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Exponentially-fitted methods and their stability functions

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ABSTRACT

We investigate the properties of stability functions of exponentially-fitted Runge–Kutta methods, and we show that it is possible (to some extent) to determine the stability function of a method without actually constructing the method itself. To focus attention, examples are given for the case of one-stage methods. We also make the connection with so-called integrating factor methods and exponential collocation methods. Various approaches are given to construct these methods.

1. Introduction

For the numerical integration of systems of first order ODEs

$$y' = f(x, y), \tag{1.1}$$

the family of Runge–Kutta methods is an excellent tool. In the case of initial value problems, the stability of such an RK method plays an important role and the stability properties of the methods should be examined. Therefore, the method is applied to the linear equation

$$y' = \lambda\, y \tag{1.2}$$

giving rise to a relation of the form $y_{n+1} = R(z)\, y_n$ with $z := \lambda\, h$ and whereby $R(z)$ is called the stability function of the method. For some families of RK methods the stability function can be written down without actually constructing the method: e.g.

- for an explicit $s \le 4$ stage method of order s, $R(z) = \sum_{j=0}^{s} \frac{z^s}{s!}$,
- for an s-stage Gauss method, $R(z) = \hat{R}_s^s(z)$, where $\hat{R}_s^s(z)$ is the Padé-approximant of order $[s/s]$ of e^z,
- also for methods of Lobotto type and Radau type the function $R(z)$ is a Padé-approximant of e^z,
- ....

On the other hand, several authors have developed exponentially fitted Runge–Kutta (EFRK) methods. These methods are designed to solve problems whose solutions are weighted sums of polynomials and exponentials (or in the complex case trigonometric functions). So far, the properties of their stability functions have not been studied in detail. In this paper, the question is raised if it is possible, given the set of functions to which the method is fitted, to predict the exact expression of its stability function without actually constructing the method.

As we are mainly interested in the stability function of the methods, we will restrict ourselves in this paper to scalar problems. However, we want to emphasize that the methods that will be discussed, can also be reformulated to solve systems of equations.

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In the next section, we start with a short introduction to EFRK methods. In Section 3, we show that if the set of functions to be integrated exactly by the method is known, then also the conditions to impose on the stability function are known. In Section 4, we consider some examples of such stability functions of one stage methods. In Sections 5 and 6, we make a connection with some very special types of exponentially-fitted methods: so-called integrating factor methods and exponential-collocation methods. Both kinds of methods have been successfully applied in solving semi-linear systems. An excellent overview of exponential integrators of both families is given in [1]. Both types of methods start from a common idea, which will be very important in our discussion: the problem (1.1) is rewritten in the form

$$y' - \omega y = \tilde{f}(x, y) = f(x, y) - \omega y, \tag{1.3}$$

which can be rewritten as

$$(e^{-\omega x} y)' = e^{-\omega x} \tilde{f}(x, y). \tag{1.4}$$

2. Exponentially-fitted Runge–Kutta methods

Exponentially-fitted Runge–Kutta methods for the solution of a first order problem (1.1) have been discussed by several authors, e.g. [2–11]. The most general form of such a method is

$$y_{n+1} = \gamma\, y_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i)$$

whereby

$$Y_i = \gamma_i\, y_n + h \sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j), \quad i = 1, \dots, s.$$

With such a method, a generalized Butcher tableau can be associated:

$$\begin{array}{c|c|cccc} c_1 & \gamma_1 & a_{11} & \dots & a_{1s} \\ c_2 & \gamma_2 & a_{21} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & \gamma_s & a_{s1} & \dots & a_{ss} \\ \hline & \gamma & b_1 & \dots & b_s \end{array}$$

or

$$\begin{array}{c|c|c} c & \Gamma & A \\ \hline & \gamma & b^T \end{array}$$

The coefficients of these EFRK methods in general depend upon the product $z_0 := \omega h$ (some authors would explicitly denote A, b and c as $A(z_0)$, $b(z_0)$ and $c(z_0)$), where ω is a parameter that can be related to the solution of the problem to be solved. In fact, EF methods are designed to solve problems which have an exponential behaviour or (in the case ω is purely imaginary) a periodic behaviour. To construct such a EFRK method, a set of linear functionals can be introduced [12]:

$$\mathcal{L}_i[y(x); h] = y(x + c_i h) - \gamma_i\, y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h), \quad i = 1, \dots, s$$

and

$$\mathcal{L}[y(x); h] = y(x + h) - \gamma\, y(x) - h \sum_{i=1}^{s} b_i y'(x + c_i h).$$

Next, conditions are imposed onto these functionals. For each stage of the method, a so-called fitting space is determined. In this paper, we will mainly consider the construction of implicit methods. In that case, each stage contains $s + 1$ parameters and for each stage the same fitting space $\mathcal{S}$ of dimension $s + 1$ can be considered.

It is well-known that collocation offers an alternative way to construct such methods: a function $P(x) \in \mathcal{S}$ is constructed such that

$$\begin{cases} P(x_n) = y_n \\ P(x_n + c_i h)' = f(x_n + c_i h, P(x_n + c_i h)), \quad i = 1, \dots, s. \end{cases} \tag{2.5}$$

The method is then defined by imposing $y_{n+1} := P(x_n + h)$.

Vanden Berghe et al. [7] have constructed methods for which

$$\mathcal{S} = \{x^q\, e^{\pm\omega x} | q = 0, 1, \dots, P\} \cup \{x^q | q = 0, 1, \dots, K\}$$

where $2\,(P+1)+K+1=s+1$. Calvo et al. [3] have studied methods for which

$$\mathcal{S}=\{\mathrm{e}^{\pm q\omega x}|q=1,\dots,P+1\}\cup\{x^q|q=0,1,\dots,K\}.$$

Note that a generalisation of both approaches is to consider a space

$$\mathcal{S}=\{\mathrm{e}^{\omega_q x}|q=1,\dots,s+1\},$$

where $\omega_1,\dots,\omega_{s+1}$ take different values.

Independently of the specific choice for the space $\mathcal{S}$, the stability function of an EFRK method can be written as

$$R(z,z_0)=\gamma+z\,b^T\,(I-z\,A)^{-1}\,\Gamma$$

where R is a rational function in z with coefficients that depend upon z_0. Further, Γ is the column matrix with entries $\gamma_i,\ i=1,\dots,s$. It is also interesting to know that, when the parameter(s) of an EFRK method tend to 0, the classical RK method of collocation type is found. Its stability function is then given by (we omit the second argument, since it is not present in the expression)

$$R(z)=1+z\,b^T\,(I-z\,A)^{-1}\,e=\mathrm{e}^z+\mathcal{O}(z^{p+1}),$$

where e is the vector of length s with unit entries and $s\le p\le 2\,s$.

Let us consider some examples of 1 stage methods and their stability functions.

- $\mathcal{S}_{2,0}(\omega)=\mathrm{Span}\{1,x\}$

$$\begin{array}{c|c|c} c_1 & 1 & c_1 \\ \hline & 1 & 1 \end{array} \tag{2.6}$$

$$R_{2,0}^{c_1}(z)=\frac{1+(1-c_1)\,z}{1-c_1\,z}$$

- $\mathcal{S}_{1,1}(\omega)=\mathrm{Span}\{1,\mathrm{e}^{\omega x}\}$

$$\begin{array}{c|c|c} c_1 & 1 & \frac{1-\mathrm{e}^{-c_1 z_0}}{z_0} \\ \hline & 1 & \frac{\mathrm{e}^{(1-c_1)\,z_0}-\mathrm{e}^{-c_1 z_0}}{z_0} \end{array} \tag{2.7}$$

$$R_{1,1}^{c_1}(z,z_0)=\frac{1+\frac{\mathrm{e}^{(1-c_1)\,z_0}-1}{z_0}z}{1-\frac{1-\mathrm{e}^{-c_1 z_0}}{z_0}z}$$

- $\mathcal{S}_{0,2}(\omega)=\mathrm{Span}\{\mathrm{e}^{\omega x},x\,\mathrm{e}^{\omega x}\}$

$$\begin{array}{c|c|c} c_1 & \frac{\mathrm{e}^{c_1 z_0}}{1+c_1 z_0} & \frac{c_1}{1+c_1 z_0} \\ \hline & \frac{1-(1-c_1)\,z_0}{1+c_1 z_0}\mathrm{e}^{z_0} & \frac{\exp((1-c_1)\,z_0)}{1+c_1 z_0} \end{array} \tag{2.8}$$

$$R_{0,2}^{c_1}(z,z_0)=\mathrm{e}^{z_0}\,\frac{1+(1-c_1)\,(z-z_0)}{1-c_1\,(z-z_0)}.$$

One immediately notices the relation between $R_{2,0}^{c_1}(z)$ and $R_{0,2}^{c_1}(z,z_0)$:

$$R_{0,2}^{c_1}(z,z_0)=\mathrm{e}^{z_0}\,R_{2,0}^{c_1}(z-z_0).$$

This leads to a very nice result for the corresponding order stars, since

$$\left|\frac{R_{0,2}^{c_1}(z,z_0)}{\mathrm{e}^z}\right|=\left|\frac{R_{2,0}^{c_1}(z-z_0)}{\mathrm{e}^{z-z_0}}\right|,$$

which means that the order stars are (apart from a shift over a distance z_0) equal to each other.

In fact, a much more general relation exists for the stability functions and the order stars: suppose a method $M_{k,l}$ (the number of stages does not really matter here) is built to integrate exactly all functions in the space

$$\mathcal{S}_{k,l}(\omega)=\mathrm{Span}\{1,x,\dots,x^{k-1},\mathrm{e}^{\omega x},x\,\mathrm{e}^{\omega x},\dots,x^{l-1}\mathrm{e}^{\omega x}\}.$$

For Eq. (1.2), this gives rise to $y_{n+1}=R_{k,l}(z,z_0)\,y_n$.

On the other hand, following Lawson [13] and defining $u(x)=\mathrm{e}^{-\omega x}y(x)$ the Eq. (1.2) becomes $u'=(\lambda-\omega)\,u$. If $y\in\mathcal{S}_{k,l}(\omega)$, then $u\in\mathcal{S}_{l,k}(-\omega)$, and this then leads to $u_{n+1}=R_{l,k}(z-z_0,-z_0)\,u_n$, from which $y_{n+1}=\mathrm{e}^{z_0}\,R_{l,k}(z-z_0,-z_0)\,y_n$ is obtained. In general, we thus have

$$R_{k,l}(z,z_0)=\mathrm{e}^{z_0}\,R_{l,k}(z-z_0,-z_0).$$

For the corresponding order star, this then means

$$\left|\frac{R_{k,l}(z,z_0)}{\mathrm{e}^z}\right|=\left|\frac{R_{l,k}(z-z_0,-z_0)}{\mathrm{e}^{z-z_0}}\right|.$$

These relations will be illustrated in Section 4.

3. Which conditions to impose

There are several ways to obtain the set of conditions that one should impose on the stability function $R(z)$ of an EFRK-method. One of the possible approaches is the following: suppose we have at our disposal an EFRK method that is fitted (at least) to the functions $\{x^q\, e^{\omega x} | q = 0, 1, \ldots, P\}$. We apply this method to

$$\begin{cases} y_0' = \omega\, y_0 \\ y_1' = \omega\, y_1 + y_0 \\ y_2' = \omega\, y_2 + y_1 \\ \vdots \\ y_P' = \omega\, y_P + y_{P-1} \end{cases} \qquad \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_P \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for which the solution is given by $y_q(x) = \frac{x^q}{q!} e^{\omega x},\ q = 0, 1, \ldots, P$.

The different equations of this system can be solved analytically one after the other, starting from the first one. This is also the case for the numerical solution of the EFRK. Let Y_q denote the vector of length s containing the internal stages for equation $q + 1$, $q = 0, \ldots, P$. Further we define Y_{-1} as a vector of length s with zero entries. Then, denoting the numerical solution of y_q at x_n as $y_{q,n}$:

$$Y_q = y_{q,n}\, \Gamma + h\, A\, (Y_{q-1} + \omega\, Y_q)$$

from which, with $z_0 := \omega\, h$,

$$Y_q = (I - z_0 A)^{-1}\, (y_{q,n}\, \Gamma + h\, A\, Y_{q-1}).$$

This then leads to

$$\begin{aligned} y_{q,n+1} &= \gamma\, y_{q,n} + h\, b^T (\omega\, Y_q + Y_{q-1}) \\ &= \gamma\, y_{q,n} + h\, b^T (\omega\, (I - z_0 A)^{-1}\, (y_{q,n}\, \Gamma + h\, A\, Y_{q-1}) + Y_{q-1}) \\ &= y_{q,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h\, b^T (I - z_0\, A)^{-1} Y_{q-1}. \end{aligned}$$

This successively leads to

$$\begin{aligned} y_{0,n+1} &= y_{0,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) \\ y_{1,n+1} &= y_{1,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h b^T (I - z_0\, A)^{-1}\, Y_0 \\ &= y_{1,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h\, b^T (I - z_0\, A)^{-2}\, \Gamma\, y_{0,n} \\ y_{2,n+1} &= y_{2,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h\, b^T (I - z_0\, A)^{-1}\, Y_1 \\ &= y_{2,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h\, b^T (I - z_0\, A)^{-2} (y_{1,n} \Gamma + h\, A Y_0) \\ &= y_{2,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h\, b^T (I - z_0\, A)^{-2} \Gamma y_{1,n} + h^2 b^T (I - z_0\, A)^{-3}\, A\, \Gamma\, y_{0,n} \\ &\vdots \\ y_{P,n+1} &= y_{P,n}\, (\gamma + z_0 b^T (I - z_0\, A)^{-1}\, \Gamma) + h b^T (I - z_0\, A)^{-2} \Gamma y_{P-1,n} + h^2 b^T (I - z_0\, A)^{-3}\, A\, \Gamma\, y_{P-2,n} \\ &\quad + \cdots + h^P b^T (I - z_0\, A)^{-P-1}\, A^{P-1}\, \Gamma\, y_{0,n}. \end{aligned}$$

This formula can be rewritten in a more elegant way, if we make use of the stability function $R(z, z_0)$ and its derivatives. Indeed, suppose the non-singular matrix B depends upon z, then from $B^{-1}\, B = I$ one obtains $\frac{\partial B^{-1}}{\partial z} = -B^{-1} \frac{\partial B}{\partial z} B^{-1}$.

For $B = I - z\, A$, one thus finds

$$\frac{\partial}{\partial z} (I - zA)^{-1} = -(I - zA)^{-1} (-A) (I - zA)^{-1} = (I - zA)^{-2}\, A$$

such that

$$\frac{\partial}{\partial z} R(z, z_0) = b^T (I - z\, A)^{-1}\, \Gamma + z\, b^T (I - zA)^{-2}\, A\Gamma = b^T (I - z\, A)^{-2}\, \Gamma.$$

For $B = (I - z\, A)^2$ we then find

$$\frac{\partial}{\partial z} (I - zA)^{-2} = -(I - zA)^{-2} 2\, (I - z\, A)\, (-A) (I - zA)^{-2} = 2\, (I - zA)^{-3}\, A$$

such that

$$\frac{\partial^2}{\partial^2 z} R(z, z_0) = 2\, b^T (I - z\, A)^{-3}\, A\, \Gamma.$$

Continuing in this way, we find

$$\frac{\partial^q}{\partial^q z} R(z, z_0) = q!\, b^T (I - z\,A)^{-q-1} A^{q-1}\, \Gamma,$$

such that we can write that, for $q = 0, 1, \dots, P$:

$$y_{q,n+1} = y_{q,n} R(z_0, z_0) + y_{q-1,n}\, h\, \frac{\partial}{\partial z} R(z, z_0)\bigg|_{z=z_0} + y_{q-2,n}\, \frac{h^2}{2!}\, \frac{\partial^2}{\partial^2 z} R(z, z_0)\bigg|_{z=z_0} + \cdots$$
$$+ y_{0,n}\, \frac{h^q}{q!}\, \frac{\partial^q}{\partial^q z} R(z, z_0)\bigg|_{z=z_0}.$$

If we now put $x_n = 0$, we find for the problem at hand that

$$\frac{h^q}{q!} \mathrm{e}^{z_0} = \frac{h^q}{q!}\, \frac{\partial^q}{\partial^q z} R(z, z_0)\bigg|_{z=z_0} \quad q = 0, 1, \dots, P.$$

We can thus conclude: a method is fitted to the functions $x^q\, \mathrm{e}^{\omega x},\ q = 0, 1, \dots, P$ iff

$$\frac{\partial^q}{\partial^q z} R(z, z_0)\bigg|_{z=z_0} = \mathrm{e}^{z_0} \quad q = 0, 1, \dots, P. \tag{3.9}$$

Some remarks can be made here. First of all, in the special case $\omega = 0$, i.e. $z_0 = 0$, we obtain the classical conditions $R^{(q)}(0) = 1,\ q = 0, 1, \dots, P$, which means that $R(z) - \exp(z) = \mathcal{O}(z^{P+1})$.

Secondly, the results can be extended to methods that are fitted so several parameters ω. For instance, suppose that a method is fitted for two values ω and ω'. We can then denote the corresponding stability function as $R(z, \{z_0, z_0'\})$ where $z_0 := \omega\, h$ and $z_0' = \omega'\, h$ and the method will be fitted to $\{x^q\, \mathrm{e}^{\omega x}, x^q\, \mathrm{e}^{\omega' x}\},\ q = 0, \dots, P$ iff

$$\frac{\partial^q}{\partial^q z} R(z, \{z_0, z_0'\})\bigg|_{z=z_0} = \mathrm{e}^{z_0} \quad \text{and} \quad \frac{\partial^q}{\partial^q z} R(z, \{z_0, z_0'\})\bigg|_{z=z_0'} = \mathrm{e}^{z_0'}, \quad q = 0, \dots, P.$$

In particular, we can consider here the cases that have been considered in [7]: an EFRK method that is fitted to the space of functions $\{1, x, \dots, x^K\} \cup \{x^q\, \mathrm{e}^{\pm\omega x} | q = 0, 1, \dots, P\}$ has to satisfy:

$$\begin{cases} \dfrac{\partial^q}{\partial^q z} R(z, \{z_0, -z_0\})\bigg|_{z=\pm z_0} = \mathrm{e}^{\pm z_0} & q = 0, 1, \dots, P \\ \dfrac{\partial^q}{\partial^q z} R(z, \{z_0, -z_0\})\bigg|_{z=0} = 1 & q = 0, 1, \dots, K. \end{cases}$$

Finally, it should also be noted that the result (3.9) can also be obtained in a much more elegant way. For instance, suppose $R(z_0, \{z_0, z_0'\}) = \mathrm{e}^{z_0}$ and $R(z_0', \{z_0, z_0'\}) = \mathrm{e}^{z_0'}$. Then, when $z_0' \to z_0$, one obtains

$$\frac{\partial}{\partial z} R(z, z_0)\bigg|_{z=z_0} = \lim_{z_0' \to z_0} \frac{R(z_0, \{z_0, z_0'\}) - R(z_0', \{z_0, z_0'\})}{z_0 - z_0'} = \lim_{z_0' \to z_0} \frac{\mathrm{e}^{z_0} - \mathrm{e}^{z_0'}}{z_0 - z_0'} = \mathrm{e}^{z_0}.$$

4. The one-stage case

For a one-stage method, the stability function will be a rational approximation of degree one in both the numerator and the denominator, i.e.

$$R(z, z_0) = \frac{a_0 + a_1\, z}{1 + b_1\, z},$$

where a_0, a_1 and b_1 can depend upon z_0. There are three degrees of freedom, so we can impose $i + j = 3$ conditions: $\frac{\partial^q}{\partial^q z} R(z, z_0)\Big|_{z=0} = 1,\ q = 1, \dots, i$ and $\frac{\partial^q}{\partial^q z} R(z, z_0)\Big|_{z=z_0} = \mathrm{e}^{z_0}, q = 1, \dots, j$, i.e. we consider the stability functions that are obtained by fitting to $\{1, x, \dots, x^{i-1}\} \cup \{\mathrm{e}^{\omega\, x}, x\, \mathrm{e}^{\omega\, x}, \dots, x^{j-1}\, \mathrm{e}^{\omega\, x}\}$. Then we obtain four different stability functions that are denoted as $R_{i,j}(z, z_0)$.

- $R_{3,0}(z, z_0) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$
- $R_{2,1}(z, z_0) = \dfrac{1+\frac{1-\mathrm{e}^{z_0}+\mathrm{e}^{z_0} z_0}{z_0\left(-1+\mathrm{e}^{z_0}\right)} z}{1+\frac{1+z_0-\mathrm{e}^{z_0}}{z_0\left(-1+\mathrm{e}^{z_0}\right)} z}$

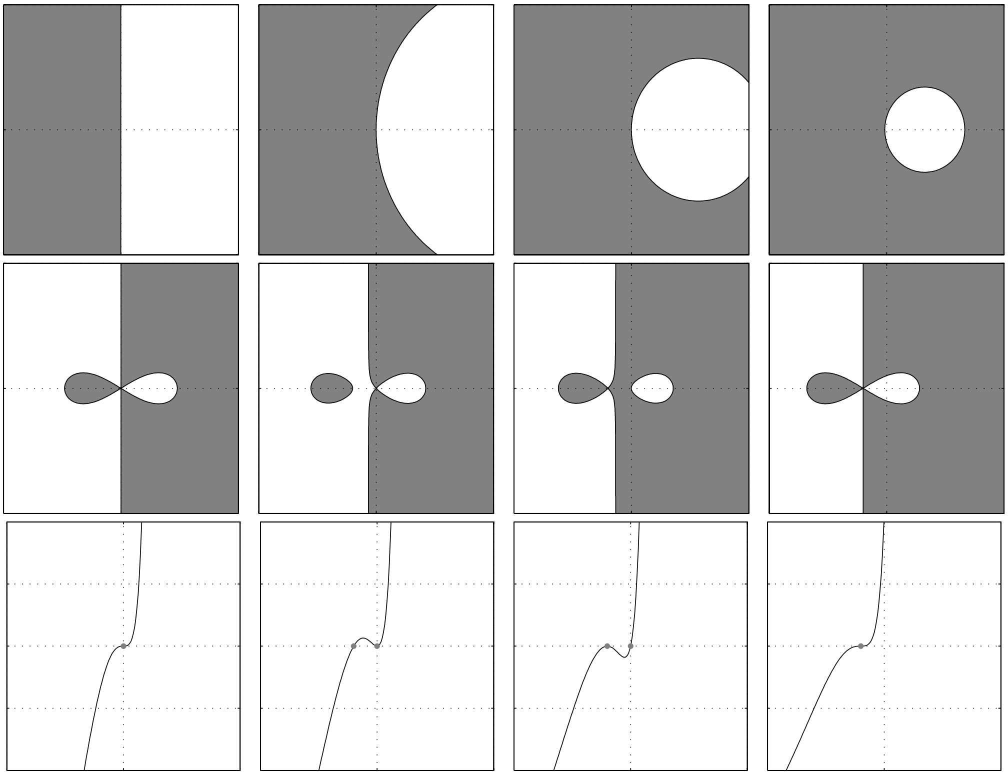

Fig. 1. The stability regions (top) and the order stars (middle) for the functions $R_{3,0}(z, z_0)$, $R_{2,1}(z, z_0)$, $R_{1,2}(z, z_0)$ and $R_{0,3}(z, z_0)$, for $z_0 = -1$. For each picture, both axes vary between -5 en $+5$. In the lower part, the difference with e^z along the real axis is shown. Again the x-axis covers the interval $[-5, 5]$, the y-axis shows the interval $[-0.1, 0.1]$.

- $R_{1,2}(z, z_0) = \dfrac{1-\frac{1+z_0-e^{z_0}}{z_0^2}z}{1-\frac{e^{-z_0}-1+z_0}{z_0^2}z}$
- $R_{0,3}(z, z_0) = e^{z_0}\dfrac{1+\frac{z-z_0}{2}}{1-\frac{z-z_0}{2}}$.

In Fig. 1 the stability regions, the order stars and the deviation from e^z along the real axis for these functions have been shown for the case $z_0 = -1$. Starting at the left side with $R_{3,0}$, which is exactly A-stable, and going to the right, we see that the stability region (i.e. the grey area) grows. From the corresponding order stars, we can learn how well the stability function approximates e^z for $z = 0$ and $z = z_0$. Indeed, we can see that an approximation of order p in $z = z_0$ or $z = 0$ results in an order star in that point with $2\,(p+1)$ equal sectors. The bottom row, which shows the differences $e^z - R_{i,j}(z, z_0)$ also shows the orders of approximation in $z = z_0$ and $z = 0$.

It should be noted that there is a connection with the functions $R^{c_1}_{i,j}(z, z_0)$ that have been considered in Section 2:

$$R_{3,0}(z, z_0) = R^{c_1}_{2,0}(z, z_0) \quad \text{iff } c_1 = \frac{1}{2}$$

$$R_{2,1}(z, z_0) = R^{c_1}_{2,0}(z, z_0) \quad \text{iff } c_1 = \frac{e^{z_0} - (1 + z_0)}{z_0\,(e^{z_0} - 1)}$$

$$R_{2,1}(z, z_0) = R^{c_1}_{1,1}(z, z_0) \quad \text{iff } c_1 = \frac{1}{z_0}\log\left(\frac{e^{z_0} - 1}{z_0}\right) \tag{4.10}$$

$$R_{1,2}(z, z_0) = R^{c_1}_{1,1}(z, z_0) \quad \text{iff } c_1 = 1 - \frac{1}{z_0}\log\left(\frac{e^{z_0} - 1}{z_0}\right) \tag{4.11}$$

$$R_{1,2}(z, z_0) = R^{c_1}_{2,0}(z) \quad \text{iff } c_1 = \frac{e^{-z_0} - (1 - z_0)}{-z_0\,(e^{-z_0} - 1)}$$

$$R_{0,3}(z, z_0) = R^{c_1}_{2,0}(z) \quad \text{iff } c_1 = \frac{1}{2}.$$

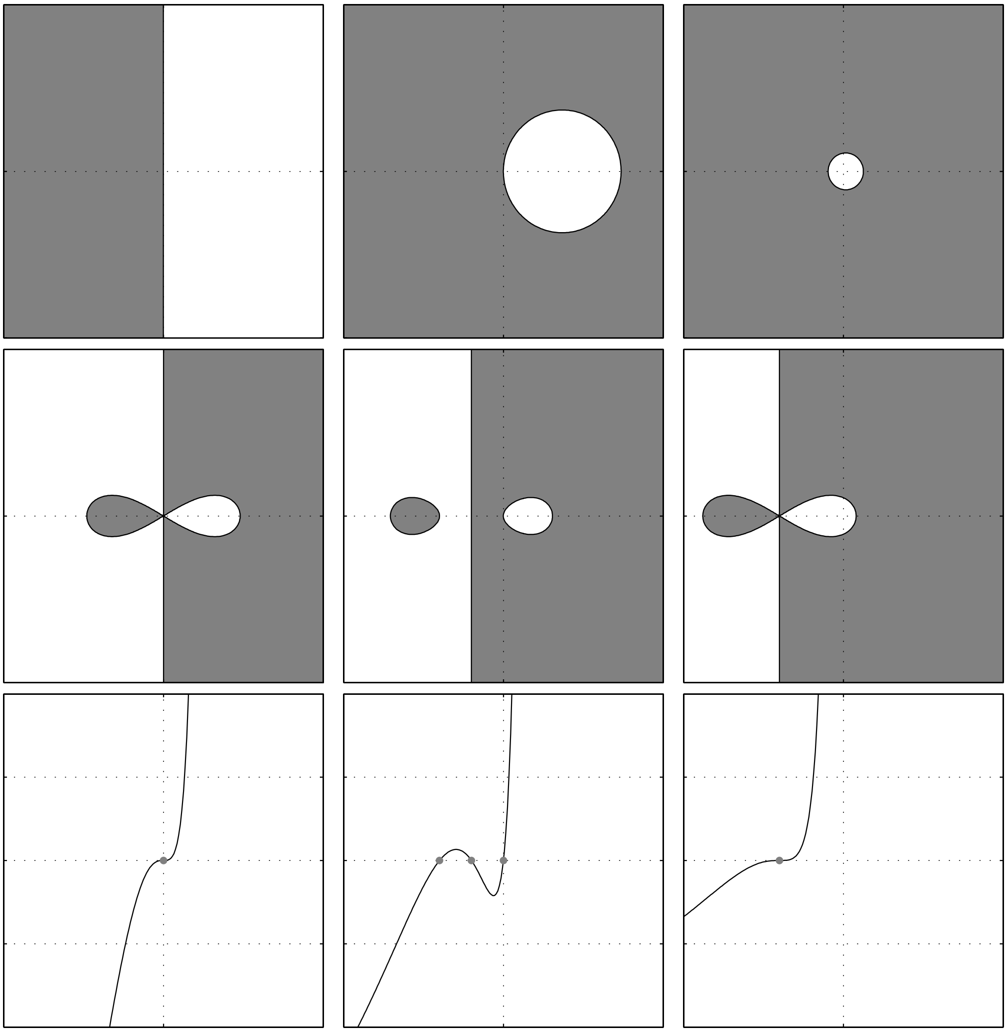

Fig. 2. The stability regions (top) and the order stars (middle) for the functions $R_{2,0}^{c_1}(z, z_0)$, $R_{1,1}^{c_1}(z, z_0)$ and $R_{0,2}^{c_1}(z, z_0)$, for $z_0 = -2$ and $c_1 = 1/2$. For each picture, both axes vary between -5 en $+5$. In the lower part, the difference with e^z along the real axis is shown. Again the x-axis covers the interval $[-5, 5]$, the y-axis shows the interval $[-0.1, 0.1]$.

One might also wonder if there exists a value $z_1 = a\, z_0$ for z, such that $R_{1,1}^{c_1}(z_1, z_0) = e^{z_1}$. It is found that this is the case if

$$c_1 = -\frac{1}{z_0} \ln\left(\frac{(1-a)\,(e^{a\,z_0}-1)}{a\,(e^{z_0}-e^{a\,z_0})}\right) = \frac{1}{z_0}\left(\ln\frac{e^{z_1}-e^{z_0}}{z_1-z_0} - \ln\frac{e^{z_1}-e^{0}}{z_1-0}\right).$$

The results given in (4.10) and (4.11) are found for the limiting cases $a = 0$ and $a = 1$. Remarkably, one also finds a nice result for $a = 1/2$: in that case one finds $c_1 = 1/2$, which means that $R_{1,1}^{(1/2)}(z, z_0)$ satisfies $R(0) = 0$, $R(z_0) = e^{z_0}$ and $R(z_0/2) = e^{z_0/2}$.

In Fig. 2 the three cases $R_{2,0}^{c_1}(z, z_0)$, $R_{1,1}^{c_1}(z, z_0)$ and $R_{0,2}^{c_1}(z, z_0)$ are compared for $c_1 = 1/2$ and $z_0 = -2$.

5. Integrating factor methods

We consider the Eq. (1.3) which we rewrite in the form (1.4). Following Lawson [13], we define $u(x) = e^{-\omega x}\, y(x)$ and $g(x, u) = e^{-\omega x}\tilde{f}(x, e^{\omega x}\, u)$ to obtain the equation $u' = g(x, u)$.

We can now apply any Runge–Kutta method defined by a matrix A and vectors b and c to $u' = g(x, u)$, which leads to

$$u_{n+1} = u_n + h \sum_{j=1}^{s} b_j\, K_j \tag{5.12}$$

where

$$K_i = g\left(x_n + c_i\,h, u_n + h\sum_{j=1}^{s} a_{ij}\,K_j\right)$$
$$= \mathrm{e}^{-\omega\,(x_n+c_i\,h)}\,\tilde{f}\left(x_n + c_i\,h, \mathrm{e}^{\omega\,(x_n+c_i\,h)}\,\left(u_n + h\sum_{j=1}^{s} a_{ij}\,K_j\right)\right).$$

Defining $k_i := \mathrm{e}^{\omega\,(x_n+c_i\,h)}\,K_i$, it then follows that

$$y_{n+1} = \mathrm{e}^{\omega\,h}\,y_n + h\,\sum_{i=1}^{s} b_i \mathrm{e}^{\omega\,(1-c_i)\,h}\,k_i \tag{5.13}$$

with

$$k_i = \tilde{f}\left(x_n + c_i\,h, \mathrm{e}^{\omega\,c_i\,h}\,y_n + h\sum_{j=1}^{s} a_{ij}\,\mathrm{e}^{\omega\,(c_i-c_j)\,h}\,k_j\right). \tag{5.14}$$

The method defined by (5.13) and (5.14) is called an integrating factor (IF) method.

As an example, let us consider the IF method that is built upon method (2.6):

$$Y_1 = \mathrm{e}^{c_1\,\omega\,h}\,y_n + h\,c_1\tilde{f}\,(x_n + c_1\,h, Y_1), \tag{5.15}$$
$$y_{n+1} = \mathrm{e}^{\omega\,h}\,y_n + h\,\mathrm{e}^{(1-c_1)\omega\,h}\tilde{f}\,(x_n + c_1\,h, Y_1). \tag{5.16}$$

Although obtained in a different way, this method is completely identical to (2.8).

To examine the general form of the stability function of an IF method, we start from (1.2). Lawson's transformation converts this equation into $u' = (\lambda - \omega)\,u$. If the method that is used consecutively is e.g. a purely polynomial method M with stability function $R_M(z)$, this gives rise to $u_{n+1} = R_M(z - z_0)\,u_n$. Then, in terms of the y-variable, this gives $y_{n+1} = \mathrm{e}^{z_0}\,R_M(z - z_0)\,y_n$, i.e.

$$R(z, z_0) = \mathrm{e}^{z_0} R_M(z - z_0),$$

which confirms the results of Section 2.

6. Exponential collocation methods

Starting from (1.4), we now construct a function $P(x) \in \mathcal{S}$ with $P(x_n) = y(x_n)$ such that for $Q(x) := \mathrm{e}^{\omega\,x}(\mathrm{e}^{-\omega\,x}P(x))'$ s-collocation conditions can be imposed:

$$Q(x_n + c_i\,h) = \tilde{f}\,(x_n + c_i\,h, P(x_n + c_i\,h)), \quad i = 1, \ldots, s. \tag{6.17}$$

The exponential collocation method is then defined by imposing $y_{n+1} := P(x_{n+1})$.

It should be noted here that the collocation function $Q(x)$ will belong to a space $\mathcal{S}_Q$ which does not, in general, coincide with $\mathcal{S}$. Also, we assume that $Q(x)$ can be written down in a Lagrangian way, i.e. in $\mathcal{S}_Q$ there exist canonical basis functions $l_j(t)$, $j = 1, \ldots, s$, such that $l_j(c_i) = \delta_{ij}$.

Since

$$\int_{x_n}^{x_n+t\,h} \mathrm{d}(\mathrm{e}^{-\omega\,x}P(x)) = \int_{x_n}^{x_n+t\,h} \mathrm{e}^{-\omega\,x}Q(x)\mathrm{d}x = h\int_0^t \mathrm{e}^{-\omega\,(x_n+\tau\,h)}Q(x_n + \tau\,h)\mathrm{d}\tau$$

we have

$$P(x_n + t\,h) = \mathrm{e}^{t\,\omega\,h}P(x_n) + h\int_0^t \mathrm{e}^{\omega\,(t-\tau)h}Q(x_n + \tau\,h)\mathrm{d}\tau.$$

Defining $k_i := \tilde{f}\,(x_n + c_i\,h, P(x_n + c_i\,h))$, $i = 1, \ldots, s$, we can then write down the interpolant as

$$Q(x_n + \tau\,h) = \sum_{j=1}^{s} l_j(\tau)k_j.$$

We then have for $t = c_i$

$$P(x_n + c_i\,h) = \mathrm{e}^{c_i\,\omega\,h}P(x_n) + h\sum_{j=1}^{s} a_{ij}\,k_j, \quad i = 1, \ldots, s,$$

and for $t = 1$

$$P(x_n + h) = e^{\omega h} P(x_n) + h \sum_{j=1}^{s} b_j k_j,$$

with

$$a_{ij} := \int_0^{c_i} e^{\omega (c_i - \tau) h} l_j(\tau) d\tau \quad \text{and} \quad b_j := \int_0^1 e^{\omega (1-\tau) h} l_j(\tau) d\tau.$$

The exponential collocation method is thus given by

$$y_{n+1} = e^{\omega h} y_n + h \sum_{i=1}^{s} b_i k_i$$

with

$$k_i = \tilde{f}\left(x_n + c_i h, e^{c_i \omega h} y_n + h \sum_{j=1}^{s} a_{ij} k_j\right), \quad i = 1, \ldots, s.$$

An alternative approach to obtain this method is to start construct a collocation method for $y' = f(x, y(x))$ directly, since the equation $e^{\omega x}(e^{-\omega x} P(x))' = \tilde{f}(x, P(x))$ is equivalent to $P(x)' = f(x, P(x))$. This will be illustrated in Sections 6.1 and 6.2.

It is also well-known that polynomial collocation methods can be derived starting from linear functionals. Well, this is the case for exponential collocation methods too. The approaches given above are equivalent to imposing that the functionals

$$\mathcal{L}_i[y(x); h] := y(x_n + c_i h) - \gamma_i y_n - h \sum_{j=1}^{s} a_{ij}(y'(x_n + c_j h) - \omega y(x_n + c_j h)), \quad i = 1, \ldots, s \tag{6.18}$$

and

$$\mathcal{L}[y(x); h] := y(x_n + h) - \gamma y_n - h \sum_{i=1}^{s} b_i(y'(x_n + c_i h) - \omega y(x_n + c_i h))$$

identically vanish for $y(x) = e^{\omega x}$ and for each function in the s dimensional space $\mathcal{S}_Q$.

6.1. Example 1: Polynomial interpolation

Suppose $\mathcal{S}$ is the space Π_{s-1} of polynomials of degree at most $s - 1$. In that case, the solution $P(x)$ of

$$P' - \omega P = Q(x)$$

is of the form $P(x) = \alpha e^{\omega x} + \tilde{Q}(x)$ where α is a constant and $\tilde{Q} \in \Pi_{s-1}$.

Then from

$$\tau^q = \sum_{j=1}^{s} l_j(\tau) c_j^q$$

we have for $q = 0, \ldots, s - 1$:

$$\int_0^{c_i} e^{\omega (c_i - \tau) h} \tau^q d\tau = \sum_{j=1}^{s} \int_0^{c_i} e^{\omega (c_i - \tau) h} l_j(\tau) d\tau c_j^q = \sum_{j=1}^{s} a_{ij} c_j^q. \tag{6.19}$$

For $q = 0$, we find

$$\sum_{j=1}^{s} a_{ij} = \frac{e^{c_i \omega h} - 1}{\omega h} \tag{6.20}$$

and, using partial integration, it follows from (6.19) for $q = 1, \ldots, s - 1$ that

$$\sum_{j=1}^{s} a_{ij} c_j^q = \int_0^{c_i} e^{\omega (c_i - \tau) h} \tau^q d\tau = -\frac{c_i^q}{\omega h} + \frac{q}{\omega h} \int_0^{c_i} e^{\omega (c_i - \tau) h} \tau^{q-1} d\tau = -\frac{c_i^q}{\omega h} + \frac{q}{\omega h} \sum_{j=1}^{s} a_{ij} c_j^{q-1}. \tag{6.21}$$

On the other hand formula (6.18) gives for $y(x) = e^{\omega x}$ that $\gamma_i = e^{c_i \omega h}$, for $q = 0$ one finds (6.20) and for $q = 1, \ldots, s - 1$ a relation equivalent to (6.21) is found.

As an example, let us consider the one-stage method that is obtained in this way :

$$Y_1 = e^{c_1 \omega h} y_n + h \frac{e^{c_1 \omega h} - 1}{\omega h} \tilde{f}(x_n + c_1 h, Y_1),$$
$$y_{n+1} = e^{\omega h} y_n + h \frac{e^{\omega h} - 1}{\omega h} \tilde{f}(x_n + c_1 h, Y_1).$$

Remark that this method, which solves (1.3) has also been found (see (2.7)) as an example of an EFRK method that is meant to solve (1.1). As a collocation method for (1.1), the method is found if the collocation function $P(x) \in \mathcal{S} = \text{Span}\{1, e^{\omega x}\}$ (such that $Q(x) = e^{\omega x}(e^{\omega x} P(x))'$ is a constant).

In general, the s-stage exponential collocation method applied to (1.3) with $Q(x) \in \Pi_{s-1}$ is equivalent to the collocation method (2.5) applied to (1.1) with $P(x) \in \Pi_{s-1} \cup \text{Span}\{e^{\omega x}\}$. We can thus conclude that in this case the function $e^{\omega x}$ has an additive character.

6.2. Example 2: Exponential interpolation

Suppose $\mathcal{S}$ is the space $e^{\omega x}\Pi_{s-1}$ of functions of the form $e^{\omega x} p_{s-1}(x)$ where $p_{s-1}(x) \in \Pi_{s-1}$. In that case, the solution $P(x)$ of

$$P' - \omega P = Q(x)$$

is of the form $e^{\omega x} p_s(x)$ with $p_s(x) \in \Pi_s$ (in fact, $p_s'(x) = p_{s-1}(x)$).

The collocation conditions (6.17) then become

$$p_s'(x) = e^{-\omega x} f(x, e^{\omega x} p_s(x)),$$

i.e. we are constructing a classical polynomial collocation method for a problem of the form $z' = g(x, z)$ with $g(x, z) = e^{-\omega x} f(x, e^{\omega x} y(x))$. The resulting exponential collocation method will be the same as the IF factor method that is built upon the polynomial collocation method.

As an example, we again consider the one-stage method that is obtained in this way. We then find (5.15), (5.16). Again, this method has been found (see (2.8)) as an example of an EFRK method which has been fitted to $\mathcal{S} = \text{Span}\{e^{\omega x}, x\, e^{\omega x}\}$ and that is meant to solve (1.1).

In general, the s-stage exponential collocation method applied to (1.3) with $Q(x) \in e^{\omega x}\Pi_{s-1}$ is equivalent to the collocation method (2.5) applied to (1.1) with $P(x) \in e^{\omega x}\Pi_s$. For this case the function $e^{\omega x}$ has a multiplicative character.

7. Conclusions

In this paper, we have analysed properties of stability functions of EFRK methods. Whereas purely polynomial methods impose conditions on the stability function $R(z)$ for $z = 0$ solely, EFRK that are fitted for parameter values $\omega_1, \omega_2, \ldots, \omega_n$ impose conditions for $z = \omega_1 h, \ldots, z = \omega_n h$ on the stability function $R(z, \{z_1, \ldots, z_n\})$. It was shown that nice relations exist between the different stability functions and, more in particular, between the corresponding order stars. Finally, the stability functions of integrating factor methods and exponential collocation methods were considered. It was also shown that exponential-fitting, integrating factor and exponential collocation can lead to the same method.

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