Expected Number of Tosses for the First Occurrence of GPPG

Solution via Markov Chains (Gaussian Elimination)

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1 Problem Statement

We are tasked with finding the expected number of tosses of a fair coin required to observe the sequence **GPPG** (Head-Tail-Tail-Head) for the first time. Since the coin is fair, the probability of a Head (G) is $p = 0.5$ and the probability of a Tail (P) is $q = 0.5$.

2 Markov Chain Model

We define 5 states as follows:

- S_0: Start state, no prefix matched.
- S_1: Last symbol was G.
- S_2: Last symbols were GP.
- S_3: Last symbols were GPP.
- S_4: Sequence GPPG matched (absorbing).

3 State Automaton

4 System of Equations

Let E_i be the expected number of additional tosses from state S_i. Using the transitions:

$$\begin{aligned}
E_0 &= 1 + 0.5E_0 + 0.5E_1 \\
E_1 &= 1 + 0.5E_1 + 0.5E_2 \\
E_2 &= 1 + 0.5E_1 + 0.5E_3 \\
E_3 &= 1 + 0.5E_0 \\
E_4 &= 0 \quad \text{(absorbing)}
\end{aligned}$$

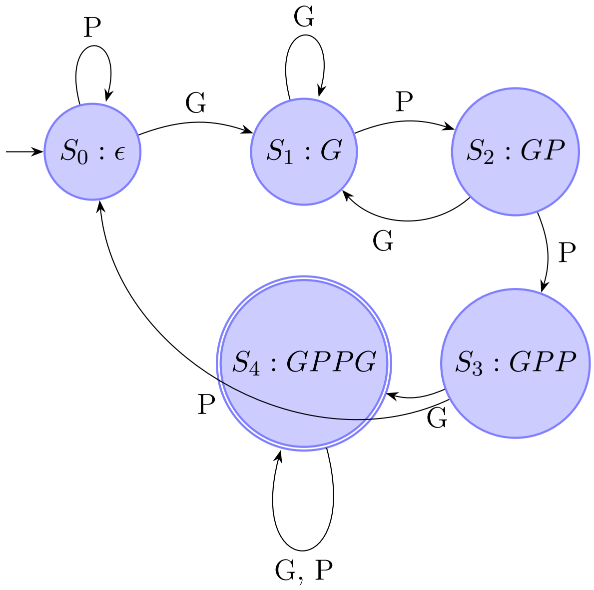

Figure 1: Markov automaton for detecting GPPG.

Rewriting in linear form:

$$\begin{aligned}
0.5E_0 - 0.5E_1 &= 1 \\
0.5E_1 - 0.5E_2 &= 1 \\
-0.5E_1 + E_2 - 0.5E_3 &= 1 \\
-0.5E_0 + E_3 &= 1
\end{aligned}$$

5 Gaussian Elimination

$$\left[\begin{array}{rrrr|r}
0.5 & -0.5 & 0 & 0 & 1 \\
0 & 0.5 & -0.5 & 0 & 1 \\
0 & -0.5 & 1 & -0.5 & 1 \\
-0.5 & 0 & 0 & 1 & 1
\end{array}\right]$$

Step-by-step row operations (omitted here for brevity) lead to:

$$\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 18 \\
0 & 1 & 0 & 0 & 16 \\
0 & 0 & 1 & 0 & 14 \\
0 & 0 & 0 & 1 & 10
\end{array}\right]$$

6 Conclusion

- $E_0 = \mathbf{18}$

- $E_1 = 16$
- $E_2 = 14$
- $E_3 = 10$

Therefore, the expected number of coin tosses to first observe the sequence **GPPG** is:

$$\boxed{18}$$