Harbour Space Midterm 1

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Problem 1 find order of (S_n), where $S_n = \eta_1 + \eta_2 + \dots + \eta_i$, where η = r.v with following distribution $P(\eta_i = 1) = \frac{1}{2} = P(\eta_i = -1)$

We know that variance (S_n) = second central moment $= D(S_n)$, S_n is itself a random variable.

$\underline{D(S_n) = E[S_n^2] - E[S_n]^2}$

We know, that $E[S_n] = 0$ since mean of a random walk is the origin itself. Thus, need to compute $E[S_n^2]$.

$S_n^2 = \sum_j \sum_i \eta_i \eta_j$

$E[\sum_j \sum_i \eta_i \eta_j] \overset{\text{by linearity}}{=} \sum_i \sum_j E[\eta_i \eta_j]$.

Consider $i = j$, then $\eta_i \cdot \eta_j = 1 \ \forall \eta_i, \eta_j$

thus, $E[\sum\sum \eta_i \eta_i] = n \cdot 1 = \boxed{n}$

when consider $i \neq j$. then we may want to consider four cases $(1,1)$ $(1,-1)$, $(-1,1)$ & $(-1,1)$

Each case has $P(\eta_i \eta_j) = (\pm 1, \pm 1) = \frac{1}{4}$, and thus is equiprobable.

thus by symmetricity $E(\sum_j \sum_i \eta_i \eta_i)$, $i \neq j$ =

$$= \sum_i \sum_j E(\eta_i \eta_i) =$$

$$= \sum_i \sum_j \left(\frac{1}{4}\cdot(1\cdot 1) + \frac{1}{4}(1\cdot(-1)) + (-1\cdot 1) + (1\cdot(-1))\right)$$

$$= \sum_i \sum_j (0) = \boxed{0}$$

$\therefore \sum_j \sum_i E[\eta_i \eta_j] = n$, since

$\forall i = j$, $E[\eta_i, \eta_j] = 1 \Rightarrow$

$\Rightarrow \quad E[S_n^2] = \sum_i^n E[\eta_i \eta_j] = n$

$\underline{\text{and } \forall i \neq j \ E[S_n^2] = 0} \quad \to E[S_n^2] = n$

and $\sigma(S_n) = \sqrt{Var(S_n)} = \sqrt{n}$

for number of steps $= 2n$

$$\boxed{\sigma(S_{2n}) = \sqrt{2n}}$$

end

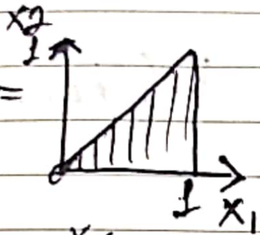
Problem 2

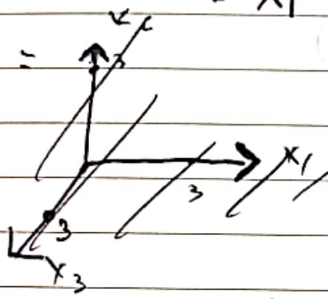
Denote $A: \{X_1 > X_2\}$

$B: \{X_1 + X_2 + X_3 < 1\}$

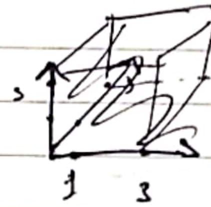
$C: \{X_1^2 + X_2^2 + X_3^2 < \frac{1}{4}\}$

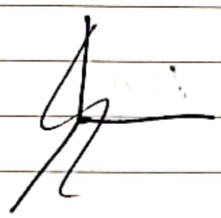
$D: \{X_3 < \frac{1}{2}\}$

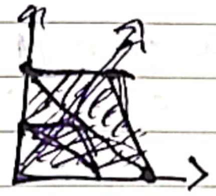
$P(A) =$  = Area of this triangle $= \boxed{\frac{1}{2}}$

$P(B) =$  denote $x_1 + x_2 + x_3 = y$, y $\Omega(y) = [0;3]$

geometrically:







tired to compute

Idea to find

V of these

$\boxed{\text{three}}$

$P(\text{sum}(x_1 x_2 x_3) \leq 1) =$

= Volume of entire pyramid − Volume of smaller pyramid

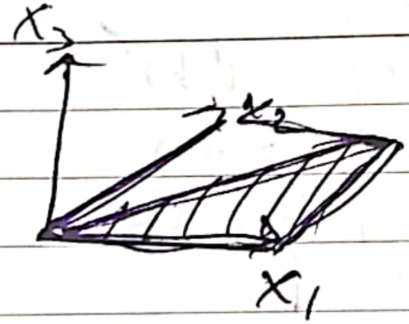
$= \quad A\ base = (3\sqrt{2})^2 \cdot \frac{\sqrt{3}}{4} = \boxed{\frac{9\sqrt{3}}{2}} = \ldots$

$h' = \frac{a\sqrt{3}}{2} = \frac{3\sqrt{6}}{2} \quad \frac{3\sqrt{6}}{2}$

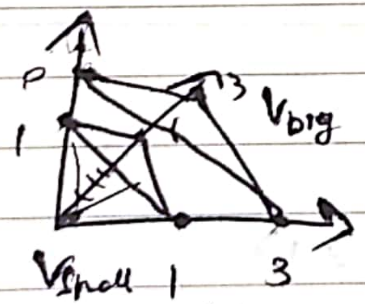
$h = \sqrt{3^2 - \left(\frac{3\sqrt{6}}{2}\right)^2} = 9 - \frac{36}{4}$

I Assume Geometrical Models

$P(A) =$

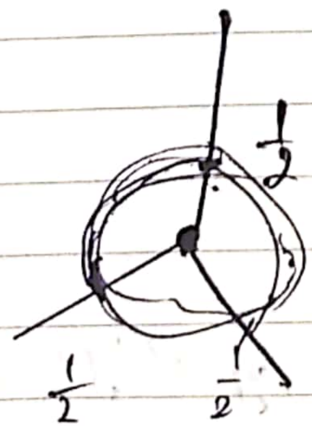


$P(B) =$



$\cancel{\boxed{V_{big} - V_{small}}}$ $\quad \dfrac{V_{small}}{V_{big}}$ — Geometrical

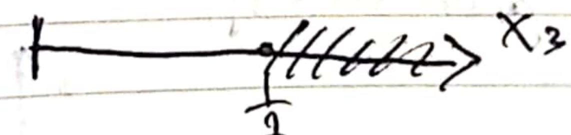
$P(C) =$



$\cancel{V(\text{Sphere with } R=1)} - V(\text{Shere with } R=\tfrac{1}{2})$

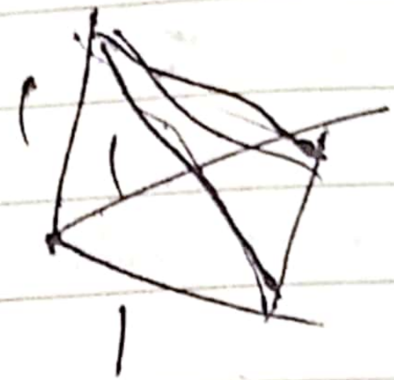
$$\frac{\frac{1}{8}\text{th } V(\text{Sphere } R=\frac{1}{2})}{\frac{1}{8}\text{th } V(\text{Sphere } R=1)}$$

$P(D) = \dfrac{1}{2}$



have no meaning

A & B $P(A) = \frac{1}{2}$ 



$P(B) = \frac{V}{V_{big\ pyramid}} =$

$= 1 \cdot 1 \cdot \frac{1}{2}$.

$V = \frac{1}{3} \cdot \text{base area} \cdot h$

$= \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$

$\boxed{P(B) = \frac{1}{27}}$

$V_{big} = \frac{1}{3} \cdot \frac{9}{2} \cdot 3 = 4.5$

$P(B) = \frac{\frac{1}{6}}{\frac{9}{2}} = \frac{1}{6} \cdot \frac{2}{9} = \frac{1}{27}$

$$P(C) = \frac{\frac{1}{8}\left(\frac{4}{3}\pi r^3\right)}{\frac{1}{8}\left(\frac{4}{3}\pi R^3\right)} \qquad r = \frac{1}{2} \qquad R = 1$$

$$= \frac{\pi\left(\frac{1}{2}\right)^3}{\pi(1)^3} = \frac{\frac{1}{8}}{1} = \frac{1}{8}$$

$\boxed{P(C) = \frac{1}{8}}$

so far

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{27}$

$P(C) = \frac{1}{8}$

$P(D) = \frac{1}{2}$

(1) A & D — $\boxed{\text{indep}}$

since $P(A \cap D) = P(A) \cdot P(D)$

(2) $P(A \cdot B) = \frac{1}{54}$ (cut pyramid in half) ($r_{small} = \frac{1}{2}$, $V_{small} = \frac{1}{2} \cdot \frac{1}{6}$ and $\frac{1}{12}$)

given (A·B)

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{27}$

$P(A) \cdot P(B) = P(A \cdot B) = \frac{1}{54} \Rightarrow$ $P(A \cdot B) = \frac{\frac{1}{12}}{\frac{9}{2}} = \frac{1}{12} \cdot \frac{2}{9} = \frac{1}{6} \cdot \frac{1}{9} = \boxed{\frac{1}{54}}$

$\Rightarrow$ $\boxed{\text{A \& B inden}}$

(3) $P(A \cdot C)$ = cutting $\frac{1}{8}$th of sphere in half diagonally

$$= \frac{\frac{1}{2}\cdot\frac{1}{8} \cdot V(\text{small})}{\frac{1}{8} \cdot V(\text{big})} = \frac{1}{2} P(C) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$$

$P(A) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ $\boxed{\text{independent}}$

(4) (B & C) $\quad P(B \cdot C) = P(C)$ since

geometrically, ~~8th of pyramid~~ C

since $\frac{1}{8}$ of sphere with $r = \frac{1}{2}$ C pyramid $\Rightarrow$

$\Rightarrow P(B \cdot C) = P(C) = \frac{1}{8}$

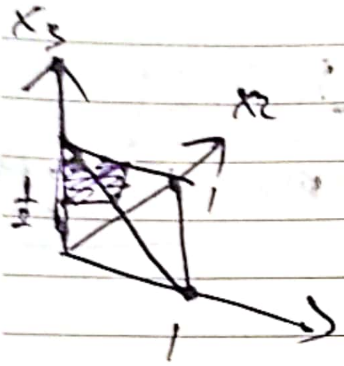
$P(B) \cdot P(C) = \frac{1}{8} \cdot \frac{1}{27} \neq \frac{1}{8} \Rightarrow$ B & C $\Rightarrow$ $\boxed{\text{dependent}}$

(5) P_2 B & D

$P(B) = \frac{1}{27}$

$P(D) = \frac{1}{2}$

$P(B \cdot D)$ - is a small pyramid heig cut from Big pyramid. (shadded)



V(small pyramid) =

$= \frac{1}{3} h \cdot A_{base} =$

$= \frac{1}{3} \cdot \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) =$

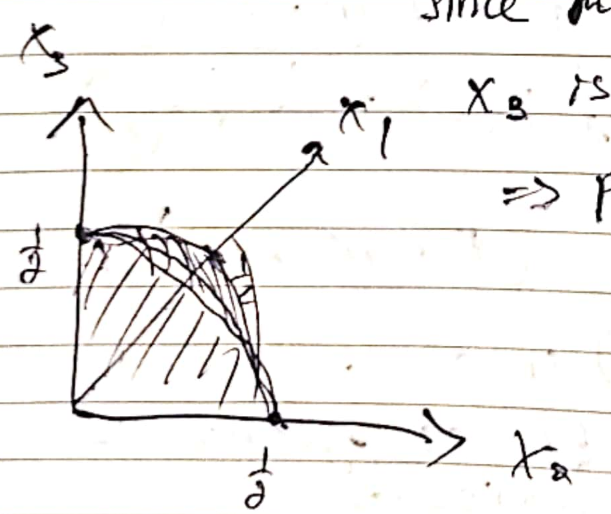
$= \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}$

$P(B) \cdot P(D) = \frac{1}{27} \cdot \frac{1}{2} = \boxed{\frac{1}{54}}$ not equal =>

=> B & D - depen

(6) $\boxed{C \& D}$ $P(C \cdot D) = 0$

since no intersection,



x_3 is limited at $\boxed{\frac{1}{2}}$.

$\Rightarrow P(C \cdot D) = 0 \neq P(C) \cdot P(D)$

$\Rightarrow$ Dependent

Finally

- A & B – indep
- A & C – indep
- A & D – indep
- B & C – dependent
- B & D – ~~indep~~ dependent
- C & D – dependent

end

Problem 3 | $\boxed{\forall x_i \ , \ x_i \geq 0}$

ξ	x_1	x_2	x_3	x_4
	p_1	p_2	p_3	p_4

distribution function

$$F_\xi : \mathbb{R} \to \mathbb{R} \cup \{+\infty, -\infty\}$$

$$F_\xi(x) = \begin{cases} P(\xi = x) & \text{if } x \in \{x_1, x_2, x_3, x_4\} \\ 0 & \text{, otherwise} \end{cases}$$

$$E\xi = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4$$

$$\int_0^{\infty}[1-F_\xi(x)]\,dx = \int_0^{x_1}[1-F_\xi(x)]\,dx + \int_{x_1}^{x_2}[1-F_\xi(x)]\,dx$$

$$+ \int_{x_2}^{x_3}[1-F_\xi(x)]\,dx + \int_{x_3}^{x_4}[1-F_\xi(x)]\,dx$$

$$+\ 0$$

$$= \cancel{(1-p_1)x_1} + \cancel{(1-p_2)x_2} - \cancel{(1-p_1)x_1}$$

$$+ \cancel{(1-p_3)x_3} - \cancel{(1-p_2)x_2} + (1-p_4)x_4 - \cancel{(1-p_3)x_3}$$

$$= x_4 - p_4x_4$$

$$E\xi = \int_0^{\infty} = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 - x_4 - p_4x_4 =$$

$$= \boxed{p_1x_1 + p_2x_2 + p_3x_3 - x_4}$$

Our Simplified Prob model is that month are equiprobable
Problem 4 | Real Approach is harder and more computations required

a) P(of different bdays in different years) =

$$= \frac{\text{num of cases needed}}{\text{num of all cases}}$$

Consider having a bday at month n is the same as taking the chair

so when $n = 4$ & 12 month (chairs), we have:

$12 \cdot 11 \cdot 10 \cdot 9$ – different variations

Assumption: Every month is equiprobable, and P(bday in Jan) = P(all other month)

So we have 12 chairs. 1st Person has 12 choices, 2nd Person has 11 (since he does not want to sit in the same) and so on we will have $12 \cdot 11 \cdot 10 \cdot 9$ outcomes, for a needed A. thus $|A| = 12 \cdot 11 \cdot 10 \cdot 9$

Now we want to calculate $|\Omega|$

here we have $|\Omega| = 12^4$

∴ So P(ppl have bdays in different months) = $\boxed{\frac{12 \cdot 11 \cdot 10 \cdot 9}{12^4}}$

b) Consider Again Probability Model where each month is equiprobable, and we will look at this problem from the „chair" perspective.

So P(at least two in one month) =

= P(num ≥ 2) = P(num = 2) + P(num = 3) + P(num = 4) and P(num = 5)

• P(num = 2) = look at this as from the perspective two people already sit on the same chair, and remaining three can sit on Any chair =

= $|A| = 11 \cdot 11 \cdot 11 \cdot 12 / |\Omega|$

• Same for P(num = 3) : $|A| = 11 \cdot 11 \cdot 12 / |\Omega|$

$$P(num = 4) = 12 \cdot 11 \cdot / |\Omega|$$

$$P(num = 5) = 12 / |\Omega|$$

$$\Rightarrow P(A) = \frac{12 \cdot 11 \cdot 11 \cdot 11 + 12 \cdot 11 \cdot 11 + 12 \cdot 11 + 12}{12^4}$$

$$= \frac{12(11^3 + 11^2 + 11^1 + 11^0)}{12^4} =$$

$$= \boxed{\frac{(11^3 + 11^2 + 11^1 + 11^0)}{12^4}}$$