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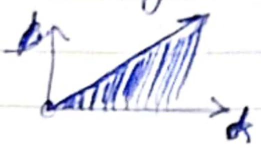
Harbour Space

Final Exam

Problem 1

1) $\gamma = \frac{\beta}{\alpha}$, $\Rightarrow \gamma < 1 = \frac{\beta}{\alpha} < 1 \to \beta < \alpha$

geometrically we can represent this as



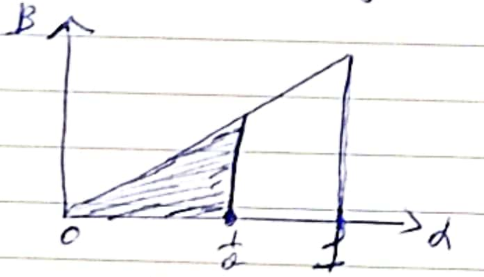
(no point at (0,0))

to compute $P(\gamma<1)$, we can geometrically observe, that this is the area under curve $\alpha = \beta$ from 0 to 1

the Area of this is $\frac{1}{2}$

thus $P(\gamma<1) = \frac{1}{2}$

- Since the distribution is uniform, $P(\alpha < \frac{1}{2}) = \frac{1}{2}$
- Now we want to find $P(\gamma < 1 \text{ and } \alpha < \frac{1}{2})$



this is an Area of triangle, an can be found via integral/geometry (i choose 2) $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ $\left(\frac{a \cdot b}{2}\right)$

$\therefore$ So far

$P(\gamma<1) = \frac{1}{2}$

$P(\alpha < \frac{1}{2}) = \frac{1}{2}$

$P(\gamma<1 \text{ and } \alpha<\frac{1}{2}) = \frac{1}{8}$

since $P(\gamma<1 \text{ and } \alpha<\frac{1}{2}) \neq P(\gamma<1) \cdot P(\alpha<\frac{1}{2})$

γ and α are $\boxed{\text{dependent}}$

(b) $F_Y(x) = P(Y \le x) = P(\beta \le x\alpha) =$
$= E_\alpha[P(\beta \le x\alpha | \alpha)] = E_\alpha[F_\beta(x\alpha)]$

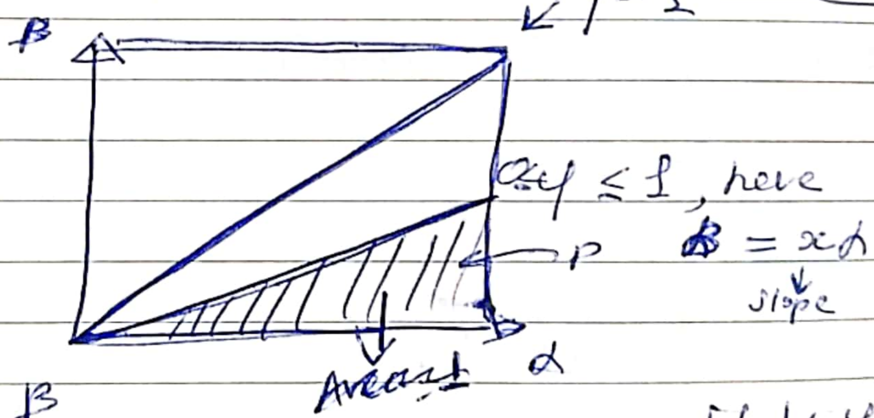
$$F_Y(x) = \int_0^1 \begin{cases} x\alpha & 0 < \alpha < \frac{1}{x} \\ 1 & \alpha > \frac{1}{x} \end{cases} dx$$

$$= \frac{x}{2}\left(\frac{1}{\max(x,1)}\right)^2 + \left(1 - \frac{1}{\max(x,1)}\right)^2$$

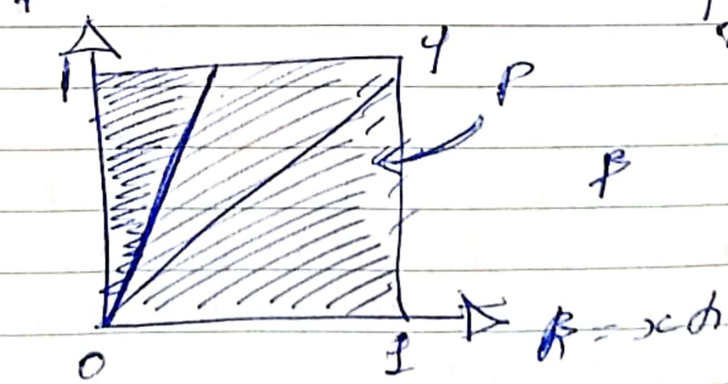
$$= \begin{cases} \frac{x}{2} & \text{if } 0 < x \le 1 \\ 1 - \frac{1}{2x} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Geometrically

$\boxed{\text{if } 0 < Y \le 1}$



if $1 \le Y \le$



So far $F_Y(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le Y \le 1 \\ 1 - \frac{1}{2x} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(x) = (F_Y(x))' = \begin{cases} \frac{1}{2} & \text{if } 0 \le Y \le 1 \\ 1 - \frac{1}{2x^2} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$

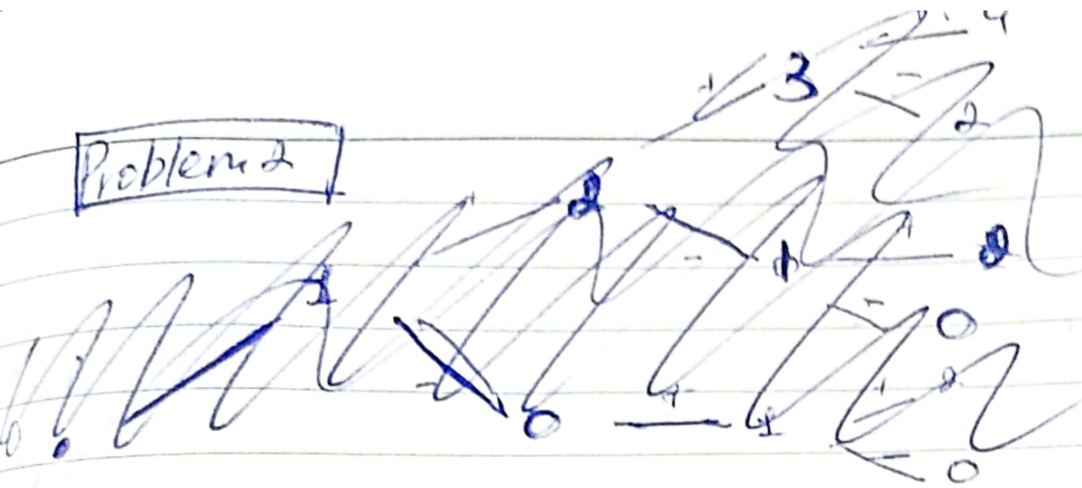
$$E(y) = \int_{-\infty}^{\infty} y \cdot f_y(x) =$$

$$= \int_0^1 y \cdot \frac{1}{2}\, dy + \int_1^{\infty} y\left(1 - \frac{1}{2y^2}\right) dy$$

$$= \left[y^2\right]_0^1 + \left[\frac{y^2}{2} - \frac{1}{2}\ln|y|\right]_1^{\infty} \Rightarrow \underline{\text{diverges}}$$

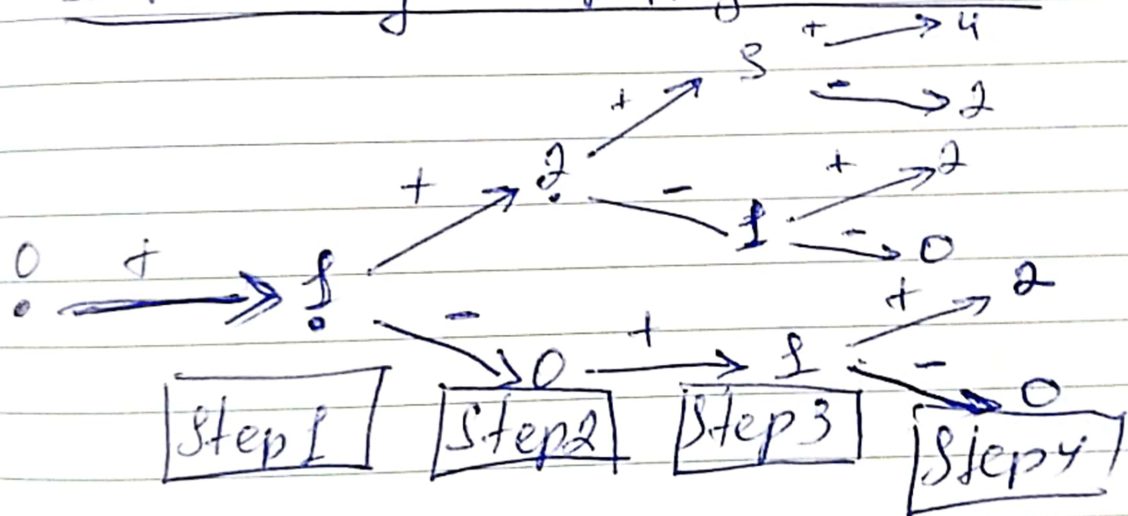
thus, $E(y)$ <u>is not defined</u>

Problem 2



Step 1 | Step 2 | Step 3 | Step 4

so manually computing we have



in total we have 6 steps,
2 of each is receiving zero

so $P(S_n = 0 \mid S_i \geq 0 \ \forall i \in [0,n]) = \frac{1}{3}$

3 Problem

$$f(x) = (1-p)^{x-1} p$$

↳ number of trials failures before we get a success

$$P(\alpha+\beta=n \mid \alpha=k) \quad 0 \le k \le n$$

is actually

$$P(k+\beta=n) \text{ or}$$

$$P(\beta=n-k) \quad 0 \le k \le n =$$

$$= ((1-p))^{n-k} p$$

Problem 4

$= (1-p)^{n-k} p$

Problem 4

Let $\cancel{x}\, \psi = \max(\phi, B)$

$P(\psi \le x) = P(\max(\phi, B) \le x) =$

$= P(\phi \le x)$ and $P(B \le x) =$

$= P(\phi \le x) \cdot P(B \le x)$

so $P(\phi \le x) = \frac{x+1}{2}$. since cdf of ϕ & B = $= \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$, $a = -1$, $b = 1$

Since $P(B \le x) = \frac{x+1}{2}$

So $F_\psi(x) = \begin{cases} 0 & \text{if } x < -1 \\ \left(\frac{x+1}{2}\right)^2 & \text{if } x \in [-1; 1] \\ 1 & \text{if } x > 1 \end{cases}$

Final Answer

$$F_\psi(x) \begin{cases} 0 & \text{if } x < -1 \\ \frac{(x+1)^2}{4} & \text{if } x \in [-1; 1] \\ 1 & \text{if } x > 1 \end{cases}$$

$\boxed{\psi = \max(\phi, B)}$

Problem 5

$$P(\alpha = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$E(\alpha) = \lambda$

$Var(\alpha) = \lambda$

By Chebyshev inequality

$$P(|\cancel{x} - E\alpha| \leq \varepsilon) \geq 1 - \frac{D\alpha}{\varepsilon^2}$$

let ε = 1 var or 1 sigma

$$P(|\alpha - E\alpha| \leq \sqrt{\lambda}) \geq 1 - \frac{D\alpha}{\lambda}$$

or P(α lies in range of ± 6) is

$$P(|\alpha - \lambda| \leq \sqrt{\lambda}) \geq 1 - \frac{\lambda}{\lambda} = 0$$

for $2 6$

$$P(|\alpha - \lambda| \leq 2\sqrt{\lambda}) \geq 1 - \frac{\lambda}{4\lambda} = \frac{3}{4}$$

$$P(|\alpha - \lambda| \leq 3\sqrt{\lambda}) \geq 1 - \frac{\lambda}{9\lambda} = \frac{8}{9}$$

$$P(|\alpha - \lambda| \leq 3\sqrt{\lambda}) =$$

$= \Big[P(\alpha - \lambda \leq 3\sqrt{\lambda})$. cdf of poisson distribution is

$P(\lambda - \alpha \leq 3\sqrt{\lambda}$

$\Downarrow$

~~$P(\alpha - \lambda$~~ $P(\alpha \leq 3\sqrt{\lambda} + \lambda$

$P(\alpha \cancel{\geq} - 3\sqrt{\lambda} + \lambda)$ $\Rightarrow$

$\Rightarrow \quad P(\lambda - 3\sqrt{\lambda} \leq \alpha \leq \lambda + 3\sqrt{\lambda})$

$$= P\left(\sqrt{\lambda}(\sqrt{\lambda} - 3) \leq \alpha \leq \sqrt{\lambda}(\sqrt{\lambda} + 3)\right)$$

$$= \sum_{j=0}^{\lfloor \lambda + 3\sqrt{\lambda} \rfloor} \frac{\lambda^j}{j!} - \sum_{j=0}^{\lfloor \lambda + 3\sqrt{\lambda} \rfloor} \frac{\lambda^j}{j!} =$$

$$= \sum_{j=\lambda - 3\sqrt{\lambda}}^{\lfloor \lambda + 3\sqrt{\lambda} \rfloor} \frac{\lambda^j}{j!}$$

~~Assume~~ $\cancel{\alpha > \lambda}$

$$\cancel{P(\alpha - \lambda \leq 3\sqrt{\lambda}) =}$$

$$\cancel{= P(\alpha \leq \lambda + 3\sqrt{\lambda}) = \sum_{0}^{\lfloor \lambda + 3\sqrt{\lambda} \rfloor} \frac{\lambda^j}{j!}}$$

So for the Poissonian R.v. Chebyshev 3-sigma rule can be expressed as

$$P(|\alpha - \lambda| \leq 3\sqrt{\lambda}) = \sum_{i=\lfloor \lambda - 3\sqrt{\lambda} \rfloor}^{\lfloor \lambda + 3\sqrt{\lambda} \rfloor} \frac{\lambda^i}{i!} \geq \frac{8}{9}$$