Diffusion along mean motion resonance for the restricted planar three body problem

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1 Introduction and main result

One central problem in the field of dynamical systems is to understand how stable or unstable the Solar System is. This is a difficult question, even in the Newtonian approximation in the plane:

$$\ddot{q}_i = \sum_{j \neq i} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_i \in \mathbf{R}^2, \quad i = 0, 1, ..., n, \tag{1}$$

with one mass m_0 (thought of as that of the Sun) much larger than the others.

KAM theory asserts that if the masses of planets are small enough, there is a nowhere dense Cantor set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions, in the neighborhood of circular, coplanar, Keplerian motions [Arn63, Féj04]. This is a remarkable confirmation of the idea that Lagrange and Laplace had about the stability of the Solar System, but only for a nowhere dense set, which, from the point of view of topology, is a small set.

According to Kolmogorov and Herman [Her98], the following conjecture is the oldest open problem in dynamical systems.

Conjecture 1.1 ([Her98])**.** *Consider the Hamiltonian System of the n-body problem restricted to an arbitrary level of energy, and let us reparameterize it so that the new flow takes infinite time to go to collisions. Then, the non wandering set of this new Hamiltonian is nowhere dense in the level of energy.*

This conjecture would imply that bounded orbits form a *nowhere dense set* and no topological stability is possible. It is largely confirmed by numerical experiments of Laskar for instance (see [Las10] for a recent account), who has shown that in our solar system collisions between Mars and Venus could occur within a few billion years. The conjectured coexistence of a nowhere dense set of positive measure of bounded quasiperiodic motions with an open and dense set of initial condition with unbounded orbits is remarkable.

1.1 The Restricted elliptic planar three-body problem

Consider (1) with $n = 2$ and one mass equal to zero. We obtain the restricted planar three-body problem. Call the three bodies the Sun (mass of order 1), Jupiter (mass of order $\mu \ll 1$) and the Asteroid (zero mass). The primaries (Sun and Jupiter) are not influenced by the Asteroid. So, if their energy is negative, the first Kepler Law asserts that they describe ellipses with the same eccentricity. Assume that these bodies describe ellipses of eccentricity $e_0 > 0$. Taking a limit as mass of the Asteroid goes to zero we obtain that the motion of the Asteroid under the influence of Sun and Jupiter, is given by the Hamiltonian

$$K(q, p, t) = \frac{\|p\|^2}{2} - \frac{1 - \mu}{\|q + \mu q_0(t)\|} - \frac{\mu}{\|q - (1 - \mu) q_0(t)\|} \tag{2}$$

where $q, p \in \mathbb{R}^2$ and $-\mu q_0(t)$ and $(1 - \mu) q_0(t)$ correspond to elliptic motions of the Sun and Jupiter respectively. Without loss of generality one can assume that $q_0(t)$ has semi major axis equal to 1 and period 2π. For $e_0 > 0$ this system has two and a half degrees of freedom.

For $e_0 = 0$ this is so-called the restricted planar circular three-body problem. Since it becomes autonomous in a frame rotating with the primaries, this system has 2 degrees of freedom and has a conserved quantity called the Jacobi constant, defined by

$$J = \frac{\|p\|^2}{2} - \frac{1-\mu}{\|q+\mu q_0(t)\|} - \frac{\mu}{\|q-(1-\mu)q_0(t)\|} - \frac{\|q\|^2}{2}. \tag{3}$$

Aforementioned KAM theory applies to both the circular and the elliptic problems [SM95] and asserts that if the mass of Jupiter is small enough, there is a set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions.

If Jupiter has a circular motion, since the system has only 2 degrees of freedom, KAM invariant tori are 2-dimensional and separate the 3-dimensional energy surfaces. But in the elliptic problem, a priori 3-dimensional KAM tori do not prevent orbits to wander on a 5-dimensional phase space. The existence of a wide enough set of wandering orbits is a first step towards a proof of conjecture 1.1 for the restricted planar elliptic three-body problem. It is the goal of this paper to achieve such a step.

Let us write Hamiltonian (2) as

$$K(q,p,t) = K_0(q,p) + K_1(q,p,t,\mu)$$

with

$$\begin{aligned} K_0(q,p) &= \frac{\|p\|^2}{2} - \frac{1}{\|q\|} \\ K_1(q,p,t,\mu) &= \frac{1}{\|q\|} - \frac{1-\mu}{\|q+\mu q_0(t)\|} - \frac{\mu}{\|q-(1-\mu)q_0(t)\|}. \end{aligned}$$

One can see that $K_1 = \mathcal{O}(\mu)$ uniformly away from collisions. Then, notice that there is a competition between integrability K_0 and non-integrability K_1, without which there would be no wandering. In this present work we consider a realistic value of the mass ratio, that is $\mu = 10^{-3}$.

Here is the main result of this paper

Theorem 1. *Let us consider the restricted planar elliptic three-body problem with ratio of masses $\mu = 10^{-3}$ and eccentricity of Jupiter e_0. Then, for e_0 small enough there exist $T > 0$ and a trajectory whose osculating eccentricity $e(t)$ satisfies that*

$$e(0) < 0.53 \quad \textit{and} \quad e(T) > 0.67$$

while its osculating semi major axis $a(t)$ remains almost constant, that is,

$$|a(t) - 7^{2/3}| \le C\mu \quad \textit{for } t \in [0,T]$$

for certain constant $C > 0$. In other words,

$$|J(T) - J(0)| > 0.1,$$

where J is the Jacobi constant defined in (3).

When Jupiter performs circular motion the Jacobi constant is an integral of motion and then KAM theory prevents global instabilities. We consider the eccentricity e_0 as a small parameter so that we can compare the dynamics of the elliptic problem with the dynamics of the circular one.

In Theorem 1, we do not know what would happen asymptotically if we let $\mu \to 0$ (our estimates worsen). On the other hand, Theorem 1 holds for realistic values of μ, which is out of reach of many qualitative results of perturbation theory where parameters are convenient assumed to be as small as needed.

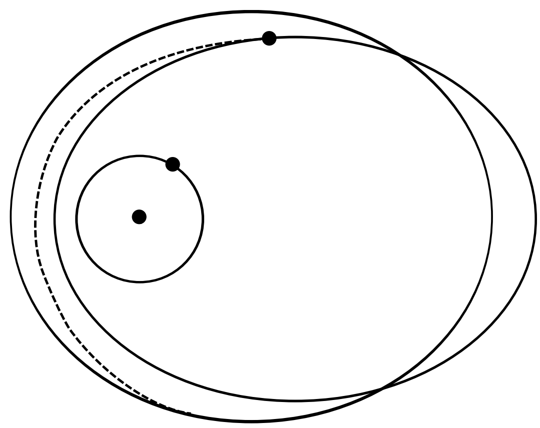

Figure 1: Transition from the osculating ellipse of eccentricty $e = 0.53$ to the ellipse of eccentricty $e = 0.67$. The dashed line schematically shows the transition. Nevertheless, the actual diffusing orbit is very complicated and diffusion is very slow.

1.2 Mechanisms of instability

The result obtained in Theorem 1 gives an example of large instability for this mechanical system. It can be interpreted as an example of Arnold diffusion (see [Arn64]). Nevertheless, Arnold diffusion usually refers to nearly integrable systems whereas Hamiltonian (2) cannot be considered as close to integrable since $\mu = 10^{-3}$. The mechanism of diffusion used in this paper is somewhat similar to the so-called Mather problem ([Mat96, BT99, DdlLS00, Kal03]). This analogy will be specified in Section 3.2.

Arguably, the main source of the existence of instabilities are *resonances*. One of the most natural resonances in the restricted planar elliptic three-body problem is the *mean motion orbital resonances*[1]. For these resonances, from time to time, Jupiter and the Asteroid will be in nearly the same relative position. Thus Jupiter's kicks could pile up and modify the eccentricity of the Asteroid, instead of averaging out. According to third Kepler's Law, these resonances take place when $a^{3/2}$ is close to a rational, where a is the semi major axis of instant ellipse of the Asteroid. In our case we consider $a^{3/2}$ *close to 7*. This resonance is convenient for the proof. Nevertheless, one should expect that the same mechanism takes place for a large number of mean motion orbital resonances.

The semi-major axis a and the eccentricity e describe completely an instant ellipse of the Asteroid (up to orientation). Therefore, geometrically Theorem 1 says that the Asteroid evolves from a (osculating) ellipse of eccentricity $e = 0.53$ to one of eccentricity $e = 0.67$ without changing much the semi major axis (see Figure 1). In Figure 2 we consider the plane (a, e), which describe the ellipse of the Asteroid. Then diffusing orbits given by Theorem 1 correspond to an (almost) horizontal line which indicates the change of shape of the Asteroid's ellipse.

1.3 Main steps of the proof

Diffusing orbit, we are looking for, lie in a neighborhood of a normally hyperbolic invariant cylinder Λ, which is defined by the resonance. The vertical ahd horizontal components of the cylinder can be parameterized by the eccentricity and the mean longitude of the Asteroid.

If the stable and unstable invariant manifolds of Λ intersect transversally, the restricted elliptic three-body problem induces two different dynamics on the cylinder (see Sections 4.5 and 4.6): *the inner and the outer ones*. The inner dynamics is simply the restriction of the Newtonian flow to Λ. The outer dynamics is harder to see. Intuitively it can be observed asymptotically by starting very close from the

[1]The mean motions are the frequencies of the Keplerian revolution of Jupiter and the Asteroid around the Sun: Jupiter make one full revolution when the Asteroid make some integer number of revolutions.

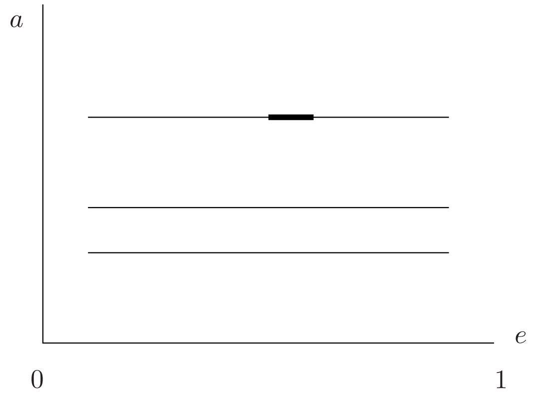

Figure 2: In this graphic we show the diffusion path that we study in the (a, e) plane. The horizontal lines represent the resonances along which we drift. The thicker line is the diffusion path whose existence we are able to prove in this paper.

cylinder and from its unstable manifold, traveling all the way up to a homoclinic intersection, and coming back close to the cylinder and along its stable manifold.

The proof consists in the following four steps:

1. Proving the existence of the normally hyperbolic invariant cylinder Λ.
2. Establishing the transversality of the stable and unstable invariant manifolds of this cylinder.
3. Compare the two types of dynamics on Λ: inner and outer.
4. Construct diffusing orbits by shadowing a carefully chosen composition of the outer and inner maps.

This program faces difficulties at every step. The first one comes from the proper degeneracy of the Newtonian potential: at the limit $\mu = 0$ (no Jupiter), the Asteroid has a one-frequency, Keplerian motion, whereas symplectic geometry would allow for a three-frequency motion (as with any potential other than the Newtonian potential $1/r$ and the elastic potential r^2). Due to this degeneracy, switching to $\mu > 0$ is a singular perturbation. In particular, hyperbolicity is small with respect to μ.

The second step, establishing the transversality of the invariant manifolds of Λ is a delicate problem. If one considers $\mu \ll 1$, the difference between the invariant manifolds becomes exponentially small with respect to μ, that is of order $\exp(-c/\sqrt{\mu})$ for a certain constant $c > 0$. Despite the inordinate amount of time and energy mathematicians have spent trying to estimate the splitting of separatrices in various problems, none of the known techniques apply here. It turns out that computing the splitting of the invariant manifolds we need to evaluate a Poincaré-Melnikov integral. Nevertheless, in this problem certain functions involved in the computation of this integral are not algebraic whereas this is crucial for all existing techniques.

Considering $\mu = 10^{-3}$ avoids this difficulty, since for this value of the parameter one can see that the splitting of separatrices is not extremely small and therefore, can be detected by means of a computer. Moreover, $\mu = 10^{-3}$ is a realistic value of for the Sun-Jupiter model.

Now we turn to the third step. Using classical perturbation methods and the properties of the underlying system one can reduce the inner and outer dynamics to two two-dimensional symplectic maps, which can be written in the following form

$$\mathcal{F}_{e_0}^{\mathrm{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \left(A^+(I,\mu)e^{it} + A^-(I,\mu)e^{-it} \right) + \mathcal{O}\left(\mu e_0^2\right) \\ t + \mu \mathcal{T}_0(I,\mu) + \mathcal{O}(\mu e_0) \end{pmatrix} \tag{4}$$

and

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \left(\Omega^+(I,\mu)e^{it} + \Omega^-(I,\mu)e^{-it} \right) + \mathcal{O}\left(\mu e_0^2\right) \\ t + \mu \omega_0(I,\mu) + \mathcal{O}(\mu e_0) \end{pmatrix}, \tag{5}$$

where (I,t) are conjugate variables which parameterize the 3-dimensional normally hyperbolic invariant cylinder Λ intersected with a certain transversal Poincaré section and $A^{\pm}, \mathcal{T}_0, \Omega^{\pm}, \omega_0$ are analytic functions. Note that these maps are real-analytic and therefore A^- and Ω^- are the complex conjugates of A^+ and Ω^+ respectively.

Then, as we will see in Section 5, the existence of diffusing orbits can be established provided the analytic function

$$\mathcal{K}^+(I,\mu) = \Omega^+(I,\mu) - \frac{e^{i\mu\omega_0(I,\mu)} - 1}{e^{i\mu\mathcal{T}_0(I,\mu)} - 1} A^+(I,\mu).$$

does not vanish for $I \in [I_-, I_+]$. Since $A^{\pm}$ and $\Omega^{\pm}$ are complex conjugate, we do not write the complex conjugate $\mathcal{K}^-(I,\mu)$. Numerically, one can check that $\mathcal{K}^+(I,\mu) \neq 0$ in an interval $I \in [I_-, I_+]$, which reduces to the proof of Theorem 1 to shadowing, made in step 4.

In fact, the complex function $\mathcal{K}^+(I,\mu)$ can be regarded as a 2-dimensional real-valued function depending analytically on (I,μ). If the dependence in μ is non-trivial, $\mathcal{K}^+(I,\mu)$ does not vanish in any point of the interval $[I_-, I_+]$ except for a finite number of μ's.

2 Setting of the problem and notations

The model of the Sun, Jupiter and a massless Asteroid in cartesian coordinates is given by the Hamiltonian (2). First, let us consider the case $\mu = 0$, that is, we consider Jupiter with zero mass. In that case, Jupiter and the Asteroid do not make influence on each other and therefore the system is reduced to two uncoupled 2-body problems, the Sun-Jupiter and the Sun-Asteroid, which are integrable.

We want to study the existence of instability in one particular resonance of this system, which appears when the period of Asteroid is seven times the period of Jupiter. To understand the system better, one can consider action-angle variables for the Sun-Asteroid two-body problem, that is the so-called Delaunay variables, which we denote by $(\ell, L, \hat{g}, G)$. The variable ℓ is the mean anomaly, L is the square of the semi major axis, $\hat{g}$ is the argument of the perihelion and G is the angular momentum. These variables can be obtained from the cartesian coordinates as follows. We define polar coordinates for the position. Namely,

$$q = (r\cos\phi, r\sin\phi).$$

Then, the actions of Delaunay coordinates are defined implicitly by

$$-\frac{1}{2L^2} = \frac{\|p\|^2}{2} - \frac{1}{\|q\|} \tag{6}$$

$$G = -J - \frac{1}{2L^2} \tag{7}$$

Using these actions, the eccentricity of the Asteroid can be expressed as

$$e = \sqrt{1 - \frac{G^2}{L^2}}. \tag{8}$$

To define the angles, we set

$$\phi = v + \hat{g} \tag{9}$$

where v is the true anomaly and $\hat{g}$ is the argument of the perihelion. Then, from v one can obtain the eccentric anomaly u using

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}. \tag{10}$$

From the eccentric anomaly, the mean anomaly is given by the Kepler equation

$$u - e\sin u = \ell. \tag{11}$$

In Delaunay variables Hamiltonian (2) has the form

$$\hat{H}(\ell, L, \hat{g}, G, t) = -\frac{1}{2L^2} + \mu\Delta H_{\mathrm{circ}}(\ell, L, \hat{g} - t, G, t, \mu) + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, \hat{g} - t, G, t, \mu, e_0) \tag{12}$$

It turns out that for $e_0 = 0$, the circular problem only depends on $\hat{g} - t$. To simplify the comparison with the circular problem, we consider rotating Delaunay coordinates, in which ΔH_{circ} is autonomous. This means to define the new angle $g = \hat{g} - t$ and a new variable I conjugate to time t. We have

$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu\Delta H_{\mathrm{circ}}(\ell, L, g, G, \mu) + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, g, G, t, \mu, e_0) + I, \tag{13}$$

In these new variables, the difference of number of degree of freedom of the elliptic and circular problems becomes more apparent. When $e_0 = 0$ the system is autonomous and then I is constant, which corresponds to the conservation of the Jacobi constant (3). Therefore, the circular problem reduces to 2 degrees of freedom.

Recall that we consider the mean motion orbit resonance of the Asteroid and Jupiter. That is, the period of the Asteroid being approximately seven times the period of Jupiter. In rotating Delaunay variables, this corresponds to

$$\dot{\ell} \sim \frac{1}{7} \quad \text{and} \quad \dot{g} \sim -1.$$

This takes place when $L \sim 7^{1/3}$. We study the dynamics in a large neighbourhood of this resoance and we will see that one can drift along it. We find trajectories that keep L close to $7^{1/3}$ while G component changes noticeably. Using (8), one can see that e also changes by an order of one. In this setting, Theorem 1 can be rephrased as follows.

Theorem 2. *There exist $e_0^* > 0$ such that for $0 < e_0 < e_0^*$, there exist $T > 0$ and an orbit of the Hamiltonian System with Hamiltonian* (13) *which satisfies*

$$G(0) < G_0 \text{ and } G(T) > G_1$$

whereas

$$\left| L(t) - 7^{1/3} \right| \le C\mu.$$

By definition the Hamiltonian (13) is autonomous. Therefore, we will restrict ourselves to a level of energy which, without loss of generality can be taken as $H = 0$. Therefore, since $|I - G| = \mathcal{O}(\mu)$, for orbits satisfying $\left|L(t) - 7^{1/3}\right| \le C\mu$, drift in G is equivalent to drift in I. For reasons that will be clear later, in order to prove Theorem 2, we will obtain orbits whose I-component undergoes a change of order 1.

The proof of this theorem is structured as follows.

In Section 3, we study the dynamics in the circular problem, that is $e_0 = 0$ and the underlying Hamiltonian (13) becomes

$$H_{\mathrm{circ}}(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu\Delta H_{\mathrm{circ}}(\ell, L, g, G, \mu). \tag{14}$$

Theorem 3 says that for an interval of Jacobi energies $[J_-, J_+]$ the circular problem has a smooth family of hyperbolic periodic orbits γ_J, whose stable and unstable manifolds intersect transversally for each $J \in [J_-, J_+]$. This theorem implies (Corollary 3.1) existence of a normally hyperbolic invariant cylinder. Later in the section (Subsections 3.1 and 3.2) we calculate the aforementioned circular outer and inner maps (see (4) and (5)).

Then in Section 4 we consider the elliptic case $e_0 > 0$ as a perturbation of the circular case.

Theorem 4 says that the family of periodic orbits $\{\gamma_J\}_{J \in [J_- . J_+]}$ give rise to a normally hyperbolic invariant cylinder Λ_{e_0} whose stable and unstable manifolds intersect transversally for each $J \in [J_- + \delta, J_+ - \delta]$ with small $\delta > 0$. These objects give rise to the inner and outer maps. *Theorem* 5 provides expansions for the inner and outer maps (see formulas (32) and (35) respectively).

Finally, in Section 5 in *Theorem* 6 we complete the proof of Theorem 2. This is done by comparing the inner and outer maps in Lemma 5.1 and constructing a transition chain of tori. It turns out that there are **no large gaps**. This a priori is surprising and contrasts a typical situation of dynamics near a resonance (see e.g. [DdlLS06a]).

Notation 2.1. *From now on, we will omit the dependence on μ. Recall that we are taking a realistic value of $\mu = 10^{-3}$.*

3 The circular problem

The circular problem is given by the Hamiltonian (13) with $e_0 = 0$. Since it does not depend on t, I is an integral of motion. Moreover, since we are studying the dynamics in the energy surface $H = 0$, we have $I = -H_{\mathrm{circ}}(\ell, L, g, G)$. Therefore, the variable I equals the opposite of the Jacobi constant (3). For each level $I =$ constant, one can study the dynamics close to the resonance $7\dot{\ell} + \dot{g} \sim 0$. Since t is a cyclic variable, one can consider the two degree of freedom Hamiltonian of the circular problem for which the conservation of energy corresponds to the conservation of the Jacobi constant (3).

Theorem 3. *Consider the Hamiltonian* (14) *with $\mu = 10^{-3}$. Then, in each energy level $J_- \le J \le J_+$, there exists a hyperbolic periodic orbit $\gamma_J = (L(t), \ell(t), G(t), g(t))$ of period T_J which satisfies*

$$|T_J - 14\pi| < C\mu,$$

smooth in J, and

$$\left| L_J(t) - 7^{1/3} \right| < C\mu$$

for certain constant $C > 0$ and all $t \in \mathbb{R}$.

Moreover, its stable and unstable invariant manifolds $W^s(\gamma_J)$ and $W^u(\gamma_J)$ intersect transversally for each $J \in [J_-, J_+]$.

Proof. Based on convincing numerical data. See Appendix B. □

We will study the elliptic problem as a perturbation of the circular one. Therefore we do not reduce the dimension of the phase space while studying the inner and outer dynamics of the circular problem. Namely, we consider the *Extended Circular Problem* given by the Hamiltonian (13) with $e_0 = 0$. In other words, we keep the conjugate variables (t, I) even if t is a cyclic variable. Consider the energy level $H = 0$. In this setting the conservation of the Jacobi constant corresponds to the conservation of I. Therefore, the periodic orbits obtained in Theorem 3 become invariant two-dimensional tori which belong to hyperplanes $I =$ constant for any $I \in [-J_+, -J_-]$. Moreover, the union of these 2-dimensional invariant tori form a normally hyperbolic invariant manifold.

Corollary 3.1. *The Hamiltonian* (13) *with $\mu = 10^{-3}$ and $e_0 = 0$ has an analytic normally hyperbolic invariant manifold Λ_0, which is foliated by two-dimensional invariant tori. Moreover, there exists a function $\mathcal{G}_0 : \mathbb{T} \times [I_-, I_+] \times \mathbb{T} \to (\mathbb{T} \times \mathbb{R})^3$ which can be expressed in coordinates as*

$$\mathcal{G}_0(g, I, t) = \left(\widetilde{\mathcal{G}}_0(g, I), g, I, t\right) = \left(\mathcal{G}_0^L(g, I), \mathcal{G}_0^\ell(g, I), \mathcal{G}_0^G(g, I), g, I, t\right). \tag{15}$$

that parameterizes Λ_0. Namely,

$$\Lambda_0 = \{\mathcal{G}_0(g, I, t) : (g, I, t) \in \mathbb{T} \times [I_-, I_+] \times \mathbb{T}\}.$$

Moreover, the stable and unstable invariant manifolds of Λ_0 intersect transversally. Let us denote Γ_0 one of these intersections. Then, there exists a function

$$\mathcal{C}_0(g, I, t) : \mathbb{R} \times [I_-, I_+] \times \mathbb{R} \to (\mathbb{T} \times \mathbb{R})^3$$

which parameterizes it and the projection into its (g, I, t)-components are locally one-to-one

$$\Gamma_0 = \left\{ \mathcal{C}_0(g, I, t) = (\mathcal{C}_0^L(g, I), \mathcal{C}_0^\ell(g, I), \mathcal{C}_0^G(g, I), g, I, t) : (g, I, t) \in \mathbb{T} \times [I_-, I_+] \times \mathbb{T} \right\}.$$

Moreover, locally

$$\mathcal{C}_0(g, I, t) = \left(\mathcal{C}_0^L(g, I), \mathcal{C}_0^\ell(g, I), \mathcal{C}_0^G(g, I), g, I, t \right).$$

We define a global Poincaré section and deal with maps, to reduce the dimension by one. There are two natural choices: $\{t = 0\}$ and $\{g = 0\}$, since both variables satisfy $\dot{t} \neq 0$ and $\dot{g} \neq 0$. We choose the section $\{g = 0\}$ and call

$$\mathcal{P}_0 : \{g = 0\} \longrightarrow \{g = 0\} \tag{16}$$

to this Poincaré map. Later we show that one can express the inner and outer maps in the coordinates (I, t), which are conjugate. It turns out that these maps are symplectic and integrable due to the conservation of I. The variables (I, t) can be used as action-angle variables. In this way, it will be easier to understand the influence of the ellipticity. We define the cylinder

$$\widetilde{\Lambda}_0 = \Lambda_0 \cap \{g = 0\}, \tag{17}$$

which is a normally hyperbolic invariant manifold for the Poincaré map $\mathcal{P}_0$.

3.1 The inner map of the circular problem

We first obtain the inner map, which is given by the global Poincaré map restricted to the cylinder (17). Since I is an integral of motion, this inner map is of the form

$$\mathcal{F}_0^{\rm in} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mu \mathcal{T}_0(I) \end{pmatrix}, \tag{18}$$

where the function $\mathcal{T}_0$ is independent of t due to the fact that the map is symplectic. In fact, $14\pi + \mu\mathcal{T}_0(I)$ is the period of the periodic orbit obtained in Theorem 3 on the corresponding energy surface. It can be seen numerically that this map is twist.

Lemma 3.1. *The inner map $\mathcal{F}_0^{\rm in}$ defined in* (18) *is a symplectic twist map, that is*

$$\partial_I \mathcal{T}_0(I) \neq 0 \qquad \text{for } I \in [I_-, I_+].$$

Moreover, the function $\mathcal{T}_0(I)$ satisfies

$$0 < \mu \mathcal{T}_0(I) < \pi. \tag{19}$$

Proof. Based on convincing numerical data. See Appendix B. □

In Section 3.2, the function $\mathcal{T}_0(I)$ will be written by means of an integral (see (26)).

The information contained in this theorem will be crucial in Section 5 to prove the existence of a transition chain of invariant tori.

3.2 The outer map of the circular problem

The outer map has been also called scattering map (see [DdlLS00, DdlLS06a, DdlLS06b, DdlLS08]). Assume that an invariant manifold is normally hyperbolic and its stable and unstable invariant manifolds intersect transversally. Then, the outer map is defined as follows[2].

[2]This definition can be modified to generalize the outer map to any normally hyperbolic invariant manifold with transversal stable and unstable invariant manifolds

Definition 3.1. *Let $\Lambda \subset M$ be a normally hyperbolic invariant manifold for a map $\mathcal{P}$ in a manifold M. Assume that $\gamma \subset W^s_\Lambda \cap W^u_\Lambda$ is a homoclinic manifold and that the intersection of W^s_Λ and W^u_Λ is transversal along γ, that is*

$$\begin{aligned} T_z W^s_\Lambda + T_z W^u_\Lambda &= T_z M, \quad \text{for } z \in \gamma \\ T_z W^s_\Lambda \cap T_z W^u_\Lambda &= T_z \gamma, \quad \text{for } z \in \gamma. \end{aligned}$$

Then, there exists a constant $\lambda > 1$ such that for any two points $x_\pm \in \Lambda$ we say that $\mathcal{F}^{\mathrm{out}}(x_-) = x_+$, if there exists a point $z \in \gamma$ such that

$$\mathrm{dist}\left(\mathcal{P}^n(z), \mathcal{P}^n(x_\pm)\right) < C\lambda^{-|n|} \qquad \text{for all } n \in \mathbb{Z}^\pm.$$

Remark 3.1. *Even if for a fixed x_-, z is not unique, the point x_+ is unique. This makes the outer map well defined.*

In the circular problem, the outer map has the form

$$\mathcal{F}_0^{\mathrm{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mu\omega_0(I) \end{pmatrix}. \tag{20}$$

since I is constant and the outer map is always exact symplectic (see [DdlLS08]).

The outer map can be defined either with discrete or continuous time. Since the Poincaré-Melnikov Theory is considerably simpler for flows than for maps, we compute $\mathcal{F}_0^{\mathrm{out}}$ using continuous time. Moreover, in Section 4.6 we will use also flows to study the outer map of the elliptic problem as a perturbation of (20). The outer map on Λ_0 given by the Hamiltonian (13) with $e_0 = 0$ does not preserve the section $\{g = 0\}$ but the inner map does. In order to fix this problem, we reparameterize the flow so that the inner and outer map preserve this section.

This reparameterization corresponds to identify the variable g with time and is given by,

$$\begin{array}{llll} \frac{d}{ds}\ell = & \frac{\partial_L H}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}} & \frac{d}{ds}L = & -\frac{\partial_\ell H}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}} \\ \frac{d}{ds}g = & 1 & \frac{d}{ds}G = & -\frac{\partial_g H}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}} \\ \frac{d}{ds}t = & \frac{1}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}} & \frac{d}{ds}I = & 0 \end{array} \tag{21}$$

where H is Hamiltonian (13) with $e_0 = 0$. Notice that this reparameterization implies the change of direction of time. However, the geometric objects stay the same. In particular, the new flow still possesses the normally invariant cylinder obtained in Corollary 3.1 and its invariant manifolds.

We will refer to this system as a *reduced circular problem.* We call it reduced because we identify g with the time s. Moreover, the t-component only depends on the other coordinates. Call

$$\Phi_0^{\mathrm{circ}}\{s, (L, \ell, G, g)\} = \left(\Phi_0^L\{s, (L, \ell, G, g)\}, \Phi_0^\ell\{s, (L, \ell, G, g)\}, \Phi_0^G\{s, (L, \ell, G, g)\}, g + s\right), \tag{22}$$

then the outer map can be computed as follows.

Lemma 3.2. *The function $\omega_0(I)$ involved in the definition of the outer map in (20) can be defined as*

$$\omega_0(I) = \omega_0^+(I) - \omega_0^-(I)$$

with

$$\omega_0^\pm(I) = -\lim_{N\to\pm\infty}\left(\int_0^{14N\pi} \frac{\partial_G \Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma, (\mathcal{C}_0^L(I,0), \mathcal{C}_0^\ell(I,0), \mathcal{C}_0^G(I,0)), 0\}\right)}{-1 + \mu\partial_G \Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma, (\mathcal{C}_0^L(I,0), \mathcal{C}_0^\ell(I,0), \mathcal{C}_0^G(I,0), 0)\}\right)} d\sigma + N\mathcal{T}_0(I)\right). \tag{23}$$

where $\mathcal{C}_0$ is the parameterization of the intersection of the invariant manifolds of Λ_0, given in Corollary 3.1 and $\mathcal{T}_0(I)$ is the function in (18).

Note that the minus sign that appears in the definition of the functions $\omega_0^{\pm}(I)$ is due to the fact that the reparameterized flow (21) reverses time.

The geometric interpretation of $\omega_0(I)$ is that the t-shift occurs since the homoclinic orbits approach different points of the same invariant curve in the future and in the past. This shift is equivalent to the shift in t that appears in the Mather Problem [Mat96]. See, for instance, formula (2.1) in Theorem 2.1 of [DdlLS00] and the constants a and b used in formula (1.4) of [BT99].

Proof. Since the t component of the reduced circular system (21) does not depend on t, its behavior is given by

$$\begin{aligned}\Phi_0^t\{s,(L,\ell,G,g,t)\} &= t+\widetilde{\Phi}_0\{s,(L,\ell,G,g)\}\\ &= t+\int_0^s \frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(L,\ell,G,g)\}\right)}d\sigma\end{aligned} \tag{24}$$

Note that, using this flow, the inner map on (18) is just the 14π-time map in the time s. Then, the original period of the periodic orbits obtained in Theorem 3, can be expressed using this new flow as

$$14\pi+\mu\mathcal{T}_0(I) = -\int_0^{14\pi} \frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0)\}\right)}d\sigma. \tag{25}$$

This allows us to define the function $\mathcal{T}_0(I)$ in (18) through integrals as

$$\mathcal{T}_0(I) = -\int_0^{14\pi} \frac{\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\left\{\sigma,\left(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0\right)\right\}\right)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\left\{\sigma,\left(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0\right)\right\}\right)}d\sigma. \tag{26}$$

Consider now a point $(\mathcal{C}_0^L(I,0),\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^G(I,0),0,I,t)$ in $W^u_{\Lambda_0}\cap W^s_{\Lambda_0}\cap\{g=0\}$. Since the first four components are independent of t, $(\mathcal{C}_0^L(I,0),\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^G(I,0),0,I,t)$ is forward asymptotic to a point

$$\left(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0,I,t+\omega_0^+(I)\right)$$

and backward asymptotic to a point

$$\left(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0,I,t+\omega_0^-(I)\right).$$

Using (24), the functions $\omega_0^{\pm}(I)$ can be defined as

$$\begin{aligned}\omega_0^{\pm}(I) = -\lim_{T\to\pm\infty}\int_0^T \Bigg(&\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^L(I,0),\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^G(I,0),0)\}\right)}\\ &-\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0)\}\right)}\Bigg)d\sigma.\end{aligned} \tag{27}$$

Moreover, since the system is 14π-periodic in the time s due to the identification of s with g, it is more convenient to write these in integrals as

$$\begin{aligned}\omega_0^{\pm}(I) = -\lim_{N\to\pm\infty}\int_0^{14N\pi} \Bigg(&\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^L(I,0),\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^G(I,0),0)\}\right)}\\ &-\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{G}_0^L(I,0),\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^G(I,0),0)\}\right)}\Bigg)d\sigma.\end{aligned}$$

Then, taking into account (25), one obtains

$$\begin{aligned}\omega_0^{\pm}(I) = -\lim_{N\to\pm\infty}\Bigg(&\int_0^{14N\pi} \frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^L(I,0),\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^G(I,0),0)\}\right)}d\sigma\\ &+N(14\pi+\mathcal{T}_0(I))\Bigg).\end{aligned}$$

An easy computation leads to formula (23). □

4 The elliptic problem

We devote this section to study the elliptic restricted three body problem. Namely, we obtain perturbative expansions of the inner and outer maps. To this end, we use Poincaré-Melnikov techniques applied to the reduced elliptic problem, which is given by

$$\begin{aligned}
\frac{d}{ds}\ell &= \frac{\partial_L H}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}+\mu e_0\partial_G\Delta H_{\mathrm{ell}}} & \frac{d}{ds}L &= -\frac{\partial_\ell H}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}+\mu e_0\partial_G\Delta H_{\mathrm{ell}}}\\
\frac{d}{ds}g &= 1 & \frac{d}{ds}G &= -\frac{\partial_g H}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}+\mu e_0\partial_G\Delta H_{\mathrm{ell}}}\\
\frac{d}{ds}t &= \frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}+\mu e_0\partial_G\Delta H_{\mathrm{ell}}} & \frac{d}{ds}I &= -\frac{\mu e_0\partial_t\Delta H_{\mathrm{ell}}}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}+\mu e_0\partial_G\Delta H_{\mathrm{ell}}}
\end{aligned} \tag{28}$$

This system is a perturbation of (21). The study of the inner map can be done both dealing with this system or the system associated to the Hamiltonian (12). Nevertheless, to simplify the exposition we use only (28) for both the inner and outer maps. We also consider the Poincaré map associated with this system and the section $\{g=0\}$,

$$\mathcal{P}_{e_0} : \{g=0\} \longrightarrow \{g=0\}, \tag{29}$$

which is a perturbation of (16).

There are two main results in this section:

- existence of a normally hyperbolic invariant cylinder with transversal intersections of invariant manifolds for the elliptic problem (Theorem 4)
- computation of expansions of the inner and outer maps associated to it (Theorem 5).

The above theorems stated in the next Section. Theorem 4 replies on Corollary 3.1, because we study the elliptic problem as a perturbation of the circular one. The proof of Theorem 5 consists of several steps. In Section 4.2 the linear term in e_0 of the elliptic Hamiltonian is computed. Then in Section 4.3 we do perturbative analysis of the normally hyperbolic invariant cylinder Λ_{e_0}. This leads to perturbative in e_0 formulas for the flow restricted to Λ_{e_0} derived in Section 4.4, this is exactly the inner map. Finally, in Section 4.6 we apply the above expansions and compute the outer map using Poincaré-Melnikov techniques. This expansion allows us to give perturbative formulas of the inner and outer maps of the elliptic problem, which are defined in

$$\widetilde{\Lambda}_{e_0} = \Lambda_{e_0} \cap \{g=0\} \tag{30}$$

4.1 The normally hyperbolic invariant cylinder and specific form of the inner and outer maps for the elliptic problem

For e_0 small enough the system associated to the Hamiltonian (13) has a normally hyperbolic invariant cylinder Λ_{e_0}, which is e_0-close to Λ_0 given in Corollary 3.1.

Theorem 4. *For any $\delta>0$, there exists $e_0^*>0$ such that for $0<e_0<e_0^*$ the system associated to Hamiltonian (13) has a normally hyperbolic locally invariant manifold, which is e_0-close in the $\mathcal{C}^1$ sense to Λ_0. Moreover, there exists a constant $C>0$ and a function $\mathcal{G}_{e_0} : \mathbb{T}\times[I_-+\delta, I_+-\delta]\times\mathbb{T} \to (\mathbb{T}\times\mathbb{R})^3$ which can be expressed in coordinates as*

$$\mathcal{G}_{e_0}(g,I,t) = \left(\mathcal{G}^L_{e_0}(g,I,t), \mathcal{G}^\ell_{e_0}(g,I,t), \mathcal{G}^G_{e_0}(g,I,t), g, I, t\right), \tag{31}$$

that parameterizes Λ_{e_0}. In other words Λ_{e_0} is a graph over (g,I,t) defined as

$$\Lambda_{e_0} = \{\mathcal{G}_{e_0}(g,I,t) : (g,I,t)\in\mathbb{T}\times[I_-+\delta, I_+-\delta]\times\mathbb{T}\}.$$

Moreover, the stable and unstable invariant manifolds of Λ_{e_0} intersect transversally and one of these intersections is close in the $\mathcal{C}^1$ sense to the manifold Γ_0 defined in Corollary 3.1. Denote Γ_{e_0} this intersection. There exists a function

$$\mathcal{C}_{e_0}(g,I,t) = (\mathcal{C}^L_{e_0}(g,I,t), \mathcal{C}^\ell_{e_0}(g,I,t), \mathcal{C}^G_{e_0}(g,I,t), g, I, t),$$

which parameterizes it. Namely,

$$\Gamma_{e_0} = \{\mathcal{C}_{e_0}(g, I, t) : (g, I, t) \in \mathbb{T} \times [I_- + \delta, I_+ - \delta] \times \mathbb{T}\}\,.$$

Remark 4.1. *Theorem 4 only guarantees local invariance for Λ_{e_0}. Nevertheless, later in Section 5 we will show the existence of invariant tori in Λ_{e_0}, which will play the role of boundaries of Λ_{e_0}. Thanks to these tori, one can choose Λ_{e_0} to be invariant. For that reason, from now we will refer Λ_{e_0} as a normally hyperbolic invariant manifold.*

We introduce the following notation.

Notation 4.1. *For any function f with 2π-periodic dependence on t, $\mathcal{N}(f)$ is the subset of the integers corresponding to which t-harmonics of f (which may depend on other variables) are non-zero.*

In $\widetilde{\Lambda}_{e_0}$ in (30) one can define an inner and outer map as we have done in $\widetilde{\Lambda}_0$ for the circular problem. Next sections are devoted to the perturbative analysis of these two maps. We state here their main result.

Theorem 5. *The normally hyperbolic invariant manifold $\widetilde{\Lambda}_{e_0}$ in (30) of the Poincaré map $\mathcal{P}_{e_0}$ (29) has associated inner and outer maps. Moreover*

- *The inner map is of the form*

$$\mathcal{F}^{\mathrm{in}}_{e_0} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 A_1(I,t) + e_0^2 A_2(I,t) + \mathcal{O}\left(\mu e_0^3\right) \\ t + \mu \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I,t) + e_0^2 \mathcal{T}_2(I,t) + \mathcal{O}(\mu e_0^2) \end{pmatrix}. \tag{32}$$

  and the functions A_1, A_2 $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$\mathcal{N}\left(A_1\right) = \{\pm 1\},\ \ \mathcal{N}\left(A_2\right) = \{0, \pm 1, \pm 2\} \tag{33}$$

$$\mathcal{N}\left(\mathcal{T}_1\right) = \{\pm 1\},\ \ \mathcal{N}\left(\mathcal{T}_2\right) = \{0, \pm 1, \pm 2\}. \tag{34}$$

- *The outer map is of the form*

$$\mathcal{F}^{\mathrm{out}}_{e_0} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \Omega(I,t) + \mathcal{O}\left(e_0^2\right) \\ t + \mu\omega_0(I) + \mathcal{O}(e_0) \end{pmatrix}. \tag{35}$$

  and the function Ω satisfies

$$\mathcal{N}\left(\Omega\right) = \{\pm 1\}. \tag{36}$$

4.2 The e_0-expansion of the elliptic Hamiltonian

We obtain expansions of ΔH_{ell} in (13) with respect to e_0. These expansions will be used later in Sections 4.1, 4.5 and 4.6. The most important goal is to see which harmonics in t have the e_0 and e_0^2 terms of the elliptic perturbation in (13). Note that the circular problem is independent of t.

We define the function

$$\mathcal{B}(r, v, g, t) = \frac{1}{\left|re^{i(v+g-t)} - r_0(t)e^{iv_0(t)}\right|}. \tag{37}$$

This function is the potential

$$\frac{1}{|q - q_0(t)|}$$

expressed in terms of $g = \hat{g} - t$, where $\hat{g}$ is the argument of the perihelion, the true anomaly v of the Asteroid defined in (9) and the radius r. The functions $r_0(t)$ and $v_0(t)$ are the radius and the true anomaly of Jupiter. The functions $r_0(t)$ and $v_0(t)$ are the only ones involved in the definition of $\mathcal{B}$ which depend on e_0.

Then, the perturbation in (12) can be expressed as

$$\begin{aligned}\mu\Delta H_{\mathrm{circ}}(\ell, L, g, G) + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, g, G, t) = &-\frac{1-\mu}{\mu}\mathcal{B}\left(-\frac{r}{\mu}, v, g, t\right) \\ &-\frac{\mu}{1-\mu}\mathcal{B}\left(\frac{r}{1-\mu}, v, g-t, t\right) + \frac{1}{r}\bigg|_{(r,v)=(r(\ell,L,G),v(\ell,L,G))}\end{aligned}$$

First we study the expansion of $\mathcal{B}$, and from it, we deduce the one of ΔH_{ell}.

Lemma 4.1. *Consider the expansion*

$$\mathcal{B}(r,v,g,t)=\mathcal{B}_0(r,v,g)+e_0\mathcal{B}_1(r,v,g,t)+e_0^2\mathcal{B}_2(r,v,g,t)+\mathcal{O}\left(e_0^3\right) \tag{38}$$

of the function $\mathcal{B}$ *defined in* (37). *Then,*

- $\mathcal{B}_0$ *satisfies* $\mathcal{N}(\mathcal{B}_0)=\{0\}$.
- $\mathcal{B}_1$ *satisfies* $\mathcal{N}(\mathcal{B}_1)=\{\pm 1\}$ *and is given by*

$$\mathcal{B}_1(r,v,g,t)=-\frac{1}{2\Delta^3(r,v,g)}\left(2\cos t-3r\cos(v+g+t)+r\cos(v+g-t)\right), \tag{39}$$

  where

$$\Delta(r,v,g)=\left(r^2+1-2r\cos(v+g)\right)^{1/2}.$$

- $\mathcal{B}_2$ *satisfies* $\mathcal{N}(\mathcal{B}_2)=\{0,\pm 1,\pm 2\}$.

Note that the restricted elliptic three body problem is a peculiar perturbation of the circular problem in the sense that the k-th e_0-order has t-harmonics up to order k. This fact will be crucial when we will compare the inner and outer dynamics in Section 5.

Proof. To prove this lemma we look for the e_0-expansions of the functions $r_0(t)$ and $v_0(t)$ involved in the definition of $\mathcal{B}$ in (37). We obtain them using the eccentric, true and mean anomalies of Jupiter.

From the relation $t=u_0-e_0\sin u_0$ (see (11)), one can obtain that

$$u_0(t)=t+e_0\sin t+\frac{e_0^2}{2}\sin 2t+\mathcal{O}\left(e_0^3\right).$$

Then, using $r_0=1-e_0\cos u_0$,

$$r_0(t)=1-e_0\cos t+e_0^2\sin^2 t+\mathcal{O}\left(e_0^3\right).$$

For the eccentric anomaly we can use

$$\tan\frac{v_0}{2}=\sqrt{\frac{1+e_0}{1-e_0}}\tan\frac{u_0}{2}$$

(see (66)), to obtain

$$v_0=u_0+e_0\sin u_0+e_0^2\left(\frac{9}{2}\sin u_0-2\sin 2u_0\right)+\mathcal{O}\left(e_0^3\right)$$

and then

$$v_0(t)=t+2e_0\sin t+e_0^2\left(\frac{9}{2}\sin t-\sin 2t\right)+\mathcal{O}\left(e_0^3\right).$$

Plugging $r_0(t)$ and $v_0(t)$ into (37), it can be easily seen that the expansion (38) satisfies all the properties of $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ stated in the lemma. $\square$

From the expansion of $\mathcal{B}$, one can easily deduce the expansion of ΔH_{ell} in (13).

Corollary 4.1. *Let us consider the* e_0*-expansion of the function* ΔH_{ell} *in* (13),

$$\Delta H_{\mathrm{ell}}=\Delta H_{\mathrm{ell}}^1+e_0\Delta H_{\mathrm{ell}}^2+\mathcal{O}\left(e_0^2\right).$$

Then, the first order is given by

$$\begin{aligned}\Delta H_{\mathrm{ell}}^1(L,\ell,G,g,t)=&-\frac{1-\mu}{\mu}\mathcal{B}_1\left(-\frac{r(L,\ell,G)}{\mu},v(L,\ell,G),g,t\right)\\&-\frac{\mu}{1-\mu}\mathcal{B}_1\left(\frac{r(L,\ell,G)}{1-\mu},v(L,\ell,G),g,t\right).\end{aligned} \tag{40}$$

where $\mathcal{B}_1$ *is the function defined in Lemma 4.1 and therefore it satisfies*

$$\mathcal{N}\left(\Delta H_{\mathrm{ell}}^1\right)=\{\pm 1\}.$$

The second order $\Delta H_{\mathrm{ell}}^2$ *satisfies*

$$\mathcal{N}\left(\Delta H_{\mathrm{ell}}^2\right)=\{0,\pm 1,\pm 2\}.$$

4.3 Perturbative analysis of the normally hyperbolic invariant manifold

The theory of normally hyperbolic invariant manifolds [Fen74, Fen77] ensures the existence of a map $\mathcal{G}_{e_0}$ parameterizing the normally hyperbolic manifold. It can be made unique imposing

$$\pi_g \mathcal{G}_{e_0}(g, I, t) = g,\ \pi_I \mathcal{G}_{e_0}(g, I, t) = I,\ \pi_t \mathcal{G}_{e_0}(g, I, t) = t,$$

where π_* is the projection with respect to the corresponding component of the function. Moreover, it satisfies the invariance equation

$$\mathcal{H}_{e_0} \circ \mathcal{G}_{e_0} = D\mathcal{G}_{e_0} \mathcal{R}_{e_0} \tag{41}$$

where $\mathcal{H}_{e_0}$ is the vector field given in (28) and $\mathcal{R}_{e_0}$ is the vector field induced in Λ_{e_0}.

Since we have regularity with respect to parameters, the invariance equation allows us to obtain expansions of both the parameterization of Λ_{e_0} and the induced flow $\mathcal{R}_{e_0}$ with respect to e_0. Let us expand $\mathcal{G}_{e_0}$ and $\mathcal{R}_{e_0}$ as

$$\mathcal{G}_{e_0} = \mathcal{G}_0 + e_0 \mathcal{G}_1 + e_0^2 \mathcal{G}_2 + \mathcal{O}\left(e_0^3\right) \tag{42}$$

$$\mathcal{R}_{e_0} = \mathcal{R}_0 + e_0 \mathcal{R}_1 + e_0^2 \mathcal{R}_2 + \mathcal{O}\left(e_0^3\right). \tag{43}$$

Then, $\mathcal{G}_0$ is the function defined in (15) and $\mathcal{R}_0$ is the flow induced by the reduced circular problem in Λ_0. Note that the e_0-expansion of $\mathcal{H}_{e_0}$ can be deduced from the one of ΔH_{ell} obtained in Section 4.2. The first orders of $\mathcal{R}_{e_0}$ will play a significant role in Sections 4.4 and 4.5.

Lemma 4.2. *The expansion of the function $\mathcal{R}_{e_0}$ in (43) satisfies that:*

- *The e_0-order $\mathcal{R}_1$ satisfies $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$. Moreover, its I-component is given by*

$$\mathcal{R}_1^I(g, I, t) = -\left.\frac{\mu \partial_t \Delta H_{\mathrm{ell}}^1(L, \ell, G, g, t)}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(L, \ell, G, g)}\right|_{(L,\ell,G)=\widetilde{\mathcal{G}}_0(g,I)} \tag{44}$$

  where $\widetilde{\mathcal{G}}_0$ is the function given in Corollary 3.1.

- *The e_0^2-order term $\mathcal{R}_2$ satisfies $\mathcal{N}(\mathcal{R}_2) = \{0, \pm 1, \pm 2\}$.*

Proof. Expanding equation (41) with respect to e_0, we have that the first terms satisfy

$$\mathcal{H}_0 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_0 \tag{45}$$

$$\left(D\mathcal{H}_0 \circ \mathcal{G}_0\right) \mathcal{G}_1 + \mathcal{H}_1 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_1 + D\mathcal{G}_1 \mathcal{R}_0 \tag{46}$$

$$\left(D\mathcal{H}_0 \circ \mathcal{G}_0\right) \mathcal{G}_2 + \frac{1}{2}\left(D^2\mathcal{H}_0 \circ \mathcal{G}_0\right) \mathcal{G}_1^{\otimes 2} + \left(D\mathcal{H}_1 \circ \mathcal{G}_0\right) \mathcal{G}_1 + \mathcal{H}_2 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_2 + D\mathcal{G}_1 \mathcal{R}_1 + D\mathcal{G}_2 \mathcal{R}_0 \tag{47}$$

By the uniqueness condition, $\mathcal{G}_1$ is of the form

$$\mathcal{G}_1(g, I, t) = \left(\widetilde{\mathcal{G}}_1(g, I, t), 0, 0, 0\right)$$

with $\widetilde{\mathcal{G}}_1(g, I, t) = (\mathcal{G}_1^L(g, I, t), \mathcal{G}_1^\ell(g, I, t), \mathcal{G}_1^G(g, I, t))$.

Taking into account (21) and (45), $\mathcal{R}_0$ is given by

$$\mathcal{R}_0(g, I) = \begin{pmatrix} \mathcal{R}_0^g(g, I) \\ \mathcal{R}_0^I(g, I) \\ \mathcal{R}_0^t(g, I) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \left.\dfrac{1}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(L, \ell, G, g)}\right|_{(L,\ell,G)=\widetilde{\mathcal{G}}_0(g,I)} \end{pmatrix} \tag{48}$$

where $\widetilde{\mathcal{G}}_0$ is the function defined in Corollary 3.1.

The other orders can be solved iteratively. Since

$$D\mathcal{G}_0 = \begin{pmatrix} D\widetilde{\mathcal{G}}_0 \\ \mathrm{Id} \end{pmatrix} \text{ and } D\mathcal{G}_i = \begin{pmatrix} D\widetilde{\mathcal{G}}_i \\ 0 \end{pmatrix} \quad \text{for } i \geq 1, \tag{49}$$

we have that

$$\mathcal{R}_1^* = \pi_* \left((D\mathcal{H}_0 \circ \mathcal{G}_0)\, \mathcal{G}_1 + \mathcal{H}_1 \circ \mathcal{G}_0 \right)$$

Replacing this into (46) we obtain an equation for $\mathcal{G}_1$. The equation for every Fourier t-coefficient is uncoupled, and therefore, since in Section 4.2 we have seen that $\mathcal{N}(\mathcal{H}_1) = \{\pm 1\}$, using the uniqueness of $\mathcal{G}_1$, one can deduce that $\mathcal{N}(\mathcal{G}_1) = \{\pm 1\}$. As a consequence we have that $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$.

Reasoning analogously, since in Section 4.2 we have seen that $\mathcal{N}(\mathcal{H}_2) = \{0, \pm 1, \pm 2\}$, one can see that $\mathcal{N}(\mathcal{G}_2) = \{0, \pm 1, \pm 2\}$ and $\mathcal{N}(\mathcal{R}_2) = \{0, \pm 1, \pm 2\}$.

Finally, taking into account (45), (49) , the e_0-expansion of the Hamiltonian given in Corollary 4.1 and (28), one can easily see that (44). □

4.4 Perturbative analysis of the inner flow

To study the inner map perturbatively, we need first to study the first orders of the flow restricted to the cylinder Λ_{e_0}. Namely, the flow $\Phi_{e_0}^{\mathrm{in}}\{s, (g, I, t)\}$ associated to the vector field $\mathcal{R}_{e_0}$. We use the perturbative expansions of $\mathcal{G}_{e_0}$ and $\mathcal{R}_{e_0}$ studied in Section 4.1.

Lemma 4.3. *The flow $\Phi_{e_0}^{\mathrm{in}}\{s, (g, I, t)\}$ has a perturbative expansion*

$$\Phi_{e_0}^{\mathrm{in}}\{s, (g, I, t)\} = \Phi_0^{\mathrm{in}}\{s, (g, I, t)\} + e_0 \Phi_1^{\mathrm{in}}\{s, (g, I, t)\} + e_0^2 \Phi_2^{\mathrm{in}}\{s, (g, I, t)\} + \mathcal{O}\left(e_0^3\right),$$

which satisfies

$$\mathcal{N}\left(\Phi_1^{\mathrm{in}}\{s, (g, I, t)\}\right) = \{\pm 1\} \tag{50}$$

$$\mathcal{N}\left(\Phi_2^{\mathrm{in}}\{s, (g, I, t)\}\right) = \{0, \pm 1, \pm 2\}. \tag{51}$$

Proof. The zero order $\Phi_0^{\mathrm{in}}\{s, (g, I, t)\}$ associated to $\mathcal{R}_0$ corresponds to the flow of the reduced circular problem (21) restricted to Λ_0. Taking into account (48), it is of the form

$$\Phi_0^{\mathrm{in}}\{s, (g, I, t)\} = \left(g + s, I, \pi_t \Phi_0^{\mathrm{in}}\{s, (g, I, t)\}\right). \tag{52}$$

Its first two components are independent of t and its third component satisfies

$$\pi_t \Phi_0^{\mathrm{in}}\{s, (g, I, t)\} = t + \widetilde{\Phi}_0^{\mathrm{in}}\{s, (g, I)\} \tag{53}$$

with $\widetilde{\Phi}_0^{\mathrm{in}}$ independent of t.

We first prove (50). The e_0-order Φ_1^{in} is a solution of the ordinary differential equation

$$\frac{d}{ds}\xi = D\mathcal{R}_0\left(\Phi_0^{\mathrm{in}}\{s, (g, I, t)\}\right)\xi + \mathcal{R}_1\left(\Phi_0^{\mathrm{in}}\{s, (g, I, t)\}\right)$$

with initial condition $\xi(0) = (0, 0, 0)$. $\mathcal{R}_0$ is independent of t and therefore, by (52),

$$D\mathcal{R}_0\left(\Phi_0^{\mathrm{in}}\{s, (g, I, t)\}\right) = D\mathcal{R}_0\left(g + s, I\right),$$

which is independent of t too. In Lemma (4.2), we have seen $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$ and therefore $\mathcal{R}_1$ can be written as

$$\mathcal{R}_1(g, I, t) = \mathcal{R}_1^+(g, I)e^{it} + \mathcal{R}_1^-(g, I)e^{-it},$$

Therefore, using (53),

$$\mathcal{R}_1\left(\Phi_0^{\mathrm{in}}\{s, (g, I, t)\}\right) = \left(\mathcal{R}_1^+\left(g + s, I\right) e^{i\widetilde{\Phi}_0^{\mathrm{in}}\{s,(g,I)\}}\right) e^{it} + \left(\mathcal{R}_1^-\left(g + s, I\right) e^{i\widetilde{\Phi}_0^{\mathrm{in}}\{s,(g,I)\}}\right) e^{-it}.$$

To prove (50), it is enough to use variation of constants formula. Let us consider $M_{g,I}(s)$ the fundamental matrix of the linear ode

$$\frac{d}{ds}\xi = D\mathcal{R}_0(g + s, I)\xi,$$

then

$$\Phi_1^{\rm in}\{s,(g,I,t)\} = \Phi_1^{{\rm in},+}\{s,(g,I)\}e^{it} + \Phi_1^{{\rm in},-}\{s,(g,I)\}e^{-it}$$

with

$$\Phi_1^{{\rm in},\pm}\{s,(g,I)\} = M_{g,I}(s)\int_0^s M_{g,I}(\sigma)\left(\mathcal{R}_1^{\pm}(g+\sigma,I)\,e^{\pm i\widetilde{\Phi}_0^{\rm in}\{s,(g,I)\}}\right)d\sigma.$$

The proof of (51) follows the same lines. Indeed, $\Phi_2^{\rm in}$ is solution of an ode of the form

$$\frac{d}{ds}\xi = D\mathcal{R}_0\left(g+s,I\right)\xi + \Xi(s,g,I,t)$$

with initial condition $\xi(0)=(0,0,0)$. The function Ξ is given in terms of the previous orders of $\mathcal{R}_{e_0}$ and $\Phi_{e_0}^{\rm in}$ as

$$\Xi = \frac{1}{2}D^2\mathcal{R}_0\left(\Phi_0^{\rm in}\right)\left(\Phi_1^{\rm in}\right)^{\otimes 2} + D\mathcal{R}_0\left(\Phi_0^{\rm in}\right)\Phi_2^{\rm in} + D\mathcal{R}_1\left(\Phi_0^{\rm in}\right)\Phi_1^{\rm in} + \mathcal{R}_2\left(\Phi_0^{\rm in}\right),$$

and then it satisfies $\mathcal{N}(\Xi)=\{0,\pm1,\pm2\}$.

Since the homogeneous linear ode is the same as for $\Phi_1^{\rm in}$ and does not depend on t, one can easily see that (51) is satisfied. $\square$

4.5 The inner map of the elliptic problem

In this section we study the inner map of the elliptic problem perturbatively from (18), taking e_0 as a small parameter. We use the e_0-expansions obtained in Sections 4.1 and 4.4.

The inner map is defined in the cylinder $\widetilde{\Lambda}_{e_0}$ in (30). We call it $\mathcal{F}_{e_0}^{\rm in}:\mathcal{C}_{e_0}\to\mathcal{C}_{e_0}$, and is defined as the (-14π)-Poincaré map of the inner flow $\Phi_{e_0}^{\rm in}$ defined in Lemma 4.3. Namely,

$$\mathcal{F}_{e_0}^{\rm in}\begin{pmatrix} I \\ t\end{pmatrix} = \begin{pmatrix} \Phi_{e_0}^{{\rm in},I}\{-14\pi,(0,I,t)\} \\ \Phi_{e_0}^{{\rm in},t}\{-14\pi,(0,I,t)\}\end{pmatrix} \tag{54}$$

We want to see which t-harmonics the first orders of the map has and also we want to compute the first order of the I component. The results stated in Theorem 5 about the inner map are summarized in the next lemma. We also show how to compute the first order term.

Lemma 4.4. *The inner map* (54) *is of the form,*

$$\mathcal{F}_{e_0}^{\rm in}:\begin{pmatrix} I \\ t\end{pmatrix}\mapsto\begin{pmatrix} I + e_0A_1(I,t) + e_0^2A_2(I,t) + \mathcal{O}\left(\mu e_0^3\right) \\ t + \mu\mathcal{T}_0(I) + e_0\mathcal{T}_1(I,t) + e_0^2\mathcal{T}_2(I,t) + \mathcal{O}(\mu e_0^2)\end{pmatrix}. \tag{55}$$

and functions A_1, A_2 $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$\mathcal{N}\left(A_1\right)=\{\pm1\},\ \ \mathcal{N}\left(A_2\right)=\{0,\pm1,\pm2\} \tag{56}$$

$$\mathcal{N}\left(\mathcal{T}_1\right)=\{\pm1\},\ \ \mathcal{N}\left(\mathcal{T}_2\right)=\{0,\pm1,\pm2\}. \tag{57}$$

Moreover A_1 can be split as,

$$A_1(I,t) = A_1^+(I)e^{it} + A_1^-(I)e^{-it},$$

with

$$A^{\pm}(I) = -\mu\int_0^{-14\pi}\frac{\partial_t\Delta H_{\rm ell}^{1,\pm}\left(\widetilde{\mathcal{G}}_0(s,I),s\right)}{-1+\mu\partial_G\Delta H_{\rm circ}\left(\widetilde{\mathcal{G}}_0(s,I),s\right)}e^{\pm i\widetilde{\Phi}_0^{\rm in}\{s,(0,I)\}}ds, \tag{58}$$

where the functions $\Delta H_{\rm ell}^{1,\pm}$ are defined as

$$\Delta H_{\rm ell}^1(L,\ell,G,g,t) = \Delta H_{\rm ell}^{1,+}(L,\ell,G,g)e^{it} + \Delta H_{\rm ell}^{1,\pm}(L,\ell,G,g)e^{-it},$$

and $\widetilde{\mathcal{G}}_0$ and $\widetilde{\Phi}_0^{\rm in}$ have been defined in Theorem 3 and (53) *respectively.*

Proof. Using the e_0-expansion of the inner flow given in Lemma 4.3, we have that

$$A_1(I,t) = \Phi_1^{\mathrm{in},I}\{-14\pi,(0,I,t)\},\ A_2(I,t) = \Phi_2^{\mathrm{in},I}\{-14\pi,(0,I,t)\}$$
$$\mathcal{T}_1(I,t) = \Phi_1^{\mathrm{in},t}\{-14\pi,(0,I,t)\},\ \mathcal{T}_2(I,t) = \Phi_2^{\mathrm{in},t}\{-14\pi,(0,I,t)\}.$$

Therefore, by (50) and (51), these functions satisfy (56) (57).

It only remains to compute explicitly $A_1(I,t)$. Using fundamental theorem of calculus and using the expansions of $\mathcal{R}_{e_0}^I$ and $\Phi_{e_0}^{\mathrm{in},I}$ given in Lemmas 4.2 and 4.3, one can see that

$$A_1(I,t) = \int_0^{-14\pi} \mathcal{R}_1^I\left(\Phi_0^{\mathrm{in}}\{s,(0,I,t)\}\right) ds.$$

Moreover,taking into account (52) and (53),

$$A_1(I,t) = \int_0^{-14\pi} \mathcal{R}_1^I\left(s, I, t+\widetilde{\Phi}_0^{\mathrm{in}}\{s,(0,I)\}\right) ds.$$

Since $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$, it can be split as

$$\mathcal{R}_1^I(g,I,t) = \mathcal{R}_1^{I,+}(g,I)e^{it} + \mathcal{R}_1^{I,-}(g,I)e^{-it}.$$

Therefore, A_1 can also be split as,

$$A_1(I,t) = A_1^+(I)e^{it} + A_1^-(I)e^{-it}.$$

To compute each harmonic, it is enough to compute the following integrals which are independent on t,

$$A^{\pm}(I) = \int_0^{-14\pi} \mathcal{R}_1^{I,+}(s,I)e^{\pm i\widetilde{\Phi}_0^{\mathrm{in}}\{s,(0,I)\}} ds.$$

Taking into account the definition of $\mathcal{R}_1^I$ in (44) one obtains (58).

□

4.6 The outer map of the elliptic problem

We devote this section to study the outer map

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \widetilde{\Lambda}_{e_0} \longrightarrow \widetilde{\Lambda}_{e_0} \tag{59}$$

for $e_0 > 0$.

Theorem 4 gives the existence of $\mathcal{C}_{e_0}$, a transversal intersection of the invariant manifolds of Λ_{e_0}. The Poincaré map (29) possesses then also a normally hyperbolic invariant manifold $\widetilde{\Lambda}_{e_0} = \Lambda_{e_0}$ and its invariant manifolds intersect transversally at $\widetilde{\Gamma}_{e_0} = \Gamma_{e_0} \cap \{g = 0\}$.

Then, Definition 3.1 also applies in this setting and we can define the outer map $\mathcal{F}_{e_0}^{\mathrm{out}}$. We want to study it as a perturbation of the outer map of the circular problem given in (20). To this end we use Poincaré-Melnikov techniques. As we have explained in Section 3.2, the original flow associated to Hamiltonian (12) does not allow us to study perturbatively $\mathcal{F}_{e_0}^{\mathrm{out}}$. Instead, we use the reduced elliptic problem defined in (28).

The results stated in Theorem 5 about the outer map are summarized in the next lemma. We also show how to compute the first order term.

Lemma 4.5. *The outer map defined in* (59) *has the following expansion with respect to* e_0*,*

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0\left(\Omega^+(I)e^{it} + \Omega^-(I)e^{-it}\right) + \mathcal{O}\left(e_0^2\right) \\ t + \mu\omega_0(I) + \mathcal{O}(e_0) \end{pmatrix}. \tag{60}$$

Moreover, the functions $\Omega^{\pm}(I)$ can be defined as

$$\Omega^{\pm}(I) = -\lim_{T\to\infty}\int_0^T \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\widetilde{\Phi}_0)\left\{s,\left(\mathcal{C}_0^L(0,I),\mathcal{C}_0^\ell(0,I),\mathcal{C}_0^G(0,I),0\right)\right\}}$$
$$- \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm i\left(t+\omega_0^+(I)\right)}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\widetilde{\Phi}_0)\left\{s,\left(\mathcal{G}_0^L(0,I),\mathcal{G}^\ell(0,I),\mathcal{G}_\ell^G(0,I),0\right)\right\}} ds$$
$$+\lim_{T\to-\infty}\int_0^T \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\widetilde{\Phi}_0)\left\{s,\left(\mathcal{C}_0^L(0,I),\mathcal{C}_0^\ell(0,I),\mathcal{C}_0^G(0,I),0\right)\right\}}$$
$$- \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm i\left(t+\omega_0^-(I)\right)}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\widetilde{\Phi}_0)\left\{s,\left(\mathcal{G}_0^L(0,I),\mathcal{G}^\ell(0,I),\mathcal{G}_\ell^G(0,I),0\right)\right\}} ds, \tag{61}$$

where

$$\Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G,t) = \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)e^{it} + \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)e^{-it}$$

has been defined in Corollary 4.1 and Φ_0^{circ} and $\widetilde{\Phi}_0$ have been defined in (22) *and* (24) *respectively.*

Proof. To prove the lemma we use the Definition 3.1 of outer map. Let us consider points $z \in \widetilde{\Gamma}_{e_0}$ and $x_\pm \in \widetilde{\Lambda}_{e_0}$ such that

$$\mathrm{dist}\left(\mathcal{P}_{e_0}^n(z),\mathcal{P}_{e_0}^n(x_\pm)\right) < C\lambda^{-|n|} \qquad \text{for } n\in\mathbb{Z}^{\pm}$$

for certain constants $C>0$ and $\lambda>1$. Using the parameterizations of $\widetilde{\Gamma}_{e_0}$ and $\widetilde{\Lambda}_{e_0}$ given in Theorem 4, we can write the points z and $x_\pm$ in coordinates as $z=\mathcal{C}_{e_0}(0,I^*,t^*)$ and $x_\pm = \mathcal{G}_{e_0}(0,I_\pm^*,t_\pm^*)$. Then, the I-component of the outer map is just given by the relation

$$\mathcal{F}_{e_0}^{\mathrm{out},I}(I_-^*,t_-^*) = I_+^* = I_-^* + (I_+^* - I_-^*).$$

To measure $I_+^*-I_-^*$ we first consider $I^*-I_\pm^*$. Define the flow Φ_{e_0} associated to the reduced elliptic problem (28). Then, applying Fundamental Theorem of Calculus

$$I^*-I_\pm^* = -\lim_{T\to\pm\infty}\int_T^0 \left(\frac{d}{ds}\Phi_{e_0}\left\{s,\mathcal{C}_{e_0}(0,I^*,t^*)\right\} - \frac{d}{ds}\Phi_{e_0}\left\{s,\mathcal{G}_{e_0}(0,I_\pm^*,t_\pm^*)\right\}\right) ds$$

Note that the minus sign in front of the integral comes from the fact that system (28) has time reversed.

Using the perturbative expansions of $\mathcal{C}_{e_0}$ and Λ_{e_0} given in Theorem 4, (28) and also the perturbative expansion of the Hamiltonian (12) given in Corollary 4.1, one can see that

$$I^*-I_\pm^* = e_0\lim_{T\to\pm\infty}\int_T^0 \left(\frac{\mu\partial_t\Delta H_{\mathrm{ell}}^1(\ell,L,g,G,t)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\left\{s,\mathcal{C}_0(0,I_-^*,t_-^*)\right\}}\right.$$
$$\left. - \frac{\mu\partial_t\Delta H_{\mathrm{ell}}^1(\ell,L,g,G,t)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}(\ell,L,g,G)}\Big|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\left\{s,\mathcal{G}_0\left(0,I_-^*,t_-^*+\omega_0^{\pm}(I_-^*)\right)\right\}}\right) ds + \mathcal{O}\left(e_0^2\right).$$

where Φ_0^{circ} and $\widetilde{\Phi}_0$ have been defined in (22) and (24) respectively.

Taking into account that in Corollary 4.1 we have proved that $\Delta H_{\mathrm{ell}}^1$ satisfies $\mathcal{N}\left(\Delta H_{\mathrm{ell}}^1\right) = \{\pm 1\}$, one can easily obtain (60). □

5 Existence of diffusing orbits

The main result of this section is the following

Theorem 6. *For any $\delta > 0$ there exists $e_0^* > 0$ and $C > 0$ such that for any $0 < e_0 < e_0^*$ there is a collection of invariant 2-dimensional tori $\{\tau_i\}_{i=1}^N \subset \tilde{\Lambda}_{e_0}$ such that*

- *$\tau_1 \cap \{I = I_- + \delta\} \neq \emptyset$ and $\tau_N \cap \{I = I_+ - \delta\} \neq \emptyset$.*
- *Hausdorff dist$(\tau_i, \tau_{i+1}) < C e_0^{3/2}$*
- *These tori form a transition chain. Namely, $W^u_{\tau_i} \pitchfork W^s_{\tau_{i+1}} \neq \emptyset$ for each $i = 1, \ldots, N-1$.*

Proof. Once we have computed the first orders in e_0 of both the outer and inner maps, we want to understand their properties and compare their dynamics. To make this comparison we first perform two steps of averaging [AKN88, LM88]. This change of coordinates straightens the I-component of the inner map at order $\mathcal{O}(e_0^3)$ in such a way that with the new system of coordinates is easier to compare the dynamics of both maps.

Lemma 5.1. *There exists a e_0-close to the identity symplectic change of variables*

$$(I, t) = (\overline{I}, \overline{t}) + e_0 \varphi(\overline{I}, \overline{t}, e_0) \tag{62}$$

defined on $\widetilde{\Lambda}_{e_0}$, which transforms the inner and outer maps $\mathcal{F}^{\mathrm{in}}_{e_0}$ in (32) and $\mathcal{F}^{\mathrm{out}}_{e_0}$ in (35) into

$$\widetilde{\mathcal{F}}^{\mathrm{in}}_{e_0} : \begin{pmatrix} \overline{I} \\ \overline{t} \end{pmatrix} \mapsto \begin{pmatrix} \overline{I} + \mathcal{O}\left(\mu e_0^3\right) \\ \overline{t} + \mu \mathcal{T}_0\left(\overline{I}\right) + e_0^2 \widetilde{\mathcal{T}}_2\left(\overline{I}\right) + \mathcal{O}\left(\mu e_0^3\right) \end{pmatrix} \tag{63}$$

and

$$\widetilde{\mathcal{F}}^{\mathrm{out}}_{e_0} : \begin{pmatrix} \overline{I} \\ \overline{t} \end{pmatrix} \mapsto \begin{pmatrix} \overline{I} + e_0 \widetilde{\Omega}(\overline{I}, \overline{t}) + \mathcal{O}\left(\mu e_0^2\right) \\ \overline{t} + \mu \omega_0(\overline{I}, \mu) + \mathcal{O}(\mu e_0) \end{pmatrix}, \tag{64}$$

where

$$\widetilde{\Omega}\left(\overline{I}, \overline{t}\right) = \widetilde{\Omega}^+\left(\overline{I}\right) e^{i\overline{t}} + \widetilde{\Omega}^-\left(\overline{I}\right) e^{-i\overline{t}}$$

with

$$\widetilde{\Omega}^\pm\left(\overline{I}\right) = \Omega^\pm\left(\overline{I}\right) - \frac{e^{\pm i\omega_0(\overline{I})} - 1}{e^{\pm i \mathcal{T}_0(\overline{I})} - 1} A_1^\pm\left(\overline{I}\right).$$

Moreover, these functions satisfy

$$\widetilde{\Omega}^\pm\left(\overline{I}\right) \neq 0 \qquad \text{for } \overline{I} \in [I_-, I_+]. \tag{65}$$

Note that we can do two steps of averaging globally in the whole cylinder $\widetilde{\Lambda}_{e_0}$ due to the absence of resonances in the first orders in e_0. Namely, there are no *big gaps*. This is quite in contrast with the typical situation in Arnold diffusion (see e.g. [DdlLS06a]).

Proof. We perform two steps of averaging. To this end we consider a generating function of the form

$$\mathcal{S}(\overline{I}, t) = \overline{I} t + e_0 \mathcal{S}_1(\overline{I}, t) + e_0^2 \mathcal{S}_2(\overline{I}, t),$$

which defines the change (62) implicitly as

$$\begin{aligned} I &= \overline{I} + e_0 \partial_t \mathcal{S}_1(\overline{I}, t) + e_0^2 \partial_t \mathcal{S}_2(\overline{I}, t) \\ \overline{t} &= t + e_0 \partial_{\overline{I}} \mathcal{S}_1(\overline{I}, t) + e_0^2 \partial_{\overline{I}} \mathcal{S}_2(\overline{I}, t). \end{aligned}$$

By Theorem 3.1 we have that (19) and by Theorem 5 we know which t-harmonics have the functions A_i and $\mathcal{T}_i$. Then, it is easy to see that the functions $\mathcal{S}_i$ that correspond to two steps of averaging are globally defined in $\widetilde{\Lambda}_{e_0}$. Moreover, in these new variables, taking into account that the inner map is exact symplectic, one can see that the inner map is of the form (63).

To obtain a perturbative expression for the outer map $\widetilde{\mathcal{F}}^{\mathrm{out}}_{e_0}$, we need to know $\mathcal{S}_1$ explicitly. It is given by

$$\mathcal{S}_1(\overline{I}, t) = -\frac{i A_1^+(\overline{I})}{e^{i\mu\mathcal{T}_0(\overline{I})} - 1} e^{it} + \frac{i A_1^-(\overline{I})}{e^{-i\mu\mathcal{T}_0(\overline{I})} - 1} e^{-it}.$$

If we apply this change to the outer map $\mathcal{F}^{\mathrm{out}}_{e_0}$ in (60), we obtain (64).

The statement (65) is based on convincing numerical data (???). □

In the new coordinates $(\overline{I},\overline{t})$ the inner map $\widetilde{\mathcal{F}}_{e_0}^{\rm in}$ in (63) is a e_0^3-close to integrable map. Moreover, thanks to Theorem 3.1 is twist and therefore we can apply KAM theory to proof the existence of invariant curves in $\widetilde{\Lambda}_{e_0}$. We use a version of the KAM Theorem from [DdlLS00] (see also [Her83]).

KAM theorem. *Let $f:[0,1]\times\mathbb{T}\to[0,1]\times\mathbb{T}$ be an exact symplectic $\mathcal{C}^l$ map with $l>4$. Assume that $f=f_0+\delta f_1$, where $f_0(I,\psi)=(I,\psi+A(I))$, A is $\mathcal{C}^l$, $|\partial_I A|>M$ and $\|f_1\|_{\mathcal{C}^l}\leq 1$. Then, if $\delta^{1/2}M^{-1}=\rho$ is sufficiently small, for a set of ω of Diophantine numbers of exponent $\theta=5/4$, we can find invariant tori which are the graph of $\mathcal{C}^{l-3}$ functions u_ω, the motion on them is $\mathcal{C}^{l-3}$ conjugate to the rotation by ω, and $\|u_\omega\|_{\mathcal{C}^{l-3}}\leq C\delta^{1/2}$.*

Applying this theorem to the map $\widetilde{\mathcal{F}}_{e_0}^{\rm in}$ we obtain the KAM tori. Moreover, this theorem ensures that the distance between these tori is no bigger than $e_0^{3/2}$. Then, the results of Lemma 5.1 and KAM theorem lead to the existence of a transition chain of invariant tori.

The transition chain is obtained comparing the outer and inner dynamics. We do this comparison in the coordinates $(\overline{I},\overline{t})$ given by Lemma 5.1 and therefore we deal with the maps $\widetilde{\mathcal{F}}_{e_0}^{\rm in}$ and $\widetilde{\mathcal{F}}_{e_0}^{\rm out}$ in (63) and (64) respectively.

KAM theorem ensures that there exists a torus τ_1 such that $\tau_1\cap\{I=I_--\delta\}\neq\emptyset$. Then, thanks to (65), $\mathcal{F}_{e_0}^{\rm out}(\tau_1)$ satisfies

$$\mathrm{dist}\left(\tau_1,\mathcal{F}_{e_0}^{\rm out}(\tau_1)\right)\geq Ce_0$$

for a constant $C>0$ independent of e_0. Then, KAM theorem ensures that there exists a torus τ_2 such that $\tau_2\cap\mathcal{F}_{e_0}^{\rm out}(\tau_1)\neq 0$. Iterating this procedure, one obtains the transition chain. $\square$

To finish the proof of Theorem 2 it only remains to prove the existence of a diffusing orbit using a shadowing method. We use a result in [FM00].

Lemma 5.2. *Let f be a $\mathcal{C}^1$ sympletic map in a $2(d+1)$ symplectic manifold. Assume that the map leaves invariant a $\mathcal{C}^1$ d-dimensional torus τ and the motion on the torus is an irrational rotation. Let Γ be a $d+1$ manifold intersecting W_τ^u transversally. Then,*

$$W_\tau^s\subset\overline{\bigcup_{i>0}f^{-i}(\Gamma)}.$$

An immediate consequence of this lemma is that any finite transtion chain can be shadowed by a true orbit, which finishes the proof of Theorem 2.

A From cartesian to Delaunay and computation of $\partial_G\Delta H_{\rm circ}$

In this appendix we explain an easy way to obtain the rotating Delaunay coordinates from rotating cartesian (or polar) coordinates in the circular problem. First recall that G can be computed as

$$G=r\left(-p_x\sin\phi+p_y\cos\phi\right).$$

In a fixed level of energy J, knowing G one can obtain L. Recall that

$$J=-\frac{1}{2L^2}-G+\mu\Delta H_{\rm circ}$$

The term $\mu\Delta H_{\rm circ}$ in cartesian only depends on x and y, and can be easily computed. Then, one can use this equation to obtain L. Knowing L and G we can obtain the eccentricity e by

$$e=\sqrt{1-\frac{G^2}{L^2}}.$$

Using that $r=L^2(1-e\cos u)$ one can obtain u and from here ℓ using the formula $u-e\sin u=\ell$. On the other hand, from u we can obtain v using

$$\tan\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}.$$

Finally, we can deduce g using that $\phi = v + g$.

We devote the rest of the section to compute $\partial_G \Delta H_{\mathrm{circ}}$. Let us define

$$N(r, v, g) = \frac{1}{(r^2 + 1 - 2r\cos(v+g))^{1/2}}.$$

Then,

$$\Delta H_{\mathrm{circ}}(L, \ell, G, g) = -\frac{1-\mu}{\mu} N\left(\frac{r(L,\ell,G)}{\mu}, v(L,\ell,G), g\right) - \frac{\mu}{1-\mu} N\left(\frac{r(L,\ell,G)}{1-\mu}, v(L,\ell,G), g\right) + \frac{1}{r(L,\ell,G)}.$$

Therefore

$$\begin{aligned}
\partial_G \Delta H_{\mathrm{circ}}(L,\ell,G,g) = -&\left[\frac{1-\mu}{\mu^2}\partial_r N\left(\frac{r(L,\ell,G)}{\mu}, v(L,\ell,G), g\right)\right.\\
&\left.+\frac{\mu}{(1-\mu)^2}\partial_r N\left(\frac{r(L,\ell,G)}{1-\mu}, v(L,\ell,G), g\right)\right]\partial_G r(L,\ell,G)\\
-&\left[\frac{1-\mu}{\mu}\partial_v N\left(\frac{r(L,\ell,G)}{\mu}, v(L,\ell,G), g\right)\right.\\
&\left.+\frac{\mu}{1-\mu}\partial_v N\left(\frac{r(L,\ell,G)}{1-\mu}, v(L,\ell,G), g\right)\right]\partial_G v(L,\ell,G)\\
-&\frac{1}{r^2}\partial_G r(L,\ell,G).
\end{aligned}$$

In only remains to compute $\partial_G r$ and $\partial_G v$. First, let us point out that

$$\partial_G e = -\frac{G}{eL^2} = \frac{e^2-1}{eG}.$$

On the other hand, using that $\ell = u - e\sin u$, one has that

$$\partial_e u = \frac{\sin u}{1 - e\cos u}.$$

Then, since $r(L, e, \ell) = L^2(1 - e\cos u(e, \ell))$, using that

$$\cos v = \frac{\cos u - e}{1 - e\cos u}, \tag{66}$$

we have that

$$\partial_e r(L, e, \ell) = L^2 \cos v$$

and therefore,

$$\partial_G r(L, \ell, G) = -\frac{G\cos v}{e}.$$

To compute $\partial_G v$ let us point out that $\partial_G v = \partial_e v \partial_G e$. Therefore it only remains to compute $\partial_e v$, we obtain it using formula (66) and

$$\sin v = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}.$$

Then,

$$\partial_e v = \frac{\sin v}{1-e^2}\left(2 + e\cos v\right).$$

and therefore,

$$\partial_G v = -\frac{\sin v}{eG}\left(2 + e\cos v\right).$$

B Numerical study of the normally hyperbolic invariant manifold of the circular problem.

We devote this appendix to show the numerical study of the hyperbolic invariant manifold of the circular problem given in Corollary 3.1 and its invariant manifolds. In other words, we show numerical results which justify the properties of the circular problem stated in Theorem 3.

B.1 Computation of the periodic orbits

To compute the periodic orbit we use rotating cartesian coordinates, since the Hamiltonian is explicit in these coordinates

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{67}$$

where r_1 and r_2 are the distance between Asteroid and Sun and Jupiter respectively.

To look for the periodic orbit we first fix the energy level $H = -1.6$ and use as a first approximation the resonant periodic orbit of the Hamiltonian (14) with $\mu = 0$. Note that the change from (rotating) Delaunay to (rotating) cartesian has been explained in Appendix A. Then, we use a continuation method to obtain the periodic orbit for the other energy levels.

We consider the Poincaré section $\{y = 0\}$. In rotating coordinates, this section corresponds to collinear configurations of the three bodies. Since we have conservation of energy and $\partial_{p_y} H \neq 0$, we can eliminate the variable p_y. In fact, given x, p_x , y and H, we just need to solve a quadratic equation for p_y. We obtain a 2-dimensional symplectic map at each energy level. Then, looking for the periodic orbit is reduced to obtain a fixed point of the map $P_J(x, p_x)$, which is the 12-th iterate of the Poincaré map (recall that we are in rotating coordinates).

A point is assumed to be on the Poincare section if it is within distance 10^{-15} to the section. We ask for an accuracy of 10^{-14} in finding the fixed point, i.e. a point $p = (x, p_x)$ is accepted as a fixed point if the distance between the point and its Poincaré iterate is $\mathrm{dist}(P_J(p), p) < 10^{-14}$. Since the RPC3BP in rotating cartesian is symmetric with respect to the horizontal axis (see (67)), it is only necessary to integrate for half a period of the periodic orbit. We look for the fixed point using a Newton-like method. Using this procedure we are able to obtain the resonant periodic orbit for energy levels $H \in [-2.05, -1.5]$

As a side product, we also compute the integration time needed to reach the Poincaré section. This number is close to the period of the resonant periodic orbit of the unperturbed system, that is $14\pi = 43.98229715025710533816$ and its properties will be studied in Appendix D.

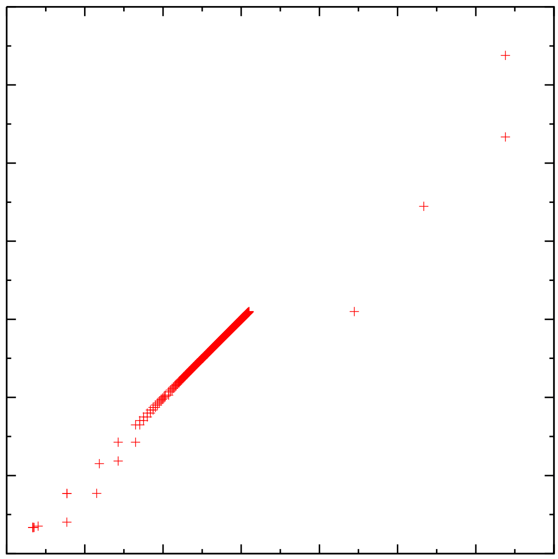

Figure 3: In this figure we show the convergence of the Newton method for obtaining the periodic orbit. It corresponds to the level of energy $H = -1.6$ and it is shown in a $\log - \log$ scale.

Finally, as a test, we check the convergence of the iterative method, which should be quadratic. The Newton-like method looks for a zero of the distance function $d(p) = P_J(p) - p$. A zero of this distance function corresponds to a fixed point in the Poincaré section, i.e. a periodic orbit for the RC3BP flow. Let d_k be the value of the distance function at the k-th step of the Newton iteration. In Figure 3 we show the convergence of d_k towards 0 in a $\log - \log$ scale for the level of energy $H = -1.6$. The slope gives the rate of convergence. The figure shows quadratic convergence for the first few iterates. After the first few iterates, for precisions $d_k < 10^{-8}$, the rate of convergence is not quadratic anymore. Most probably, this is due to the fact that, for very high precision, roundoff error dominates over the error corrected by Newton method.

B.2 Computation of the invariant manifolds and the transversal homoclinic points

In this section, we compute the stable and unstable invariant manifolds associated to the periodic orbits found in the previous section. We continue using the Poincaré map and therefore we are looking for (one dimensional) invariant manifolds of a hyperbolic fixed point at each energy level.

We first look for them at the energy level $H = -1.6$. Then, we use continuation techniques to obtain them for the other energy levels. We compute also the angle between the invariant manifolds at one of the transversal intersections.

In the level of energy $H = -1.6$, the eigenvalues of the saddle of the Poincaré map obtained in the previous section are λ and λ^{-1} with $\lambda = 2.041166124027443$. Note that one would expect that we are in a nearly integrable regime since μ is small. Then one would expect the eigenvalues to be close to 1. Nevertheless, in this problem seems that the non-integrability appears very fast when one increases μ so that for $\mu = 10^{-3}$ the eigenvalues are already far from 1.

Let us now compute the unstable manifold $W^u(p)$. Let h be a small displacement in the unstable direction. We will grow the manifold from the linear segment between the points $p+hv_u$ and $P(p+hv_u)$, where v_u is the unstable eigenvalue of the saddle. We call fundamental domain to this segment. We discretize it into a mesh of points and iterate them by the Poincare map.

Remark B.1. *In practice, we choose the small increment h such that the estimated error commited in the linear approximation of the manifold is very smaller than 10^{-10}.*

Points in the unstable manifold are iterated forwards by the Poincaré map. Points in the stable manifold are iterated forwards by the inverse Poincare map. In Figure 4 we show the invariant manifolds in the level of energy $H = -1.6$. Notice that both manifolds have a sharp fold. For clarity, we only grow the manifolds up to this fold. Figure 5 shows a closeup near the "turning point".

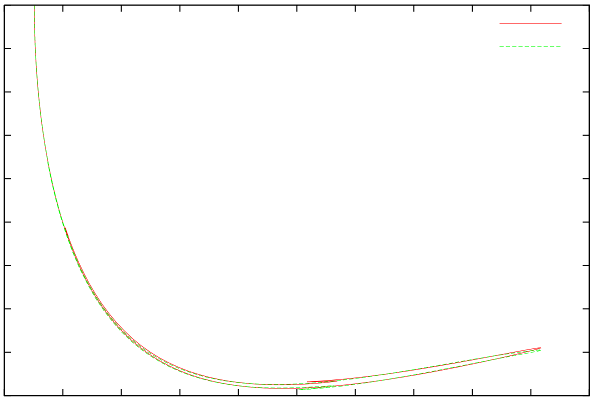

Figure 4: Invariant manifolds associated to the fixed point of the Poincaré map of the level of energy $H = -1.6$: unstable in red, stable in green.

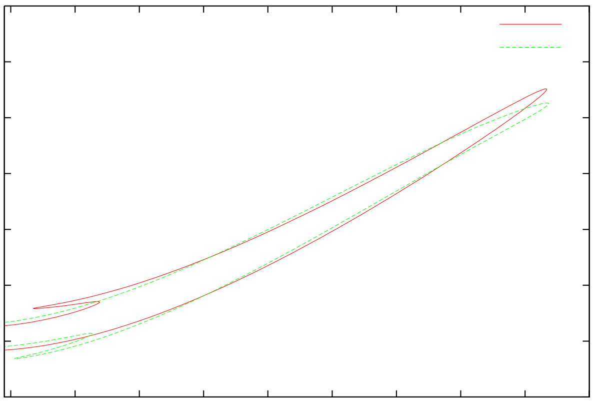

Figure 5: Zoom invariant manifolds associated to the fixed point of the Poincaré map of the level of energy $H = -1.6$ close to the turning point: unstable in red, stable in green.

The estimated error of the manifold is computed in the following way. Let λ_u be the unstable eigenvalue, and v_u the corresponding eigenvector. Let P be the Poincaré map and its inverse. Let p be the fixed point. If h is a small displacement in the unstable direction, then

$$P(p + hv_u) = p + \lambda_u h v_u + \mathcal{O}\left(h^2\right).$$

The size of the term $\mathcal{O}(h^2)$ can be estimated as

$$||P(p + hv_u) - p - \lambda_u h v_u||.,$$

and similarly, in the stable case, as

$$||P(p + hv_s) - p - \frac{1}{\lambda_s} h v_s||.$$

Then, in the level of energy $H = -1.6$ we make an estimate error $4.842651 \cdot 10^{-9}$.

Next step is to compute the intersection point of the unstable manifold with the stable manifold. We look for the intersection point that is closer to the "turning point" of the manifolds seen in Figure 5. This should be the most numerically stable one to compute, since it is "half way" in the homoclinic orbit.

The method goes as follows. We consider first the unstable manifold. We consider the fundamental segment between $p + hv_u$ and $P(p + hv_u)$ defined above. We look for the first natural n such that the two points $P^n(p + hv_u)$ and $P^{n+1}(p + hv_u)$ are at different sides of the fold (see Figure 5). In our case we need $n = 20$ iterations for the unstable direction and $m = 20$ iterations for the stable direction from the respective fundamental domains.

Then, we use a standard numerical method (Newton-like multidimensional root finding) to find two points p_u, p_s in the respective fundamental segments such that $P^n(p_u) = P^{-m}(p_s)$ within a 10^{-7} tolerance.

Finally, to obtain the homoclinic orbit by concatenating the forward orbit of z (which we iterate $n = 10$ times by the Poincaré map), and the backwards orbit of z (which we iterate $m = 10$ times) (see Figures 6 and 7).

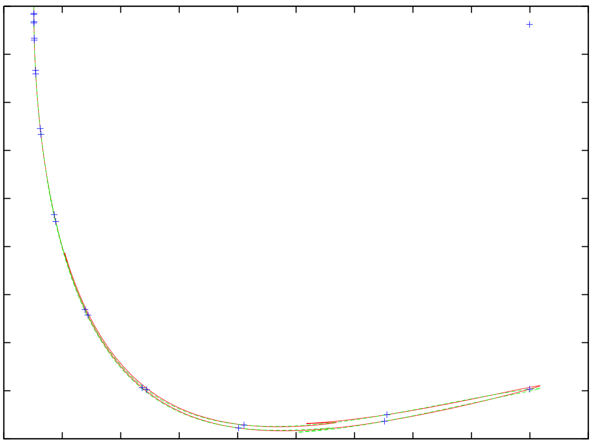

Figure 6: Homoclinic orbit superimposed to the invariant manifolds.

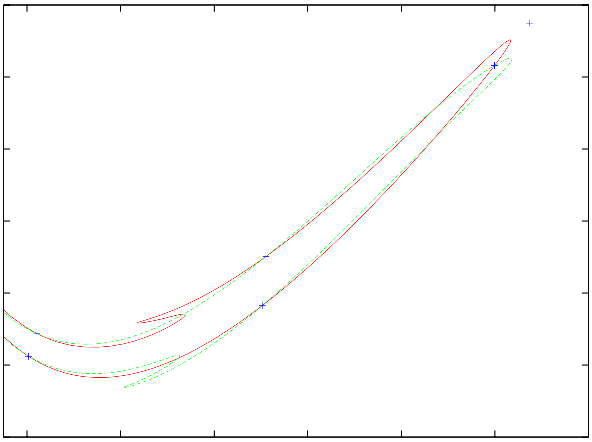

Figure 7: Closeup of homoclinic orbit superimposed to the invariant manifolds.

Once we have obtained the transversal homoclinic point for the energy level $H = -1.6$, we use continuation techniques. In Figures 8,9 and 10 we show the angle between the invariant manifolds at the homoclinic point as the energy changes. It can be seen that it is non-zero for the range in energy $H \in [-1.74, -1.56]$.

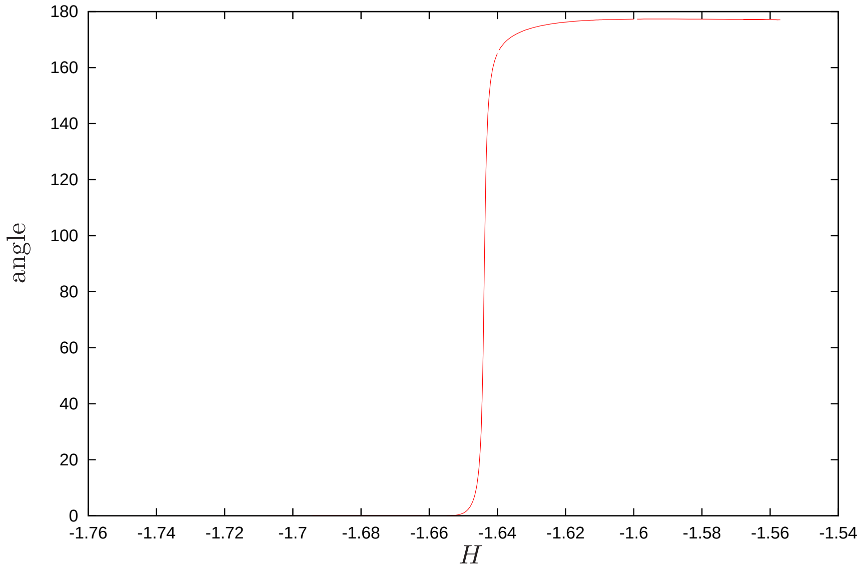

Figure 8: In this figure we depict how the angle between the invariant manifolds changes as we change the energy. Note that the angle changes drastically when the energy decreases, so that for energies below to $H = -1.65$ is impossible to detect whether is positive. Look at Figures 9 and 10 to see closeups of this graphic

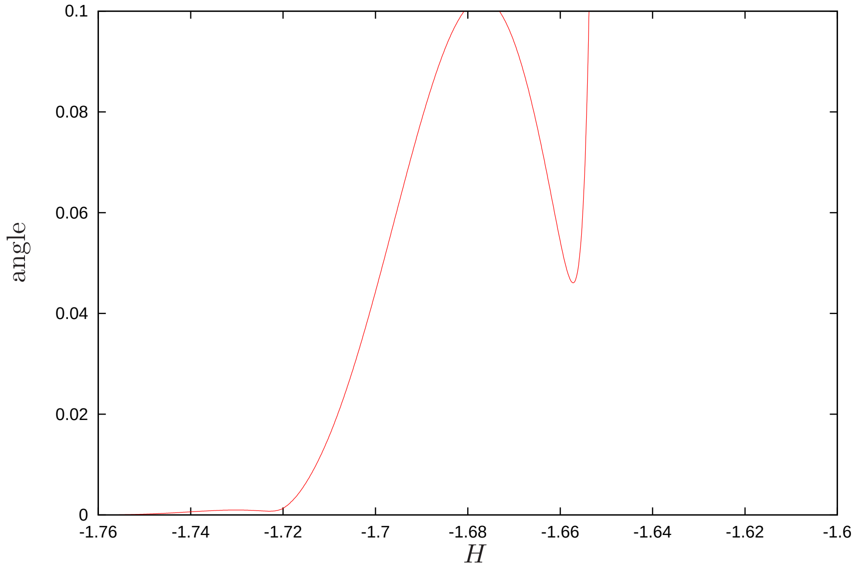

Figure 9: First closeup fo the angle between the invariant manifolds at the homoclinic point.

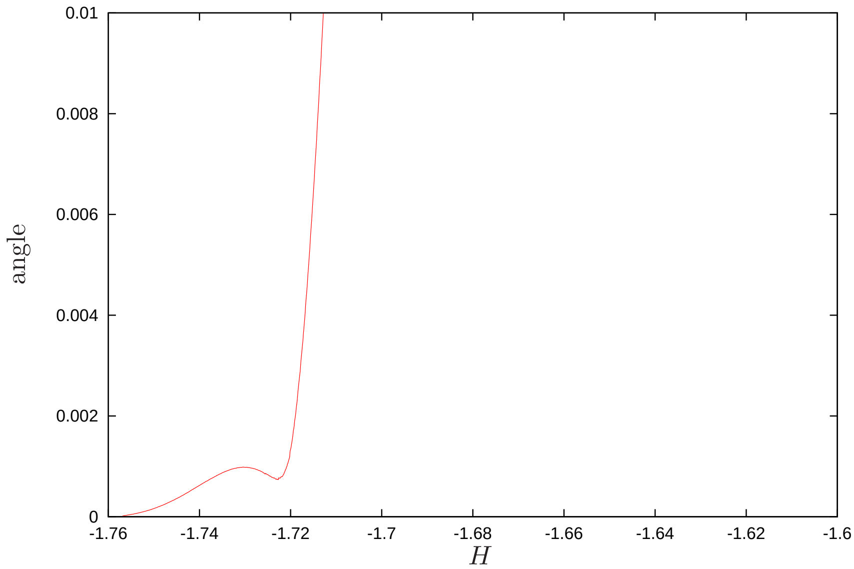

Figure 10: Second closeup fo the angle between the invariant manifolds at the homoclinic point.

C Numerical study of the inner dynamics of the circular problem

In Lemma 3.1 there are stated certain properties of the inner map of the circular problem (18). In Section 3.1 it has been explained that $14\pi + \mu\mathcal{T}_0(I)$ is the period of the periodic orbit obtained in Theorem 3. Therefore, the properties of $\mathcal{T}_0(I)$ are an outcome of the study of the periodic orbits done in Appendix B.

In Figure 11, it can be clearly seen that in the range $I \in (1.5, 1.9)$, $\mathcal{T}_0(I)$ satisfies the properties stated in Lemma 3.1.

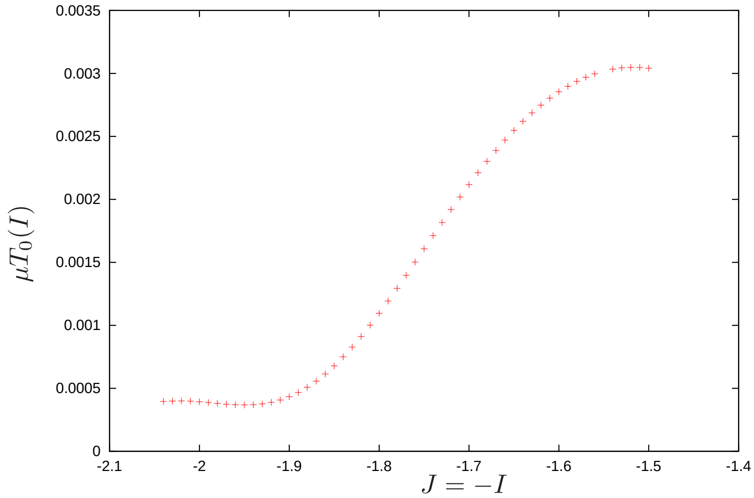

Figure 11: Graph of $\mathcal{T}_0(I)$. In the horizontal axis there is the Jacobi constant J in (3). Since we are in the level energy $H = 0$, it corresponds to $J = -I$. In the vertical axis there is the function $\mu\mathcal{T}_0(I)$ involved in the definition in the inner map (18).

D Comparison of the inner and outer dynamics of the elliptic problem

In Lemma 5.1 is stated the non-degeneracy condition (65), which is checked numerically. To this end, we compute the functions ω_0, A_1^+ and Ω^+. These functions are defined by means of Melnikov-like integrals in (23), (58) and (61) respectively.

Concerning ω_0, note that it is defined through a limit of a function which has exponential decay. This fact is more apparent if one considers formula (27). The homoclinic orbit is tending exponentially fast to the periodic orbit, and therefore, the two terms in the integral converge to each other exponentially fast. Then, to obtain a numerical approximation of it, it suffices to consider formula (23) and choose a number N sufficiently large. The remainder can be easily bounded.

The integral (58), which defines A_1^+, is over a finite time of integration and through the homoclinic orbits of the circular problem, and then is easy to compute numerically. To compute Ω (61), we proceed as for ω_0.

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