^1 dr\, r \cdot \delta(r^2 - \rho) = \text{const} \cdot \int_0^1 dr\, r \cdot \frac{\delta(r - \sqrt{\rho})}{2\sqrt{\rho}} \\ &= \text{const} \end{aligned}$$

TO BE CONTINUED

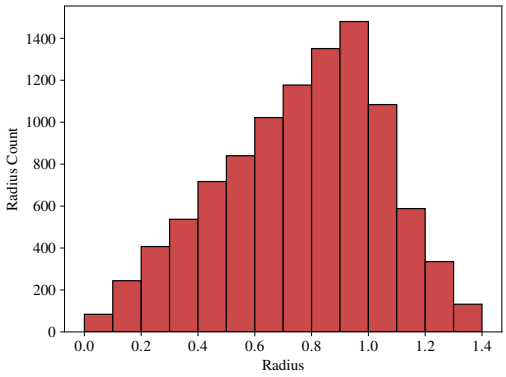

Figure 1: Histogram of radii from all the generated points of one big experiment with $P = 10\,000$.

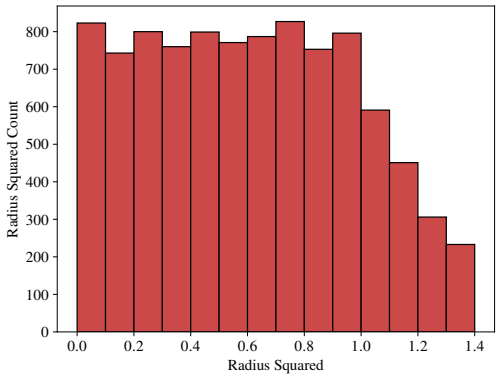

Figure 2: Histogram of the squared radii from all the generated points of one big experiment with $P = 10\,000$.

2 Rearranging Calculations

The number of calculations is now split into $P = 100$ and $X = 100$. In this case we get a value of

$$\pi_\mathrm{f} = 3.15 \pm 0.16. \tag{5}$$

This value has a much smaller error (5.1 %) compared to section 1. A histogram for the distribution of π_x is shown in fig. 4. In that case the π_x histogram resembles a gaussian distribution.

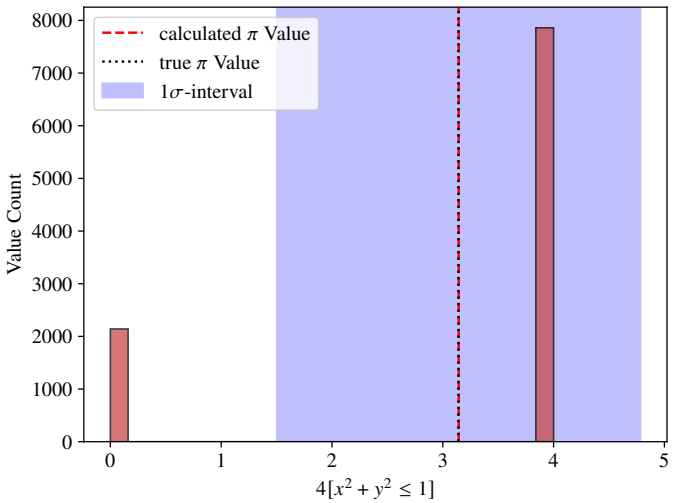

Figure 3: Histogram of the indicator variable $[X^2 + Y^2 \leq 1]$ for an experiment with $P = 10\,000$.

With $P = 1$ and $X = 10\,000$ we get a value of

$$\pi_\mathrm{f} = 3.2 \pm 1.7, \tag{6}$$

which is identical to the calculation from section 1. In fact, they have to be the same because they are mathematically the same. In the special case where $X = 1$, the final π value is given by

$$\pi_\mathrm{f} = \frac{4}{P}\sum_{p=1}^{P}[r_p^2 \leq 1] \qquad \Delta\pi_\mathrm{f} = \sqrt{\frac{1}{P-1}\sum_{p=1}^{P}(4[r_p^2 \leq 1] - \pi_\mathrm{f})^2}$$

with using the definition of the variance from above. For the special case of $P = 1$ and the uncertainty from eq. (2) we get

$$\pi_\mathrm{f} = \frac{4}{X}\sum_{x=1}^{X}[r_x^2 \leq 1] \qquad \Delta\pi_\mathrm{f} = \sqrt{\frac{1}{X-1}\sum_{x=1}^{X}(4[r_x^2 \leq 1] - \pi_\mathrm{f})^2}.$$

For $X = P$ it is clear that they are the same, so similar results are expected. Since we can compare the results for both extreme cases, one can make the assumption that the definition of eq. (3) for $X = 1$ is as good as the definition in 2 for the general case.

3 Uncertainty Dependency of X and P

Now we focus on how the uncertainty changes as a function of X and P. The π values were calculated for 16 different combinations of point- and experiment count. The results are shown in table 1 with the corresponding histograms for the π_x distributions. It is easy to see that for a constant P the uncertainty

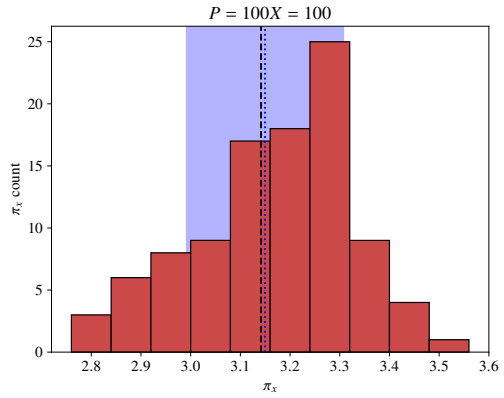

Figure 4: Histogram of calculated π values with $P = 100$ and $X = 100$.

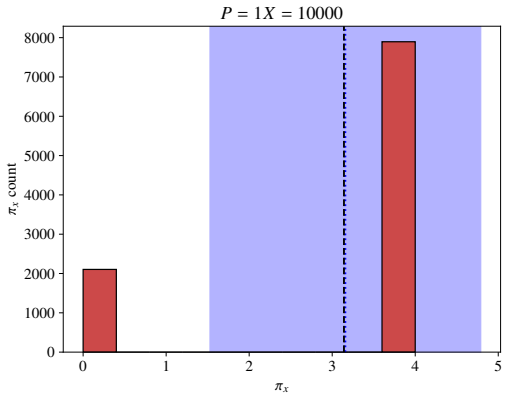

Figure 5: Histogram of calculated π values with $P = 1$ and $X = 10\,000$.

Table 1: Calcuated π values for different P and X.

P X	10	100	1000	10000
10	3.20 ± 0.62	3.104 ± 0.136	3.109 ± 0.047	3.143 ± 0.016
100	3.072 ± 0.518	3.171 ± 0.171	3.140 ± 0.053	3.144 ± 0.017
1000	3.125 ± 0.529	3.136 ± 0.164	3.141 ± 0.052	3.141 ± 0.017
10000	3.144 ± 0.521	3.142 ± 0.164	3.142 ± 0.052	3.142 ± 0.016

does not change with higher X which is surprising, because it means that for a given number of generated points the π value cannot be made more precise with higher experiment count. The uncertainties for every P are shown in fig. 7. On the other hand are correlation between the uncertainty and P for constant X is visible. For the log-log plot in fig. 8 a linear function would be a good approximation and thus we can write

$$\log \mathrm{err}(P) = -n \log P + b$$

with $b, n > 0$. This results in the relation

$$\mathrm{err}(P) = \frac{\mathrm{e}^b}{P^n}$$

for the uncertainty. This function will not be fitted to the data but from visual inspections it is clear that the same relation with same parameters b and n will hold for every X.

With this information we can conclude that the most efficient way to calculate pi with a given PX is to maximize P and keep X small, although From section 1 it is known that X has to be atleast bigger than 1.

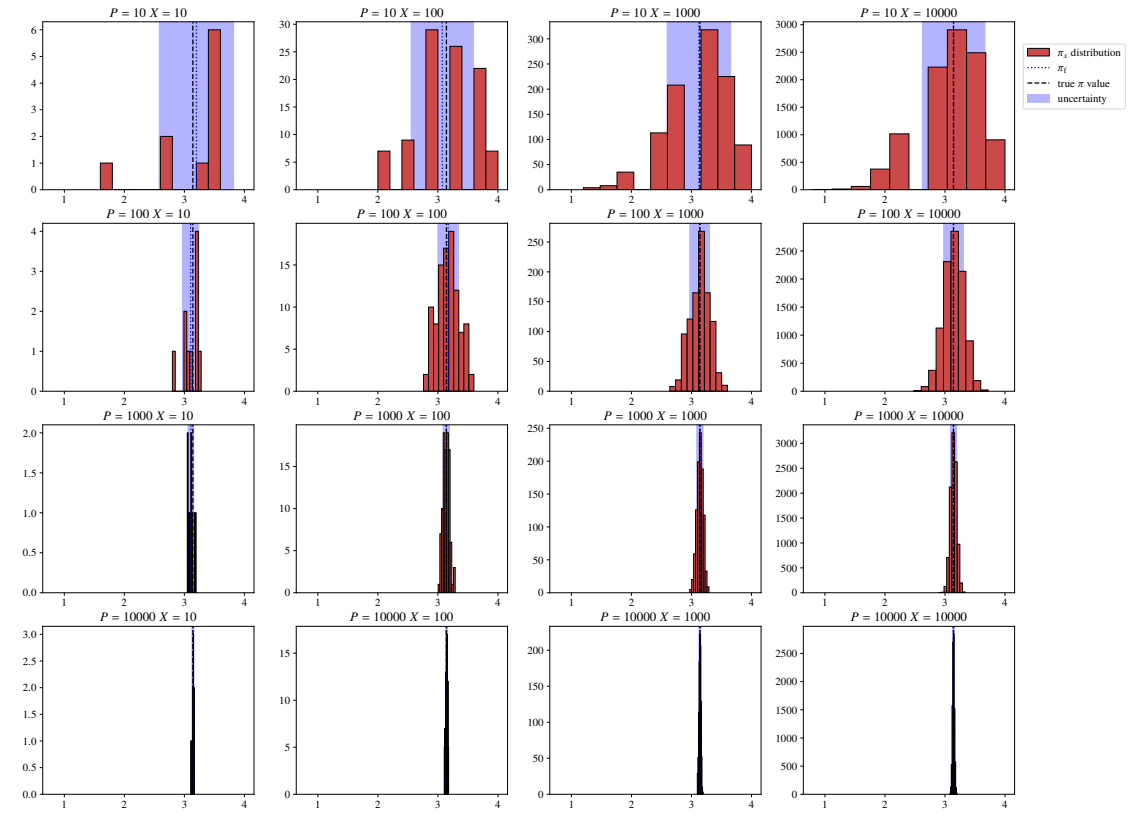

Figure 6: π_x distributions for different combinations of P and X.

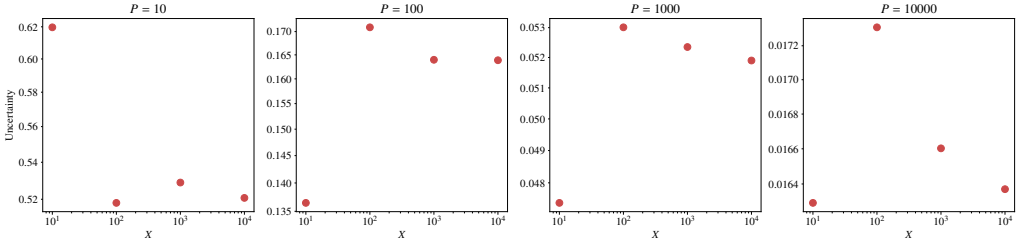

Figure 7: Uncertainty as a function of X for different P values.

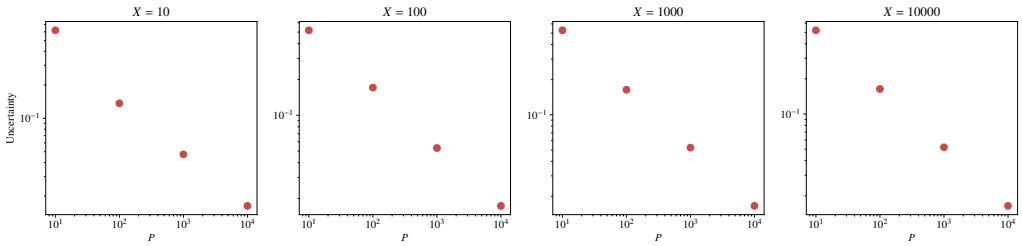

Figure 8: Uncertainty as a function of P for different X values.