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1 Euler's theorem on homogeneous functions

Let $(x, y) \mapsto f(x, y)$ be a homogeneous function of degree n, that is, for any $\lambda > 0$,

$$f(\lambda x, \lambda y) = \lambda^n f(x, y). \tag{1}$$

By differentiating the left-hand side of (1) with respect to λ, we obtain

$$\frac{\partial}{\partial \lambda} f(\lambda x, \lambda y) = \frac{\partial f}{\partial x}\frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial f}{\partial y}\frac{\partial(\lambda y)}{\partial \lambda} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

From the right-hand side of (1),

$$\frac{\partial}{\partial \lambda}\lambda^n f(x, y) = n\lambda^{n-1} f(x, y).$$

Since the derivatives of both sides must coincide, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\lambda^{n-1} f(x, y)$$

Let $\lambda = 1$, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

We have proved the following theorem.
Theorem 1.1. *Let $(x, y) \mapsto f(x, y)$ be a homogeneous function of degree n. It holds that*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

The case of $n = 1$ is the most important for economics.
Corollary 1.1. Let $(x, y) \mapsto f(x, y)$ be a homogeneous function of degree 1. It holds that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$$

2 Constant returns to scale and zero profit

Let $(K, L, A) \mapsto F(K, AL)$ be a production function. It is usually assumed that F has constant returns to scale; i.e., it is homogeneous of degree 1 (linearly homogeneous). For $\lambda > 0$, it holds that

$$F(\lambda K, \lambda AL) = \lambda F(K, AL).$$

When all factors of input (K and AL) are increased by a common factor, the output is increased by the same factor.

Note that from the chain rule,

$$\frac{\partial F}{\partial L} = \frac{\partial F}{\partial (AL)}\frac{d(AL)}{dL} = \frac{\partial F}{\partial (AL)}A.$$

By Euler's theorem, we have

$$F(K, AL) = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial (AL)}AL = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L.$$

The first order conditions for the firm's profit maximization give us

$$\frac{\partial F}{\partial K} = r + \delta$$
$$\frac{\partial F}{\partial L} = w,$$

where $r + \delta$ is the gross rental rate and w is the wage rate.

Recall that the cost for production is $(r + \delta)K + wL$. That is, the linear homogeneity and factor market competition equate output and cost:

$$F(K, AL) = (r + \delta)K + wL.$$