A few classical methods of estimating the dipole moment of a neodymium magnet in an external magnetic field

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Introduction: The magnetic moment

The torque τ on a magnet in an external magnetic field $\mathbf{B}$ is aligned according to a relation between the field vector, and a vector quantity that is defined as the magnetic moment of said magnet, μ:

$$\tau = \mu \times \mathbf{B}. \tag{1}$$

— This definition is practical to work with, as the torque on a body is related to a number of phenomena; such as equilibrium conditions of dynamics, in which the net sum of torques applied on some body is zero, represented,

$$\sum \tau = 0; \tag{2}$$

and is related to the rate of change in the body's angular momentum by the following relation,

$$\sum \tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}. \tag{3}$$

The equivalence of the torque-definition of the magnetic moment with the magnetic moment as that equivalent vector quantity (according to definition) of a current loop allows equation (1) to relate the force on the body with the rate of change of a varying external magnetic field in the following relation:

$$F_z = \mu \frac{\mathrm{d}B_z}{\mathrm{d}z}, \tag{4}$$

where F_z is the vertical (z-direction) force component on the magnet, B_z is the vertical field component of the external magnetic field and μ is the magnitude of the magnetic moment. All these relations will be exploited to measure the magnetic moment with varying degree of accuracy.
The neodymium magnet in question, in a shape of a disc, is at times enclosed within a billiard-ball. The ball is to sit inside a hemisphere, on a surface made up of air currents in an attempt to minimize the forces of friction between the ball and hemisphere. The position of the hemisphere is between the two rings of a Helmholtz-coil providing a (uniform, or varying) external magnetic field at the position of the magnet inside the ball. Alternatively, a magnet will also be used without a billiard ball-encasement, together with the Helmholtz-coil.

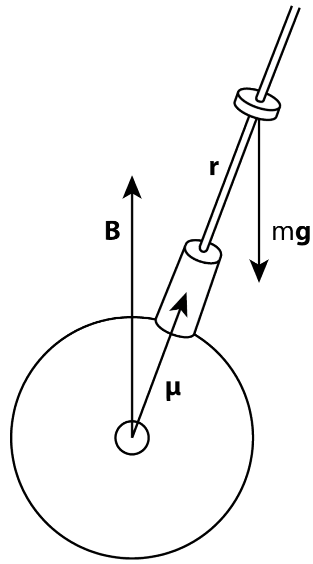

Figure 1: The billard-ball with its magnetic moment and a weight on an extension, schematic of experiment 1's set-up.

Pre-determining neccesary quantities

For quantities related to the "magnetic cueball" we'll be using the subscript cb.
Its mass is measured to be: $M_{cb} = 141 \pm 1$ g
Radius is measured as well: $R_{cb} = 2.65 \pm 0.5$ cm
By assuming the ball to be a perfect sphere allows us to estimate its moment of inertia I, by the classical moment of inertia relation for a solid sphere, namely, $I_{cb} = \frac{2}{5} M_{cb} R_{cb}^2 = 3.99 \cdot 10^{-5}$ kg m^2 and by knowing the cueballs spin frequency ω,

concequently allows us to estimate its angular momentum L by the relation $L = I_{cb}\omega$

For gravitational acceleration we'll be using a numerical value of $g = 9.83$ m/s^2

The strength of the magnetic field produced by the magnetic coil is directly proportional to the current I flowing through the coil, by a proportionality factor 0.00137 T/A, so the B-field can be estimated as $B = 0.00137I$ [T]. Similarly, the magnets vertical gradient at the center with respect to the two coaxial coils, and the distance between them, has been pre-measured to be proportional to I by $\partial B_z/\partial z \simeq 1.69 \cdot 10^{-2} I$

Part I
Magnetic and gravitational torque equilibrium

Model

The torque due to gravity of the ball around its principal axis can be considered zero, due to the gravitational centre of the ball coinciding with the origin of the ball-coordinate system. Thus, an extension of the ball (conviniently aligned in the direction of the magnetic moment) of a rod with a weight on it will cause the gravitational centre to shift along the length of the extension, and a gravitational torque works on the ball-system:

$$\tau = \sum_i \tau_i, \quad \tau_i = \mathbf{r}_i \times m_i \mathbf{g},$$

where $\mathbf{r}_i$ is the position of system component i, m_i its mass and $\mathbf{g}$ is the acceleration of gravity. When the system is at rest, with a zero net-torque, the magnitude of the torque produced by the magnetic moment must equal the magnitude of the gravitational torque;

$$\mu \times \mathbf{B} = \sum_i \mathbf{r}_i \times m_i \mathbf{g}. \tag{5}$$

The angle between the two vectors involved in the (only) two non-zero gravitational torque components of the system is the same; so the sum of the two components can be expressed with a single vector cross product. Furthermore, as is displayed in figure 1, the resulting torque is along the same direction as that of the magnetic moment torque, allowing equation (5) to be expressed as a scalar equation

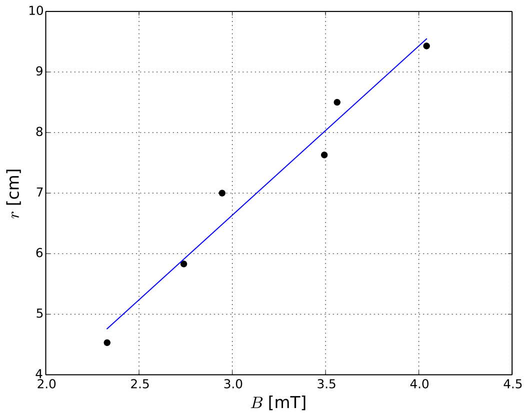

Figure 2: Location of the extra weight, r is plotted against the B-fields strength

$$\mu B = r' m' g,$$

or

$$r' = \frac{\mu}{m' g} B, \tag{6}$$

where the contribution to r' and m' are composite from rod and mass. However, by varying only the position of the mass, a plot of equation (6) will retain the same slope as a plot

$$r = \frac{\mu}{mg} B, \tag{7}$$

for r and m corresponding to position and mass respectively of the weight only; a slope that allows for the calculation of μ.

Results

Plotting r against B and exploiting the relation in equation (7), a numerical value of the slope $\kappa = \mu/mg$ of the regressed line through the data-points, as depicted in Fig. (2) enables us to estimate the magnetic dipole moment, which yields,

$$\mu = 0.374 \text{ J/T}$$

Part II
Spherical pendulum

Model

The restoring force of the torque in equation (1) inserted into equation (3) gives rise to an equation of motion describing (for small angular displacements of magnetic moment) harmonic oscillation behaviour. From the combination of equations (1) and (3) mentioned,

$$\mu \times \mathbf{B} = \dot{\mathbf{L}}, \tag{8}$$

where $\mathbf{L}$ is the angular momentum. In other words,

$$-\mu \times \mathbf{B} = \mathbf{I}\ddot{\theta}, \tag{9}$$

where $\mathbf{I}$ is the moment of inertia of the ball and θ the angular displacement of the magnetic moment w.r.t. the external magnetic field $\mathbf{B}$, (quantities related to the angular momentum through $\mathbf{L} = \mathbf{I}\dot{\theta} = \mathbf{I}\omega$.) This has a scalar form masking the assumption that θ is small, replacing $\sin\theta \to \theta$. Therefore,

$$-\mu B\theta = I\ddot{\theta}, \tag{10}$$

which is a harmonic oscillator differential equation. Solving it for θ and then expressed with the period of oscillation T, it can take the following form,

$$T^2 = \frac{4\pi^2 I}{\mu}\frac{1}{B} \tag{11}$$

which can be plotted, so that a line of slope κ can be obtained by plotting T^2 against B^{-1}, allowing us to calculate μ.

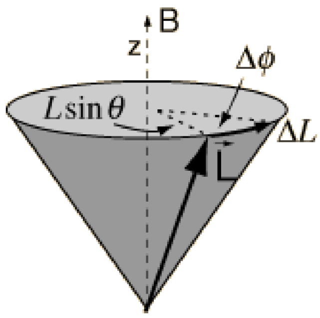

Figure 3: Precessing angular momentum $\mathbf{L}$ as described in the text for experiment 3. Figure adapted from http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/larmor.html

Results

With a stopwatch, we measure the interval of time required for 5 periods of oscillation. This is done for various B-field strengths by adjusting the current through the coils. Using the relationship obtained in eq. (11), we can plot the period squared against the reciprocal of the B-fields strength. The regressed line in Fig. (4) yields a slope $\kappa = 4\pi^2 I/\mu$, where we have already obtained $I = I_{cb}$, allowing us to calculate μ to be,

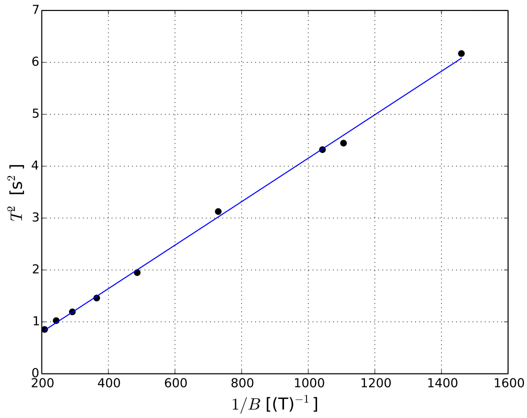

Figure 4: The cueballs period squared T^2 plotted against the reciprocal of the magnetic field B

$$\mu = \frac{4\pi^2 I_{cb}}{\kappa} = 0.376 \ \text{J/T}$$

this is in agreement with the magnetic moment value obtained in **Part I**

Part III
Precessional motion

Model

Allowing the sphere with its magnetic moment to spin freely at an angle θ with respect to the vertical, while being exposed to an external magnetic field, will cause the sphere to precess about the axis of the magnetic field direction, so that the angular momentum vector (and the magnetic moment vector) sweeps out a cone in space. The rate of change of the sweeping, (the precessional angular frequency Ω_p,) can be found with the equation of motion, which is that of eq. (8). Letting the torque approach its interval representation, $\tau = \Delta L/\Delta t$, and recognizing that the change in angular momentum ΔL, which directionally is at all times perpendicular to L, is related to the change in precessional angle ϕ (where $\dot{\phi} = \Omega_p$) through the common arc length formula

$$\Delta L = L\sin\theta\Delta\phi, \tag{12}$$

see figure 3. Dividing through by Δt and taking infinitesimal changes ($\Delta \to \mathrm{d}$) yields

$$\dot{L} = L\sin\theta\Omega_p. \tag{13}$$

That is to say, eq. (8) in scalar form can be written,

$$\Omega_p = \frac{\mu}{L}B \tag{14}$$

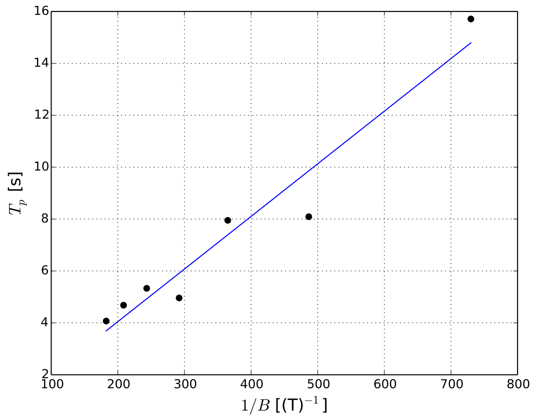

Figure 5: The precessional period T_p plotted as a function of the reciprocal of the magnetic field B

as $\sin\theta$ once again cancels.
We can also represent the precessional frequency $\Omega_p = 2\pi/T_p$, where T_p is intuitively the precessional period, we rewrite eq. (14) as

$$T_p = \frac{2\pi L}{\mu}\frac{1}{B} \tag{15}$$

Results

With a stopwatch, we measure a precessional period T_p for various currents I. Exploiting their inverse relation, we expect to obtain a straight line with slope $\kappa = 2\pi L/\mu$ enabling us to estimate the magnetic moment by,

$$\mu = \frac{2\pi L}{\kappa} \tag{16}$$

For the angular momentum $L = I_{cb}\omega$ we have already obtained the cueball's moment of inertia I_{cb}, to estimate its spin frequency ω we adjust a strobe lights frequency until a marked reference point located near the spin axis appears to remain stationary, meaning the strobe light and spin frequency are in sync, when that happens we thus have $\omega = \omega_{cb} = 2\pi f_{strobe}$. L is assumed to be a constant, but due to human error it deviates little, so we approached that using L's arithmetic mean. Then obtaining this slope κ in Fig. (5), we can finally calculate μ with the relation in eq. (16), we got,

$$\mu = 0.425 \text{ J/T}$$

Appearantely this method of measurement resulted in somewhat higher values of mu compared the ones made in **Part I** & **II**.

Part IV
Force & field gradient

Using equation (4) we can see, that a magnet, suspended in a non uniformal external magnetic field connected to a spring, will in equilibrium satisfy

$$kz = \mu\frac{\mathrm{d}B_z}{\mathrm{d}z} \tag{17}$$

where k is the spring constant, thus assuming Hook's law because the distance z from the magnets equilibrium position is small (≈ 20 mm). The equation can be plotted, using a difference quotient method to compute the derivative of B_z. Alternatively, it can be integrated with respect to. z which yields

$$z^2 = \frac{2\mu}{k}B_z + \text{Constant}, \tag{18}$$

which can be subsequently plotted, and allowing for μ to be calculated from the slope.

Results

To estimate the springs k factor we measured 10 periods with a stopwatch for a small displacement of a 20 g test mass. A harmonic springs period T_{spring} follows the relation,

$$T_{\text{spring}} = 2\pi\sqrt{\frac{m}{k}} \quad \Leftrightarrow \quad k = m\left(\frac{2\pi}{T_{\text{spring}}}\right)^2$$

resulting in $k = 1.22$ N/m

We're given a previously measured field gradient at the center of the helmholtz coil, $\partial B_z/\partial z = 1.69\cdot 10^{-2} I$. We rewrite and use eq. (17),

$$z = \frac{\mu}{k}\frac{\partial B_z}{\partial z} \tag{19}$$

Measuring the springs displacement z as for different currents I, we can estimate μ from the regressed line's slope $\kappa = \mu/k$, the measurements results can be seen in Fig. (6)

This line yields a magnetic dipole moment

$$\mu = 0.394 \text{ J/T}$$

a value somewhat in agreement with the ones obtained in **Part I, II**, and not that much further from the one obtained in **Part III**.

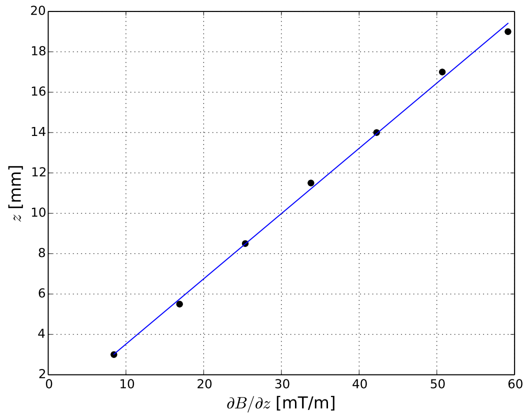

Figure 6:

Part V

Conclusions

1 Uncertainties

All these experiments involve one or several quantities which are controlled by human hands and thus carry with them uncertainties. We deem the least uncertain experiments on two criteria.

1. That they are those in which the data points used to compute a slope confine themselves in a highest degree to a straight line, and

2. that they are those in which the setting of physical quantities by hand is most precice.

Judging from the measurements, experiments 3 and 4 give the tightest confinement to the computed slope. By our judgement, experiment 3 involves several cases where eye-coordinated handling of the position of the ball had significant (and seemingly, on this scale, unavoidable) consequences for the resulting measurements. Likewise, in experiment 1 the exact equilibrium position of the system was exceedingly hard to reach, and compromises had to be done. Those two experiments are thus deemed to be the least precise, and those with the highest uncertainty.

2 Remarks

Having now discussed the uncertainty factors of this experiment, the devation μ did not surprise us. In fact, by inspecting figures (2) through (6) we find that these classical models did a decent job of demonstrating the theories linear behaviour. Here is a summary of the values obtained of the magnetic moment μ:

Part	Case	μ [J/T]
I	Torque equilibrium	0.374
II	Spherical pendulum	0.376
III	Precessional motion	0.425
IV	Force & field gradient	0.394

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