Harbour Space HW1

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M-2 - Math Refresher for Masters

Data Science

ex 1-2 | 2.6

find sol'n to $Ax=b$

$$\left[\begin{array}{cccccc|c} 0&1&0&0&1&0&2\\ 0&0&0&1&1&0&-1\\ 0&0&0&0&0&1&1 \end{array}\right] \xrightarrow{R_3:=R_1-R_3} \left[\begin{array}{cccccc|c} 0&1&0&0&1&0&2\\ 0&0&0&1&1&0&-1\\ 0&0&0&0&1&-1&1 \end{array}\right] \xrightarrow{R_2:=R_2-R_3}$$

$$\to \left[\begin{array}{cccccc|c} 0&1&0&0&1&0&2\\ 0&0&0&1&0&1&-2\\ 0&0&0&0&1&-1&1 \end{array}\right] \xrightarrow{R_1:=R_1-R_3} \left[\begin{array}{cccccc|c} 0&1&0&0&0&1&1\\ 0&0&0&1&0&1&-2\\ 0&0&0&0&1&-1&1 \end{array}\right]$$

solution set: $\begin{cases} x_2 + x_6 = 1 \\ x_4 + x_6 = -2 \\ x_5 - x_6 = 1 \end{cases} \Rightarrow \begin{cases} x_1 - \text{free} \\ x_2 = 1 - x_6 \\ x_3 - \text{free} \\ x_4 = -2 - x_6 \\ x_5 = 1 + x_6 \\ x_6 = \text{free} \end{cases}$

Done!

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ex 2-2 | 2.7

solve $Ax = 12x \to Ax - 12x = 0$ or $(A-12I)x = 0$

$\boxed{! \sum_i^3 x_i = 1}$

$$\begin{bmatrix} 6&4&3\\ 6&0&9\\ 0&8&0 \end{bmatrix} - \begin{bmatrix} 12&&\\ &12&\\ &&12 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|c} -6&4&3&0\\ 6&-12&9&0\\ 0&8&-12&0 \end{array}\right] \xrightarrow[R_3:=\frac{R_3}{4}]{R_2:R_1+R_2} \left[\begin{array}{ccc|c} -6&4&3&0\\ 0&-8&12&0\\ 0&2&-3&0 \end{array}\right] \xrightarrow[R_3:\frac{R_2}{4}+R_3]{R_2:-\frac{R_2}{4}}$$

$$\to \left[\begin{array}{ccc|c} -6&4&3&0\\ 0&2&-3&0\\ 0&0&0&0 \end{array}\right] \xrightarrow[R_2:=\frac{R_2}{2}]{R_1: R_1-2R_2} \left[\begin{array}{ccc|c} -6&0&9&0\\ 0&1&-\frac{3}{2}&0\\ 0&0&0&0 \end{array}\right] \xrightarrow{R_1:\to\frac{R_1}{-6}}$$

$$\to \left[\begin{array}{ccc|c} 1&0&-\frac{3}{2}&0\\ 0&1&-\frac{3}{2}&0\\ 0&0&0&0 \end{array}\right]$$

general solution: $x_1 = \frac{3}{2}x_3$, $x_2 = \frac{3}{2}x_3$

! $x_1 + x_2 + x_3 = 1$

$\hookrightarrow \frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 = 1$

$\to \boxed{4x_3 = 1}$

$\boxed{x_3 = \frac{1}{4}}$

$\boxed{x_2 = \frac{3}{8} \quad x_1 = \frac{3}{8}}$

Done!

ex 3-2 | 2.8 ! there is A^{-1}, s.t. $AA^{-1} = E$ iff $\det(A) \neq 0$

a.

$$\det\left[\begin{array}{ccc|cc} 2 & 3 & 4 & 2 & 3 \\ 3 & 4 & 5 & 3 & 4 \\ 4 & 5 & 6 & 4 & 5 \end{array}\right] = 2\cdot4\cdot6 + 3\cdot5\cdot4 + 4\cdot3\cdot5 - 4\cdot4\cdot4 - 5\cdot5\cdot2 - 6\cdot3\cdot3 = 48 + 60 + 60 - 64 - 50 - 54 = 0$$

=> not invertible!

b.

$$\det\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 0\cdot\det[\ldots] + 1\cdot\det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 0\det[\ldots]$$

$$= \underbrace{\left[1\cdot0\cdot0 + 1\cdot1\cdot1 + 1\cdot1\cdot0 - 1\cdot0\cdot0 - 1\cdot1\cdot1 - 0\cdot1\cdot1\right]}_{0} + \underbrace{\left[0\cdot1\cdot0 + 1\cdot1\cdot1 + 0\cdot1\cdot1 - 1\cdot1\cdot0 - 1\cdot1\cdot0 - 0\cdot1\cdot1\right]}_{1} = 1$$

since $\det[A] \neq 0 \to A^{-1}$ exist.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[P_4 = P_1 - P_4]{P_3 = P_1 - P_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \xrightarrow[R_4 : R_2 + R_4]{R_3 : R_2 + R_3}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_4 := R_3 - 2R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \end{array}\right] \xrightarrow[R_3 := R_3 + R_4]{R_4 := -R_4}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right] \xrightarrow[\substack{R_2 = R_2 - \frac{1}{2}R_3 \\ R_1 = R_1 - \frac{1}{2}R_3}]{R_3 = \frac{1}{2}R_3} \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$

Done ✓ (checked on Python)

ex 4-1 | 2.10

a. set of vectors $\{v_1, v_2, v_3\}$ is linearly indep

iff $[v_1\, v_2\, v_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ has only trivial sol'n

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ -1 & 1 & -3 & 0 \\ 3 & -2 & 8 & 0 \end{array}\right] \xrightarrow[R_3 : 3R_1 - 2R_3]{R_2 : R_1 + 2R_2} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

sol'n

x_3 - free

$x_2 = x_3$

$x_1 = \dfrac{-x_3 - 3x_3}{2}$

nontrivial

$\Rightarrow$ linearly dependent

b.
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

1. since 3rd row of $v_1 = 1$ (or $a_{31} = 1$) and $a_{32}, a_{33} = 0$, we cannot obtain v_1 from v_2 & v_3

2. also we cannot obtain v_2 from v_1 & v_3 since $a_{21} = 2$, $a_{22} = 1$ & $a_{23} = 0$, so v_2 should be $\frac{1}{2}v_1 +$ some $b v_2$.

   ↳ but b should be $\frac{1}{2}$, as a_{41} & $a_{51} = 0$ for v_1

   $\Rightarrow$ v_2 cannot be obtained as a linear combination from v_1 & v_3

3. same logic applies for v_3

   $\Rightarrow$ $\mathbb{V} = \{v_1, v_2, v_3\}$ is lin. independent.

end

ex 5-1 | 2,4

A_1:

a. Dim (A_1) = # of vector in the $\mathcal{B}$ of A_1

↳ ro ech. form

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 = 2R_1 - R_3 \\ R_4 = R_1 - R_4}]{R_2 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 = \frac{1}{2}R_2 + R_3]{R_2 = \frac{1}{2}R_2} \begin{array}{c} \text{pivots} \\ \downarrow \ \downarrow \ \ \ \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

since we have pivots in 2 columns => dim $(A_1) = 2$

- dim(A_2)

↳ ech form

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \xrightarrow[\substack{R_2 = \frac{R_1}{3} - R_2 \\ R_3 = \frac{7R_1}{3} - R_3 \\ R_4 = R_1 - R_4}]{R_1 = \frac{1}{3}R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & -3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow[R_3, R_4 = 0]{R_2 = \frac{R_2}{-3}} \begin{array}{c} \text{pivots} \\ \downarrow \ \downarrow \ \ \ \\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

since we have pivots in 2 columns => dim$(A_2) = 2$

b. $\mathcal{B}(A_1)$ is $\left\{ \begin{matrix} 1 & 0 \\ 1 & -2 \\ 2 & 1 \\ 1 & 0 \end{matrix} \right\}$ as pivots in columns 1 & 2

$\mathcal{B}(A_2)$ is $\left\{ \begin{matrix} 3 & -3 \\ 1 & 2 \\ 7 & -5 \\ 3 & -1 \end{matrix} \right\}$

c.

5.1 | 2.14 (c)

Let v be a linear combination of $\beta(A_1)$ and $\beta(A_2)$, also $v \in U_1 \cap U_2 \Rightarrow v \in U_1$ & $v \in U_2$

thus $v = \beta(A_1)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ for some α_1, α_2

also $v = \beta(A_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ for some β_1, β_2 $\Rightarrow$

$$\Rightarrow v = \begin{bmatrix}1 & 0\\ 1 & -2\\ 2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}3 & -3\\ 1 & 2\\ 7 & -5\\ 3 & -1\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 3 & -3\\ 1 & -2 & 1 & 2\\ 2 & 1 & 7 & -5\\ 1 & 0 & 3 & -1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ -\beta_1\\ -\beta_2\end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -3 & 3\\ 1 & -2 & -1 & -2\\ 2 & 1 & -7 & 5\\ 1 & 0 & -3 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ -\beta_1\\ -\beta_2\end{bmatrix} = 0$$

$$\left[\begin{array}{cccc|c}1 & 0 & -3 & 3 & 0\\ 1 & -2 & -1 & -2 & 0\\ 2 & 1 & -7 & 5 & 0\\ 1 & 0 & -3 & 1 & 0\end{array}\right] \xrightarrow[\substack{R_3 = 2R_1 - R_3\\ R_4 = R_1 - R_4}]{R_2 = R_1 - R_2} \left[\begin{array}{cccc|c}1 & 0 & -3 & 3 & 0\\ 0 & 2 & -2 & 5 & 0\\ 0 & -1 & 1 & 1 & 0\\ 0 & 0 & 0 & 2 & 0\end{array}\right] \xrightarrow[R_4 = \frac{R_4}{2}]{R_3 = R_2 + 2R_3}$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & -3 & 3 & 0\\ 0 & 2 & -2 & 5 & 0\\ 0 & 0 & 0 & 7 & 0\\ 0 & 0 & 0 & 2 & 0\end{array}\right] \xrightarrow[R_4 = \frac{2}{7}R_3 - \frac{1}{2}R_4]{} \left[\begin{array}{cccc|c}1 & 0 & -3 & 3 & 0\\ 0 & 2 & -2 & 5 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$\alpha_1 - 3\beta_1 + 3\beta_2 = 0$

$2\alpha_2 - 2\beta_1 + 5\beta_2 = 0$

$\beta_2 = 0$

[sol'n] $\beta_2 = 0$

β_1 - free

$\alpha_1 = 3\beta_1$

$\alpha_2 = \beta_1$

let $\beta_1 = 1 \Rightarrow$

$\Rightarrow \alpha_1 = 3$

$\alpha_2 = 1$

$v = \beta(A_1)\cdot\begin{bmatrix}3\\ 1\end{bmatrix} =$

$$= 3\cdot\begin{bmatrix}1\\ 1\\ 2\\ 1\end{bmatrix} + 1\begin{bmatrix}0\\ -2\\ 1\\ 0\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 7\\ 3\end{bmatrix} \rightarrow \text{basis for } U_1 \cap U_2$$

$$\boxed{\beta(U_1 \cap U_2) = \left\{\begin{bmatrix}3\\ 1\\ 7\\ 3\end{bmatrix}\right\}}$$