In [1]:
```
import Pkg
Pkg.activate(@__DIR__)
Pkg.instantiate()
using LinearAlgebra, Plots
import ForwardDiff as FD
import MeshCat as mc
using JLD2
using Test
using Random
include(joinpath(@__DIR__,"utils/cartpole_animation.jl"))
include(joinpath(@__DIR__,"utils/basin_of_attraction.jl"))
```

```
  Activating environment at `~/OCRL/HW2_S24/Project.toml`
plot_basin_of_attraction (generic function with 1 method)
```

Note:

Some of the cells below will have multiple outputs (plots and animations), it can be easier to see everything if you do `Cell -> All Output -> Toggle Scrolling`, so that it simply expands the output area to match the size of the outputs.

Q2: LQR for nonlinear systems (40 pts)

Linearization warmup

Before we apply LQR to nonlinear systems, we are going to treat our linear system as if it's nonlinear. Specifically, we are going to "approximate" our linear system with a first-order Taylor series, and define a new set of $(\Delta x, \Delta u)$ coordinates. Since our dynamics are linear, this approximation is exact, allowing us to check that we set up the problem correctly.

First, assume our discrete time dynamics are the following:

$$x_{k+1} = f(x_k, u_k)$$

And we are going to linearize about a reference trajectory $\bar{x}_{1:N}, \bar{u}_{1:N-1}$. From here, we can define our delta's accordingly:

$$x_k = \bar{x}_k + \Delta x_k \tag{1}$$
$$u_k = \bar{u}_k + \Delta u_k \tag{2}$$

Next, we are going to approximate our discrete time dynamics function with the following first order Taylor series:

$$x_{k+1} \approx f(\bar{x}_k, \bar{u}_k) + \left[\frac{\partial f}{\partial x}\bigg|_{\bar{x}_k, \bar{u}_k}\right](x_k - \bar{x}_k) + \left[\frac{\partial f}{\partial u}\bigg|_{\bar{x}_k, \bar{u}_k}\right](u_k - \bar{u}_k)$$

Which we can substitute in our delta notation to get the following:

$$\bar{x}_{k+1} + \Delta x_{k+1} \approx f(\bar{x}_k, \bar{u}_k) + \left[\frac{\partial f}{\partial x}\bigg|_{\bar{x}_k, \bar{u}_k} \right] \Delta x_k + \left[\frac{\partial f}{\partial u}\bigg|_{\bar{x}_k, \bar{u}_k} \right] \Delta u_k$$

If the trajectory $\bar{x}, \bar{u}$ is dynamically feasible (meaning $\bar{x}_{k+1} = f(\bar{x}_k, \bar{u}_k)$), then we can cancel these equivalent terms on each side of the above equation, resulting in the following:

$$\Delta x_{k+1} \approx \left[\frac{\partial f}{\partial x}\bigg|_{\bar{x}_k, \bar{u}_k} \right] \Delta x_k + \left[\frac{\partial f}{\partial u}\bigg|_{\bar{x}_k, \bar{u}_k} \right] \Delta u_k$$

Cartpole

We are now going to look at two different applications of LQR to the nonlinear cartpole system. Given the following description of the cartpole:

No description has been provided for this image

(if this image doesn't show up, check out `cartpole.png`)
with a cart position p and pole angle θ. We are first going to linearize the nonlinear discrete dynamics of this system about the point where $p = 0$, and $\theta = 0$ (no velocities), and use an infinite horizon LQR controller about this linearized state to stabilize the cartpole about this goal state. The dynamics of the cartpole are parametrized by the mass of the cart, the mass of the pole, and the length of the pole. To simulate a "sim to real gap", we are going to design our controllers around an estimated set of problem parameters `params_est`, and simulate our system with a different set of problem parameters `params_real`.

In [2]:
```
"""
continuous time dynamics for a cartpole, the state is
x = [p, θ, ṗ, θ̇]
where p is the horizontal position, and θ is the angle
```

```
where θ = 0 has the pole hanging down, and θ = 180 is up.

The cartpole is parametrized by a cart mass `mc`, pole
mass `mp`, and pole length `l`. These parameters are loaded
into a `params::NamedTuple`. We are going to design the
controller for a estimated `params_est`, and simulate with
`params_real`.
"""
function dynamics(params::NamedTuple, x::Vector, u)
    # cartpole ODE, parametrized by params.

    # cartpole physical parameters
    mc, mp, l = params.mc, params.mp, params.l
    g = 9.81

    q = x[1:2]
    qd = x[3:4]

    s = sin(q[2])
    c = cos(q[2])

    H = [mc+mp mp*l*c; mp*l*c mp*l^2]
    C = [0 -mp*qd[2]*l*s; 0 0]
    G = [0, mp*g*l*s]
    B = [1, 0]

    qdd = -H\(C*qd + G - B*u[1])
    return [qd;qdd]

end

function rk4(params::NamedTuple, x::Vector,u,dt::Float64)
    # vanilla RK4
    k1 = dt*dynamics(params, x, u)
    k2 = dt*dynamics(params, x + k1/2, u)
    k3 = dt*dynamics(params, x + k2/2, u)
    k4 = dt*dynamics(params, x + k3, u)
    x + (1/6)*(k1 + 2*k2 + 2*k3 + k4)
end
```

```
rk4 (generic function with 1 method)
```

Part A: Infinite Horizon LQR about an equilibrium (10 pts)

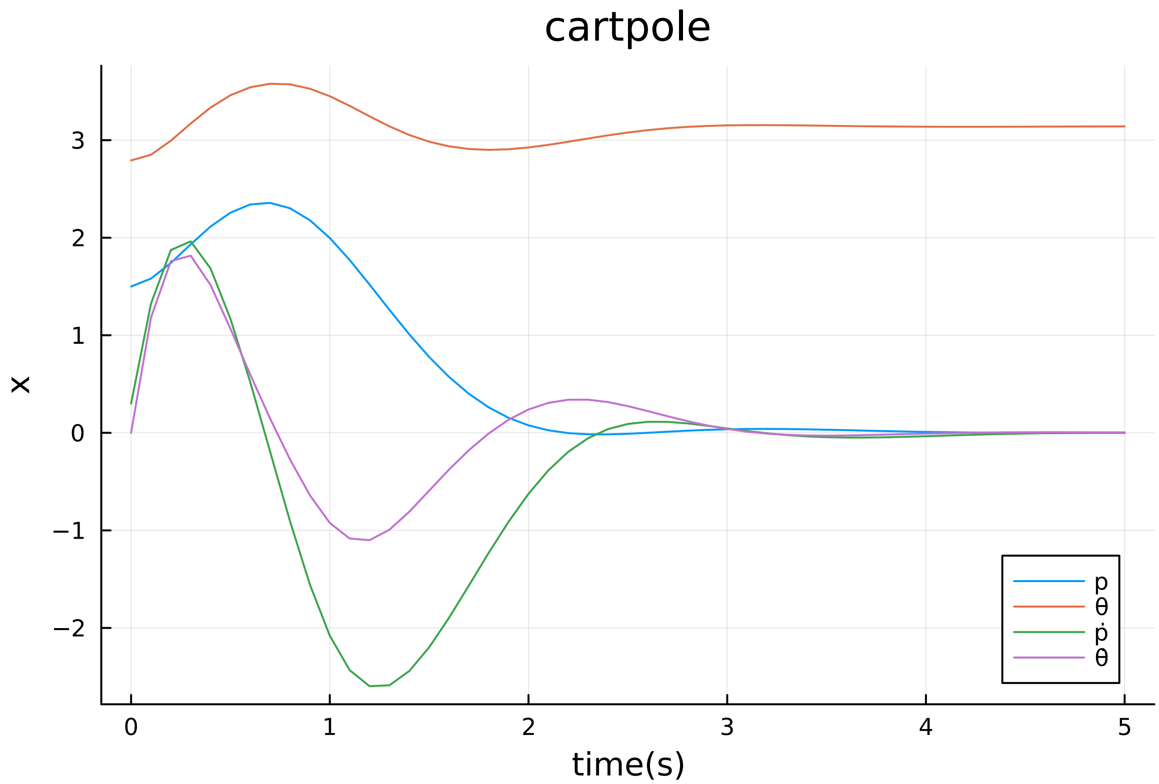
Here we are going to solve for the infinite horizon LQR gain, and use it to stabilize the cartpole about the unstable equilibrium.

In [17]:

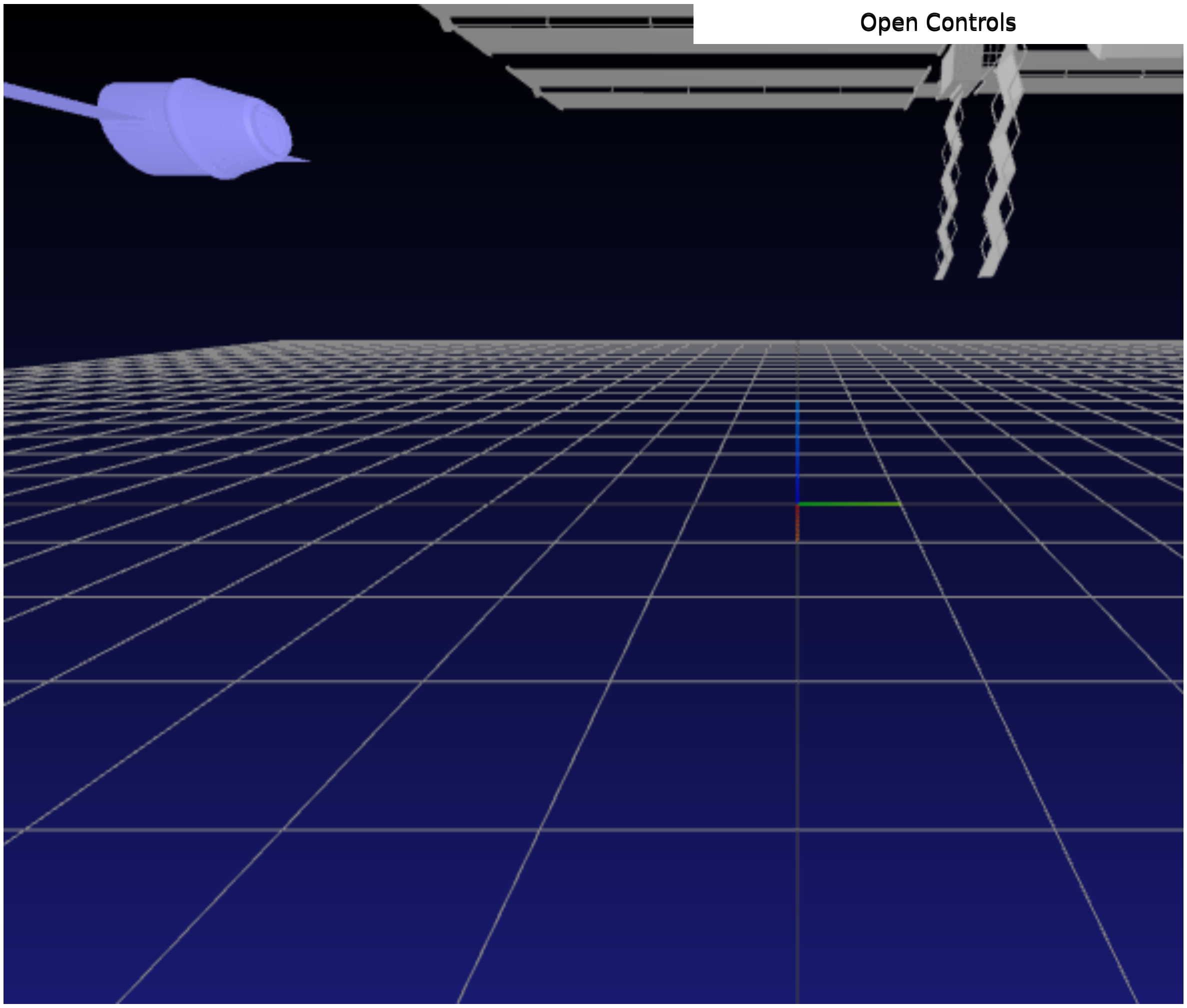
```
@testset "LQR about eq" begin

    # states and control sizes
    nx = 4
    nu = 1
```

```
# desired x and g (linearize about these)
xgoal = [0, pi, 0, 0]
ugoal = [0]

# initial condition (slightly off of our linearization point)
x0 = [0, pi, 0, 0] + [1.5, deg2rad(-20), .3, 0]

# simulation size
dt = 0.1
tf = 5.0
t_vec = 0:dt:tf
N = length(t_vec)
X = [zeros(nx) for i = 1:N]
X[1] = x0

# estimated parameters (design our controller with these)
params_est = (mc = 1.0, mp = 0.2, l = 0.5)

# real paremeters (simulate our system with these)
params_real = (mc = 1.2, mp = 0.16, l = 0.55)

# TODO: solve for the infinite horizon LQR gain Kinf

#Copied from Q1

"""
P,K = ihlqr(A,B,Q,R)

TODO: complete this infinite horizon LQR function where
you do the ricatti recursion until the cost to go matrix
P converges to a steady value |P_k - P_{k+1}| ≤ tol
"""
function ihlqr(A::Matrix,       # vector of A matrices
               B::Matrix,       # vector of B matrices
               Q::Matrix,       # cost matrix Q
               R::Matrix;       # cost matrix R
               max_iter = 1000, # max iterations for Ricatti
               tol = 1e-5       # convergence tolerance
               )::Tuple{Matrix, Matrix} # return two matrices

    # get size of x and u from B
    nx, nu = size(B)

    # initialize S with Q
    P = deepcopy(Q)
    K = [zeros(nu,nx) for i = 1:max_iter-1]

    # Ricatti
    for ricatti_iter = 1:max_iter
        # k = ricatti_iter
        P_= deepcopy(P)

        K = (R+B'*P*B)\(B'*P*A)
        P = Q+(A'*P*(A-B*K))

        if (norm(P-P_) <= tol)
```

```
                return P,K
            end
        end
        error("ihlqr did not converge")
    end

    # cost terms
    Q = diagm([1,1,.05,.1])
    R = 0.1*diagm(ones(nu))

    Kinf = zeros(1,4)
    A = zeros(nx,nx)
    B = zeros(nx, nu)

    A = FD.jacobian(dx -> rk4(params_est,dx,ugoal,dt), xgoal)
    B = FD.jacobian(du -> rk4(params_est,xgoal,du,dt), ugoal)
    P, K = ihlqr(A,B,Q,R)
    Kinf = K

    # TODO: simulate this controlled system with rk4(params_real, ...)
    for k = 1:(N-1)
        X[k+1] = rk4(params_real, X[k], -Kinf*(X[k] - xgoal), dt)
    end
    # ---------------tests and plots/animations---------------
    @test X[1] == x0
    @test norm(X[end])>0
    @test norm(X[end] - xgoal) < 0.1

    Xm = hcat(X...)
    display(plot(t_vec,Xm',title = "cartpole",
                 xlabel = "time(s)", ylabel = "x",
                 label = ["p" "θ" "ṗ" "θ̇"]))

    # animation stuff
    display(animate_cartpole(X, dt))
    # ---------------tests and plots/animations---------------
end
```




```
(1, 4)
┌ Info: Listening on: 127.0.0.1:8703, thread id: 1
└ @ HTTP.Servers /home/rsharde/.julia/packages/HTTP/1EWL3/src/Servers.jl:369
┌ Info: MeshCat server started. You can open the visualizer by visiting the
following URL in your browser:
│ http://127.0.0.1:8703
└ @ MeshCat /home/rsharde/.julia/packages/MeshCat/I6NTX/src/visualizer.jl:63
```




```
Test Summary: | Pass  Total
LQR about eq  |    3      3
Test.DefaultTestSet("LQR about eq", Any[], 3, false, false)
```

Part B: Infinite horizon LQR basin of attraction (5 pts)

In part A we built a controller for the cartpole that was based on a linearized version of the system dynamics. This linearization took place at the `(xgoal, ugoal)`, so we should only really expect this model to be accurate if we are close to this linearization point (think small angle approximation). As we get further from the goal state, our linearized model is less and less accurate, making the performance of our controller suffer. At a certain point, the controller is unable to stabilize the cartpole due to this model mismatch.

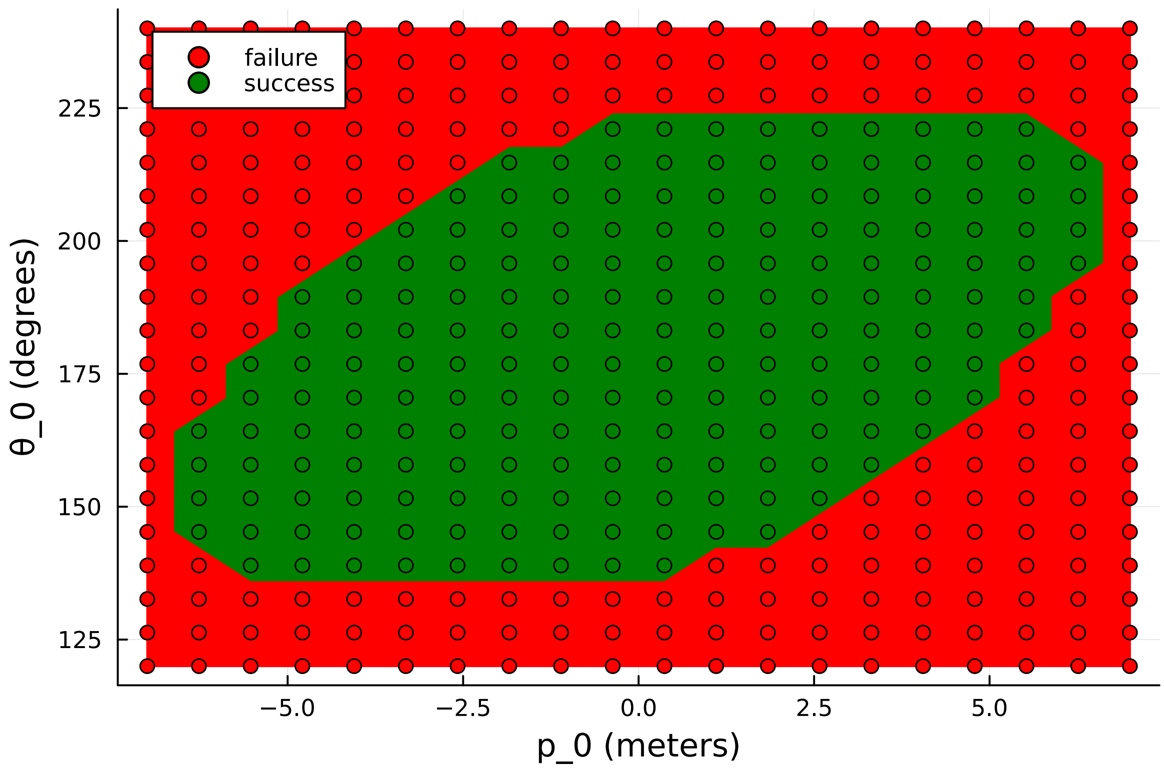
To demonstrate this, you are now being asked to take the same controller you used above, and try it for a range of initial conditions. For each of these simulations, you will determine if the controller was able to stabilize the cartpole. From here, you will plot the successes and failures on a plot and visualize a "basin of attraction", that is, a region of the state space where we expect our controller to stabilize the system.

In [22]:
```
function create_initial_conditions()
```

```
    # create a span of initial configurations
    M=20
    ps = LinRange(-7, 7, M)
    thetas = LinRange(deg2rad(180-60), deg2rad(180+60), M)

    initial_conditions = []

    for p in ps
        for theta in thetas
            push!(initial_conditions, [p, theta, 0, 0.0])
        end
    end

    return initial_conditions, ps, thetas
end

function check_simulation_convergence(params_real, initial_condition, Kinf,
    """
    args
        params_real: named tuple with model dynamics parameters
        initial_condition: X0, length 4 vector
        Kinf: IHLQR feedback gain
        xgoal: desired state, length 4 vector
        N: number of simulation steps
        dt: time between steps

    return
        is_controlled: bool
    """

    x0 = 1 * initial_condition

    is_controlled = false

    # Simulate the closed-loop (controlled) cartpole starting at the initial
    x = zeros(N, length(initial_condition))
    x[1, :] = x0
    for k in 1:N-1
        # Compute control input

        # println(size((x[k, :] - xgoal)'))
        # println(size(Kinf))
        u = -Kinf * (x[k, :] - xgoal)
        # Ensure u is a column vector
        u = reshape(u, :, 1)
        # Perform integration using rk4
        x[k+1, :] = rk4(params_real, x[k, :], u, dt)
        # Check if the state has diverged
        if norm(x[k+1, :]) > 100
            return false
        end
    end

    # Check if the final state is close to the goal
    if norm(x[end, :] - xgoal) < 0.1
        is_controlled = true
```

```
    end

    return is_controlled
end


let

    nx = 4
    nu = 1
    xgoal = [0, pi, 0, 0]
    ugoal = [0]
    dt = 0.1
    tf = 5.0
    t_vec = 0:dt:tf
    N = length(t_vec)


    # estimated parameters (design our controller with these)
    params_est = (mc = 1.0, mp = 0.2, l = 0.5)

    # real paremeters (simulate our system with these)
    params_real = (mc = 1.2, mp = 0.16, l = 0.55)

    # TODO: solve for the infinite horizon LQR gain Kinf
    # this is the same controller as part B
    #Copied from Q1

    """
    P,K = ihlqr(A,B,Q,R)

    TODO: complete this infinite horizon LQR function where
    you do the ricatti recursion until the cost to go matrix
    P converges to a steady value |P_k - P_{k+1}| ≤ tol
    """
    function ihlqr(A::Matrix,       # vector of A matrices
                B::Matrix,       # vector of B matrices
                Q::Matrix,       # cost matrix Q
                R::Matrix;       # cost matrix R
                max_iter = 1000, # max iterations for Ricatti
                tol = 1e-5       # convergence tolerance
                )::Tuple{Matrix, Matrix} # return two matrices

        # get size of x and u from B
        nx, nu = size(B)

        # initialize S with Q
        P = deepcopy(Q)
        K = [zeros(nu,nx) for i = 1:max_iter-1]

        # Ricatti
        for ricatti_iter = 1:max_iter
            # k = ricatti_iter
            P_= deepcopy(P)

            K = (R+B'*P*B)\(B'*P*A)
```

```
            P = Q+(A'*P*(A-B*K))

            if (norm(P-P_) <= tol)
                return P,K
            end
        end
        error("ihlqr did not converge")
    end

    # cost terms
    Q = diagm([1,1,.05,.1])
    R = 0.1*diagm(ones(nu))
    A = zeros(nx,nx)
    B = zeros(nx, nu)

    A = FD.jacobian(dx -> rk4(params_est, dx, ugoal, dt), xgoal)
    B = FD.jacobian(du -> rk4(params_est, xgoal, du, dt), ugoal)
    P, K = ihlqr(A,B,Q,R)

    Kinf = K
    # create the set of initial conditions we want to test for convergence
    initial_conditions, ps, thetas = create_initial_conditions()

    convergence_list = []

    for initial_condition in initial_conditions

        convergence = check_simulation_convergence(params_real,
                                                   initial_condition,
                                                   Kinf, xgoal, N, dt)

        push!(convergence_list, convergence)
    end

    plot_basin_of_attraction(initial_conditions, convergence_list, ps, rad2d


    # -------------tests------------------
    @test sum(convergence_list) < 190
    @test sum(convergence_list) > 180
    @test length(convergence_list) == 400
    @test length(initial_conditions) == 400

end
```



Test Passed

Part C: Infinite horizon LQR cost tuning (5 pts)

We are now going to tune the LQR cost to satisfy our following performance requirement:

$\|x(5.0) - x_{\text{goal}}\|_2 =$ `norm(X[N] - xgoal) < 0.1`

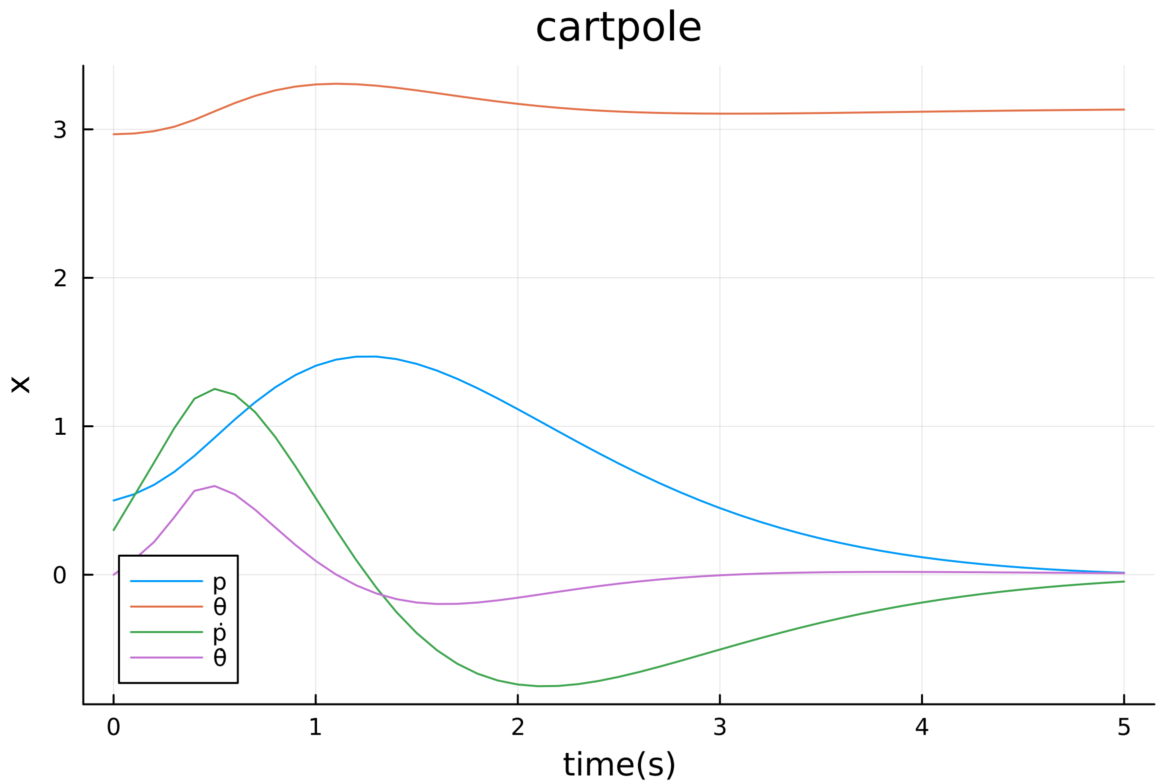
which says that the L2 norm of the state at 5 seconds (last timestep in our simulation) should be less than 0.1. We are also going to have to deal with the following actuator limits: $-3 \leq u \leq 3$. You won't be able to directly reason about this actuator limit in our LQR controller, but we can tune our cost function to avoid saturating the actuators (reaching the actuator limits) for too long. Here are our suggestions for tuning successfully:

1. First, adjust the values in Q and R to find a controller that stabilizes the cartpole. The key here is tuning our cost to keep the control away from the actuator limits for too long.
2. Now that you can stabilize the system, the next step is to tune the values in Q and R accomplish our performance goal of `norm(X[N] - xgoal) < 0.1`. Think about the individual values in Q, and which states we really want to penalize. The positions (p, θ) should be penalized differently than the velocities $(\dot{p}, \dot{\theta})$.

In [39]:
```
@testset "LQR about eq" begin

    # states and control sizes
```

```
nx = 4
nu = 1

# desired x and g (linearize about these)
xgoal = [0, pi, 0, 0]
ugoal = [0]

# initial condition (slightly off of our linearization point)
x0 = [0, pi, 0, 0] + [0.5, deg2rad(-10), .3, 0]

# simulation size
dt = 0.1
tf = 5.0
t_vec = 0:dt:tf
N = length(t_vec)
X = [zeros(nx) for i = 1:N]
X[1] = x0

# estimated parameters (design our controller with these)
params_est = (mc = 1.0, mp = 0.2, l = 0.5)

# real paremeters (simulate our system with these)
params_real = (mc = 1.2, mp = 0.16, l = 0.55)

# TODO: solve for the infinite horizon LQR gain Kinf

# cost terms
Q = diagm([100,1,10,100]) #diagm([1,1,.05,.1])
R = 100*diagm(ones(nu)) #0.1*diagm(ones(nu))

Kinf = zeros(1,4)

# vector of length 1 vectors for our control
U = [zeros(1) for i = 1:N-1]

function ihlqr(A::Matrix,       # vector of A matrices
            B::Matrix,       # vector of B matrices
            Q::Matrix,       # cost matrix Q
            R::Matrix;       # cost matrix R
            max_iter = 1000, # max iterations for Ricatti
            tol = 1e-5       # convergence tolerance
            )::Tuple{Matrix, Matrix} # return two matrices

    # get size of x and u from B
    nx, nu = size(B)

    # initialize S with Q
    P = deepcopy(Q)
    K = [zeros(nu,nx) for i = 1:max_iter-1]

    # Ricatti
    for ricatti_iter = 1:max_iter
        # k = ricatti_iter
        P_= deepcopy(P)

        K = (R+B'*P*B)\(B'*P*A)
```

```
            P = Q+(A'*P*(A-B*K))

            if (norm(P-P_) <= tol)
                return P,K
            end
        end
        error("ihlqr did not converge")
    end

    A = zeros(nx,nx)
    B = zeros(nx, nu)

    A = FD.jacobian(dx -> rk4(params_est,dx,ugoal,dt), xgoal)
    B = FD.jacobian(du -> rk4(params_est,xgoal,du,dt), ugoal)
    P, K = ihlqr(A,B,Q,R)
    Kinf = K

    # TODO: simulate this controlled system with rk4(params_real, ...)
    # TODO: make sure you clamp the control input with clamp.(U[i], -3.0, 3.
    for k = 1:(N-1)
        U[k] = -Kinf*(X[k]-xgoal)
        U[k] = clamp.(U[k],-3.0,3.0)
        X[k+1] = rk4(params_real, X[k], U[k], dt)
    end




    # ---------------tests and plots/animations---------------
    @test X[1] == x0  # initial condition is used
    @test norm(X[end])>0 # end is nonzero
    @test norm(X[end] - xgoal) < 0.1 # within 0.1 of the goal
    @test norm(vcat(U...), Inf) <= 3.0 # actuator limits are respected

    Xm = hcat(X...)
    display(plot(t_vec,Xm',title = "cartpole",
                 xlabel = "time(s)", ylabel = "x",
                 label = ["p" "θ" "ṗ" "θ̇"]))

    # animation stuff
    display(animate_cartpole(X, dt))
    # ---------------tests and plots/animations---------------


end
```




```
┌ Info: Listening on: 127.0.0.1:8718, thread id: 1
└ @ HTTP.Servers /home/rsharde/.julia/packages/HTTP/1EWL3/src/Servers.jl:369
┌ Info: MeshCat server started. You can open the visualizer by visiting the
following URL in your browser:
│ http://127.0.0.1:8718
└ @ MeshCat /home/rsharde/.julia/packages/MeshCat/I6NTX/src/visualizer.jl:63
```

Unable to connect

Firefox can't establish a connection to the server at 127.0.0.1:8718.

- The site could be temporarily unavailable or too busy. Try again in a few moments.
- If you are unable to load any pages, check your computer's network connection.
- If your computer or network is protected by a firewall or proxy, make sure that Firefox is permitted to access the web.

Try Again

```
Test Summary: | Pass  Total
LQR about eq  |    4      4
Test.DefaultTestSet("LQR about eq", Any[], 4, false, false)
```

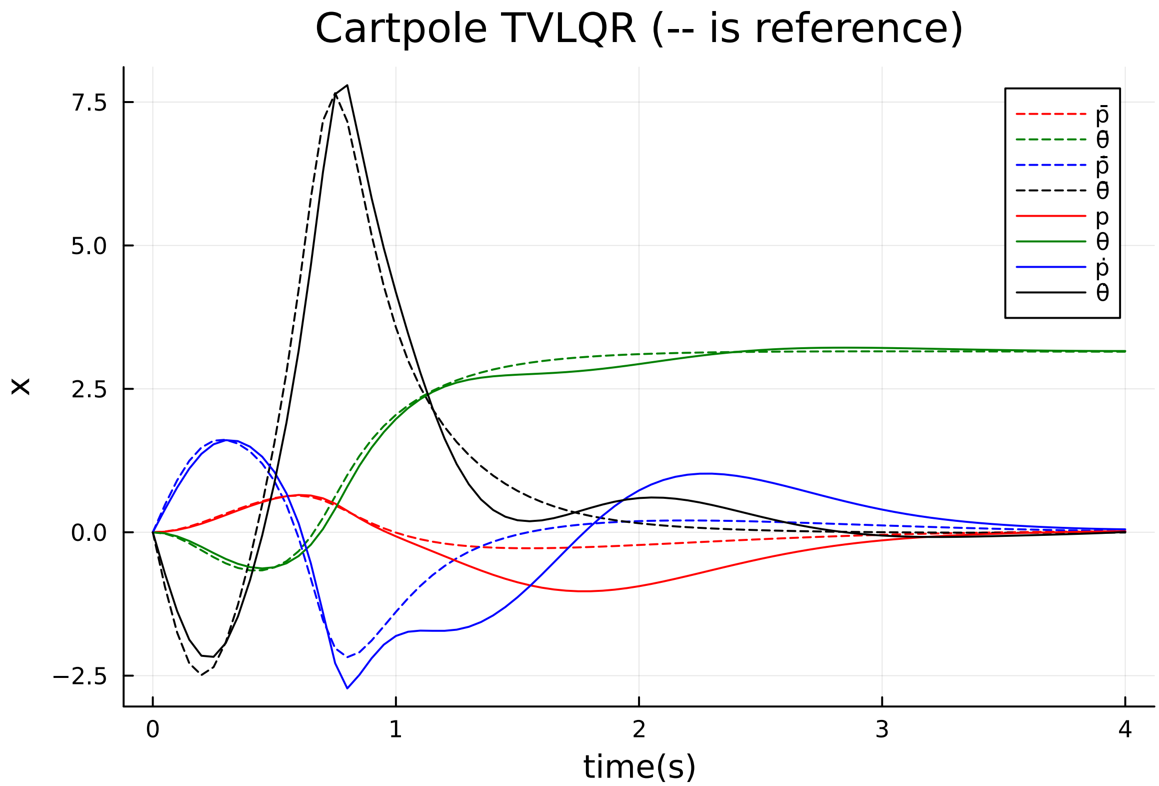
Part D: TVLQR for trajectory tracking (15 pts)

Here we are given a swingup trajectory that works for `params_est`, but will fail to work with `params_real`. To account for this sim to real gap, we are going to track this trajectory with a TVLQR controller.

In [82]:
```
@testset "track swingup" begin

    # optimized trajectory we are going to try and track
    DATA = load(joinpath(@__DIR__,"swingup.jld2"))
    Xbar = DATA["X"]
    Ubar = DATA["U"]

    # states and controls
    nx = 4
    nu = 1

    # problem size
    dt = 0.05
    tf = 4.0
```

```
t_vec = 0:dt:tf
N = length(t_vec)

# states (initial condition of zeros)
X = [zeros(nx) for i = 1:N]
X[1] = [0, 0, 0, 0.0]

# make sure we have the same initial condition
@assert norm(X[1] - Xbar[1]) < 1e-12

# real and estimated params
params_est = (mc = 1.0, mp = 0.2, l = 0.5)
params_real = (mc = 1.2, mp = 0.16, l = 0.55)

# TODO: design a time-varying LQR controller to track this trajectory
# use params_est for your control design, and params_real for the simula

# cost terms
Q = diagm([1,1,.05,.1])
Qf = 10*Q
R = 0.05*diagm(ones(nu))

# TODO: solve for tvlqr gains K
A = [FD.jacobian(dx -> rk4(params_est,dx,Ubar[k],dt), Xbar[k+1]) for k =
B = [FD.jacobian(du -> rk4(params_est,Xbar[k+1],du,dt), Ubar[k]) for k =

# TODO: simulate this controlled system with rk4(params_real, ...)
P = [zeros(nx,nu) for i = 1:N]
K = [zeros(nu,nx) for i = 1:N-1]
P[N] = deepcopy(Qf)

for k = (N-1):-1:1 #Ricatti is calculated backwards in time
    # TODO
    K[k] = (R+B[k]'*P[k+1]*B[k])\(B[k]'*P[k+1]*A[k])
    P[k] = Q+(A[k]'*P[k+1]*(A[k]-B[k]*K[k]))
end

for k = 1:N-1
    X[k+1] = rk4(params_real,X[k],Ubar[k] - K[k]*(X[k]-Xbar[k]),dt)
end

# ---------------tests and plots/animations---------------
xn = X[N]
@test norm(xn)>0
@test 1e-6<norm(xn - Xbar[end])<.2
@test abs(abs(rad2deg(xn[2])) - 180) < 5 # within 5 degrees

Xm = hcat(X...)
Xbarm = hcat(Xbar...)
plot(t_vec,Xbarm',ls=:dash, label = ["p̄" "θ̄" "ṗ̄" "θ̄̇"],lc = [:red :green
display(plot!(t_vec,Xm',title = "Cartpole TVLQR (-- is reference)",
             xlabel = "time(s)", ylabel = "x",
             label = ["p" "θ" "ṗ" "θ̇"],lc = [:red :green :blue :black]))

# animation stuff
display(animate_cartpole(X, dt))
```

```
    # ---------------tests and plots/animations---------------


end
```




```
┌ Info: Listening on: 127.0.0.1:8725, thread id: 1
└ @ HTTP.Servers /home/rsharde/.julia/packages/HTTP/1EWL3/src/Servers.jl:369
┌ Info: MeshCat server started. You can open the visualizer by visiting the
following URL in your browser:
│ http://127.0.0.1:8725
└ @ MeshCat /home/rsharde/.julia/packages/MeshCat/I6NTX/src/visualizer.jl:63
```

Unable to connect

Firefox can’t establish a connection to the server at 127.0.0.1:8725.

- The site could be temporarily unavailable or too busy. Try again in a few moments.
- If you are unable to load any pages, check your computer’s network connection.
- If your computer or network is protected by a firewall or proxy, make sure that Firefox is permitted to access the web.

Try Again

```
Test Summary: | Pass  Total
track swingup |    3      3
Test.DefaultTestSet("track swingup", Any[], 3, false, false)
```

Part E (5 pts): One sentence short answer

1. Will the LQR controller from part A be stable no matter where the cartpole starts?

Yes, because it is designed to stabilize the system around the defined operating point.

2. In order to build an infinite-horizon LQR controller for a nonlinear system, do we always need a state to linearize about?

No.

3. If we are worried about our LQR controller saturating our actuator limits, how should we change the cost?

We could change the cost associated with the control effort, the R in the cost function, to impact the control effort.