Variants of GMRES minimizing the ℓ_1 or ℓ_∞ norms of the residual

Roland Herzog

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We consider the problem of solving $Ax = b$ for some non-singular matrix $A \in \mathbb{R}^{n\times n}$. Let $r_0 = b - Ax_0$ be the initial residual and consider the Arnoldi process (with respect to the Euclidean inner product) generating the basis vectors $\{v_j\}$, starting with $v_1 = r_0/\|r_0\|_2$. These basis vectors span the Krylov subspace $\mathcal{K}_k(A; r_0) = \operatorname{span}\{r_0, A\,r_0, \ldots, A^{k-1}r_0\}$.

1 The ℓ_1 Problem

In each iteration consider the problem

$$\text{Minimize} \quad \|r_0 - AV_k u\|_1, \quad u \in \mathbb{R}^k. \tag{1}$$

This can be recast as the linear program (LP)

$$\begin{aligned} \text{Minimize} \quad & e^\top t, \quad (u,t) \in \mathbb{R}^k \times \mathbb{R}^n \\ \text{s.t.} \quad & -t \le r_0 - AV_k u \le t. \end{aligned} \tag{2}$$

The corresponding dual problem (written as a minimization problem, i.e., with negative objective) is

$$\begin{aligned} \text{Minimize} \quad & r_0^\top z, \quad z \in \mathbb{R}^n \\ \text{s.t.} \quad & \|z\|_\infty \le 1 \\ \text{and} \quad & V_k^\top A^\top z = 0. \end{aligned} \tag{3}$$

Notice that from iteration to iteration the number of variables in (2) grows by one, while in (3) the number of equality constraints grows by one. **TODO: add relation between primal and dual solutions.**

1.1 LP Solver

Let us consider a (primal) simplex method for (2). To this end, recast (2) in normal form:

$$\begin{aligned} \text{Minimize} \quad & e^\top t, \quad (t, s^+, s^-, u^+, u^-) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \\ \text{s.t.} \quad & t - s^+ + AV_k u^+ - AV_k u^- = r_0 \\ \text{and} \quad & -t + s^- + AV_k u^+ - AV_k u^- = r_0 \\ \text{as well as} \quad & t, s^+, s^-, u^+, u^- \ge 0 \end{aligned} \tag{4}$$

We thus have $|B| = 2n$ basic variables and $|N| = n + 2k$ non-basic variables. Notice that when $(t, s^{\pm}, u^{\pm})$ is feasible at iteration k, then simply replacing $u^{\pm}$ by $\binom{u^{\pm}}{0}$ will be feasible at iteration $k+1$ (with the same basis).

Let us describe what a simplex step for (4) looks like in order to be able to subsequently rewrite it directly into a step for problem (2). So let B be a basis and N be the corresponding non-basis.

- **Pricing**
  Determine the reduced cost
  $$\Delta := c - \begin{bmatrix} I & -I & 0 & AV_k & -AV_k \\ -I & 0 & I & AV_k & -AV_k \end{bmatrix}^{\top} \begin{bmatrix} I & -I & 0 & AV_k & -AV_k \\ -I & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,B)}^{-\top} c_B.$$
  If $\Delta \geq 0$, then the current iterate is an optimal solution of (4). Otherwise choose a non-basic index $k \in N$ such that $\Delta_k < 0$.

- **Determination of Step Direction**
  The step direction
  $$d_B := \begin{bmatrix} I & -I & 0 & AV_k & -AV_k \\ -I & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,B)}^{-1} \begin{bmatrix} I & -I & 0 & AV_k & -AV_k \\ -I & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,k)}$$
  describes how the basic variables are changing.

- **Ratio Test**
  Evaluate
  $$\frac{t_i}{d_i}, \quad \frac{s_i^+}{d_i}, \quad \frac{s_i^-}{d_i}, \quad \frac{u_i^+}{d_i}, \quad \frac{u_i^-}{d_i}$$
  for all respective basic indices such that $d_i > 0$. If none exist, stop with *problem unbounded* (cannot happen).

- **Basis Update**
  Update $t^+, s^{\pm}, u^{\pm}$ according to direction d_B on the basic variables etc. Update the basic and non-basic indices B and N.

2 The ℓ_∞ Problem

In each iteration consider the problem

$$\text{Minimize} \quad \|r_0 - AV_k u\|_\infty, \quad u \in \mathbb{R}^k. \tag{5}$$

This can be recast as the linear program (LP)

$$\begin{aligned} \text{Minimize} \quad & t, \quad (u,t) \in \mathbb{R}^k \times \mathbb{R}^1 \\ \text{s.t.} \quad & -t\,e \leq r_0 - AV_k u \leq t\,e \end{aligned} \tag{6}$$

The corresponding dual problem (written as a minimization problem, i.e., with negative objective) is

$$\begin{aligned} \text{Minimize} \quad & r_0^\top z, \quad z \in \mathbb{R}^n \\ \text{s.t.} \quad & \|z\|_1 \leq 1 \\ \text{and} \quad & V_k^\top A^\top z = 0. \end{aligned} \tag{7}$$

Notice that from iteration to iteration the number of variables in (6) grows by one, while in (7) the number of equality constraints grows by one. **TODO: add relation between primal and dual solutions.**

2.1 LP Solver

Let us consider a (primal) simplex method for (6). To this end, recast (6) in normal form:

$$\begin{aligned} \text{Minimize} \quad & t, \quad (t, s^+, s^-, u^+, u^-) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \\ \text{s.t.} \quad & t\,e - s^+ + AV_k u^+ - AV_k u^- = r_0 \\ \text{and} \quad & -t\,e + s^- + AV_k u^+ - AV_k u^- = r_0 \\ \text{as well as} \quad & t, s^+, s^-, u^+, u^- \geq 0 \end{aligned} \tag{8}$$

We thus have $|B| = 2n$ basic variables and $|N| = n + k + 1$ non-basic variables. Notice that when $(t, s^\pm, u^\pm)$ is feasible at iteration k, then simply replacing $u^\pm$ by $\left(\begin{smallmatrix} u^\pm \\ 0 \end{smallmatrix}\right)$ will be feasible at iteration $k+1$ (with the same basis).

Let us describe what a simplex step for (8) looks like in order to be able to subsequently rewrite it directly into a step for problem (6). So let B be a basis and N be the corresponding non-basis.

- **Pricing**
  Determine the reduced cost

  $$\Delta := c - \begin{bmatrix} e & -I & 0 & AV_k & -AV_k \\ -e & 0 & I & AV_k & -AV_k \end{bmatrix}^\top \begin{bmatrix} e & -I & 0 & AV_k & -AV_k \\ -e & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,B)}^{-\top} c_B.$$

  If $\Delta \geq 0$, then the current iterate is an optimal solution of (8). Otherwise choose a non-basic index $k \in N$ such that $\Delta_k < 0$.

- **Determination of Step Direction**
  The step direction

  $$d_B := \begin{bmatrix} e & -I & 0 & AV_k & -AV_k \\ -e & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,B)}^{-1} \begin{bmatrix} e & -I & 0 & AV_k & -AV_k \\ -e & 0 & I & AV_k & -AV_k \end{bmatrix}_{(:,k)}$$

  describes how the basic variables are changing.

- **Ratio Test**
  Evaluate

  $$\frac{t_i}{d_i}, \quad \frac{s_i^+}{d_i}, \quad \frac{s_i^-}{d_i}, \quad \frac{u_i^+}{d_i}, \quad \frac{u_i^-}{d_i}$$

  for all respective basic indices such that $d_i > 0$. If none exist, stop with *problem unbounded* (cannot happen).

- **Basis Update**
  Update $t^+, s^\pm, u^\pm$ according to direction d_B on the basic variables etc. Update the basic and non-basic indices B and N.

References