Chebyshev's Inequality. *Let ξ be a nonnegative random variable. Then for every $\varepsilon > 0$*

$$\mathsf{P}(\xi \geq \varepsilon) \leq \frac{\mathsf{E}\xi}{\varepsilon}. \tag{21}$$

The proof follows immediately from

$$\mathsf{E}\xi \geq \mathsf{E}[\xi \cdot I_{(\xi \geq \varepsilon)}] \geq \varepsilon \mathsf{E} I_{(\xi \geq \varepsilon)} = \varepsilon \mathsf{P}(\xi \geq \varepsilon).$$

From (21) we can obtain the following variant of Chebyshev's inequality: If ξ is any random variable then

$$\mathsf{P}(\xi \geq \varepsilon) \leq \frac{\mathsf{E}\xi^2}{\varepsilon^2} \tag{22}$$

and

$$\mathsf{P}(|\xi - \mathsf{E}\xi| \geq \varepsilon) \leq \frac{\mathsf{V}\xi^2}{\varepsilon^2}, \tag{23}$$

where $\mathsf{V}\xi = \mathsf{E}(\xi - \mathsf{E}\xi)^2$ is the variance of ξ.

The Cauchy–Bunyakovskii Inequality. *Let ξ and η satisfy $\mathsf{E}\xi^2 < \infty, \mathsf{E}\eta^2 < \infty$. Then $\mathsf{E}|\xi\eta| < \infty$ and*

$$(\mathsf{E}|\xi\eta|)^2 \leq \mathsf{E}\xi^2 \cdot \mathsf{E}\eta^2. \tag{24}$$

PROOF. Suppose that $\mathsf{E}\xi^2 > 0$, $\mathsf{E}\eta^2 > 0$. Then, with $\tilde{\xi} = \xi/\sqrt{\mathsf{E}\xi^2}$, $\tilde{\eta} = \eta/\sqrt{\mathsf{E}\eta^2}$, we find, since $2|\tilde{\xi}\tilde{\eta}| \leq \tilde{\xi}^2 + \tilde{\eta}^2$, that

$$2\mathsf{E}|\tilde{\xi}\tilde{\eta}| \leq \mathsf{E}\tilde{\xi}^2 + \mathsf{E}\tilde{\eta}^2 = 2,$$

i.e. $\mathsf{E}|\tilde{\xi}\tilde{\eta}| \leq 1$, which establishes (24).

On the other hand if, say, $\mathsf{E}\xi^2 = 0$, then $\xi = 0$ (a.s.) by Property **I**, and then $\mathsf{E}\xi\eta = 0$ by Property **F**, i.e. (24) is still satisfied.

Jensen's Inequality. *Let the Borel function $g = g(x)$ be convex downward and $\mathsf{E}|\xi| < \infty$. Then*

$$g(\mathsf{E}\xi) \leq \mathsf{E}g(\xi). \tag{25}$$

PROOF. If $g = g(x)$ is convex downward, for each $x_0 \in R$ there is a number $\lambda(x_0)$ such that

$$g(x) \geq g(x_0) + (x - x_0) \cdot \lambda(x_0) \tag{26}$$

for all $x \in R$. Putting $x = \xi$ and $x_0 = \mathsf{E}\xi$, we find from (26) that

$$g(\xi) \geq g(\mathsf{E}\xi) + (\xi - \mathsf{E}\xi) \cdot \lambda(\mathsf{E}\xi),$$

and consequently $\mathsf{E}g(\xi) \geq g(\mathsf{E}\xi)$.

A whole series of useful inequalities can be derived from Jensen's inequality. We obtain the following one as an example.

Lyapunov's Inequality. *If* $0 < s < t$,

$$(\mathsf{E}\,|\xi|^s)^{1/s} \le (\mathsf{E}\,|\xi|^t)^{1/t}. \tag{27}$$

To prove this, let $r = t/s$. Then, putting $\eta = |\xi|^s$ and applying Jensen's inequality to $g(x) = |x|^r$, we obtain $|\mathsf{E}\eta|^r \le \mathsf{E}\,|\eta|^r$, i.e.

$$(\mathsf{E}\,|\xi|^s)^{t/s} \le \mathsf{E}\,|\xi|^t,$$

which establishes (27).

The following chain of inequalities among absolute moments in a consequence of Lyapunov's inequality:

$$\mathsf{E}|\xi| \le (\mathsf{E}\,|\xi|^2)^{1/2} \le \cdots \le (\mathsf{E}\,|\xi|^n)^{1/n}. \tag{28}$$

Hölder's Inequality. *Let* $1 < p < \infty$, $1 < q < \infty$, *and* $(1/p) + (1/q) = 1$. *If* $\mathsf{E}\,|\xi|^p < \infty$ *and* $\mathsf{E}\,|\eta|^q < \infty$, *then* $\mathsf{E}\,|\xi\eta| < \infty$ *and*

$$\mathsf{E}\,|\xi\eta| \le (\mathsf{E}\,|\xi|^p)^{1/p}(\mathsf{E}\,|\eta|^q)^{1/q}. \tag{29}$$

If $\mathsf{E}\,|\xi|^p = 0$ or $\mathsf{E}\,|\eta|^q = 0$, (29) follows immediately as for the Cauchy–Bunyakovskii inequality (which is the special case $p = q = 2$ of Hölder's inequality).

Now let $\mathsf{E}\,|\xi|^p > 0$, $\mathsf{E}\,|\eta|^q > 0$ and

$$\tilde{\xi} = \frac{\xi}{(\mathsf{E}\,|\xi|^p)^{1/p}}, \qquad \tilde{\eta} = \frac{\eta}{(\mathsf{E}\,|\eta|^q)^{1/q}}.$$

We apply the inequality

$$x^a y^b \le ax + by, \tag{30}$$

which holds for positive x, y, a, b and $a + b = 1$, and follows immediately from the concavity of the logarithm:

$$\ln[ax + by] \ge a \ln x + b \ln y = \ln x^a y^b.$$

Then, putting $x = |\tilde{\xi}|^p$, $y = |\tilde{\eta}|^q$, $a = 1/p$, $b = 1/q$, we find that

$$|\tilde{\xi}\tilde{\eta}| \le \frac{1}{p}|\tilde{\xi}|^p + \frac{1}{q}|\tilde{\eta}|^q,$$

whence

$$\mathsf{E}|\tilde{\xi}\tilde{\eta}| \le \frac{1}{p}\mathsf{E}|\tilde{\xi}|^p + \frac{1}{q}\mathsf{E}|\tilde{\eta}|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

This establishes (29).