Existence of instabilities in the restricted elliptic three-body problem

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1 Introduction and main result

One central problem in the field of dynamical systems is to understand how stable or unstable the Solar System is. This is a difficult question, even in the Newtonian approximation in the plane:

$$\ddot{q}_i = \sum_{j \neq i} m_j \frac{q_j - q_i}{\|q_j - q_i\|^2}, \quad q_i \in \mathbf{R}^2, \quad i = 0, 1, ..., n, \tag{1}$$

with one mass m_0 (thought of as that of the Sun) much larger than the others.

KAM theory asserts that if the masses of planets are small enough, there is a nowhere dense Cantor set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions, in the neighborhood of circular, coplanar, Keplerian motions [Arn63, Féj04, ?]. This is a remarkable confirmation of the idea that Lagrange and Laplace had about the stability of the Solar System, but only for a nowhere dense set, which, from the point of view of topology, is a small set.

According to Kolmogorov and Herman [Her98], the proof of the following conjecture is the oldest open problem in dynamical systems.

Conjecture 1.1 ([Her98])**.** *Consider the Hamiltonian System of the n-body problem restricted to an arbitrary level of energy, and let us reparameterize it so that the new flow takes infinite time to go to collisions. Then, the non wandering set of this new Hamiltonian is nowhere dense in the level of energy.*

This conjecture would imply that bounded orbits form a *nowhere dense set* and no topological stability is possible. It is largely confirmed by numerical experiments of Laskar for instance (see [Las10] for a recent account), who has shown that in our solar system collisions between Mars and Venus could occur within a few billion years. The conjectured coexistence of a nowhere dense set of positive measure of bounded quasiperiodic motions with an open and dense set of initial condition with unbounded orbits is truly remarkable.

1.1 The Restricted elliptic planar three-body problem

Consider (1) with $n = 2$ and one mass equal to zero. We obtain the restricted planar three-body problem. Call the three bodies the Sun (mass of order 1), Jupiter (mass of order $\mu \ll 1$) and the Asteroid (zero mass). Due to absence of effect of the Asteroid on the other two the first Kepler law says that the Sun and Jupiter can follow either circular or elliptic orbits. Assume that these bodies describe ellipses of the same eccentricity $e_0 > 0$. Taking a limit as mass of the Asteroid goes to zero we obtain that the motion of the Asteroid under the influence of Sun and Jupiter, is given by the Hamiltonian

$$K(q, p, t) = \frac{\|p\|^2}{2} - \frac{1 - \mu}{\|q + \mu q_0(t)\|} - \frac{\mu}{\|q - (1 - \mu)q_0(t)\|} \tag{2}$$

where $q, p \in \mathbb{R}^2$ and $-\mu q_0(t)$ and $(1 - \mu)q_0(t)$ correspond to elliptic motions of the Sun and Jupiter respectively. Without loss of generality one can assume that $q_0(t)$ has semi major axis equal to 1 and period 2π. For $e_0 > 0$ this system has two and a half degrees of freedom.

For $e_0 = 0$ this is so-called the restricted planar circular 3 Body Problem. It turns out that this system has 2 degrees of freedom and has a conserved quantity called the Jacobi constant, defined by

$$J = \frac{\|p\|^2}{2} - \frac{1-\mu}{\|q + \mu q_0(t)\|} - \frac{\mu}{\|q - (1-\mu)q_0(t)\|} - \frac{\|q\|^2}{2}. \tag{3}$$

Aforementioned KAM theory applies to both the circular and the elliptic problems [SM95] and asserts that if the mass of Jupiter is small enough, there is a set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions.

If Jupiter has a circular motion, since the system has only 2 degrees of freedom, KAM invariant tori are 2-dimensional and separate the 3-dimensional energy surfaces. But in the elliptic problem, a priori 3-dimensional KAM tori do not prevent orbits to wander on a 5-dimensional phase space. The existence of a wide enough set of wandering orbits is a first step towards a proof of conjecture 1.1 for the restricted planar elliptic three-body problem. In this paper we do a first step.

Let us write Hamiltonian (2) as

$$K(q, p, t) = K_0(q, p) + K_1(q, p, t, \mu)$$

with

$$K_0(q, p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}$$

$$K_1(q, p, t, \mu) = \frac{1}{\|q\|} - \frac{1-\mu}{\|q + \mu q_0(t)\|} - \frac{\mu}{\|q - (1-\mu)q_0(t)\|}.$$

One can see that $K_1 = \mathcal{O}(\mu)$ uniformly away from collisions. Then, notice that there is a competition between integrability K_0 and non-integrability K_1, without which there would be no wandering. In this present work we consider a realistic value of the mass ratio, that is $\mu = 10^{-3}$.

Here is the main result of this paper

Theorem 1.2. *Let us consider the restricted planar elliptic three-body problem with ratio of masses $\mu = 10^{-3}$ and eccentricity of Jupiter e_0. Then, for e_0 small enough there exist $T > 0$ and a trajectory whose osculating eccentricity $e(t)$ satisfies that*

$$e(0) < 0.55 \quad and \quad e(T) > 0.65$$

while its osculating semi major axis $a(t)$ remains almost constant, that is,

$$|a(t) - a(0)| \leq C\mu \quad for\ t \in [0, T]$$

for certain constant $C > 0$. In other words,

$$|J(T) - J(0)| > 0.1,$$

where J is the Jacobi constant defined in (3).

When Jupiter performs circular motion the Jacobi constant is an integral of motion and then KAM theory prevents global instabilities. We consider the eccentricity e_0 as a small parameter so that we can compare the dynamics of the elliptic problem with the dynamics of the circular one.

1.2 Mechanisms of instability

The result obtained in Theorem 1.2 gives an example of large instability for this mechanical system. It can be interpreted as an example of Arnold diffusion (see [Arn64]). Nevertheless, Arnold diffusion usually refers to nearly integrable systems whereas Hamiltonian (2) cannot be considered as close to integrable since $\mu = 10^{-3}$. The mechanism of diffusion used in this paper is somewhat similar to the so-called Mather problem ([Mat96, BT99, DdlLS00]). This analogy will be specified in Section ??.
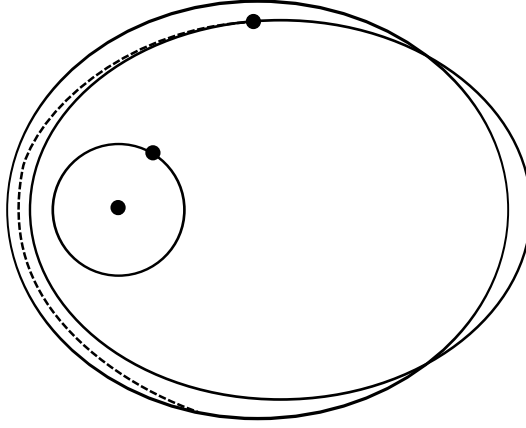
Figure 1: Transition from the osculating ellipse of eccentricty $e = 0.55$ to the ellipse of eccentricty $e = 0.65$.

Arguably, the main source of the existence of instabilities are *resonances*. One of the most natural resonances in the planar restricted elliptic three-body problem is the *mean motion orbital resonances*[1]. For these resonances, from time to time, Jupiter and the Asteroid will be in nearly the same relative position. Thus Jupiter's kicks could pile up and modify the eccentricity of the Asteroid, instead of averaging out. According to third Kepler's Law, these resonances take place when $a^{3/2}$ is close to a rational, where a is the semi major axis of instant ellipse of the Asteroid. In our case we consider $a^{3/2}$ *close to 7*. Nevertheless, one should expect that the same mechanism takes place for a large number of mean motion orbital resonances.

The semi-major axis a and the eccentricity e describe completely an instant ellipse of the Asteroid (up to orientation). Therefore, geometrically Theorem 1.2 says that the Asteroid evolves from a (osculating) ellipse of eccentricity $e = 0.55$ to one of eccentricity $e = 0.65$ without changing much the semi major axis (see Figure ??). In Figure ?? we consider the plane (a, e), which describe the ellipse of the Asteroid. Then diffusing orbits given by Theorem 1.2 correspond to an (almost) horizontal line which indicates the change of shape of the Asteroid's ellipse.
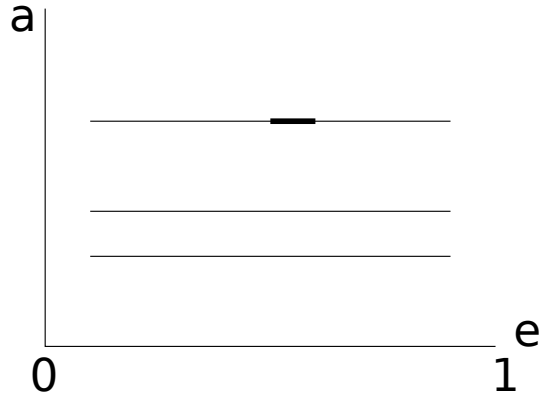

Figure 2: In this graphic we show the diffusion path that we study in the (a, e) plane. The horizontal lines represent the resonances along which we drift. The thicker line is the diffusion path whose existence we are able to prove in this paper.

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[1]The mean motions are the frequencies of the Keplerian revolution of Jupiter and the Asteroid around the Sun: Jupiter make one full revolution when the Asteroid make some integer number of revolutions.

1.3   Main steps of the proof

Diffusing orbit, we are looking for, lie in a neighborhood of a normally hyperbolic invariant cylinder Λ, which is defined by the resonance. The vertical component of the cylinder can be parameterized parameterized by the eccentricity of the Asteroid.

If the stable and unstable invariant manifolds of Λ intersect transversally, the restricted elliptic three-body problem induces two different dynamics on the cylinder (see Sections 4.4 and 4.5): *the inner and the outer ones*. The inner dynamics is simply the restriction of the Newtonian flow to Λ. The outer dynamics is harder to see. Intuitively it can be observed asymptotically by starting very close from the cylinder and from its unstable manifold, traveling all the way up to a homoclinic intersection, and coming back close to the cylinder and along its stable manifold.

The proof consists in the following steps:

1. Proving the existence of the normally hyperbolic invariant cylinder Λ.

2. Establishing the transversality of the stable and unstable invariant manifolds of this cylinder.

3. Compare the two types of dynamics on Λ: inner and outer.

4. Construct diffusing orbits by shadowing a carefully chosen composition of the outer and inner maps.

This program faces difficulties at every step. The first one comes from the proper degeneracy of the Newtonian potential: at the limit $\mu = 0$ (no Jupiter), the Asteroid has a one-frequency, Keplerian motion, whereas symplectic geometry would allow for a three frequency motion (as with any potential other than the Newtonian potential $1/r$ and the elastic potential r^2). Due to this degeneracy, switching to $\mu > 0$ is a singular perturbation. In particular, hyperbolicity is small with respect to μ.

Concerning the second step, establishing the transversality of the invariant manifolds of Λ is a delicate problem. If one considers $\mu \ll 1$, the difference between the invariant manifolds becomes exponentially small with respect to μ, that is of order $\exp(-c/\sqrt{\mu})$ for a certain constant $c > 0$. Despite the inordinate amount of time and energy mathematicians have spent trying to estimate the splitting of separatrices in various problems, none of the known techniques apply here. It turns out that computing the splitting of the invariant manifolds we need to evaluate a Poincaré-Melnikov integral. Nevertheless, in this problem certain functions involved in the computation of this integral are not algebraic whereas this is crucial for all existing techniques.

Considering $\mu = 10^{-3}$ avoids this difficulty, since for this value of the parameter one can see that the splitting of separatrices is not extremely small and therefore, can be detected by means of a computer. Moreover, $\mu = 10^{-3}$ is a realistic value of for the Sun-Jupiter model.

Now we turn to the third step. Using classical perturbation methods and the properties of the underlying system one can reduce the inner and outer dynamics to two two-dimensional symplectic maps, which can be written in the following form

$$\mathcal{F}_{e_0}^{\mathrm{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \left(A^+(I)e^{it} + A^-(I)e^{-it} \right) + \mathcal{O}\left(\mu e_0^2 \right) \\ t + \mu \mathcal{T}_0(I) + \mathcal{O}(\mu e_0) \end{pmatrix} \tag{4}$$

and

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \left(\Omega^+(I)e^{it} + \Omega^-(I)e^{-it} \right) + \mathcal{O}\left(\mu e_0^2 \right) \\ t + \mu \omega_0(I) + \mathcal{O}(\mu e_0) \end{pmatrix}, \tag{5}$$

where (I, t) are conjugate variables which parameterize the 3-dimensional normally hyperbolic invariant cylinder Λ intersected with a certain transversal Poincaré section and $A^\pm, \mathcal{T}_0, \Omega^\pm, \omega_0$ are analytic functions.

Then, as we will see in Section 5, the existence of diffusing orbits can be established provided the analytic function

$$\mathcal{K}^+(I) = \Omega^+(I) - \frac{e^{i\omega_0(I)} - 1}{e^{i\mathcal{T}_0(I)} - 1} A^+(I).$$

does not vanish for $I \in [I_-, I_+]$. Since $A^\pm$ and $\Omega^\pm$ are complex conjugate do not write the complex conjugate $\mathcal{K}^+(I)$. Numerically, one can check that $\mathcal{K}^+(I) \neq 0$ in an interval $I \in [I_-, I_+]$, which reduces to the proof of Theorem 1.2 to shadowing.

In fact, the complex function $\mathcal{K}^+(I)$ can be regarded as a 2-dimensional real-valued function depending analytically on (I, μ). If the dependence in μ is non-trivial, $\mathcal{K}^+(I)$ does not vanish in any point of the interval $[I_-, I_+]$ except for a finite number of μ's.

2 Setting of the problem and notation

The model of the Sun, Jupiter and a massless Asteroid in cartesian coordinates is given by the Hamiltonian (2). First, let us consider the case $\mu = 0$, that is, we consider Jupiter with zero mass. In that case, Jupiter and the Asteroid do not make influence on each other and therefore the system is reduced to two uncoupled 2 Body problems, the Sun-Jupiter and the Sun-Asteroid, which are integrable.

We want to study the existence of instability in one particular resonance of this system, which appears when the period of Jupiter is seven times the period of the Asteroid. To understand the system better, one can consider action-angle variables for the Sun-Asteroid two-body problem, that is the so-called Delaunay variables, which we denote by $(\ell, L, \hat{g}, G)$. The variable ℓ is the mean anomaly, L is the square of the semi major axis, $\hat{g}$ is the argument of the perihelion and G is the angular momentum. These variables can be obtained from the cartesian coordinates as follows. We define polar coordinates for the position. Namely,

$$q = (r \cos \phi, \sin \phi).$$

Then, the actions of Delaunay coordinates are defined implicitly by

$$-\frac{1}{2L^2} = \frac{\|p\|^2}{2} - \frac{1}{\|q\|} \tag{6}$$

$$G = \frac{\|q\|^2}{2} \tag{7}$$

Using these actions, the eccentricity of the Asteroid can be expressed as

$$e = \sqrt{1 - \frac{G^2}{L^2}}. \tag{8}$$

To define the angles, we set

$$\phi = v + \hat{g} \tag{9}$$

where v is the true anomaly and $\hat{g}$ is the argument of the perihelion. Then, from v one can obtain the eccentric anomaly u using

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}. \tag{10}$$

From the eccentric anomaly, the mean anomaly is given by the implicit relation

$$u - e \sin u = \ell. \tag{11}$$

In Delaunay variables Hamiltonian (2) has the form

$$\hat{H}(\ell, L, \hat{g}, G, t) = -\frac{1}{2L^2} + \mu \Delta H_{\text{circ}}(\ell, L, \hat{g} - t, G, t, \mu) + \mu e_0 \Delta H_{\text{ell}}(\ell, L, \hat{g} - t, G, t, \mu, e_0) \tag{12}$$

It turns out that for $e_0 = 0$, the circular problem only depends on $\hat{g} - t$. To understand better the appearance of instabilities it is more convenient to consider rotating Delaunay coordinates and make Hamiltonian autonomous. This means to define the new angle $g = \hat{g} - t$ and a new variable I conjugate to time t. We have

$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu \Delta H_{\text{circ}}(\ell, L, g, G, \mu) + \mu e_0 \Delta H_{\text{ell}}(\ell, L, g, G, t, \mu, e_0) + I, \tag{13}$$

In these new variables, the difference of number of degree of freedom of the elliptic and circular problems becomes more apparent. When $e_0 = 0$ the system is autonomous and then I is constant, which corresponds to the conservation of the Jacobi constant (3) and, therefore, the circular problem reduces to 2 degrees of freedom.

In rotating Delaunay variables, the mean motions orbit resonance of the Asteroid and Jupiter we consider, that is the period of the Asteroid being approximately seven times the period of Jupiter, corresponds to

$$\dot{\ell} \sim \frac{1}{7} \quad \text{and} \quad \dot{g} \sim -1,$$

which takes place when $L \sim 7^{1/3}$. We study the dynamics in this region of the phase space and we will see that one can drift along this resonance. That is, keeping L close to constant, one can find trajectories whose G component changes drastically, in such a way that, using (8), one can see that e also changes drastically. In this setting, Theorem 1.2 can be rephrased as follows.

Theorem 2.1. *There exist $e_0^* > 0$ such that for $0 < e_0 < e_0^*$, there exist $T > 0$ and an orbit of the the Hamiltonian System with Hamiltonian (13) which satisfies*

$$G(0) < G_0 \ and \ G(T) > G_1$$

whereas

$$\left| L(t) - 7^{1/3} \right| \leq C\mu.$$

By definition the Hamiltonian ((13)) is autonomous. Therefore, we will restrict ourselves to a level of energy which, without loss of generality can be taken as $H = 0$. Therefore, in the orbits satisfying $\left| L(t) - 7^{1/3} \right| \leq C\mu$, to obtain drift in G is equivalent to obtain drift in I since $|I - G| = \mathcal{O}(\mu)$. For reasons that will be clear later, in order to prove Theorem 2.1, we will obtain orbits whose I-component undergoes a change of order 1.

The proof of this theorem is structured as follows. First in Section 3, we study the dynamics in the circular problem, that is $e_0 = 0$ and the underlying Hamiltonian ((13)) becomes

$$H_{\text{circ}}(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu \Delta H_{\text{circ}}(\ell, L, g, G, \mu). \tag{14}$$

In this section we obtain the mentioned outer and inner maps, which turn out to be integrable symplectic maps defined on the cylinder. Then in Section 4 we consider the elliptic case $e_0 > 0$ and we study the inner and outer dynamics as a perturbation of the ones obtained in for the circular problem. Comparing these two dynamics allows us to prove Theorem 2.1.

3   The circular problem

The circular problem is given by the Hamiltonian (13) with $e_0 = 0$. Since it does not depend on t, I is an integral of motion. Moreover, since we are studying the dynamics in the energy surface $H = 0$, $I = -H_{\text{circ}}(\ell, L, g, G)$. Therefore, it is the Jacobi constant (3) with opposite sign. For each level $I = \text{constant}$, one can study the dynamics close to the resonance $7\ell + \dot{g} \sim 0$. Since t is a cyclic variable, one can consider the 2 degree of freedom Hamiltonian of the circular problem for which the conservation of energy corresponds to the conservation of the Jacobi constant (3).

Theorem 3.1. *Consider the Hamiltonian (14) with $\mu = 10^{-3}$. Then, in each energy level $J_- \leq J \leq J_+$, there exists a hyperbolic periodic orbit $(L(t), \ell(t), G(t), g(t))$ of period T which satisfies*

$$|T - 14\pi| < C\mu$$

and

$$\left| L(t) - 7^{1/3} \right| < C\mu$$

for certain constant $C > 0$ and all $t \in \mathbb{R}$ and has period.

Moreover its stable and unstable invariant manifolds intersect transversally.

Proof. Computer assisted □

We will study the elliptic problem as a perturbation of the circular one. Therefore we do not reduce the dimension of the phase space while studying the inner and outer dynamics of the circular problem. Namely, we consider the *Extended Circular Problem* given by the Hamiltonian (13) with $e_0 = 0$. In other words, we keep the conjugate variables (t, I) even if t is a cyclic variable. Consider the energy level $H = 0$. In this setting the conservation of the Jacobi constant corresponds to the conservation of I. Therefore, the periodic orbits obtained in Theorem 3.1 become invariant two-dimensional tori which belong to hyperplanes $I = $ constant for any $I \in [-J_+, -J_-]$. Moreover, the union of these 2-dimensional invariant tori form a normally hyperbolic invariant manifold.

Corollary 3.2. *The Hamiltonian* (13) *with* $\mu = 10^{-3}$ *and* $e_0 = 0$ *has an analytic normally hyperbolic invariant manifold* Λ_0, *which is foliated by two-dimensional invariant tori. Moreover, there exists a function* $\mathcal{G}_0 : \mathbb{T}^2 \times [I_-, I_+] \to (\mathbb{T} \times \mathbb{R})^3$ *which can be expressed in coordinates as*

$$\mathcal{G}_0(g, t, I) = \left(\widetilde{\mathcal{G}}_0^\ell(g, I), g, t, I \right) = \left(\mathcal{G}_0^\ell(g, I), \mathcal{G}_0^L(g, I), g, \mathcal{G}_0^G(g, I), t, I \right). \tag{15}$$

that parameterizes Λ_0. *Namely,*

$$\Lambda_0 = \{ \mathcal{G}_0(g, t, I) : (g, t, I) \in \mathbb{T} \times [I_-, I_+] \times \mathbb{T} \}.$$

Moreover, the stable and unstable invariant manifolds of Λ_0 *intersect transversally. Let us call* Γ_0 *to one of these intersections. Then, there exists a function*

$$\mathcal{C}_0(g, t, I) = (\mathcal{C}_0^L(g, I), \mathcal{C}_0^\ell(g, I), g, \mathcal{C}_0^G(g, I), t, I),$$

which parameterizes it. Namely,

$$\Gamma_0 = \left\{ \mathcal{C}_0(g, t, I) = (\mathcal{C}_0^L(g, I), \mathcal{C}_0^\ell(g, I), g, \mathcal{C}_0^G(g, I), t, I) : (g, t, I) \in \mathbb{T}^2 \times [I_-, I_+] \right\}.$$

We define a global Poincaré section and deal with maps, to reduce the dimension by one. There are two natural choices: $\{t = 0\}$ and $\{g = 0\}$, since both variables satisfy $\dot{t} \neq 0$ and $\dot{g} \neq 0$. We choose the section $\{g = 0\}$ and call

$$\mathcal{P}_0 : \{g = 0\} \longrightarrow \{g = 0\} \tag{16}$$

to this Poincaré map. Later we show that one can express the inner and outer maps in the coordinates (I, t), which are conjugate. It turns out that these maps are symplectic and integrable due to the conservation of I. The variables (I, t) can be used as action-angle variables. In this way, it will be easier to understand the influence of the ellipticity. We define the cylinder

$$\widetilde{\Lambda}_0 = \Lambda_0 \cap \{g = 0\}, \tag{17}$$

which is a normally hyperbolic invariant manifold for the Poincaré map $\mathcal{P}_0$.

3.1 The inner map of the circular problem

We first obtain the inner map, which is given by the global Poincaré map restricted to the cylinder (17). Since I is an integral of motion, this inner map is of the form

$$\mathcal{F}_0^{\text{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mu \mathcal{T}_0(I) \end{pmatrix}, \tag{18}$$

where the function $\mathcal{T}_0$ is independent of t due to the fact that the map is symplectic. In fact, $14\pi + \mu \mathcal{T}_0(I, \mu)$ is the period of the periodic orbit obtained in Theorem 3.1 on the corresponding energy surface. It can be seen numerically that this map is twist.

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Theorem 3.3. *The inner map $\mathcal{F}_0^{\text{in}}$ defined in (18) is a symplectic twist map, that is*

$$\partial_I T_0(I,\mu) \neq 0 \qquad for \ I \in [I_-, I_+].$$

Moreover, the function $\mathcal{T}_0(I,\mu)$ satisfies

$$0 < \mu \mathcal{T}_0(I,\mu) < \pi. \tag{19}$$

Proof. Computer assisted □

The information contained in this theorem will be crucial in Section 5 to prove the existence of a transition chain of invariant tori.

3.2 The outer map of the circular problem

The outer map has been also called scattering map (see [DdlLS00, DdlLS06a, DdlLS06b, DdlLS08]). Suppose that an invariant manifold is normally hyperbolic and its stable and unstable invariant manifolds intersect transversally. Then, the outer map is defined as follows as follows[2].

Definition 3.4. *Let $\Lambda \subset M$ be a normally hyperbolic invariant manifold for a map $\mathcal{P}$ in a manifold M. Assume that $\gamma \subset W_\Lambda^s \cap W_\Lambda^u$ is a homoclinic manifold and that the intersection of W_Λ^s and W_Λ^u is transversal along γ, that is*

$$T_z W_\Lambda^s + T_z W_\Lambda^u = T_z M, \quad for \ z \in \gamma$$
$$T_z W_\Lambda^s \cap T_z W_\Lambda^u = T_z \gamma, \quad for \ z \in \gamma.$$

Then, there exists a constant $\lambda > 1$ such that for any two points $x_\pm \in \Lambda$ we say that $\mathcal{F}^{\text{out}}(x_-) = x_+$, if there exists a point $z \in \gamma$ such that

$$\text{dist}\left(\mathcal{P}^n(z), \mathcal{P}^n(x_\pm)\right) < C\lambda^{-|n|} \qquad for \ all \ n \in \mathbb{Z}^\pm.$$

In the circular problem, the outer map has the form

$$\mathcal{F}_0^{\text{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mu\omega_0(I) \end{pmatrix}. \tag{20}$$

since I is constant and the outer map is always symplectic (see [DdlLS08]).

The outer map can be defined either with discrete or continuous time. Since the Poincaré-Melnikov Theory is considerably simpler for flows than for maps, we compute $\mathcal{F}_0^{\text{out}}$ using continuous time. Moreover, in Section 4.5 we will use also flows to study the outer map of the elliptic problem as a perturbation of (20). The outer map on Λ_0 given by the Hamiltonian (13) with $e_0 = 0$ does not preserve the section $\{g = 0\}$, which the inner map does. In order to fix this problem, we reparameterize the flow so that the inner and outer map preserve this section.

This reparameterization corresponds to identify the variable g with time and is given by,

$$
\begin{array}{llll}
\frac{d}{ds}\ell = & \frac{\partial_L H(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\text{circ}}(\ell,L,g,G)} & \frac{d}{ds}L = & -\frac{\partial_\ell H(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\text{circ}}(\ell,L,g,G)} \\
\frac{d}{ds}g = & 1 & \frac{d}{ds}G = & -\frac{\partial_g H(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\text{circ}}(\ell,L,g,G)} \\
\frac{d}{ds}t = & \frac{1}{-1+\mu\partial_G \Delta H_{\text{circ}}(\ell,L,g,G)} & \frac{d}{ds}I = & 0
\end{array}
\tag{21}
$$

where H is Hamiltonian (13) with $e_0 = 0$. Notice that this reparameterization implies the change of direction of time. However, the geometric objects stay the same. In particular, the new flow still possesses the normally invariant cylinder obtained in Corollary 3.2 and its invariant manifolds.

We will refer to this system as a *reduced circular problem*. We call it reduced because we identify g with the time s. Moreover, the t-component only depends on the other coordinates. Call

$$\Phi_0^{\text{circ}}\{s, (\ell, L, g, G)\} = \left(\Phi_0^\ell\{s, (\ell, L, g, G)\}, \Phi_0^L\{s, (\ell, L, g, G)\}, g+s, \Phi_0^G\{s, (\ell, L, g, G)\}\right), \tag{22}$$

then the outer map can be computed as follows.

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[2]This definition can be modified to generalize the outer map to any normally hyperbolic invariant manifold with transversal stable and unstable invariant manifolds

Lemma 3.5. *The function $\omega_0(I)$ involved in the definition of the outer map in* (20) *can be defined as*

$$\omega_0(I) = \omega_0^+(I) - \omega_0^-(I)$$

with

$$\omega_0^\pm(I) = -\lim_{N\to\pm\infty}\left(\int_0^{14N\pi}\frac{\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0))\}\right)}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0))\}\right)}d\sigma + N\mathcal{T}_0(I)\right).$$
$$(23)$$

where $\mathcal{C}_0$ is the parameterization of the intersection of the invariant manifolds of Λ_0, given in Corollary 3.2

Note that the minus sign that appears in the definition of the functions $\omega^\pm(I)$ is due to the fact that the reparameterized flow (21) reverses time.

Proof. Since the t component of the reduced circular system (21) does not depend on t, its behavior is given by

$$\Phi_0^t\{s,(\ell,L,g,G,t)\} = t + \int_0^s \frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\ell,L,g,G)\}\right)}d\sigma. \qquad (24)$$

Consider a point $(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0),t,I)$ in $W_{\Lambda_0}^u \cap W_{\Lambda_0}^s \cap \{g=0\}$. Since the first four components are independent of t, $(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0),t,I)$ is forward asymptotic to a point $(\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^L(I,0),0,\mathcal{G}_0^G(I,0),t+\omega_0^+(I),I)$ and backward asymptotic to a point $(\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^L(I,0),0,\mathcal{G}_0^G(I,0),t+\omega_0^-(I),I)$. Using (24), the functions $\omega_0^\pm(I)$ can be defined as

$$\omega_0^\pm(I) = -\lim_{T\to\pm\infty}\int_0^T\left(\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0))\}\right)}\right.$$
$$\left. -\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^L(I,0),0)\}\right)}\right)d\sigma.$$
$$(25)$$

Moreover, since the system is 14π-periodic in the time s due to the identification of s with g, it is more convenient to write these in integrals as

$$\omega_0^\pm(I) = -\lim_{N\to\pm\infty}\int_0^{14N\pi}\left(\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0))\}\right)}\right.$$
$$\left. -\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{G}_0^\ell(I,0),\mathcal{G}_0^L(I,0),0,\mathcal{G}_0^G(I,0))\}\right)}\right)d\sigma.$$

The flow (21) induces the same inner map (18) as the original one but now it can be interpreted as the 14π-Poincaré map in the s-time. Then, the second part of the integral is just the evolution of the t-component under the flow restricted to the cylinder Λ_0. Then, using the inner map defined in (18), the functions $\omega_0^\pm(I)$ can be defined as

$$\omega_0^\pm(I) = -\lim_{N\to\pm\infty}\left(\int_0^{14N\pi}\frac{1}{-1+\mu\partial_G\Delta H_{\mathrm{circ}}\left(\Phi_0^{\mathrm{circ}}\{\sigma,(\mathcal{C}_0^\ell(I,0),\mathcal{C}_0^L(I,0),0,\mathcal{C}_0^G(I,0))\}\right)}d\sigma\right.$$
$$\left. + N(14\pi + \mathcal{T}_0(I))\right).$$

An easy computation leads to formula (23). $\qquad\qquad\square$

4 The elliptic problem

We devote this section to study the elliptic restricted three body problem and prove the existence of a diffusing orbit. To obtain the perturbative expansions of the inner and outer maps of the elliptic problem, we use the reduced elliptic problem, which is given by

$$
\begin{aligned}
\tfrac{d}{ds}\ell &= \frac{\partial_L H}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}} + \mu e_0 \partial_G \Delta H_{\mathrm{ell}}} & \tfrac{d}{ds}L &= -\frac{\partial_\ell H}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}} + \mu e_0 \partial_G \Delta H_{\mathrm{ell}}} \\
\tfrac{d}{ds}g &= 1 & \tfrac{d}{ds}G &= -\frac{\partial_g H}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}} + \mu e_0 \partial_G \Delta H_{\mathrm{ell}}} \\
\tfrac{d}{ds}t &= \frac{1}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}} + \mu e_0 \partial_G \Delta H_{\mathrm{ell}}} & \tfrac{d}{ds}I &= -\frac{\mu e_0 \partial_t \Delta H_{\mathrm{ell}}}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}} + \mu e_0 \partial_G \Delta H_{\mathrm{ell}}}
\end{aligned}
\tag{26}
$$

This system is a perturbation of (21). Of course, the study of the inner map can be done both dealing with this system or the system associated to the Hamiltonian (12). Nevertheless, to simplify the exposition we use only (26) for both the inner and outer maps. We also consider the Poincaré map associated with this system and the section $\{g = 0\}$,

$$
\mathcal{P}_{e_0} : \{g = 0\} \longrightarrow \{g = 0\}, \tag{27}
$$

which is a perturbation of (16).

First in Section 4.1 we study analytic properties of the elliptic perturbation, which will be used in the following sections. In Section 4.2 we study the normally hyperbolic invariant manifold Λ_{e_0} that (26) possesses. We study it as a perturbation of Λ_0 given in Corollary 3.2. In Section 4.3 we study the e_0-expansion of the flow restricted to Λ_{e_0}. This expansion allows us to give perturbative formulas of the inner and outer maps of the elliptic problem, which are defined in

$$
\widetilde{\Lambda}_{e_0} = \Lambda_{e_0} \cap \{g = 0\} \tag{28}
$$

We study first the inner map in Section 4.4. In Section 4.5 we give perturbative formulas of the outer map of the elliptic problem. We obtain them using Poincaré-Melnikov techniques. Finally, in Section 5 we compare the dynamics given by both maps. This comparison allows us to obtain a transition chain of invariant tori. Using shadowing methods, we can see that there exists a diffusing orbit close to this transition chain.

4.1 The e_0-expansion of the elliptic Hamiltonian

We obtain expansions of ΔH_{ell} in (13) with respect to e_0. These expansions will be used later in Sections 4.2, 4.4 and 4.5. The most important goal is to see which harmonics in t have the e_0 and e_0^2 terms of the elliptic perturbation in (13). Note that the circular problem is independent of t.

We introduce the following notation.

Notation 4.1. *For any function f with 2π-periodic dependence on t, $\mathcal{N}(f)$ is the subset of the integers corresponding to which t-harmonics of f (which may depend on other variables) are non-zero.*

We define the function

$$
\mathcal{B}(r, v, g, t) = \frac{1}{\left| re^{i(v+g-t)} - r_0(t)e^{iv_0(t)} \right|}. \tag{29}
$$

This function is the potential

$$
\frac{1}{|q - q_0(t)|}
$$

expressed in terms of $g = \hat{g} - t$, where $\hat{g}$ is the argument of the perihelion, the true anomaly v of the Asteroid defined in (9) and the radius r. The functions $r_0(t)$ and $v_0(t)$ are the radius and the true anomaly of Jupiter. The functions $r_0(t)$ and $v_0(t)$ are the only ones involved in the definition of $\mathcal{B}$ which depend on e_0.

Then, the perturbation in (12) can be expressed as

$$\mu \Delta H_{\text{circ}}(\ell, L, g, G) + \mu e_0 \Delta H_{\text{ell}}(\ell, L, g, G, t) = -\frac{1-\mu}{\mu} \mathcal{B}\left(-\frac{r}{\mu}, v, g, t\right)$$

$$-\frac{\mu}{1-\mu} \mathcal{B}\left(\frac{r}{1-\mu}, v, g-t, t\right) + \frac{1}{r}\bigg|_{(r,v)=(r(\ell,L,G),v(\ell,L,G))}$$

First we study the expansion of $\mathcal{B}$, and from it, we deduce the one of ΔH_{ell}.

Lemma 4.2. *Consider the expansion*

$$\mathcal{B}(r, v, g, t) = \mathcal{B}_0(r, v, g) + e_0 \mathcal{B}_1(r, v, g, t) + e_0^2 \mathcal{B}_2(r, v, g, t) + \mathcal{O}\left(e_0^3\right) \qquad (30)$$

of the function $\mathcal{B}$ defined in (29). Then,

- $\mathcal{B}_0$ *satisfies* $\mathcal{N}(\mathcal{B}_0) = \{0\}$.

- $\mathcal{B}_1$ *satisfies* $\mathcal{N}(\mathcal{B}_1) = \{\pm 1\}$ *and is given by*

$$\mathcal{B}_1(r, v, g, t) = -\frac{1}{2\Delta^3(r, v, g)} \left(2\cos t - 3r\cos(v + g + t) + r\cos(v + g - t)\right), \qquad (31)$$

  where

$$\Delta(r, v, g) = \left(r^2 + 1 - 2r\cos(v + g)\right)^{1/2}.$$

- $\mathcal{B}_2$ *satisfies* $\mathcal{N}(\mathcal{B}_2) = \{0, \pm 1, \pm 2\}$.

Proof. To proof this lemma we look for the the e_0-expansions of the functions $r_0(t)$ and $v_0(t)$ involved in the definition of $\mathcal{B}$ in (29). We obtain them using the eccentric, true and mean anomalies of Jupiter.

From the relation $t = u_0 - e_0 \sin u_0$ (see (11)), one can obtain that

$$u_0(t) = t + e_0 \sin t + \frac{e_0^2}{2} \sin 2t + \mathcal{O}\left(e_0^3\right).$$

Then, using $r_0 = 1 - e_0 \cos u_0$,

$$r_0(t) = 1 - e_0 \cos t + e_0^2 \sin^2 t + \mathcal{O}\left(e_0^3\right).$$

For the eccentric anomaly we can use

$$\tan \frac{v_0}{2} = \sqrt{\frac{1 + e_0}{1 - e_0}} \tan \frac{u_0}{2}$$

(see (58)), to obtain

$$v_0 = u_0 + e_0 \sin u_0 + e_0^2 \left(\frac{9}{2} \sin u_0 - 2 \sin 2u_0\right) + \mathcal{O}\left(e_0^3\right)$$

and then

$$v_0(t) = t + 2e_0 \sin t + e_0^2 \left(\frac{9}{2} \sin t - \sin 2t\right) + \mathcal{O}\left(e_0^3\right).$$

Plugging $r_0(t)$ and $v_0(t)$ into (29), it can be easily seen that the expansion (30) satisfies all the properties of $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ stated in the lemma. $\qquad \square$

From the expansion of $\mathcal{B}$, one can easily deduce the expansion of ΔH_{ell} in (13).

Lemma 4.3. *Let us consider the e_0-expansion of the function ΔH_{ell} in (13),*

$$\Delta H_{\mathrm{ell}} = \Delta H_{\mathrm{ell}}^1 + e_0 \Delta H_{\mathrm{ell}}^2 + \mathcal{O}\left(e_0^2\right).$$

Then, the first order is given by

$$
\begin{aligned}
\Delta H_{\mathrm{ell}}^1(L,\ell,G,g,t) = &-\frac{1-\mu}{\mu}\mathcal{B}_1\left(-\frac{r(\ell,L,G)}{\mu}, v(\ell,L,G), g-t, t\right) \\
&-\frac{\mu}{1-\mu}\mathcal{B}_1\left(\frac{r(\ell,L,G)}{1-\mu}, v(\ell,L,G), g-t, t\right).
\end{aligned}
\tag{32}
$$

where $\mathcal{B}_1$ is the function defined in 4.2 and therefore it satisfies

$$\mathcal{N}\left(\Delta H_{\mathrm{ell}}^1\right) = \{\pm 1\}.$$

The second order $\Delta H_{\mathrm{ell}}^2$ satisfies

$$\mathcal{N}\left(\Delta H_{\mathrm{ell}}^2\right) = \{0, \pm 1, \pm 2\}.$$

4.2 The normally hyperbolic invariant cylinder for the elliptic problem

For e_0 small enough the system associated to the Hamiltonian (13) has a normally hyperbolic invariant cylinder Λ_{e_0}, which is e_0-close to Λ_0 given in Corollary 3.2.

Lemma 4.4. *There exists $e_0^* > 0$ such that for $e_0 < e_0^*$ the system associated to Hamiltonian (13) has a normally hyperbolic locally invariant manifold, which is e_0-close in the $\mathcal{C}^0$ sense to Λ_0. Moreover, there exists a constant $C > 0$ and a function $\mathcal{G}_{e_0} : \mathbb{T} \times [I_- + Ce_0, I_+ - Ce_0] \times \mathbb{T} \to (\mathbb{T} \times \mathbb{R})^3$ which can be expressed in coordinates as*

$$\mathcal{G}_{e_0}(g,t,I) = \left(\mathcal{G}_{e_0}^\ell(g,I,t), \mathcal{G}_{e_0}^L(g,t,I), g, \mathcal{G}_{e_0}^G(g,t,I), I, t\right).
\tag{33}$$

that parameterizes Λ_{e_0}. Namely,

$$\Lambda_{e_0} = \{\mathcal{G}_{e_0}(g,t,I) : (g,I,t) \in \mathbb{T} \times [I_- + Ce_0, I_+ - Ce_0] \times \mathbb{T}\}.$$

Moreover, the stable and unstable invariant manifolds of Λ_{e_0} intersect transversally and one of these intersections is close in the $\mathcal{C}^0$ sense to the manifold Γ_0 defined in Corollary 3.2. Call Γ_{e_0} to this intersection. There exists a function

$$\mathcal{C}_{e_0}(g,I,t) = (\mathcal{C}_{e_0}^L(g,I,t), \mathcal{C}_{e_0}^\ell(g,I,t), g, \mathcal{C}_{e_0}^G(g,I,t), I, t),$$

which parameterizes it. Namely,

$$\Gamma_{e_0} = \{\mathcal{C}_{e_0}(g,I,t) : (g,I,t) \in \mathbb{T} \times [I_- + Ce_0, I_+ - Ce_0] \times \mathbb{T}\}.$$

The theory of normally hyperbolic invariant manifolds [Fen74, Fen77] ensures the existence of a map $\mathcal{G}_{e_0}$ parameterizing the normally hyperbolic manifold. It can be made unique imposing

$$\pi_g \mathcal{G}_{e_0}(g,I,t) = g, \ \pi_I \mathcal{G}_{e_0}(g,I,t) = I, \ \pi_t \mathcal{G}_{e_0}(g,I,t) = t,$$

where π_* is the projection with respect to the corresponding component of the function. Moreover, it satisfies the invariance equation

$$\mathcal{H}_{e_0} \circ \mathcal{G}_{e_0} = D\mathcal{G}_{e_0} \mathcal{R}_{e_0}
\tag{34}$$

where $\mathcal{H}_{e_0}$ is the vector field given in (26) and $\mathcal{R}_{e_0}$ is the vector field induced in Λ_{e_0}.

Since we have regularity with respect to parameters, the invariance equation allows us to obtain expansions of both the parameterization of Λ_{e_0} and the induced flow $\mathcal{R}_{e_0}$ with respect to e_0. Let us expand $\mathcal{G}_{e_0}$ and $\mathcal{R}_{e_0}$ as

$$\mathcal{G}_{e_0} = \mathcal{G}_0 + e_0 \mathcal{G}_1 + e_0^2 \mathcal{G}_2 + \mathcal{O}\left(e_0^3\right)
\tag{35}$$

$$\mathcal{R}_{e_0} = \mathcal{R}_0 + e_0 \mathcal{R}_1 + e_0^2 \mathcal{R}_2 + \mathcal{O}\left(e_0^3\right)
\tag{36}$$

Then, $\mathcal{G}_0$ is the function defined in (15) and $\mathcal{R}_0$ is the flow induced by the reduced circular problem in Λ_0. Note that the e_0-expansion of $\mathcal{H}_{e_0}$ can be deduced from the one of ΔH_{ell} obtained in Section 4.1. The first orders of $\mathcal{R}_{e_0}$ will play a significant role in Sections 4.3 and 4.4.

Lemma 4.5. *The expansion of the function $\mathcal{R}_{e_0}$ in (36) satisfies that:*

- *The e_0-order $\mathcal{R}_1$ satisfies $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$. Moreover, its I-component is given by*

$$\mathcal{R}_1^I = -\left.\frac{\mu \partial_t \Delta H_{\mathrm{ell}}^1(L, \ell, G, g, t)}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(L, \ell, G, g)}\right|_{(L, \ell, G) = \widetilde{\mathcal{G}}_0(g, I, t)} \tag{37}$$

  where $\widetilde{\mathcal{G}}_0$ is the function given in Corollary 3.2.

- *The e_0^2-order term $\mathcal{R}_2$ satisfies $\mathcal{N}(\mathcal{R}_2) = \{0, \pm 1, \pm 2\}$.*

Proof. Expanding equation (34) with respect to e_0, we have that the first terms satisfy

$$\mathcal{H}_0 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_0 \tag{38}$$

$$(D\mathcal{H}_0 \circ \mathcal{G}_0)\,\mathcal{G}_1 + \mathcal{H}_1 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_1 + D\mathcal{G}_1 \mathcal{R}_0 \tag{39}$$

$$(D\mathcal{H}_0 \circ \mathcal{G}_0)\,\mathcal{G}_2 + \frac{1}{2}\left(D^2\mathcal{H}_0 \circ \mathcal{G}_0\right)\mathcal{G}_1^{\otimes 2} + (D\mathcal{H}_1 \circ \mathcal{G}_0)\,\mathcal{G}_1 + \mathcal{H}_2 \circ \mathcal{G}_0 = D\mathcal{G}_0 \mathcal{R}_2 + D\mathcal{G}_1 \mathcal{R}_1 + D\mathcal{G}_2 \mathcal{R}_0 \tag{40}$$

By the uniqueness condition, the expansion of $\mathcal{G}_{e_0}$ is of the form

$$\mathcal{G}_i(g, I, t) = \left(\widetilde{\mathcal{G}}_i(g, I, t), 0, 0, 0\right) \quad \text{for } i \geq 1$$

with $\widetilde{\mathcal{G}}_i(g, I, t) = (\mathcal{G}_1^L(g, I, t), \mathcal{G}_1^\ell(g, I, t), \mathcal{G}_1^G(g, I, t))$.

Taking into account (21) and (38), $\mathcal{R}_0$ is given by

$$\mathcal{R}_0(g, I) = \begin{pmatrix} \mathcal{R}_0^g(g, I) \\ \mathcal{R}_0^I(g, I) \\ \mathcal{R}_0^t(g, I) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \left.\dfrac{1}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(L, \ell, G, g)}\right|_{(L, \ell, G) = \widetilde{\mathcal{G}}_0(g, I)} \end{pmatrix} \tag{41}$$

The other orders can be solved iteratively. Since

$$D\mathcal{G}_0 = \begin{pmatrix} D\widetilde{\mathcal{G}}_0 \\ \mathrm{Id} \end{pmatrix} \quad \text{and} \quad D\mathcal{G}_i = \begin{pmatrix} D\widetilde{\mathcal{G}}_i \\ 0 \end{pmatrix} \quad \text{for } i \geq 1, \tag{42}$$

we have that

$$\mathcal{R}_1^* = \pi_* \left((D\mathcal{H}_0 \circ \mathcal{G}_0)\,\mathcal{G}_1 + \mathcal{H}_1 \circ \mathcal{G}_0\right)$$

Replacing this into (39) we obtain an equation for $\mathcal{G}_1$. The equation for every Fourier t-coefficient is uncoupled, and therefore, since in Section 4.1 we have seen that $\mathcal{N}(\mathcal{H}_1) = \{\pm 1\}$, using the uniqueness of $\mathcal{G}_1$, one can deduce that $\mathcal{N}(\mathcal{G}_1) = \{\pm 1\}$. As a consequence we have that $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$.

Reasoning analogously, since in Section 4.1 we have seen that $\mathcal{N}(\mathcal{H}_2) = \{0, \pm 1, \pm 2\}$, one can see that $\mathcal{N}(\mathcal{G}_2) = \{0, \pm 1, \pm 2\}$ and $\mathcal{N}(\mathcal{R}_2) = \{0, \pm 1, \pm 2\}$.

Finally, taking into account (38), (42) , the e_0-expansion of the Hamiltonian given in Lemma 4.3 and (26), one can easily see that (37). □

4.3 e_0-expansion of the inner flow

To study the inner map perturbatively, we need first to study the first orders of the flow restricted to the cylinder Λ_{e_0}. Namely, the flow $\Phi_{e_0}^{\mathrm{in}}\{s, (g, I, t)\}$ associated to the vector field $\mathcal{R}_{e_0}$. To do this study, we use the perturbative expansions of $\mathcal{G}_{e_0}$ and $\mathcal{R}_{e_0}$ studied in Section 4.2.

Lemma 4.6. *The flow $\Phi^{\text{in}}_{e_0}\{s, (g, I, t)\}$ has a perturbative expansion*

$$\Phi^{\text{in}}_{e_0}\{s, (g, I, t)\} = \Phi^{\text{in}}_0\{s, (g, I, t)\} + e_0\Phi^{\text{in}}_1\{s, (g, I, t)\} + e_0^2\Phi^{\text{in}}_2\{s, (g, I, t)\} + \mathcal{O}\left(e_0^3\right),$$

which satisfies

$$\mathcal{N}\left(\Phi^{\text{in}}_1\{s, (g, I, t)\}\right) = \{\pm 1\} \tag{43}$$

$$\mathcal{N}\left(\Phi^{\text{in}}_2\{s, (g, I, t)\}\right) = \{0, \pm 1, \pm 2\}. \tag{44}$$

Proof. The zero order $\Phi^{\text{in}}_0\{s, (g, I, t)\}$ associated to $\mathcal{R}_0$ corresponds to the flow of the reduced circular problem (21) restricted to Λ_0. Taking into account (41), it is of the form

$$\Phi^{\text{in}}_0\{s, (g, I, t)\} = \left(g + s, I, \pi_t\Phi^{\text{in}}_0\{s, (g, I, t)\}\right). \tag{45}$$

Its first two components are independent of t and its third component satisfies

$$\pi_t\Phi^{\text{in}}_0\{s, (g, I, t)\} = t + \widetilde{\Phi}^{\text{in}}_0\{s, (g, I)\} \tag{46}$$

with $\widetilde{\Phi}^{\text{in}}_0$ independent of t.

We first prove (43). The e_0-order Φ^{in}_1 is a solution of the ordinary differential equation

$$\frac{d}{ds}\xi = D\mathcal{R}_0\left(\Phi^{\text{in}}_0\{s, (g, I, t)\}\right)\xi + \mathcal{R}_1\left(\Phi^{\text{in}}_0\{s, (g, I, t)\}\right)$$

with initial condition $\xi(0) = (0, 0, 0)$. $\mathcal{R}_0$ is independent of t and therefore, by (45),

$$D\mathcal{R}_0\left(\Phi^{\text{in}}_0\{s, (g, I, t)\}\right) = D\mathcal{R}_0\left(g + s, I\right),$$

which is independent of t too. In Lemma (4.5), we have seen $\mathcal{N}(\mathcal{R}_1) = \{\pm 1\}$ and therefore $\mathcal{R}_1$ can be written as

$$\mathcal{R}_1(g, I, t) = \mathcal{R}_1^+(g, I)e^{it} + \mathcal{R}_1^-(g, I)e^{-it},$$

Therefore, using (46),

$$\mathcal{R}_1\left(\Phi^{\text{in}}_0\{s, (g, I, t)\}\right) = \left(\mathcal{R}_1^+\left(g + s, I\right)e^{i\widetilde{\Phi}^{\text{in}}_0\{s, (g, I)\}}\right)e^{it} + \left(\mathcal{R}_1^-\left(g + s, I\right)e^{i\widetilde{\Phi}^{\text{in}}_0\{s, (g, I)\}}\right)e^{-it}.$$

To proof (43), it is enough to use variation of constants formula. Let us consider $M_{g,I}(s)$ the fundamental matrix of the linear ode

$$\frac{d}{ds}\xi = D\mathcal{R}_0(g + s, I)\xi,$$

then

$$\Phi^{\text{in}}_1\{s, (g, I, t)\} = \Phi^{\text{in},+}_1\{s, (g, I)\}e^{it} + \Phi^{\text{in},-}_1\{s, (g, I)\}e^{-it}$$

with

$$\Phi^{\text{in},\pm}_1\{s, (g, I)\} = M_{g,I}(s)\int_0^s M_{g,I}(\sigma)\left(\mathcal{R}_1^\pm\left(g + \sigma, I\right)e^{\pm i\widetilde{\Phi}^{\text{in}}_0\{s, (g, I)\}}\right)d\sigma.$$

The proof of (44) follows the same lines. Indeed, Φ^{in}_2 is solution of an ode of the form

$$\frac{d}{ds}\xi = D\mathcal{R}_0\left(g + s, I\right)\xi + \Xi(s, g, I, t)$$

with initial condition $\xi(0) = (0, 0, 0)$. The function Ξ is given in terms of the previous orders of $\mathcal{R}_{e_0}$ and $\Phi^{\text{in}}_{e_0}$ as

$$\Xi = \frac{1}{2}D^2\mathcal{R}_0\left(\Phi^{\text{in}}_0\right)\left(\Phi^{\text{in}}_1\right)^{\otimes 2} + D\mathcal{R}_0\left(\Phi^{\text{in}}_0\right)\Phi^{\text{in}}_2 + D\mathcal{R}_1\left(\Phi^{\text{in}}_0\right)\Phi^{\text{in}}_1 + \mathcal{R}_2\left(\Phi^{\text{in}}_0\right),$$

and then it satisfies $\mathcal{N}(\Xi) = \{0, \pm 1, \pm 2\}$.

Since the homogeneous linear ode is the same as for Φ^{in}_1 and does not depend on t, one can easily see that (44) is satisfied. $\qquad\square$

4.4 The inner map of the elliptic problem

In this section we study the inner map of the elliptic problem perturbatively from (18), taking e_0 as a small parameter. We use the e_0-expansions obtained in Sections 4.2 and 4.3.

The inner map is defined in the cylinder $\widetilde{\Lambda}_{e_0}$ in (28). We call it $\mathcal{F}_{e_0}^{\mathrm{in}} : \mathcal{C}_{e_0} \to \mathcal{C}_{e_0}$, and is defined as the (-14π)-Poincaré map of the inner flow $\Phi_{e_0}^{\mathrm{in}}$ defined in Lemma 4.6. Namely,

$$
\mathcal{F}_{e_0}^{\mathrm{in}} \left(\begin{array}{c} I \\ t \end{array} \right) = \left(\begin{array}{c} \Phi_{e_0}^{\mathrm{in},I}\{-14\pi, (0, I, t)\} \\ \Phi_{e_0}^{\mathrm{in},t}\{-14\pi, (0, I, t)\} \end{array} \right) \tag{47}
$$

We want to see which t-harmonics the first orders of the map has and also we want to compute the first order of the I component.

Lemma 4.7. *The inner map* (47) *is of the form,*

$$
\mathcal{F}_{e_0}^{\mathrm{in}} : \left(\begin{array}{c} I \\ t \end{array} \right) \mapsto \left(\begin{array}{c} I + e_0 A_1(I, t) + e_0^2 A_2(I, t) + \mathcal{O}\left(\mu e_0^3\right) \\ t + \mu \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I, t) + e_0^2 \mathcal{T}_2(I, t) + \mathcal{O}(\mu e_0^2) \end{array} \right). \tag{48}
$$

and functions A_1, A_2 $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$
\mathcal{N}\left(A_1\right) = \{\pm 1\}, \ \ \mathcal{N}\left(A_2\right) = \{0, \pm 1, \pm 2\} \tag{49}
$$
$$
\mathcal{N}\left(\mathcal{T}_1\right) = \{\pm 1\}, \ \ \mathcal{N}\left(\mathcal{T}_2\right) = \{0, \pm 1, \pm 2\}. \tag{50}
$$

Moreover A_1 can be split as,

$$
A_1(I, t) = A_1^+(I)e^{it} + A_1^-(I)e^{-it},
$$

with

$$
A^{\pm}(I) = -\mu \int_0^{-14\pi} \left. \frac{\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(L, \ell, G, g, G)}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(L, \ell, G, g, G)} e^{\pm it} \right|_{(L,\ell,G,g,G,t)=\Phi_0^{\mathrm{in}}\{s,(0,I,t)\}} ds, \tag{51}
$$

where the functions $\Delta H_{\mathrm{ell}}^{1,\pm}$ are defined as

$$
\Delta_{\mathrm{ell}}^1(L, \ell, G, g, t) = \Delta H_{\mathrm{ell}}^{1,+}(L, \ell, G, g)e^{it} + \Delta H_{\mathrm{ell}}^{1,\pm}(L, \ell, G, g)e^{-it}
$$

and Φ_0^{in} is the inner flow of the circular problem, which has been defined in Lemma 4.6.

Proof. Using the e_0-expansion of the inner flow given in Lemma 4.6, we have that

$$
A_1(I, t) = \Phi_1^{\mathrm{in},I}\{-14\pi, (0, I, t)\}, \ \ A_2(I, t) = \Phi_2^{\mathrm{in},I}\{-14\pi, (0, I, t)\}
$$
$$
\mathcal{T}_1(I, t) = \Phi_1^{\mathrm{in},t}\{-14\pi, (0, I, t)\}, \ \ \mathcal{T}_2(I, t) = \Phi_2^{\mathrm{in},t}\{-14\pi, (0, I, t)\}.
$$

Therefore, by (43) and (44), these functions satisfy (49) (50).

It only remains to compute explicitly $A_1(I, t)$. Using fundamental theorem of calculus and using the expansions of $\mathcal{R}_{e_0}^I$ and $\Phi_{e_0}^{\mathrm{in},I}$ given in Lemmas 4.5 and 4.6, one can see that

$$
A_1(I, t) = \int_0^{-14\pi} \mathcal{R}_1^I\left(\Phi_0^{\mathrm{in}}\{s, (0, I, t)\}\right) ds.
$$

Moreover, taking into account (45) and (46),

$$
A_1(I, t) = \int_0^{-14\pi} \mathcal{R}_1^I\left(s, I, t + \widetilde{\Phi}_0^{\mathrm{in}}\{s, (0, I)\}\right) ds.
$$

Since $\mathcal{N}\left(\mathcal{R}_1\right) = \{\pm 1\}$, it can be split as

$$
\mathcal{R}_1^I(g, I, t) = \mathcal{R}_1^{I,+}(g, I)e^{it} + \mathcal{R}_1^{I,-}(g, I)e^{-it}.
$$

Therefore, A_1 can also be split as,

$$A_1(I,t) = A_1^+(I,t)e^{it} + A_1^-(I,t)e^{-it}.$$

To compute each harmonic, it is enough to compute the following integrals which are independent on t,

$$A^{\pm}(I) = \int_0^{-14\pi} \mathcal{R}_1^{I,+}(s,I)e^{i\widetilde{\Phi}_0^{\mathrm{in}}\{s,(0,I)\}}ds$$

Taking into account the definition of $\mathcal{R}_1^I$ in (37) one can obtain (51). $\qquad\qquad\qquad\qquad\qquad\square$

4.5 The outer map of the elliptic problem

We devote this section to study the outer map

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \widetilde{\Lambda}_{e_0} \longrightarrow \widetilde{\Lambda}_{e_0} \tag{52}$$

for $e_0 > 0$.

Lemma 4.4 gives the existence of $\mathcal{C}_{e_0}$, a transversal intersection of the invariant manifolds of Λ_{e_0}. The Poincaré map (27) possesses then also a normally hyperbolic invariant manifold $\widetilde{\Lambda}_{e_0} = \Lambda_{e_0}$ and its invariant manifolds intersect transversally at $\widetilde{\Gamma}_{e_0} = \Gamma_{e_0} \cap \{g = 0\}$.

Then, Definition 3.4 also applies in this setting and we can define the outer map $\mathcal{F}_{e_0}^{\mathrm{out}}$. We want to study it as a perturbation of the outer map of the circular problem given in (20). To this end we use Poincaré-Melnikov techniques. As we have explained in Section 3.2, the original flow associated to Hamiltonian (12) does not allow us to study perturbatively $\mathcal{F}_{e_0}^{\mathrm{out}}$. Instead, we use the reduced elliptic problem defined in (26).

Lemma 4.8. *The outer map defined in (52) has the following expansion with respect to e_0,*

$$\mathcal{F}_{e_0}^{\mathrm{out}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 \left(\Omega^+(I)e^{it} + \Omega^-(I)e^{-it} \right) + \mathcal{O}\left(e_0^2\right) \\ t + \mu\omega_0(I) + \mathcal{O}(e_0) \end{pmatrix}. \tag{53}$$

Moreover, the functions $\Omega^{\pm}(I)$ can be defined as

$$\Omega^{\pm}(I) = -\lim_{T\to\infty} \int_0^T \left. \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\{s,\mathcal{C}_0(0,I,t)\}}$$

$$-\left. \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\{s,\mathcal{G}_0(0,I,t+\omega_0^+(I))\}} ds$$

$$+\lim_{T\to-\infty} \int_0^T \left. \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\{s,\mathcal{C}_0(0,I,t)\}}$$

$$-\left. \frac{\mu\partial_t \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)}{-1+\mu\partial_G \Delta H_{\mathrm{circ}}(\ell,L,g,G)} e^{\pm it} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}},\Phi_0^t)\{s,\mathcal{G}_0(0,I,t+\omega_0^-(I))\}} ds,$$

where

$$\Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G,t) = \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)e^{it} + \Delta H_{\mathrm{ell}}^{1,\pm}(\ell,L,g,G)e^{-it}$$

has been defined in Lemma 4.3 and $(\Phi_0^{\mathrm{circ}},\Phi_0^t)$ has been defined in (22) and (24).

Proof. To prove the lemma we use the Definition 3.4 of outer map. Let us consider points $z \in \widetilde{\Gamma}_{e_0}$ and $x_{\pm} \in \widetilde{\Lambda}_{e_0}$ such that

$$\mathrm{dist}\left(\mathcal{P}_{e_0}^n(z), \mathcal{P}_{e_0}^n(x_{\pm})\right) < C\lambda^{-|n|} \qquad \text{for } n \in \mathbb{Z}^{\pm}$$

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for certain constants $C > 0$ and $\lambda > 1$. Using the parameterizations of $\widetilde{\Gamma}_{e_0}$ and $\widetilde{\Lambda}_{e_0}$ given in Lemma 4.4, we can write the points z and $x_\pm$ in coordinates as $z = \mathcal{C}_{e_0}(0, I^*, t^*)$ and $x_\pm = \mathcal{G}_{e_0}(0, I^*_\pm, t^*_\pm)$. Then, the I-component of the outer map is just given by the relation

$$\mathcal{F}^{\mathrm{out},I}_{e_0}(I^*_-, t^*_-) = I^*_+ = I^*_- + (I^*_+ - I^*_-).$$

To measure $I^*_+ - I^*_-$ we first consider $I^* - I^*_\pm$. Define the flow Φ_{e_0} associated to the reduced elliptic problem (26). Then, applying Fundamental Theorem of Calculus

$$I^* - I^*_\pm = -\lim_{T \to \pm\infty} \int_T^0 \left(\frac{d}{ds} \Phi_{e_0}\left\{s, \mathcal{C}_{e_0}(0, I^*, t^*)\right\} - \frac{d}{ds} \Phi_{e_0}\left\{s, \mathcal{G}_{e_0}(0, I^*_\pm, t^*_\pm)\right\} \right) ds$$

Note that the change in front of the integral comes from the fact that system (26) has time reversed.

Using the perturbative expansions of $\mathcal{C}_{e_0}$ and Λ_{e_0} given in Lemma 4.4, (26) and also the perturbative expansion of the Hamiltonian (12) given in Lemma 4.3, one can see that

$$I^* - I^*_\pm = e_0 \lim_{T \to \pm\infty} \int_T^0 \left(\left. \frac{\mu \partial_t \Delta H^1_{\mathrm{ell}}(\ell, L, g, G, t)}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(\ell, L, g, G)} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}}, \Phi_0^t)\{s, \mathcal{C}_{e_0}(0, I^*_-, t^*_-)\}} \right.$$
$$\left. - \left. \frac{\mu \partial_t \Delta H^1_{\mathrm{ell}}(\ell, L, g, G, t)}{-1 + \mu \partial_G \Delta H_{\mathrm{circ}}(\ell, L, g, G)} \right|_{(\ell,L,g,G,t)=(\Phi_0^{\mathrm{circ}}, \Phi_0^t)\{s, \mathcal{G}_{e_0}(0, I^*_-, t^*_- + \omega_0^\pm(I^*_-))\}} \right) ds + \mathcal{O}\left(e_0^2\right).$$

Taking into account that in Lemma 4.3 we have proved that $\Delta H^1_{\mathrm{ell}}$ satisfies $\mathcal{N}\left(\Delta H^1_{\mathrm{ell}}\right) = \{\pm 1\}$, one can easily obtain (53). $\qquad\square$

5 Existence of diffusing orbits

Once we have computed the first orders in e_0 of both the outer and inner maps, we want to compare its dynamics. To make this comparison we first perform two steps of averaging [AKN88, LM88]. This change of coordinates straightens the I-component of the inner map at order $\mathcal{O}(e_0^3)$ in such a way that with the new system of coordinates is easier to compare the dynamics of both maps.

Lemma 5.1. *There exists a e_0-close to the identity symplectic change of variables*

$$(I, t) = (\overline{I}, \overline{t}) + e_0 \varphi(\overline{I}, \overline{t}, e_0) \tag{54}$$

defined on $\widetilde{\Lambda}_{e_0}$, which transforms the inner and outer maps $\mathcal{F}^{\mathrm{in}}_{e_0}$ in (48) and $\mathcal{F}^{\mathrm{out}}_{e_0}$ in (53) into

$$\widetilde{\mathcal{F}}^{\mathrm{in}}_{e_0} : \begin{pmatrix} \overline{I} \\ \overline{t} \end{pmatrix} \mapsto \begin{pmatrix} \overline{I} + \mathcal{O}\left(\mu e_0^3\right) \\ \overline{t} + \mu \mathcal{T}_0\left(\overline{I}\right) + e_0^2 \widetilde{\mathcal{T}}_2\left(\overline{I}\right) + \mathcal{O}\left(\mu e_0^3\right) \end{pmatrix} \tag{55}$$

and

$$\widetilde{\mathcal{F}}^{\mathrm{out}}_{e_0} : \begin{pmatrix} \overline{I} \\ \overline{t} \end{pmatrix} \mapsto \begin{pmatrix} \overline{I} + e_0 \widetilde{\Omega}(\overline{I}, \overline{t}) + \mathcal{O}\left(\mu e_0^2\right) \\ \overline{t} + \mu \omega_0(\overline{I}, \mu) + \mathcal{O}(\mu e_0) \end{pmatrix}, \tag{56}$$

where

$$\widetilde{\Omega}\left(\overline{I}, \overline{t}\right) = \widetilde{\Omega}^+\left(\overline{I}\right) e^{i\overline{t}} + \widetilde{\Omega}^-\left(\overline{I}\right) e^{-i\overline{t}}$$

with

$$\widetilde{\Omega}^\pm\left(\overline{I}\right) = \Omega^\pm\left(\overline{I}\right) - \frac{e^{\pm i\omega_0(I)} - 1}{e^{\pm i\mathcal{T}_0(I)} - 1} A_1^\pm\left(\overline{I}\right).$$

Moreover, these functions satisfy

$$\widetilde{\Omega}^\pm\left(\overline{I}\right) \neq 0 \qquad for \ \overline{I} \in [I_-, I_+] \tag{57}$$

Note that we can do two steps of averaging globally in the whole cylinder $\widetilde{\Lambda}_{e_0}$ due to the absence of resonances in the first orders in e_0. Namely, there are no *big gaps* (see [DdlLS06a, Tre04, CY04]).

Proof. We perform two steps of averaging. To this end we consider a generating function of the form

$$\mathcal{S}(\overline{I}, t) = \overline{I}t + e_0 \mathcal{S}_1(\overline{I}, t) + e_0^2 \mathcal{S}_2(\overline{I}, t),$$

which defines the change (54) implicitly as

$$I = \overline{I} + e_0 \partial_t \mathcal{S}_1(\overline{I}, t) + e_0^2 \partial_t \mathcal{S}_2(\overline{I}, t)$$
$$\overline{t} = t + e_0 \partial_{\overline{I}} \mathcal{S}_1(\overline{I}, t) + e_0^2 \partial_{\overline{I}} \mathcal{S}_2(\overline{I}, t).$$

By Theorem 3.3 we have that (19) and by Lemma 4.7 we know which t-harmonics have the functions A_i and $\mathcal{T}_i$. Then, it is easy to see that the functions $\mathcal{S}_i$ that correspond to two steps of averaging are globally defined in $\widetilde{\Lambda}_{e_0}$. Moreover, in these new variables, taking into account that the inner map is exact symplectic, one can see that the inner map is of the form (55).

To obtain a perturbative expression for the outer map $\widetilde{\mathcal{F}}_{e_0}^{\mathrm{out}}$, we need to know $\mathcal{S}_1$ explicitly. It is given by

$$\mathcal{S}_1(\overline{I}, t) = -\frac{i A_1^+(\overline{I})}{e^{i\mu \mathcal{T}_0(\overline{I})} - 1} e^{it} + \frac{i A_1^-(\overline{I})}{e^{-i\mu \mathcal{T}_0(\overline{I})} - 1} e^{-it}.$$

If we apply this change to the outer map $\mathcal{F}_{e_0}^{\mathrm{out}}$ in (53), we obtain (56).

The proof of (57) is computer assisted. □

In the new coordinates $(\overline{I}, \overline{t})$ the inner map $\mathcal{F}_{e_0}^{\mathrm{in}}$ in (55) is a e_0^3-close to integrable map. Moreover, thanks to Theorem 3.3 is twist and therefore we can apply KAM theory to proof the existence of invariant curves in $\widetilde{\Lambda}_{e_0}$.

Theorem 5.2. *Vadim: which KAM theorem shall we use? In Delshams-Llave-Seara use one by Herman which is for $\mathcal{C}^k$ maps. Would be nice to use one for analytic maps?*

...one can see that any invariant curve Γ that it must satisfy

$$|\Gamma - I^*| = \mathcal{O}\left(e_0^{3/2}\right),$$

for certain $I^* \in (I_-, I_+)$.

The results of Lemma 5.1 and Theorem 5.2 lead to the existence of a transition chain of invariant tori.

Lemma 5.3. *The Poincaré map (16) has a sequence of invariant tori $\{\tau_i\}_{i=1}^N \subset \widetilde{\Lambda}_{e_0}$ such that*

- $\tau_1 \cap \{I = I_-\} \neq \emptyset$ *and* $\tau_N \cap \{I = I_+\} \neq \emptyset$.

- *It forms a transition chain. Namely*

$$W_{\tau_i}^u \pitchfork W_{\tau_{i+1}}^s \neq \emptyset.$$

Proof. The transition chain is obtained comparing the outer and inner dynamics. We do this comparison in the coordinates $(\overline{I}, \overline{t})$ given by Lemma 5.1 and therefore we deal with the maps $\widetilde{\mathcal{F}}_{e_0}^{\mathrm{in}}$ and $\widetilde{\mathcal{F}}_{e_0}^{\mathrm{out}}$ in (55) and (56) respectively.

Theorem 5.2 ensures that there exists a torus τ_1 such that $\tau_1 \cap \{I = I_-\} \neq \emptyset$. Then, thanks to (57), $\mathcal{F}_{e_0}^{\mathrm{out}}(\tau_1)$,

$$\mathrm{dist}\left(\tau_1, \mathcal{F}_{e_0}^{\mathrm{out}}(\tau_1)\right) \geq C e_0$$

for a constant $C > 0$ independent of e_0. Then, Theorem 5.2 ensures that there exists a torus τ_2 such that $\tau_2 \cap \mathcal{F}_{e_0}^{\mathrm{out}}(\tau_1) \neq 0$. Iterating this procedure, one obtains the transition chain. □

Vadim: Here we should talk about shadowing. I'm not sure which shadowing methods fits better.

6 From cartesian to Delaunay and computation of $\partial_G \Delta H_{\text{circ}}$

Here we explain an easy way to obtain the rotating Delaunay coordinates from rotating cartesian (or polar) coordinates in the circular problem. First recall that G can be computed as

$$G = r \left(-p_x \sin \phi + p_y \cos \phi \right).$$

In a fixed level of energy J, knowing G one can obtain L. Recall that

$$J = -\frac{1}{2L^2} - G + \mu \Delta H_{\text{circ}}$$

The term $\mu \Delta H_{\text{circ}}$ in cartesian only depends on x and y, and can be easily computed. Then, one can use this equation to obtain L. Knowing L and G we can obtain the eccentricity e by

$$e = \sqrt{1 - \frac{G^2}{L^2}}.$$

Using that $r = L^2(1 - e \cos u)$ one can obtain u and from here ℓ using the formula $u - e \sin u = \ell$. On the other hand, from u we can obtain v using

$$\tan \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2}$$

Finally, we can deduce g using that $\phi = v + g$.

We devote the rest of the section to compute $\partial_G \Delta H_{\text{circ}}$. Let us define

$$N(r, v, g) = \frac{1}{(r^2 + 1 - 2r \cos(v + g))^{1/2}}.$$

Then,

$$\Delta H_{\text{circ}}(L, \ell, G, g) = -\frac{1 - \mu}{\mu} N \left(\frac{r(L, \ell, G)}{\mu}, v(L, \ell, G), g \right) - \frac{\mu}{1 - \mu} N \left(\frac{r(L, \ell, G)}{1 - \mu}, v(L, \ell, G), g \right) + \frac{1}{r(L, \ell, G)}.$$

Therefore

$$\partial_G \Delta H_{\text{circ}}(L, \ell, G, g) = - \left[\frac{1 - \mu}{\mu^2} \partial_r N \left(\frac{r(L, \ell, G)}{\mu}, v(L, \ell, G), g \right) + \frac{\mu}{(1 - \mu)^2} \partial_r N \left(\frac{r(L, \ell, G)}{1 - \mu}, v(L, \ell, G), g \right) \right] \partial_G r(L, \ell, G)$$

$$- \left[\frac{1 - \mu}{\mu} \partial_v N \left(\frac{r(L, \ell, G)}{\mu}, v(L, \ell, G), g \right) + \frac{\mu}{1 - \mu} \partial_v N \left(\frac{r(L, \ell, G)}{1 - \mu}, v(L, \ell, G), g \right) \right] \partial_G v(L, \ell, G)$$

$$- \frac{1}{r^2} \partial_G r(L, \ell, G).$$

In only remains to compute $\partial_G r$ and $\partial_G v$. First, let us point out that

$$\partial_G e = -\frac{G}{eL^2} = \frac{e^2 - 1}{eG}.$$

On the other hand, using that $\ell = u - e \sin u$, one has that

$$\partial_e u = \frac{\sin u}{1 - e \cos u}.$$

Then, since $r(L, e, \ell) = L^2(1 - e \cos u(e, \ell))$, using that

$$\cos v = \frac{\cos u - e}{1 - e \cos u}, \tag{58}$$

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we have that

$$\partial_e r(L, e, \ell) = L^2 \cos v$$

and therefore,

$$\partial_G r(L, \ell, G) = -\frac{G \cos v}{e}.$$

To compute $\partial_G v$ let us point out that $\partial_G v = \partial_e v \partial_G e$. Therefore it only remains to compute $\partial_e v$, we obtain it using formula (58) and

$$\sin v = \frac{\sqrt{1 - e^2} \sin u}{1 - e \cos u}.$$

Then,

$$\partial_e v = \frac{\sin v}{1 - e^2} \left(2 + e \cos v\right).$$

and therefore,

$$\partial_G v = -\frac{\sin v}{eG} \left(2 + e \cos v\right).$$

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