Determining the viscosity of three unknown fluids via Stoke's law and a rotational pendulum viscometer.

Magnus F. Ivarsen (1) and Róbert K. Lárusson (2)
University of Iceland. 1) mfi2@hi.is 2) rkl1@hi.is

April 23, 2014

Abstract: *In this experiment we aim to determine the viscosity of three unknown 'oil-like' fluids. First, we will put Stoke's law to the test by measuring the dragforce subjected to different sized spheres in these fluids. Second, we will re-determine the viscosity of each fluid with a rotational cylinder pendulum viscometer and compare the results obtained.*

Viscosity

Viscosity is a characteristic of fluids that manifests in its inherent resistance to deforming forces and is due to the fluid's individual particles' surface resistance, which in turn is dependent on dimensionality and elemental constituents. It is defined as the ratio of the shearing stress to the velocity gradient:

$$\mu = -\frac{\frac{F}{A}}{\frac{\mathrm{d}v_x}{\mathrm{d}y}}, \tag{1}$$

for a sheared three-dimensional fluid as per fig. 1, where F is the force directed onto the fluid by the movement of the lower plate, A is the area of the fluid perpendicular to the plates, v_x the velocity of the fluid and x, y are coordinates parallel to the plates and vertical respectively. Furthermore, the kinematic viscosity

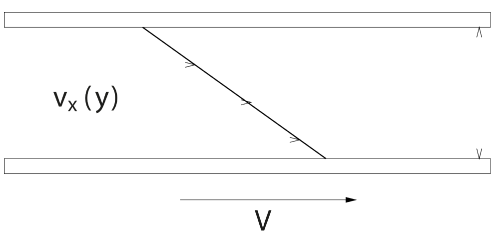

Figure 1: title

ν is defined as the ratio of viscosity to fluid density ρ

$$\nu = \frac{\mu}{\rho}. \tag{2}$$

Viscosity and an object's travel through a fluid

A tool kit to investigate viscosity is developed through looking at some of the governing mechanisms of fluid dynamics.

Mass continuity equation

In an arbitrary packet of volume in the fluid, the accumulation of mass must equal the flow of mass into the packet minus the flow of mass out of the packet. That is to say,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v}) = -\rho \nabla \cdot \boldsymbol{v} - \boldsymbol{v} \nabla \rho, \tag{3}$$

where $\boldsymbol{v}$ is the velocity field which the fluid density $\rho(x, y, z)$ is subject to. This corresponds to

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho \equiv \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{v}, \tag{4}$$

where the Lagrangian derivative is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \nabla.$$

For incompressible fluids ρ is constant, and eq. (4) reduces to

$$\rho \nabla \cdot \boldsymbol{v} = 0. \tag{5}$$

The Navier-Stokes equation

The motion of fluids is described by the Navier-Stokes equation. It is, for an incompressible fluid,

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g}, \tag{6}$$

where p is the pressure scalar field and $\boldsymbol{g}$ the acceleration of gravity. Note that ∇^2 here is the vector Laplacian operator. As each term in eq. (6) has units of density $\times$ acceleration, one renders the equation unitless by factoring in a constant $L/\rho V^2$ on both sides, where L is the characteristic length (to be specified more specifically later) and V the mean velocity of the fluid:

$$\left(\frac{L}{V}\frac{\partial}{\partial t}\frac{\boldsymbol{v}}{V}+\frac{\boldsymbol{v}}{V}\cdot(L\nabla)\frac{\boldsymbol{v}}{V}\right)=$$
$$=-L\nabla\frac{p}{\rho V^2}+(L\nabla)^2\frac{\boldsymbol{v}}{V}\left(\frac{\mu}{LV\rho}\right)+\left(\frac{Lg}{V^2}\right)\frac{\boldsymbol{g}}{g}. \quad (7)$$

Then, define the following dimensionless quantities,

$$\boldsymbol{v}^*\equiv\frac{1}{V}\boldsymbol{v},\quad p^*\equiv\frac{1}{\rho V^2}p,$$
$$\nabla^*\equiv L\nabla \text{ and}$$
$$\frac{\mathrm{D}^*}{\mathrm{D}t^*}\equiv\frac{L}{V}\frac{\partial}{\partial t}+\boldsymbol{v}^*\cdot\nabla^*,$$

after which eq. (7) becomes (asterisks removed for ease of reading)

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=-\nabla p+\frac{\mu}{\rho LV}\nabla^2\boldsymbol{v}+\frac{gL}{V^2}\frac{\boldsymbol{g}}{g}=-\nabla p+\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{v}+\frac{1}{\mathrm{Fr}}\frac{\boldsymbol{g}}{g}, \quad (8)$$

where the appearing constant are defined as

$$\mathrm{Re}\equiv\frac{\rho LV}{\mu}, \quad (9)$$

and

$$\mathrm{Fr}\equiv\frac{V^2}{gL}, \quad (10)$$

with Re and Fr being called Reynold's and Froude numbers respectively. Their relative value has an impact on simplifying measures in solving eq. (8), as terms can be neglected when the constants take on suitably sized numerical values.
For the case of an uncompressable liquid with an object going through it at a steady velocity, the Reynolds number can be written,

$$\mathrm{Re}=\frac{au\rho_{\mathrm{Oil}}}{\mu} \quad (11)$$

Stokes flow

In the case of an incompressible fluid, the Navier-Stokes equation and the mass continuity equation are referred to as the Stokes equations for incompressible Newtonian fluids, and the ensuing flow is referred to as Stokes flow. Steady Stokes flow cause the said equations to be representable by Laplace equations, and those can then be solved to yield Stokes' law, which in turn allows the calculation of viscosity by certain trivial measurements. However, stokes flow requires a relatively low Reynold's number, which must be confirmed retrospectively by using the definition. Stokes' law for the frictous force exerted of a sphere falling through a viscous fluid is

$$F_k=6\pi\mu au, \quad (12)$$

where F_k is the force of friction that the sphere encounters, a is the sphere's radius and u_∞ is the velocity of the sphere. Furthermore, at terminal velocity ($u=u_\infty$) The frictous force is

$$F_k=F_g=(\rho_s-\rho_f)g\frac{4}{3}\pi a^3,$$

where $\rho_{s,f}$ are the densities of sphere and fluid respectively, so that the viscosity is expressed as

$$\mu=\frac{2}{9}\frac{ga^2}{u_\infty}(\rho_s-\rho_f) \quad (13)$$

for a falling sphere at terminal velocity through a fluid. Now, the radius a and the terminal velocity u_∞ take on the role of the characteristic length and mean velocity respectively, in eq. (9), the value of which must be calculated post-experimentally to confirm the validity of the Stokes flow assumption. The quantities on the right side in eq. (13) are all measurable in an experiment where spheres are allowed to fall through a large cylinder of fluid.

Correction

Due to the assumptive nature of the theoretical musings described above, corrective measures may be required. The correction to the damping force of friction due to the neglected Re term is

$$F_k=6\pi\mu au_\infty\left(1+\frac{3}{16}\mathrm{Re}-\frac{19}{1280}\mathrm{Re}^2+\dots\right). \quad (14)$$

Furthermore, the size of the cylinder, which has diameter d, has an effect on the viscosity experienced by the ball as per the Faxen and Ladenburg terms:

$$\mu=\mu_0\left(1-\left[2.104\frac{a}{d}-2.09\frac{a^3}{d^3}+0.95\frac{a^5}{d^5}\right]\right), \quad (15)$$

where μ_0 is the viscosity as calculated from measurements. A value of $a/d<0.01$ allows one to ignore the viscosity correction.

Measuring the viscosity with a turning pendulum

A rotational viscometer is an apparatus that takes advantage of the fact that the torque required to rotate an object in a fluid is proportional to the fluids viscosity. The apparatus comes with - usually - a disc that can spin coaxial to the cylinder weight. Observing how much the disc spins can be related to the fluids viscosity.

Experimental results and calculations

Stokes

We started out grouping different-sized balls, measured their masses and radii in order to calculate their densities ρ. The 3 different sizes we used turned out to be of same density. Table 1 lists the values obtained for these balls. Next we weighed a small amount of each oil in order to obtain their densities. We also measure the

Ball	ρ [g/cm^3]	a [cm]
1	8.23	0.085
2	8.23	0.1275
3	8.23	0.135

Table 1: The densities and the radii of the balls used.

Oil	ρ [g/cm^3]	D [cm]
1	0.725	4
2	0.775	4
3	0.925	3.25

Table 2: The densities of the oils and the radii of their containers.

radii of the containing vials that were used in the experiment. The results are listed in table 2 Next we put tape reference markers on all the vials with and measure the distance between the markers on each vial. We drop 10-15 balls of each size in each oil and measure the time it takes for each to go the known distance, assuming that terminal velocity has been reached before the individual ball passes the first reference marker. Having done that we can compute u_∞ and finally calculate the viscosity μ with the relation from equation 13. The values are summarized in table 3. It's logical to then

Ball \Oil	1	2	3
1	6.08	8.86	
2	3.64	3.83	19.8
3	2.02	2.11	10.7

Table 3: Calculations of the viscosity μ in units of Poise (= 0.1 Pa·s) summarized for all the balls and oils. Measuring the time for ball 1 (the smallest) in oil 3 turned out to be impractical as oil 3 (old motor oil) was almost opaque.

calculate the Reynolds number given the relationship in equation (11), where we have obtained all the necessary quantities. The results of the Reynolds number for all balls and oils are summarized in table 5.

Having obtained the Reynolds constant for each combination of ball and fluid, we can estimate the frictional force F_k. It's evident that some of the Reynolds numbers obtained fulfil the criteria of needing correctional terms due to inertial forces. We compare and contrast the forces with and without correction. Comparing the data of tables 6 and 7 with respect to the data of table 5, it's evident that for small Reynolds numbers the correction yields neglectable difference. For ball #3, the Reynolds number is relatively large and we deem it outside our judgement whether approximation is justified.

Rotating pendulum viscometer

Now for the second method, we use three different sized cylinder weights hooked up to a rotational pendulum viscometer in order to determine the viscosity of oil 3.

Ball \Oil	1	2	3
1	5.94	8.86	
2	3.52	3.70	18.9
3	1.95	2.04	10.2

Table 4: Calculations of the viscosity μ in units of Poise (= 0.1 Pa·s) after using the correctional formula for μ as displayed in equation (15)

Ball \Oil	1	2	3
1	2.02	1.01	
2	19.2	18.5	0.817
3	73.9	72.1	3.33

Table 5: The dimensionless quantity Reynold's constant ($\times 10^3$) for all the balls and oils.

Ball \Oil	1	2	3
1	1.85	1.84	
2	6.19	6.15	5.95
3	7.33	7.28	7.05

Table 6: Frictional forces F_k [N $\times 10^4$] without correction

Ball \Oil	1	2	3
1	1.86	1.85	
2	6.41	6.3	5.96
3	8.29	8.21	7.09

Table 7: Frictional forces F_k [N $\times 10^4$] after correction due to the Reynolds number

We begin by measuring and noting down the diameter and mass of each cylinder. We had to be assured that each cylinder was totally submerged the oil. The disc coaxial to the cylinders are to be turned 360°C and slowly released and then observed by how much the disc spins ϕ. We do this procedure 3 times for each cylinder and take note of ϕ. Each cylinder yielded similar values of ϕ for each measurement, so we take the arithmetic mean to represent our result. The results of these measurements are all summarized in table 8. Then to finally determine the viscosity with each cylin-

Cylinder	m [g]	d [cm]	ϕ [°C]
1	52.00	1.22	360+293
2	138.0	2.83	360+170
3	288.0	4.11	342

Table 8: Mass, diameter and the arithmetic mean of the measurements of ϕ for each cylinder.

der, we wanted to consult a chart made by the apparatus's manufacturers; *however*, the chart is unreadable.