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Risk-sharing and crises. Global games of regime change with endogenous wealth ☆

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Abstract

I add heterogeneous agents and risk-sharing opportunities to a global game of regime change. The novel insight is that when there is a risk-sharing motive, fundamentals drive not only individual behavior, but also select which individuals are more relevant for the likelihood of a crisis because of endogenous shifts in wealth. If attacking is relatively safe, attack behavior in the global game and trade in state-contingent assets feed back into each other. This feedback implies that multiple equilibria may exist even if signal noise becomes arbitrarily small. In addition, heterogeneity in risk-aversion within the population amplifies the influence of the state of the economy on the probability of a crisis.

JEL classification: D8; G2; E3

Keywords: Global games; Risk-aversion; Heterogeneous agents; Risk-sharing; Financial crises; Strategic risk

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1. Introduction

Global games of regime change are coordination games of incomplete information in which a status quo is abandoned once a sufficiently large fraction of agents attacks it [1,18]. These games have been used to study phenomena, such as currency attacks, bank runs, and debt crises, where the abandonment of the status quo is referred to as a "crisis". In the literature on global games the effect of the aggregate state of the economy on the occurrence of a crisis is modeled through a random variable representing fundamentals. Fundamentals are low in bad times and high in good times. A crisis occurs if the aggregate attack by the population is large relative to fundamentals.

So far, the literature has mostly dealt with a population which is homogeneous in terms of preferences. I show that if agents who differ in their preferences towards risk can trade in state-contingent assets, then there may be multiple equilibria even in the limit in which signal noise becomes arbitrarily small. Moreover, I show that the effect of the state of the economy on the likelihood of a crisis is enhanced.

I model a population which is diverse in terms of risk-aversion and faces exogenous aggregate endowment risk. This population is allowed to trade claims to the aggregate endowment. Once the endowment has been realized, agents play a global game of regime change. In the global game I assume that attacking is less risky than not attacking. The application I have in mind is that of a bank run game. Attacking corresponds to withdrawing money from the bank, which is safer than leaving it in the bank. If a large enough fraction of the population withdraws their money, the attack on the bank succeeds.

The intuition for what happens in the model is the following. Agents, who face aggregate endowment risk and differ in their risk-aversion, benefit from trade. More risk-averse agents have a higher willingness to pay for consumption in low aggregate endowment states and, in equilibrium, end up holding a larger fraction of claims that pay off in those states. Thus, if a low endowment state is realized, the economy's wealth will be concentrated in the hands of the more risk-averse segment of the population. Agents who are more risk-averse are also more likely to use their wealth to attack the status quo. In consequence, it is more likely that the attack succeeds in a low endowment state. In turn, a higher probability of the attack succeeding lowers the return of transferring wealth into such a state. Agents who are less risk-averse, and therefore less concerned with equalizing consumption across states, shift wealth out of the low endowment state, leaving a still larger fraction of wealth in the hands of the more risk-averse.

The feedback between the risk-sharing decision and the attack behavior in the global game is the driver behind the results of the paper. Even if the signal noise in the global game is arbitrarily small, endogenous shifts in wealth may be strong enough to overturn the uniqueness result that is usually obtained. Moreover, the endogenous shifts in wealth amplify the effect of the state of the world on the probability that an attack succeeds.

Heterogeneity in risk-aversion is an ubiquitous phenomenon which has been documented in several studies, for different time frames, using varying empirical techniques, and for several population groups in different countries. Barsky et al. [3] and Chiappori et al. [5], for example, respectively report significant dispersions in the degree of risk-aversion in populations as different as US households and Thai villagers. Mazzocco [17] finds heterogeneity in risk-aversion even within households. Adding heterogeneity in risk-aversion to a global game of regime change is therefore a modification backed by empirical evidence.

My model builds on the previous literature. Guimarães and Morris [13] study a related model of regime change in which agents are risk-averse. Most of their article deals with agents who share the same level of risk-aversion. In one of the sections they do, however, consider the case of an ex-ante heterogeneous population and find that the threshold for fundamentals that trigger an attack can be elegantly expressed as a weighted average of the thresholds that would prevail in separate economies for each ex-ante type. If their result were applied to a population with differing parameters in a CRRA utility function, then the weights would be the wealth owned by each type scaled by each type's size in the population. I show that allowing for wealth to be endogenously determined, modifies the results qualitatively. Once the wealth distribution at the global game stage becomes endogenous, the weights, which are fixed in [13], interact with the determination of the threshold as there is feedback between the risk-sharing decision and the decision to attack in the global game. In consequence, allowing signals to become arbitrarily precise no longer provides a sufficient condition for equilibrium uniqueness.

The models presented in [2,15,24] are similar to my model in that agents participate in a financial market before playing the global game. Those models are concerned with how the existence of public information affects the multiple vs. unique equilibrium question. Therefore, the model of the financial market is different from mine. They model an asset that serves the purpose of improving the amount of public information in the model: the asset traded in the financial market depends on the fundamentals. The equilibrium price therefore helps in the inference problem and eases coordination, increasing the degree of complementarities. In this way it helps in restoring multiplicity. In my model, the reason for having a financial market before participating in the global game is unrelated to the inference problem. The financial market is used exclusively for risk-sharing between ex-ante heterogeneous agents.

Goldstein and Pauzner [10] construct a model in which the same agents participate in two consecutive coordination games. The action of each agent coupled with the occurrence of a crisis in the first game affects individual wealth. Given their assumption of decreasing absolute risk-aversion, individual wealth affects the degree of individual (absolute) risk-aversion, and consequently, individual behavior of agents at the second coordination game. In my paper, individual risk appetite (of which the relevant measure given my setup is relative risk-aversion) remains stable throughout the game. It is a change in "market risk-aversion" embedded in equilibrium asset prices at the first stage, and the selection of a risk-aversion profile through a cross-sectional shift in wealth at the second stage which delivers the feedback between the two stages.

I set up the model in Section 2, then goes through an example in Section 3, before considering the general case in Section 4. Section 5 applies the model to a bank run game adapted from [8] and briefly discusses how the main assumptions of the model can be relaxed.

2. The model

There are three periods $t = 0, 1, 2$. In period $t = 0$, agents are allowed to trade in state-contingent assets. These assets are complete with respect to a set of aggregate states S. In $t = 1$ the aggregate state $s \in S$ is realized. State s determines the aggregate endowment through the function $e : S \to \mathbb{R}$. Without loss of generality, I order the states of the world from lowest to highest according to the aggregate endowment they are mapped into. In the final period $t = 2$ agents play a global game of regime change. The specifics of this game may depend on the realization s.

Agents derive utility exclusively from consuming in period $t = 2$. There is a single consumption good, so that utility can be expressed in terms of terminal wealth w_T. Agents are risk-averse and have CRRA preferences.

$$u(w_T, \rho) = \frac{w_T^{1-\rho}}{1-\rho}, \quad \rho \neq 1,\ \rho > 0 \tag{1}$$

and $u(w_T, 1) = \log w_T$. A type is a level of relative risk-aversion. The population consists of I types $\{\rho_1, \ldots, \rho_I\}$. For notational convenience, order types in increasing levels of risk-aversion, i.e. $0 < \rho_1 < \cdots < \rho_I$. There is a continuum of measure one of each type. These types differ only in terms of their preferences and are equal in all other respects. In particular, they all share equally in the aggregate endowment available at $t = 1$. Each agent is entitled to $e(s)/I$.[1]

Let g be the probability density defined over S. At $t = 0$, agents are allowed to trade in state contingent claims that pay off at $t = 1$. Let $q(s)$ be the price of a state-contingent claim that pays off in state s. Instead of explicitly introducing notation for the set of state-contingent claims, I write the budget constraint in terms of wealth $w(s, \rho)$ taken into state s by an agent of type ρ:

$$\int_S q(s)w(s,\rho)\,ds \leqslant \frac{1}{I}\int_S q(s)e(s)\,ds. \tag{2}$$

2.1. Global game of regime change

In period $t = 2$ agents know which aggregate state s has been realized and have to decide how much of their wealth they use to attack the regime. Let a be the amount of wealth (and, for later use, let α be the *fraction* of wealth) that is used to attack the regime by an individual agent. Denote as K the aggregation of a over the whole population, and let θ be the "fundamentals" of the economy. Writing e for the level of the aggregate endowment, an attack succeeds and the status quo is abandoned if and only if

$$\frac{K}{e} \geqslant \theta. \tag{3}$$

The left hand side measures the intensity of an attack. Individual decisions a are a function of individual wealth. Since individual wealth scales up with the aggregate endowment, dividing by e measures an "intensity" of attack rather than an absolute value.[2] I refer to the abandonment of the status quo (i.e. the realization of (3)) as a "crisis".

Fundamentals are a random variable with a Normal distribution.

$$\theta \sim N\big(\mu(s), \sigma_\theta\big), \quad \frac{d\mu(s)}{ds} \geqslant 0. \tag{4}$$

If $\frac{d\mu(s)}{ds} = 0$, then the global game is the same in all states, as fundamentals do not depend on the aggregate state. If the derivative is positive, then fundamentals are stronger in better states of the world as the distribution shifts upward. Agents know the distribution of θ but not its realization. They individually observe a private signal x.

[1] An anonymous referee notes that the assumption that there is a continuum of agents of each type and that they share equally in the aggregate endowment is not as restrictive as it may seem. To see why define a function F such that the proportion of the endowment held by agents with risk-aversion coefficients below ρ is $F(\rho)$. Now partition the continuum of agents into a sequence of I sets, each containing the same aggregate endowment, and replace the risk-aversion coefficients of all agents in a given set i by the mean or median risk-aversion coefficient within set i. The result is functionally equivalent to my model. The only qualitative restriction that is imposed is that F is independent of the state.

[2] Notice that there is an additional degree of freedom, as I allow fundamentals on the right hand side to depend on s. Because the distribution of θ depends on s, the interpretation of an intensity can be converted into an absolute value interpretation by an appropriate choice of $\mu(s)$ (defined below). Also, focusing on an intensity of attack is natural in the case of bank runs, since what matters is the fraction of deposits withdrawn from the system, not its absolute magnitude.

$$x = \theta + \sigma_\varepsilon \varepsilon, \quad \varepsilon \sim N(0, 1). \tag{5}$$

The parameter σ_ε is a measure of how precise the signal is. In several parts of the paper I will focus on the limiting case in which $\sigma_\varepsilon \to 0$, and signals become arbitrarily precise. Notice that after observing the signal agents differ in two dimensions: their type of risk-aversion ρ and their informational type, x. A strategy is a map $a(x, \rho, s)$ with $a : (x, \rho, s) \mapsto [0, w(s, \rho)]$. Attacking is a safe choice and yields a gross return of 1. Not attacking yields an uncertain return. If a crisis occurs, then the return is f. If no crisis occurs, then the return is R. These parameters satisfy

$$R > 1 > f > 0. \tag{6}$$

Notice that returns are asymmetric in terms of the riskiness of the available actions. Not attacking the status quo yields a risky return. Attacking earns a safe return.[3] Agents are allowed to choose their action on a continuum. Because of risk-aversion, alternatives between the corner solutions of a full attack and no attack need to be included.

Terminal wealth w_T is generated from decisions a and start-of-period-2 wealth w. There are two outcomes, $\omega \in \{n, c\}$, where c denotes the occurrence of a crisis and n the opposite.

$$w_T(\omega) = \begin{cases} a + R(w - a) & \text{if } \omega = n, \\ a + f(w - a) & \text{if } \omega = c. \end{cases} \tag{7}$$

2.2. *Equilibrium*

An equilibrium of the model consists of profiles of wealth choices $\mathbf{w}^* = \{w^*(s, \rho_i)\}_{i=1}^I$, of strategies $\mathbf{a}^* = \{a^*(x, \rho_i, s)\}_{i=1}^I$ and state-prices $\mathbf{q}^* = \{q^*(s)\}$ such that

(i) given the probability of a crisis induced by $\mathbf{a}^*$, $(\mathbf{w}^*, \mathbf{q})^*$ is a Walrasian equilibrium of the market in state-contingent claims at $t = 0$,
(ii) given $\mathbf{w}^*$, and for all realizations of $s \in S$, the strategy profile $\mathbf{a}^*$ is a Bayesian Nash equilibrium of the global game at $t = 2$.

The steps used to find an equilibrium are the following. Start at date $t = 2$, the global game stage. At date $t = 2$ a particular state $\tilde{s}$ will have realized. The Walrasian equilibrium of date $t = 0$ together with the realization of the state imply $\{w^*(\tilde{s}, \rho_i)\}_{i=1}^I$, a realized wealth distribution at date $t = 2$. Agents of all types need to choose an attack action $a(x, \rho_i, \tilde{s}) \in [0, w(\tilde{s}, \rho_i)]$ to maximize the expectation of (1) taking into account (7) and the crisis condition (3). They play a Bayesian Nash equilibrium. This means that the beliefs used in the maximization are derived using Bayes rule from the information structure (4) and (5) together with equilibrium actions. The solution method used so far is standard in global games models. The only difference is that the Bayesian Nash equilibrium is solved for all $\tilde{s} \in S$ and depends on $\mathbf{w}^*$ which is, at this point, still unspecified.

[3] This modeling choice holds in several interpretations of what the global game is. For example, in a bank run, withdrawing is safer than leaving deposits in the bank, in a currency crisis, switching to foreign currency is safer than holding local assets for the local population which is long in the local currency, in a debt crisis, not refinancing is safer than refinancing, in a run on the payments system, withdrawing and resorting to storage is safer than leaving funds in the compromised system. See [13] for a setup in which the relative riskiness of attacking is endogenous. Notice also that agents cannot "escape" from playing the global game and must, by assumption, participate in it.

The profile of actions from the global game at date $t = 2$ implies a probability of a crisis. At date $t = 0$ agents face a decision problem. They maximize the expectation of (1) (where the expectation is taken over states) subject to the budget constraint in (2). The solutions of this decision problem together with the aggregate consistency condition

$$\sum_{i=1}^{I} w^*(s, \rho_i) \leqslant e(s), \quad \forall s \in S, \tag{8}$$

yield a Walrasian equilibrium $(\mathbf{w}^*, \mathbf{q})^*$.[4]

3. An example

I will consider a simplified example featuring only two states and two types of risk-aversion in the population. I return to the general case later. The example in this section highlights two of the features of equilibria of the general case. It shows that (1) multiple equilibria are possible even if signal noise σ_ε vanishes, and that (2) the effect of changes in fundamentals on the probability of a crisis is amplified by shifts in the wealth distribution. An additional benefit of the example is that equilibria are explicitly solved for.

In the example there are two equally likely states, state H with a high endowment $e(H)$ and state L with a low endowment $e(L)$. Define the difference as $\Delta_e \equiv e(H) - e(L) > 0$. The population consists of only two types: a risk-neutral type ($\rho = 0$) and an infinitely risk-averse type ($\rho = \infty$). Both types are entitled to one-half of the aggregate endowment.

To characterize the equilibrium it is necessary to figure out the intensity of an attack by the two types in the population. I will denote the intensity of an attack (defined as the fraction of wealth used to attack) by type ρ as A_ρ.

The decision to attack for risk-averse types is trivial. Infinitely risk-averse types will always attack as long as there is some chance of the status quo being abandoned. If p is the probability of the status quo being maintained, then $a(s, \infty) = w(s, \infty)$ as long as $p < 1$. Since any finite signal entails a positive probability of abandoning the status quo, $p < 1$ for all infinitely risk-averse agents. The intensity of the attack by risk-averse agents (i.e. the fraction of wealth in the hands of infinitely risk-averse types that is devoted to the attack) is one:

$$A_\infty = 1. \tag{9}$$

Risk neutral types, on the other hand, maximize the expected return of their wealth. This means that they compare the return from attacking to the expected return of not attacking. They will attack if and only if

$$pR + (1 - p)f < 1, \tag{10}$$

where I have assumed that they do not attack when indifferent. If they strictly prefer to attack, then they will use all their wealth in the attack. Expressing the behavior of risk-neutral types in terms of the fraction of wealth used to attack (denoted by α) rather than the level, the optimal fraction of wealth used to attack, as a function of their belief p, is

[4] Since agents of a given type ρ are identical at date $t = 0$, in the aggregate consistency condition I have already used the fact that agents of the same type will be willing to make the same choices.

$$\alpha(p,0)=\begin{cases}1 & \text{if } p\leqslant \underline{p}\equiv\frac{1-f}{R-f},\\ 0 & \text{if } p>\underline{p}\equiv\frac{1-f}{R-f}.\end{cases} \tag{11}$$

To obtain A_0, the aggregate attack intensity of risk-neutral agents, we need to integrate $\alpha(p,0)$ over p. The value of p for an individual is a function of the signal she has observed, as well as her beliefs about the actions played by other agents in equilibrium. Since a fraction of the risk-neutral types will receive signals that generate beliefs $p>\underline{p}$, not all risk-neutral types will attack. Thus, the intensity of the attack by risk-neutral types is less than one:

$$A_0(\theta)<A_\infty=1. \tag{12}$$

From (3), there will be a crisis if and only if

$$\frac{K(s)}{e(s)}\equiv\frac{w(s,0)}{e(s)}A_0(\theta)+\frac{w(s,\infty)}{e(s)}A_\infty\geqslant\theta,\quad s\in\{H,L\} \tag{13}$$

or, equivalently, using the fact that wealth shares sum to one, and the result that $A_\infty=1$ from (9),

$$\left(1-\frac{w(s,\infty)}{e(s)}\right)A_0(\theta)+\frac{w(s,\infty)}{e(s)}\geqslant\theta,\quad s\in\{H,L\}. \tag{14}$$

The threshold above which an attack succeeds is calculated by asking when the size of the attack (which depends on the threshold) is equal to the threshold.[5]

$$\theta_s^*=\left(1-\frac{w(s,\infty)}{e(s)}\right)A_0\bigl(\theta_s^*\bigr)+\frac{w(s,\infty)}{e(s)},\quad s\in\{H,L\}. \tag{15}$$

The value of a threshold therefore depends on the amount of wealth in the hands of each type of agent. The right hand side is an increasing function of the fraction of wealth in the hands of the infinitely risk-averse group in the population. Equivalently, it is a decreasing function of the fraction of wealth in the hands of the risk-neutral group in the population.

3.1. Trade in state-contingent claims and the probability of a crisis

In equilibrium the risk-averse will hold more wealth in the low state and the risk-neutral will hold more wealth in the high state. The type with infinite risk-aversion will be insured by the risk-neutral type.

The argument can be made graphically in the Edgeworth box in Fig. 1. The lower left corner represents the origin for the infinitely risk-averse type. The opposite corner is that of the risk-neutral type. Trade in state-contingent claims necessarily produces an allocation on the contract curve. Given that indifference curves for the infinitely risk-averse type are Leontief, this implies that, in this case, the contract curve is the 45° line in Fig. 1. Since both types initially start by owning one half of the aggregate endowment, trade must deliver an allocation within the upper left triangle of the Edgeworth box. In consequence, the following relationships hold.

$$\frac{w(H,\infty)}{e(H)}<\frac{1}{2}<\frac{w(H,0)}{e(H)}\quad\text{and}\quad\frac{w(L,\infty)}{e(L)}>\frac{1}{2}>\frac{w(L,0)}{e(L)}. \tag{16}$$

Because of trade in state-contingent claims, state L has a larger fraction of the endowment in the hands of the more risk-averse group. Since this group is more likely to attack than the risk-neutral group, the probability of a crisis is higher in state L than in state H.

[5] The proof that this equation characterizes a threshold equilibrium is given in Lemma 3 (for the general case).

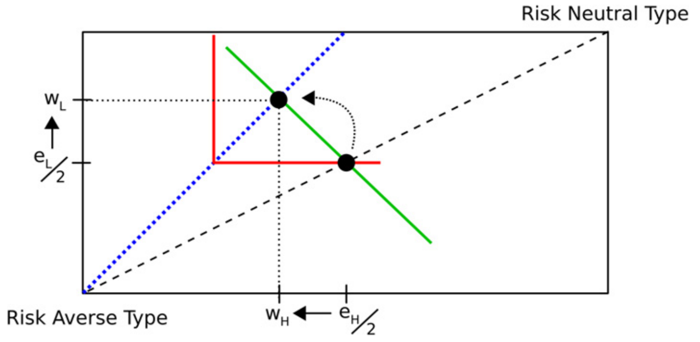


Fig. 1. The horizontal axis measures wealth transferred into state H, the vertical axis measures wealth transferred into state L. Trade in state-contingent claims produces an allocation that is in the upper left quarter of the Edgeworth box. This implies that in state L, the more risk-averse agent owns a higher fraction of the aggregate endowment. In state H this is reversed.

The probability of a crisis is equal to the probability that fundamentals are lower than the threshold values θ_L^* and θ_H^*. In terms of thresholds a higher probability of a crisis means in that θ_L^* is higher than θ_H^*. Mathematically, the value of the thresholds is a weighted average of the aggregate attack decisions by both types. The weight put on the decision of the risk-averse is larger in state L. This implies that the value of θ_s^* that solves (15) is larger in state L than in state H.[6]

3.2. Equilibrium multiplicity in the example

A standard result in the global games literature is that the equilibrium is unique if $\sigma_\varepsilon \to 0$. This was shown by [19] for the case in which agents are ex-ante homogeneous, and by [13] for the case in which they differ in their level of risk-aversion. When wealth is endogenous, and agents are able to deal with state-contingent claims, this result fails to hold. This subsection demonstrates that there may be multiple equilibria even if $\sigma_\varepsilon \to 0$.

For $\sigma_\varepsilon \to 0$ the distribution of beliefs over p for an agent who observes a signal at the threshold when $\theta = \theta_s^*$ converges to a uniform distribution. This result is formally shown for the general case in the proof of Theorem 2 and is omitted here. The fraction of wealth used to attack by the risk-neutral types is

$$\lim_{\sigma_\varepsilon \to 0} A_0(\theta) = \int_{p=0}^{1} \alpha(p,0)\,dp = \int_{p=0}^{\underline{p}} 1\,dp = \underline{p} \equiv \frac{1-f}{R-f} < 1. \tag{17}$$

The equation for the threshold in the limit, when $\sigma_\varepsilon \to 0$, thus becomes

$$\theta_s^* = \frac{1-f}{R-f} + \frac{w(s,\infty)}{e(s)}\left(\frac{R-1}{R-f}\right), \quad s \in \{H, L\}. \tag{18}$$

[6] Notice that this result does not require that the distribution of θ is shifted towards lower values in state L. The result holds even if expected fundamentals do not differ between the two states, i.e. even if $\mu(L) = \mu(H)$.

Table 1
Numerical solutions to Eqs. (18) through (21) when $R = 2$, $f = 0.8$, $\mu(H) = \mu(L) = 0.82$, $e(H) = 2$, $e(L) = 1$, $\sigma_\varepsilon \to 0$, and $\sigma_\theta = 0.01$.

	θ_H^*	θ_L^*	$\psi(H)$	$\psi(L)$	q_H/q_L
Equilibrium I	0.47923	0.79179	0.00000	0.00239	1.00120
Equilibrium II	0.49291	0.81915	0.00000	0.46621	1.30396
Equilibrium III	0.51389	0.86111	0.00000	0.99998	1.99996

The infinitely risk-averse agent will choose to take the same amount of wealth into both states:

$$w \equiv w(H, \infty) = w(L, \infty) = \frac{1}{2} \frac{\frac{q_H}{q_L} e(H) + e(L)}{\frac{q_H}{q_L} + 1}. \tag{19}$$

State prices can be solved for from the risk-neutral agent's maximization problem. When $\sigma_\varepsilon \to 0$ and when the true θ is not equal to θ_s^*, errors in predicting whether there will be a crisis or not tend to zero. This implies that payoffs when $\theta \neq \theta_s^*$ are going to be either R or 1, but almost never f. The indirect utility for the risk-neutral agent converges to

$$\big[R - \psi(H)(R-1)\big] w(H, 0) + \big[R - \psi(L)(R-1)\big] w(L, 0) \tag{20}$$

where $\psi(s) \equiv \Phi(\frac{\theta_s^* - \mu(s)}{\sigma_\theta})$. Hence, at an interior equilibrium, state-prices satisfy

$$\frac{q_H}{q_L} = \frac{R - \psi(H)(R-1)}{R - \psi(L)(R-1)}. \tag{21}$$

An interior equilibrium is the solution to the system of Eqs. (18) through (21). The solution to this system of equations is not necessarily unique. Consider the following parameterization: $R = 2$, $f = 0.8$, $\mu(H) = \mu(L) = 0.82$, $e(H) = 2$, $e(L) = 1$, $\sigma_\varepsilon \to 0$, and $\sigma_\theta = 0.01$. With these parameters there are three equilibria. The numerical solution is given in Table 1.

Notice that equilibria in the parameterized example are qualitatively very different. While the probability of a crisis in state H is close to zero in all three equilibria, the probability of a crisis in the bad state (L) ranges from close to zero in Equilibrium I to close to 1 in Equilibrium III. These changes in probabilities are also reflected in state prices, as can be seen in the last column.

Multiplicity arises in the parameterized example because the right hand side of (18) responds strongly to changes in the threshold when the state is L. This is shown in Fig. 2 which plots the left hand side and the right hand side of the equation that determines this threshold (18) for the parameterized example. The left hand side is a 45-degree line. There should, in principle, be three curves for the right hand side in panel (a) of Fig. 2: one for each value of θ_H^*. However, since the right hand side in (18) depends on the value of θ_H^* only through the probability of a crisis ψ_H and since this probability is (numerically) zero, the right hand side appears as a single curve. The three places at which the solid line crosses the 45-degree line mark the points where (18) is satisfied. These are the three values for θ_L^* in Table 1.

Panel (b) of Fig. 2 depicts the equation that determines this threshold (18) when the state is H. Since the right hand side of this equation is downward sloping in this case, each θ_L^* corresponds to a unique θ_H^*.

The economic intuition for the existence of multiple equilibria can be cast in terms of the behavior of risk-neutral agents. The probability of an attack succeeding is high in the bad state and low in the good state. This gives risk-neutral agents an incentive to transfer wealth out of the bad state and into the good state. As a consequence, a larger fraction of wealth is in the hands of

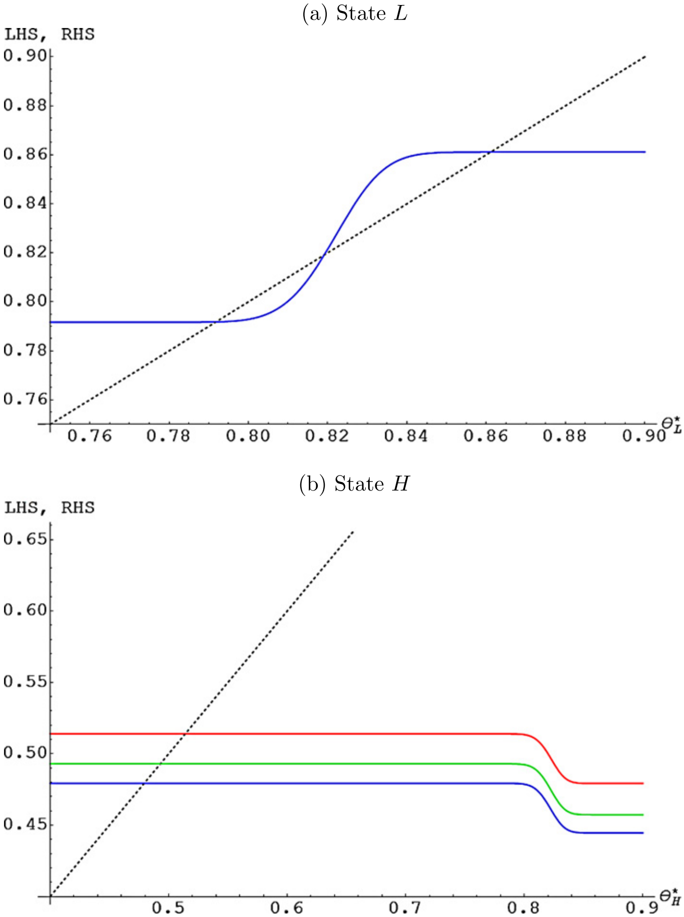


Fig. 2. Multiplicity in the determination of the threshold θ_L^*. The figure plots the left hand side and the right hand side of the equation that determines this threshold (18). Panel (a) is for state L and panel (b) for state H.

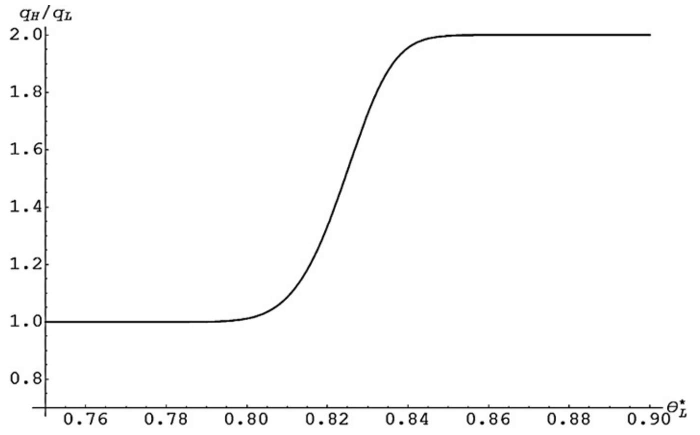
the risk-averse agents in state L and increases the probability of an attack succeeding in that state since the attack is the safe option. If the response of wealth due to changes in the probability of a crisis is sufficiently strong, then a different equilibrium with a higher probability of a successful attack may also be an equilibrium.

That multiplicity is related to risk-neutral agents' desire to transfer wealth into the good state can be verified by looking at state prices. At an interior equilibrium the ratio of state prices is determined by the marginal rate of substitution of a risk-neutral agent, as seen in (21). The last column of Table 1 shows that equilibria correspond to state-prices that differ substantially. Fig. 3 conveys the same message by plotting the price ratio and the equilibrium fraction of wealth commanded by infinitely risk-averse agents in state L.

Mathematically, the cause for multiplicity lies in the fact that the slope of the right hand side (with respect to θ_s^*) of (18) is larger than one or, equivalently,

$$\frac{d\frac{w(s,\infty)}{e(s)}}{d\theta_s^*} > \frac{R-f}{R-1}. \tag{22}$$

(a) Prices



(b) Fraction of wealth in the hands of the infinitely risk-averse in state L

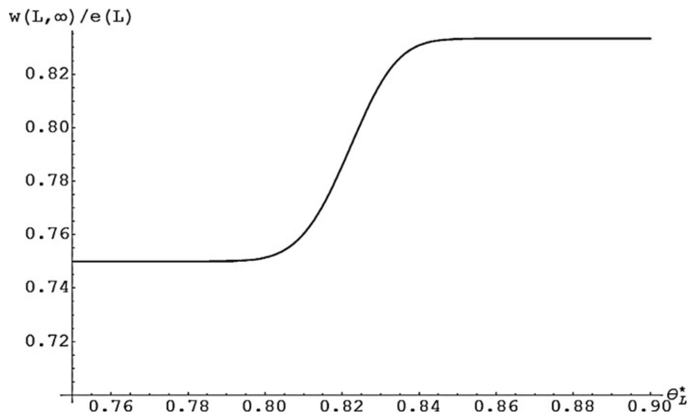


Fig. 3. State-prices and wealth in the numerical example.

The derivative of the wealth fraction with respect to the threshold is

$$\frac{d}{d\theta_s^*}\left[\frac{w(s,\rho)}{e(s)}\right] = \frac{1}{e(s)}\frac{dw(s,\rho)}{d\frac{q_H}{q_L}}\frac{d\frac{q_H}{q_L}}{d\theta_s^*}. \tag{23}$$

This derivative in state H is negative. This implies that the condition in (22) is not satisfied. The derivative for state L, however, is equal to

$$\frac{1}{e(L)}\frac{(R-1)\Delta_e}{2\sigma_\theta}\frac{\phi_L(\cdot)}{(1+\frac{q_H}{q_L})[R-\psi(L)(R-1)]^2} > 0. \tag{24}$$

Notice that for sufficiently small σ_θ this expression becomes arbitrarily large, implying that, if the right parameters are chosen (as it happens in the parameterization), (18) can hold for multiple values of θ_L^*, giving rise to equilibrium multiplicity.

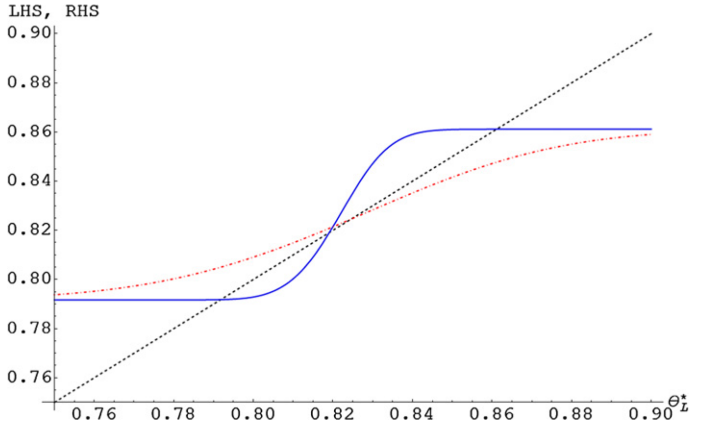


Fig. 4. The figure plots the left hand side and the right hand side of the equation that determines this threshold (18) when the state is L for $\sigma_\theta = 0.01$ (solid) and $\sigma_\theta = 0.04$ (dot-dashed).

The expression in (24) also hints at a way of restoring uniqueness: increasing σ_θ. Fig. 4 shows that uniqueness is restored if σ_θ is quadrupled and set at 0.04 instead of 0.01.[7] I will return to the relationship between σ_θ and uniqueness of equilibria when discussing the general case.

3.3. The effect of fundamentals on the probability of a crisis: feedback through state prices

The price ratio $\frac{q_H}{q_L}$ measures the relative value of transferring wealth into the two states. This price ratio must equal the marginal rate of substitution of the risk-neutral type if consumption is strictly positive. Absent the global game, since both states are equally likely, the price ratio would be unity at interior equilibria. This is not true, however, when the global game is played at date $t = 2$. The reason is that agents care about terminal wealth, which is affected by the return from not attacking, which itself depends on the probability of a crisis. Therefore, it is necessary to consider the use that the risk-neutral agent makes with wealth in the global game stage.

For a risk-neutral agent, the effect on expected terminal wealth of transferring one dollar into state s is the gross return obtained in the game that is played in that state. Since there is a higher probability of a crisis in state L, the price of a claim that pays one dollar in state L must be lower than the price of a claim that pays in H, i.e. $\frac{q_H}{q_L} > 1$. This, in turn, hurts the risk-neutral type but is a positive pure income effect for the risk-averse type, who is buying cheap claims to wealth in state L and selling expensive claims to wealth in state H. She is now able to obtain an even higher fraction of the endowment in state L, thus raising the probability of a crisis.

Fig. 5 qualitatively illustrates the change in the relative price in the form of a clockwise rotation of the budget constraint. The dashed line represents an indifference curve of the risk-neutral agent if this agent, contrary to rational expectations, did not include the change in the probability of a crisis that comes about when wealth shifts in the computation (i.e. if the probability of a crisis is held fixed). The solid line corresponds to the actual indifference curve that leads to the equilibrium since agents in this model anticipate the change in probabilities.

[7] The equilibrium values in the unique equilibrium that arises when σ_θ is set to 0.04 are $\theta_H^* = 0.495607$, $\theta_L^* = 0.824548$, $q_H/q_L = 1.37482$.

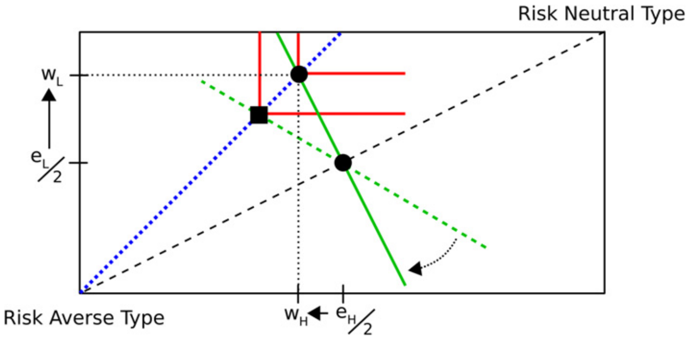


Fig. 5. The probability of a crisis is priced into state prices. The budget constraint coincides with a risk-neutral agent's indifference curve. It rotates (clockwise, because L is on the vertical axis) relative to prices that do not take into account the change in probabilities due to wealth shifts. The new equilibrium is higher up on the 45° line from the risk-averse type's perspective.

In this simple example, there exists feedback between the fraction of wealth which is (endogenously) in the hands of the most risk-averse segment of the population and the probability of a crisis in the low endowment state.

4. The general case

4.1. Regime change with complete information

When $\sigma_\varepsilon = 0$ and agents know the exact value of θ, then there are three distinct regions of interest for the global game: $\theta \leqslant 0$, $\theta > 1$ and $\theta \in (0, 1]$. Since $\frac{K}{e}$ is a fraction, when $\theta \leqslant 0$ or $\theta > 1$ there is a unique equilibrium, crisis and no crisis, respectively. Agents play a dominant action in these cases. There do however exist multiple equilibria when $\theta \in (0, 1]$. There is an equilibrium where everybody does a full attack $a = w(s, \rho)$ (which implies $\frac{K}{e} = 1 \geqslant \theta$) and a crisis occurs and also an equilibrium where nobody attacks and no crisis occurs. Risk-aversion does not play a role in the determination of a crisis since agents do not face uncertainty in any given equilibrium. Neither does the equilibrium wealth distribution derived from trade at $t = 0$.

4.2. Regime change with incomplete information

If p is the individual belief that an agent associates to the event that a crisis does not occur, then expected utility as a function of the attack decision a is

$$U_p(a) = pu\big(a + R(w - a), \rho\big) + (1 - p)u\big(a + f(w - a), \rho\big). \tag{25}$$

Given the CRRA utility function, it turns out that the utility-maximizing attack decision can be expressed as a linear function of wealth.

Lemma 1. *The individually optimal attack decision is separable into wealth and a component independent of wealth, which depends on the perceived probability of regime survival p, the agent's level of risk-aversion ρ, and the parameters R and f.*

$$a(p, \rho, w) = \alpha(p, \rho)w. \tag{26}$$

The fraction of wealth used to attack the regime is given by

$$\alpha(p,\rho)=\begin{cases}1 & \text{if } p<\underline{p},\\ \frac{R-f(\frac{p(R-1)}{(1-p)(1-f)})^{1/\rho}}{(R-1)+(1-f)(\frac{p(R-1)}{(1-p)(1-f)})^{1/\rho}} & \text{if } p\in[\underline{p},\overline{p}(\rho)],\\ 0 & \text{if } p>\overline{p}(\rho),\end{cases} \tag{27}$$

where $\underline{p}=\frac{1-f}{R-f}$ *and* $\overline{p}(\rho)=\frac{1-f}{(R-1)(\frac{f}{R})^{\rho}+(1-f)}$. *The function* $\alpha(p,\rho)$ *is continuous in both arguments, decreasing in* p*, and increasing in* ρ.

Fix a state s. Conditional on the signal x, the inference problem is the same for agents of all types. Other characteristics, such as level of risk-aversion or wealth do not play any role for this computation. The conditional CDF of θ given that signal x was observed is given by

$$H(\theta\mid x)\equiv\Pr\{\tilde{\theta}\leqslant\theta\mid\tilde{x}=x\}. \tag{28}$$

A threshold equilibrium is characterized by a critical value θ_s^* such that a crisis occurs if and only if $\theta<\theta_s^*$. There is a reason for focusing on threshold equilibria. The equilibrium of the global game may or may not be unique. If the equilibrium is not unique then it is a standard result in games of strategic complementarities that all equilibria lie between a largest and a smallest equilibrium in which agents use strategies which are monotone in their signals. In this setting, because signals are affiliated to θ, this implies that the smallest and the largest equilibrium – call them extremal equilibria – are threshold equilibria. Notice that if the equilibrium is unique, then the largest and smallest equilibrium coincide and the unique equilibrium is a threshold equilibrium as well. By studying threshold equilibria I am therefore characterizing either the unique equilibrium, or the extremal points of the equilibrium set in the case of multiple equilibria.

If a threshold equilibrium is played, then agents who observe signal x must attach probability

$$p=1-H\big(\theta_s^*\mid x\big) \tag{29}$$

to the regime's survival. An agent of type ρ performs this calculation and attacks with a fraction $\alpha(p,\rho)$ of wealth. Using a "Law of Large Numbers" argument, the total strength of the attack by the continuum of agents of risk-aversion ρ is equal to their expected behavior, where the expectation is with respect to the distribution of signals.

In a threshold equilibrium there is a one-to-one relationship between signals x and beliefs p. Eq. (29) can be inverted to express signals as a function of beliefs in a unique way. This allows to write the fraction of wealth withdrawn as an expectation with respect to p.[8]

Lemma 2. *In a threshold equilibrium, the fraction of wealth used to attack by type* ρ, $A(\mu,\rho,\theta_s^*,\theta)$, *can be expressed as an expectation with respect to* p

$$A\big(\mu,\rho,\theta_s^*,\theta\big)=\int_{p=0}^{1}\alpha(p,\rho)\,d\Gamma_\mu\big(p\mid\theta_s^*,\theta\big)$$

where $\Gamma_\mu(p\mid\theta_s^*,\theta)$ *is the proportion of the population with beliefs* p *or less given a realization of* θ *and a conjectured threshold equilibrium with threshold* θ_s^*. *Further,* $\Gamma_\mu(p\mid\theta_s^*,\theta)$ *is strictly increasing in* p *and* θ_s^*, *and strictly decreasing in* θ *and* μ.

8 This technique is used in [20] and [13].

The fraction of the endowment that is used to attack under the conjectured equilibrium by agents of all types is

$$\frac{K(s,\theta_s^*,\theta)}{e(s)} = \sum_i \frac{w(s,\rho_i)}{e(s)} A\big(\mu(s),\rho_i,\theta_s^*,\theta\big). \tag{30}$$

Lemma 3. *For any state s, threshold θ_s^* is part of a threshold equilibrium if and only if the threshold equals the size of the attack when the actual realization equals the threshold, i.e.*

$$\begin{aligned}\theta_s^* &= \frac{K(s,\theta_s^*,\theta_s^*)}{e(s)} \\ &= \sum_i \frac{w(s,\rho_i)}{e(s)} A\big(\mu(s),\rho_i,\theta_s^*,\theta_s^*\big).\end{aligned} \tag{31}$$

Lemma 3, together with the result from Lemma 2, provides a very simple characterization of the thresholds θ_s^*. With the characterization for threshold equilibria found in Lemma 3 it is straightforward to prove the existence of a threshold equilibrium. Existence follows from the fact that the right hand side of Eq. (31) is continuous and lies in the interval [0, 1].[9]

The threshold is not necessarily unique. Differentiate both sides of (31) with respect to θ_s^*. A typical sufficient condition for uniqueness is that the right hand side is less than one since the right hand side of (31) must lie in the unit interval while the left hand side can take any real value.

$$1 > \sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*} A\big(\mu(s),\rho_i,\theta_s^*,\theta_s^*\big) + \sum_i \frac{w(s,\rho_i)}{e(s)} \frac{\partial A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)}{\partial \theta_s^*}. \tag{32}$$

The second term on the right hand side of (32) can be shown to be bounded by $\frac{\sigma_\varepsilon}{\sigma_\theta^2}\frac{1}{\sqrt{2\pi}}$ following the proof in [13]. Incidentally, this is also the same bound as the one derived in [21] for a risk-neutral population and discrete actions. The magnitude of the first term depends on all the parameters of the model, that is R, f, the characteristics of the distribution of the prior and the signal, the aggregate endowment and the distribution of levels of risk-aversion in the population. Because of the existence of the first term, taking the limit as $\frac{\sigma_\varepsilon}{\sigma_\theta^2} \to 0$, i.e. improving the informativeness of the private signal relative to the public information, is not a sufficient condition for uniqueness in this setting. This was shown in the example of Section 3 which exhibited multiple equilibria.

Multiplicity due to the first term in (32) appears when, in a given state, the fraction of wealth in the hands of more risk-averse agents is an increasing function of the threshold. Then, since the threshold is a weighted average of type-dependent aggregate attack decisions where wealth shares are used as weights, a shift of wealth into the hands of risk-averse agents leads to a higher value of the threshold, producing a feedback loop.

The more general issue of uniqueness vs. multiplicity of equilibria requires solving the equilibrium at date $t = 0$ first. It is therefore postponed to Section 4.5. The following proposition collects the findings at the global game stage for a given wealth distribution.

[9] The fact that the right hand side of (31) is continuous and in the interval [0, 1] is shown as an intermediate result in Lemma 3.

Proposition 1. *A threshold equilibrium is indexed by θ_s^* (the crisis index). If the realization of θ falls below this threshold then there is a crisis (the status quo is abandoned). Otherwise, no crisis occurs. The values of θ_s^* satisfy a system of equations, one for each state $s \in S$.*

$$\theta_s^* = \sum_i \frac{w(s, \rho_i)}{e(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big).$$

The term $A(\mu, \rho, \theta^*, \theta^*)$ has the following comparative statics

$$\frac{\partial A(\mu, \rho, \theta^*, \theta^*)}{\partial \mu} < 0, \qquad \frac{\partial A(\mu, \rho, \theta^*, \theta^*)}{\partial \theta^*} > 0, \qquad \frac{\partial A(\mu, \rho, \theta^*, \theta^*)}{\partial \rho} > 0.$$

At extremal threshold equilibria

$$\frac{\partial}{\partial \theta_s^*} \sum_i \frac{w(s, \rho_i)}{e(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big) < 1.$$

The result that the slope of the aggregate attack is less than one at extremal threshold equilibria is exposed graphically in Fig. 4 for both cases: when there are multiple threshold equilibria and when there is a single threshold equilibrium.

The proposition contains all the knowledge about the probability of a crisis that can be gained purely from the analysis of the global game for a fixed wealth distribution. The thresholds can be interpreted as a crisis index. Larger values of the crisis index imply an increased likelihood of a crisis. States with higher μ will tend to decrease the crisis index. States in which risk-averse agents are weighted stronger will tend to increase the crisis index.

4.3. Probability of a crisis

For any given threshold equilibrium, the ex-ante probability of a crisis is equal to the probability that θ lies below the threshold θ_s^*. Since θ has a Normal distribution with mean $\mu(s)$ and standard deviation σ_θ, this probability is

$$\psi(s) = \Phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right) \tag{33}$$

where $\Phi(\cdot)$ is the standard normal CDF. The probability of a crisis changes with the state according to

$$\frac{d\psi(s)}{ds} = -\frac{1}{\sigma_\theta}\phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right)\frac{d\mu(s)}{ds} + \frac{1}{\sigma_\theta}\phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right)\frac{d\theta_s^*}{ds}. \tag{34}$$

The first term arises because the state determines where the distribution is centered through the mean $\mu(s)$. An increase in $\mu(s)$ reduces the probability mass below any given value for θ_s^*. The second term appears because θ_s^*, the threshold value defining the limit between the crisis region and the non-crisis region, shifts as well. The movement in θ_s^* itself is in part due to the change in $\mu(s)$, as a shift in fundamentals increases the expected value of not attacking, and thus lowers θ_s^*. The second reason of why θ_s^* shifts with the state is that $e(s)$ changes. The derivative of (31) yields

$$\frac{d\theta_s^*}{ds} = \sum_i \left(\frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} \frac{de(s)}{ds} + \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} \frac{d\mu(s)}{ds} + \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*} \frac{d\theta_s^*}{ds} \right) A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)$$
$$+ \sum_i \frac{w(s,\rho_i)}{e(s)} \left(\frac{\partial A(\mu(s), \rho_i, \theta_s^*, \theta_s^*)}{\partial \mu(s)} \frac{d\mu(s)}{ds} + \frac{\partial A(\mu(s), \rho_i, \theta_s^*, \theta_s^*)}{\partial \theta_s^*} \frac{d\theta_s^*}{ds} \right). \quad (35)$$

Solving this equation for $\frac{d\theta_s^*}{ds}$, and substituting this result in (34), produces an equation of the form

$$\frac{d\psi(s)}{ds} = \frac{1}{\sigma_\theta} \phi(\cdot) \Bigg[-\frac{d\mu(s)}{ds} + \underbrace{T_1 \frac{d\mu(s)}{ds} + T_2 \frac{de(s)}{ds}}_{\frac{d\theta_s^*}{ds}} \Bigg]. \quad (36)$$

To find the values of T_1 and T_2, trade in state-contingent claims at $t = 0$ has to be analyzed. This task is taken up in the next section.

4.4. *Trade in state-contingent claims and the probability of a crisis*

The characteristics of an equilibrium allocation are well known if the global game stage at date $t = 2$ is absent. The solution to the risk-sharing problem in this case is that risk-averse agents carry wealth predominantly into good states and less risk-averse agents carry wealth into bad states as shown, for example, by [25] and, more recently, by [14]. I find that the result that more risk-averse agents take a larger fraction of wealth into low endowment states carries over to the case in which a global game is played afterwards.[10]

Define the wealth-weighted harmonic average of risk-aversion:

$$\frac{1}{\rho_H(s)} = \sum_i \frac{w(s,\rho_i)}{e(s)} \frac{1}{\rho_i}. \quad (37)$$

This average level of risk-aversion can be interpreted as the (state-dependent) degree of risk-aversion of a “representative agent”. The terminology “representative agent” refers to the fact that an individual with this level of risk-aversion would be content with maintaining his or her initial endowment and that state prices can be read off this agent’s marginal rate of substitution.[11]

The following lemma characterizes the effect of changes in the state of the world on the wealth distribution in an equilibrium. It follows from solving for the constrained efficient allocation in a planner’s problem, and serves as an intermediate step for figuring out the signs of T_1 and T_2 in (36).

Lemma 4. *For the equilibrium allocation of state-contingent claims, the following relationships hold*:

[10] This result is not straightforward since an externality appears in the market at date $t = 0$ in the model in which the global game stage is played at $t = 2$. Wealth choices affect the equilibrium of the global game in the final stage, meaning that the indirect utility of taking wealth into a given state is affected by other agent’s choices.

[11] For details on the “representative agent” interpretation see [14].

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} = \frac{w(s,\rho_i)}{e(s)}\frac{1}{e(s)}\left(\frac{\rho_H(s)}{\rho_i} - 1\right),$$
$$0 = \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} + \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}.$$

If the population were ex-ante homogeneous, with a common level of risk-aversion ρ, then the effect of the aggregate endowment on the wealth distribution would be zero, as there are no benefits from trade. Notice that a homogeneous population implies that everybody is the "representative agent" $\rho_H(s) = \rho$, so that the first derivative in Lemma 4 is zero.[12] It is only when there is heterogeneity in risk-aversion that trade at date $t = 0$ occurs. With a heterogeneous population, an agent with a level of risk-aversion that is higher than the harmonic average ρ_H takes a bigger fraction of wealth into states of the world with a low aggregate endowment.

The second equation in Lemma 4 states that the effect of the mean $\mu(s)$ and the threshold θ_s^* on the wealth distribution are of opposite signs and of the same magnitude. If, contrary to the model, the global game at $t = 2$ was absent, then both $\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)}$ and $\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}$ would be identical to zero and the equation would trivially hold. Since the global game is present, equilibrium wealth shares depend on what happens in the global game stage. The result that the effect of the mean and the threshold are of comparable magnitudes follows from the fact that these two variables affect the problem only through the probability of a crisis. The argument that goes into the Normal distribution in (33) is $\frac{\theta_s^* - \mu(s)}{\sigma_\theta}$. Since only the net value of the threshold minus the mean enters the probability of a crisis, they must have the same effect, but be of different sign.

The characterization of wealth choices in Lemma 4 is sufficient to show that T_1 and T_2 in (36) are both negative. The result is not only for the case of a unique equilibrium. It is more general. If there are multiple equilibria, then the result applies to extremal threshold equilibria, meaning the smallest and the largest threshold equilibrium.

Theorem 1. *At an extremal threshold equilibrium, in the equation describing the probability of a crisis*

$$\frac{d\psi(s)}{ds} = \frac{1}{\sigma_\theta}\phi(\cdot)\left[-\frac{d\mu(s)}{ds} + T_1\frac{d\mu(s)}{ds} + T_2\frac{de(s)}{ds}\right]$$

the terms T_1 and T_2 are both negative if the population is heterogeneous in risk-aversion.

When the limit $\sigma_\varepsilon \to 0$ is taken an even cleaner result emerges. How does an economy with a heterogeneous population compare to an economy with a homogeneous population? Changes in the probability can be expressed as a linear combination of this change for a homogeneous population plus a term which is negative and proportional to $\frac{d\mu(s)}{ds}$ and $\frac{de(s)}{ds}$. Denote a homogeneous population by the superindex h and the limiting variables with a tilde, i.e. $\tilde{X} = \lim_{\sigma_\varepsilon \to 0} X$.

Theorem 2. *Take the limit $\sigma_\varepsilon \to 0$. Then, the effect of the aggregate state on the probability of a crisis with a heterogeneous population can be expressed in terms of this same change for a homogeneous population*

[12] Incidentally, this then implies that $T_2 = 0$, because $\frac{de(s)}{ds}$ drops out of (35) and, in consequence, also out of (36).

$$\frac{d\tilde{\psi}(s)}{ds} = \frac{d\tilde{\psi}(s)^h}{ds} + \frac{1}{\sigma_\theta}\phi(\cdot)\left[\tilde{T}_1\frac{d\mu(s)}{ds} + \tilde{T}_2\frac{de(s)}{ds}\right]$$

where $\tilde{T}_1, \tilde{T}_2 < 0$.

Theorem 2 holds regardless of the level of risk-aversion posed for the homogeneous population which is used as a comparison.[13] It states that the effect of fundamentals on the probability of a crisis is always larger when the population is heterogeneous.

4.5. Uniqueness vs. multiplicity of equilibria

In this section I derive a sufficient condition for uniqueness in the global game in the special case in which $\sigma_\varepsilon \to 0$. This extreme case highlights the difference with the uniqueness result in the previous literature. The limit $\sigma_\varepsilon \to 0$ would imply a unique equilibrium in the environments modeled by [21] and [13], where $\frac{\sigma_\varepsilon}{\sigma_\theta^2}\frac{1}{\sqrt{2\pi}} < 1$ is sufficient for uniqueness. With endogenous wealth determination, the condition does not necessarily imply uniqueness.

If $\sigma_\varepsilon \to 0$, then

$$\frac{\partial A(\mu(s), \rho_i, \theta_s^*, \theta_s^*)}{\partial \theta_s^*} = 0 \tag{38}$$

for all agents since, in the limit, the distribution of beliefs of other agent's play becomes uniform at the threshold, as shown in the proof of Theorem 2. In consequence, $A(\mu(s), \rho_i, \theta_s^*, \theta_s^*) \to \tilde{A}(\rho_i)$, which is independent of θ_s^* (and also of $\mu(s)$). Given this result, the condition for uniqueness in (32) simplifies to

$$\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}\tilde{A}(\rho_i) < 1. \tag{39}$$

This condition may, or may not, be satisfied according to the parameter values of the model.

In the example in Section 3 it was shown that increasing σ_θ restored uniqueness. The sufficient (although not necessary) condition for uniqueness when $\sigma_\varepsilon \to 0$ stated in Theorem 3 has the same flavor, as it derives a lower bound for σ_θ.

Theorem 3. *If* $\sigma_\varepsilon \to 0$ *and*

$$\sigma_\theta > \frac{1}{2\pi}\frac{R}{\underline{\rho}}$$

then the equilibrium of the global game of regime change that is played at $t = 2$ *is unique.*

[13] The intuition for why Theorem 2 holds for any level of homogeneous risk-aversion is that, if the population is homogeneous, then $\mu(s)$ and θ_s^* cannot have any effect on the threshold through wealth shifts in equilibrium since, by aggregate consistency, $w(s, \rho) = e(s)$. Only the effect of the aggregate endowment remains. But, as shown in Lemma 4, the effect of changes in $e(s)$ on the threshold through wealth shifts is zero for a homogeneous population, regardless of its level of risk-aversion. The limit $\sigma_\varepsilon \to 0$, in turn, implies that probabilities are affected exclusively through wealth shifts since it ensures that attack decisions do not vary with the value of the threshold θ_s^*. The reason is that, when $\sigma_\varepsilon \to 0$, the distribution of beliefs of an agent observing a signal equal to the threshold is uniform, and does therefore not depend on θ_s^*.

The first parameter $\frac{1}{2\pi}$ enters the equation because the Normal distribution has been used, and $\frac{1}{2\pi}$ is the maximum value taken by a normal density function. The parameter R is the payoff from attacking.

Holding everything else constant, the condition for uniqueness is easier to satisfy for greater values of σ_θ and for larger values of $\underline{\rho}$, the minimum level of risk-aversion in the population. A greater value of σ_θ makes the probability of a crisis less responsive to a given change in the threshold θ_s^*. Intuitively, the density spreads out and a given change in the threshold corresponds to a smaller area under the density. Larger values of $\underline{\rho}$ help because more curvature on the utility function implies smaller substitution effects (by the least risk-averse agent), which make wealth shifts less responsive to changes in the probability of a crisis.

5. Application and discussion

In this section I show that the global game stage described in Section 2 can be interpreted as a stylized version of a bank run game in which the government intervenes once the bank is bankrupt. For this purpose I adapt the environment of Ennis and Keister [8], who construct such a model in which the government intervenes by freezing deposits during a bank run.

It is well known that the direct application of the techniques of Carlsson and van Damme [4] and Morris and Shin [19] to bank run games is not possible. Bank run games that are specified from first principles typically do not exhibit global strategic complementarities. In a bank run model, the net benefit from running increases when more people run only up to the point where the bank is bankrupt. Once the bank is bankrupt the net benefit from running decreases when more people run. Rochet and Vives [22] and Goldstein and Pauzner [11] recognize this problem and deal with this issue in alternative ways.[14]

In reality, widespread bank runs typically provoke intervention by the government. As argued by Ennis and Keister [8], in practice, when governments intervene they typically freeze deposits. In a deposit freeze depositors are legally prevented from withdrawing their funds during a period of time. Because of the deposit freeze, the net benefit from running once the bank is bankrupt does not decrease when more people run. Therefore, a stylized version of the model by Ennis and Keister [8], in which the response of the government is taken into account, will exhibit global strategic complementarities.[15]

5.1. Bank runs and deposit freezes

According to Ennis and Keister [8] two key elements are typically present in government responses to widespread bank runs. First, deposits are frozen. Second, rescheduling or re-denomination of debt contract occurs, lowering the real return to depositors affected by the

[14] In the model of Rochet and Vives [22] global strategic complementarities are restored by assuming that the decision to withdraw is made by fund managers who face a fixed penalty when the bank fails. Goldstein and Pauzner [11], on the other hand, show that under certain assumptions the uniqueness result that is typical in the global games literature can be obtained in a modified version of the model by Diamond and Dybvig [6] even in the absence of global strategic complementarities. Besides restoring uniqueness, these models deliver the result that a bank runs depend on fundamentals. Therefore, these two models offer a way of calculating how the probability of a bank run depends on the aggregate state. A review of sunspot and risk-dominance techniques which are useful to determine the probability of bank runs is given by Ennis [7].

[15] I am not the first to recognize that the environment of Ennis and Keister [8] fits into a standard global games framework. Their model is adapted in a similar way by Sákovics and Steiner [23].

freeze. The leading example used by Ennis and Keister [8] is Argentina's bank run during the last days of November 2001. In the first days of December 2001 deposits were frozen for an initial period of 90 days. The freeze was extended several times and the last restrictions to withdraw were finally lifted in early 2003. In the case of Argentina dollar deposits were converted into peso deposits at an official exchange rate which was less favorable than the market rate, therefore imposing costs on depositors affected by the freeze.[16]

The global game stage in Section 2 can be interpreted as a stylized model of the Argentine situation in November 2001. Depositors decide how much to withdraw from the bank knowing that their deposits may be subject to a deposit freeze. In the event of a deposit freeze agents are temporarily unable to access their funds and rescheduling, or forceful conversion to local currency, occurs and the return on money left in the bank is $f < 1$. If, on the other hand, deposits are not frozen then depositors earn a gross return of $R > 1$ on the amount that is left in the bank.

The government decides whether to freeze deposits comparing the size of the attack on the banking system (the amount of deposits that were withdrawn) to the resources it has available for providing liquidity to the banking sector. If the size of the attack exceeds the amount of resources that the government is able or willing to use to stop the run, then the government is forced to decree a deposit freeze. In the case of Argentina, most deposits were dollar-denominated. The fundamental variable θ could be interpreted as the amount of foreign reserves available for extending liquidity to the banks. If θ is high enough (greater than one) the government is able to defend the banking system against an attack of any size. If θ is negative, then the government is forced to freeze deposits even for arbitrarily small attacks.[17] Depositors do not directly observe θ but observe their private signal x.

The application of the results of this paper to bank runs implies that risk-averse agents carry more wealth into low aggregate endowment states and that their behavior makes those states even worse because they are more likely to participate in a bank run. Argentina in 2001 can be characterized as being in a low endowment state. The country was facing a recession which had started in 1998. According to data from the International Monetary Fund, real GDP dropped by 8.4% from the peak in 1997 to 2001. The evolution of the stock market during that period is consistent with wealth shifts into the hands of more risk-averse agents. Stock markets are procyclical and may serve as a risk-sharing mechanism if less risk-averse agents invest in the stock market while more risk-averse do not. The stock market index at the Buenos Aires exchange (MERVAL) progressively dropped from above 800 in mid-1997 to 200 in November 2001. Stock market investors lost more than 75% on their investments in the three years preceding the events of November 2001. More risk-averse people, who are less likely to invest in the stock market, would have been able to maintain their wealth using time deposits or other safe instruments. It is therefore likely that the recession had shifted wealth from the hands of less risk-averse agents into the hands of more risk-averse agents.

[16] Deposit freezes are not unique to the Argentine episode in 2001. Argentina also imposed a deposit freeze in 1989. During the deposit freeze deposits were converted into government bonds. These bonds traded at a 50% discount in the initial months after the conversion. Other countries that used deposit freezes include Brazil in 1990, Ecuador in 1999, and Uruguay in 2002. Details on these episodes and on the costs imposed on depositors can be found in [16]. Friedman and Schwartz [9] describe how deposit freezes were regularly used in the US during banking panics up to and including the Panic of 1933.

[17] The choices of 0 and 1 for the dominance regions are not very restrictive. Notice that θ does not need to be interpreted literally as the amount of foreign reserves; θ could be a strictly increasing function of foreign reserves.

Through the lens of the model, the wealth shifts that came about because of the low payoff from stock market investments and the bank run are two phenomena that cannot be considered in isolation. If the model is taken at face value, it implies that wealth shifts played a role in amplifying the effect of the recession on the likelihood of a bank run in Argentina.

5.2. *Assumptions of the model*

The amplification mechanism described in this paper rests on two main assumptions: (1) attacking is safe while not attacking is risky, and (2) agents have access to a full set of state-contingent assets. The first assumption implies that the more risk-averse will be more likely to attack. The second assumption allows wealth to go into the hands of the risk-averse in states with a low aggregate endowment.

Both of these assumptions can be relaxed. Start with the assumption that attacking is completely safe. If attacking is risky but still less risky than not attacking, the risk-averse will be relatively more likely to attack, and the feedback mechanism will be at work. As the application to the bank run game shows, the assumption is reasonable for some real world phenomena. On the other hand, it is true that not all possible applications will have the feature that attacking is less risky than not attacking. In such cases the results of this model do not apply.

The assumption that agents can trade using a set of contracts that is complete with respect to the states can also be weakened. Even in the absence of a complete set of state-contingent assets, agents will try to exploit whatever opportunities arise to transfer wealth across states, because there are gains from trade that stem from heterogeneity in risk-aversion. If risk-sharing occurs, risk-averse agents will hold a bigger fraction of wealth in states with lower aggregate endowments.

6. Conclusion

This paper studies the effect of heterogeneity in risk-aversion on the probability of a crisis. With a heterogeneous population, the cross-sectional distribution of wealth is a key factor in driving the likelihood of a crisis. Rather than taking the wealth distribution as exogenous, I endogenize it by allowing agents to participate in trade of state-contingent claims to insure against aggregate endowment shocks.

I find that even if signals become arbitrarily precise, the equilibrium is not necessarily unique if there is heterogeneity in risk-aversion and if wealth is endogenously determined. A parameterized case where multiple equilibria arise is numerically solved.

I also find that heterogeneity in risk-aversion increases the effect of the state of the economy on the probability of a crisis. The reason is that the endogenous determination of wealth acts as an amplification mechanism by putting more wealth in the hands of more risk-averse agents during economic downturns.

The model can be applied to think about bank runs. A recession will put more wealth in the hands of more risk-averse depositors. Since they are more likely to withdraw their money when uncertain about the survival of the bank, this increases the probability of a bank run during a recession. The endogenous response of wealth shifts to the increased likelihood of bank failure puts still more resources in the hands of the more risk-averse depositors. When applied to bank runs, the model implies that the existence of wealth shifts amplifies the effect of a recession on the likelihood of a bank run.

In a world inhabited by people who differ in their risk-aversion, the existence of risk-sharing opportunities produces feedback between the reallocation of wealth and the aggregate size of an attack. A model that abstracts from wealth shifts will underestimate the effect of economic conditions on the probability of crises.

Appendix A. Proofs

Proof of Lemma 1. The optimal date 1 strategy results from state-by-state maximization of (25) with respect to the choice variable $a \in [0, w]$. The derivative of (25) is

$$U'_p(a) = -p(R-1)\big[Rw - (R-1)a\big]^{-\rho} + (1-p)(1-f)\big[fw + (1-f)a\big]^{-\rho}.$$

Depending on the value of p, the solution is either interior or on one of the boundaries. The condition for an interior solution is $U'_p(a) = 0$. If $U'_p(a) > 0$ for all $a \in [0, w]$, then the returns to increasing a are not exhausted in the domain of a and the solution calls for setting $a = w$. If, on the other hand, $U'_p(a) < 0$ for all $a \in [0, w]$, then there is a corner solution with $a = 0$.

Setting $U'_p(a) = 0$ and solving for a delivers the expression for the interior case

$$a = \frac{R - f[\frac{p}{1-p}\frac{R-1}{1-f}]^{\frac{1}{\rho}}}{R - 1 + (1-f)[\frac{p}{1-p}\frac{R-1}{1-f}]^{\frac{1}{\rho}}} w.$$

To deal with the corner at which it is optimal to choose $a = 0$ notice that it is both necessary and sufficient for $U'_p(a) < 0$ to be satisfied at $a = 0$ since $U'_p(a)$ is decreasing in a (because the objective function is concave). Using this information, the condition for a corner with $a = 0$ becomes

$$U'_p(0) = -p(R-1)R^{-\rho} + (1-p)(1-f)f^{-\rho} < 0$$

which, solved for p, establishes the value for $\overline{p}(\rho)$.

$$p > \overline{p}(\rho) \equiv \frac{1-f}{(R-1)(\frac{f}{R})^{\rho} + (1-f)}.$$

The corner solution at which $a = w$ calls for $U'_p(a) > 0$ for all a. The decreasing nature of $U'_p(a)$ makes it necessary to check the condition only for $a = w$.

$$U'_p(w) = \big[-p(R-1) + (1-p)(1-f)\big]w^{-\rho} > 0$$

yielding

$$p < \underline{p} \equiv \frac{1-f}{R-f}.$$

Notice that

$$\underline{p} = \frac{1-f}{R-f} \leqslant \frac{1-f}{(R-1)(\frac{f}{R})^{\rho} + (1-f)} = \overline{p}(\rho)$$

with equality if and only if $\rho = 0$ (risk-neutrality).

The solution can be expressed as a linear function of wealth.

$$a(p, \rho, w) = \alpha(p, \rho)w$$

by defining $\alpha(p,\rho)$ as

$$\alpha(p,\rho)=\begin{cases}1 & \text{if } p\in[0,\underline{p}),\\ \dfrac{R-f[\frac{\pi}{1-\pi}\frac{R-1}{1-f}]^{\frac{1}{\rho}}}{R-1+(1-f)[\frac{\pi}{1-\pi}\frac{R-1}{1-f}]^{\frac{1}{\rho}}} & \text{if } p\in[\underline{p},\overline{p}(\rho)],\\ 0 & \text{if } p\in(\overline{p}(\rho),1].\end{cases}$$

Proving continuity is straightforward by verifying from the above formula that

$$\alpha(\underline{p},\rho)=1 \quad\text{and}\quad \alpha\big(\overline{p}(\rho),\rho\big)=0.$$

The function does not admit derivative at $\underline{p}$ and $\overline{p}(\rho)$ but does so everywhere else. It is constant outside of the interval $[\underline{p},\overline{p}(\rho)]$. Inside that interval, the derivatives with respect to both arguments are

$$\frac{\partial\alpha(p,\rho)}{\partial p}=-\frac{(R-f)(\frac{p(R-1)}{(1-f)(1-p)})^{\frac{1}{\rho}}}{(1-p)p(R-1+(1-f)(\frac{p(R-1)}{(1-f)(1-p)})^{\frac{1}{\rho}})^2\rho}<0,$$

$$\frac{\partial\alpha(p,\rho)}{\partial\rho}=\frac{(R-f)(\frac{p(R-1)}{(1-f)(1-p)})^{\frac{1}{\rho}}\log(\frac{p(R-1)}{(1-f)(1-p)})}{(R-1+(1-f)(\frac{p(R-1)}{(1-f)(1-p)})^{\frac{1}{\rho}})^2\rho^2}>0.$$

Notice that in the above equation the sign of the derivative with respect to ρ is unambiguous since $\frac{p(R-1)}{(1-f)(1-p)}\in[1,(\frac{R}{f})^{\rho}]$ whenever $p\in[\underline{p},\overline{p}(\rho)]$. Therefore, the logarithm is positive. □

Proof of Lemma 2. The joint distribution of θ and x is

$$\begin{bmatrix}\theta\\ x_i\end{bmatrix}\sim N\left(\begin{bmatrix}\mu(s)\\ \mu(s)\end{bmatrix},\begin{bmatrix}\sigma_\theta^2 & \sigma_\theta^2\\ \sigma_\theta^2 & \sigma_\theta^2+\sigma_\varepsilon^2\end{bmatrix}\right).$$

Therefore, using standard results for the Normal distribution,

$$(\theta\mid x)\sim N\big(\mu_{\theta|x},\sigma^2_{\theta|x}\big)$$

where

$$\mu_{\theta|x}=\mu(s)+\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\varepsilon^2}\big(x-\mu(s)\big)=\frac{x\sigma_\theta^2+\mu(s)\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2},$$

$$\sigma^2_{\theta|x}=\sigma_\theta^2-\frac{\sigma_\theta^2\sigma_\theta^2}{\sigma_\theta^2+\sigma_\varepsilon^2}=\frac{\sigma_\theta^2\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2},$$

$$(\theta\mid x)\sim N\left(\frac{x\sigma_\theta^2+\mu(s)\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2},\frac{\sigma_\theta^2\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2}\right).$$

Therefore, the conditional CDF is given by

$$H(\theta\mid x)=\Phi\left(\sqrt{\frac{\sigma_\theta^2+\sigma_\varepsilon^2}{\sigma_\theta^2\sigma_\varepsilon^2}}\left(\theta-\frac{x\sigma_\theta^2+\mu(s)\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2}\right)\right).$$

This expression is increasing in θ and decreasing in x and in $\mu(s)$.

Define $\chi(p,\theta_s^*)$ as the signal such that an agent observing it assigns a value of p to the status quo surviving in an equilibrium with threshold θ_s^*. The monotonicity of $H(\theta\mid x)$ guarantees that there is only one such value. Hence, $\chi(p, \theta_s^*)$ is implicitly defined by

$$p = 1 - H\big(\theta_s^* \mid \chi\big(p, \theta_s^*\big)\big).$$

Using the value for $H(\theta \mid x)$ obtained above, find $\chi(p, \theta_s^*)$ by solving for x in

$$p = 1 - \Phi\left(\sqrt{\frac{\sigma_\theta^2 + \sigma_\varepsilon^2}{\sigma_\theta^2 \sigma_\varepsilon^2}}\left(\theta_s^* - \frac{x\sigma_\theta^2 + \mu(s)\sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}\right)\right)$$

to obtain

$$\chi\big(p, \theta_s^*\big) = \frac{\sigma_\theta^2 + \sigma_\varepsilon^2}{\sigma_\theta^2}\left[\theta_s^* - \sqrt{\frac{\sigma_\theta^2 \sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right] - \frac{\sigma_\varepsilon^2}{\sigma_\theta^2}\mu(s).$$

Now it is possible to calculate the fraction of agents with beliefs of p or lower when there is a threshold θ_s^* and θ is the true state. Accumulate all signals equal to or lower than $\chi(p, \theta_s^*)$.

$$\Gamma_\mu\big(p \mid \theta_s^*, \theta\big) = \int_{x=-\infty}^{\chi(p,\theta_s^*)} \phi\left(\frac{x-\theta}{\sigma_\varepsilon}\right) dx = \Phi\left(\frac{\chi(p, \theta_s^*) - \theta}{\sigma_\varepsilon}\right).$$

Combining the last two equations,

$$\Gamma_\mu\big(p \mid \theta_s^*, \theta\big) = \Phi\left(\frac{\theta_s^* - \theta}{\sigma_\varepsilon} + \frac{\sigma_\varepsilon}{\sigma_\theta^2}\big(\theta_s^* - \mu\big) - \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right).$$

This expression is increasing in p and θ_s^* and decreasing in θ and in μ. Calculating the expectation under the conjectured equilibrium the fraction of the endowment withdrawn by type ρ agents is equal to

$$A\big(\mu, \rho, \theta_s^*, \theta\big) = \int_{p=0}^{1} \alpha(p, \rho) d\Gamma_\mu\big(p \mid \theta_s^*, \theta\big).$$

This proves the lemma. □

Proof of Lemma 3. Notice that holding everything else fixed, $\Gamma_\mu(p \mid \theta_s^*, \theta)$ is decreasing in θ. This means that this distribution is stochastically dominant for higher realizations of θ. This in turn, coupled with the fact that the function $\alpha(p, \rho)$ is continuous and decreasing in p, implies that the per-type function $A(\mu, \rho, \theta_s^*, \theta)$ is continuous and a decreasing function of the realized value of θ. $\frac{K(s,\theta_s^*,\theta)}{e(s)}$ is a convex combination of the per-type functions $A(\mu, \rho, \theta_s^*, \theta)$ and is continuous and decreasing in θ as well. Define the function

$$Z(\theta) \equiv \theta - \frac{K(s, \theta_s^*, \theta)}{e(s)}.$$

This function is continuous and increasing in θ. If θ_s^* is to be a threshold equilibrium then $\theta < \theta_s^* \Leftrightarrow Z(\theta) < 0$ (a crisis) and $\theta > \theta_s^* \Leftrightarrow Z(\theta) > 0$ (no crisis). By continuity, $\theta = \theta_s^* \Leftrightarrow Z(\theta) = Z(\theta_s^*) = 0$. And, by definition of $Z(\cdot)$, $Z(\theta_s^*) = 0 \Leftrightarrow \theta_s^* = \frac{K(s,\theta_s^*,\theta_s^*)}{e(s)}$. □

Proof of Proposition 1. Lemmas 2 and 3 prove that θ_s^* is calculated as the weighted average of the terms $A(\mu(s), \rho, \theta_s^*, \theta_s^*)$.

The results on the derivatives of $A(\mu(s), \rho, \theta_s^*, \theta_s^*)$ follow from considering

$$A(\mu, \rho, \theta_s^*, \theta_s^*) = \int_{p=0}^{1} \alpha(p, \rho) d\Gamma_\mu(p \mid \theta_s^*, \theta_s^*)$$

with

$$\Gamma_\mu(p \mid \theta_s^*, \theta_s^*) = \Phi\left(\frac{\sigma_\varepsilon}{\sigma_\theta^2}(\theta_s^* - \mu) - \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right).$$

Changes in μ and θ_s^* shift the CDF $\Gamma_\mu(p \mid \theta_s^*, \theta_s^*)$ to the right and left, respectively. These stochastic-dominance results paired with $\alpha(p, \rho)$ being monotonically decreasing yield the results on the signs of the derivatives with respect to μ and θ_s^*. The derivative with respect to ρ follows from the fact that $\alpha(p, \rho)$ is increasing in ρ.

The result that at extremal threshold equilibrium the slope of the aggregate intensity of the attack is less than unity follows from continuity of the functions $A(\mu, \rho, \theta_s^*, \theta_s^*)$ (and therefore of $K(s, \theta_s^*, \theta_s^*)$) with respect to θ_s^* together with

$$\left[\frac{K(s, 0, 0)}{e(s)}, \frac{K(s, 1, 1)}{e(s)}\right] \subset [0, 1].$$

Notice that continuity of the functions $K(s, \theta_s^*, \theta_s^*)$ also requires that wealth shares are continuous with respect to the threshold. This is a consequence of the analysis performed Section 4.4. □

Proof of Lemma 4. Indirect utility. To characterize the Walrasian equilibrium at date $t = 0$, the first step is to construct the indirect utility function involving equilibrium behavior in the global game. Before observing x, an agent values a payoff of $w(s, \rho)$ at date 1 by correctly anticipating the distribution of p given the equilibrium value of θ_s^*, and using the knowledge of the distribution over θ. Indirect utility is the expectation with respect to the distribution of p given θ_s^* and the distribution of θ:

$$E_{(p,\theta)}\left[\frac{1}{1-\rho}\left\{p\left[w(s,\rho)\left(\alpha + (1-\alpha)R\right)\right]^{1-\rho} + (1-p)\left[w(s,\rho)\left(\alpha + (1-\alpha)f\right)\right]^{1-\rho}\right\} \mid \theta_s^*\right],$$

where $\alpha = \alpha(p, \rho)$ is the optimal intensity of attack. Since wealth is predetermined it can be taken out of the expectation. As a consequence, indirect utility has the familiar CRRA form modified by a multiplicative constant.

$$V(s, \rho)\frac{w(s, \rho)^{1-\rho}}{1-\rho}.$$

The multiplicative constant is both state- and type-dependent. It depends on the type through the function $\alpha(p, \rho)$ and the exponent ρ. It depends on the state through $\mu(s)$ and the threshold θ_s^*, but not through $e(s)$.

$$V(s, \rho) = E_{(p,\theta)}\left[p\left[\alpha + (1-\alpha)R\right]^{1-\rho} + (1-p)\left[\alpha + (1-\alpha)f\right]^{1-\rho} \mid \theta_s^*\right]. \tag{40}$$

Ex-ante indirect utility is the expectation of indirect utility over all possible states $s \in S$.

$$U^0(\rho) = \int_S g(s) V(s, \rho) \frac{w(s, \rho)^{1-\rho}}{1-\rho} \, ds.$$

Agents decide on their risk-sharing behavior by maximizing this ex-ante indirect utility. They take both $\mu(s)$ and θ_s^* as given when doing so. Therefore, they take $V(s, \rho)$ as given. They are limited in their choices of $w(s, \rho_i)$ by the budget constraint, which is common to all types.

Equilibrium wealth distribution and constrained efficiency. The expression $V(s, \rho)$ represents the value by which utility in state s is modified because of the availability of the risky investment. Agents take $V(s, \rho)$ as given and disregard the externality generated by the combination of their choices of how much wealth to carry into each state and their optimal strategy in the global game. This externality is embodied in the functions $V(s, \rho)$. But complete markets in state-contingent claims ensure that the equilibrium is efficient among those allocations that take the expressions $V(s, \rho)$ from the continuation equilibrium.

The planner's problem. A social planner with Pareto weights $\{\lambda_i\}$ maximizes

$$\sum_i \lambda_i \int_S g(s) V(s, \rho_i) \frac{w(s, \rho_i)^{1-\rho_i}}{1-\rho_i} \, ds$$

subject to

$$\sum_i w(s, \rho_i) \leqslant e(s), \quad \forall s \in S.$$

Agents are homogeneous with respect to their beliefs about states s of the world. Therefore, the objective function of the planner can be rewritten by moving the sum over types into the expectation. Thus, the objective function for the planner is

$$\int_S g(s) \sum_i \lambda_i V(s, \rho_i) \frac{w(s, \rho_i)^{1-\rho_i}}{1-\rho_i} \, ds.$$

The resource constraint will hold with equality for all states. From the rewritten maximization problem it is clear that a necessary condition for global constrained optimality is state-by-state optimal risk-sharing. This means that by fixing a state $s \in S$, the solution has to solve the within-state risk-sharing problem:

$$\max \sum_i \lambda_i V(s, \rho_i) \frac{w(s, \rho_i)^{1-\rho_i}}{1-\rho_i}$$

subject to the resource constraint for that state.

$$\sum_i w(s, \rho_i) = e(s).$$

Due to the fact that marginal utility is infinite at zero consumption, solutions will be interior. Define the modified Pareto-weight $\hat{\lambda}_i \equiv \lambda_i V(s, \rho_i)$. $\hat{\lambda}_i$ does depend on the state, but since the planner's maximization is made for a fixed state, this dependence can be safely ignored. Call the multiplier on the resource constraint γ. Then, the first order condition for $w(s, \rho_i)$ states

$$\hat{\lambda}_i w(s, \rho_i)^{-\rho_i} = \gamma.$$

Solve the first order condition for $w(s,\rho_i)$ and use it for all j in the resource constraint,

$$\sum_j \left(\frac{\hat{\lambda}_j}{\gamma}\right)^{\frac{1}{\rho_j}} - e(s) = 0.$$

Now eliminate γ with the first order condition for i to get

$$\sum_j \left(\frac{\hat{\lambda}_j}{\hat{\lambda}_i}\right)^{\frac{1}{\rho_j}} w(s,\rho_i)^{\frac{\rho_i}{\rho_j}} - e(s) = 0.$$

Calculations for the expressions in the statement of the lemma. Define the implicit function

$$F\big(w(s,\rho_i), e(s)\big) \equiv \sum_j \left(\frac{\hat{\lambda}_j}{\hat{\lambda}_i}\right)^{\frac{1}{\rho_j}} w(s,\rho_i)^{\frac{\rho_i}{\rho_j}} - e(s) = 0.$$

1. Calculation of $\frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)}$: Function $F(w(s,\rho_i), e(s))$ implicitly relates $w(s,\rho_i)$ to $e(s)$ in an optimal risk-sharing arrangement. As an intermediate result, first calculate $\frac{dw(s,\rho_i)}{de(s)}$. Using the Implicit Function Theorem,

$$\begin{aligned}
\frac{dw(s,\rho_i)}{de(s)} &= -\frac{\frac{\partial F}{\partial e(s)}}{\frac{\partial F}{\partial w(s,\rho_i)}} = -\frac{-1}{\sum_j \frac{\rho_i}{\rho_j}(\frac{\hat{\lambda}_j}{\hat{\lambda}_i})^{\frac{1}{\rho_j}} w(s,\rho_i)^{\frac{\rho_i}{\rho_j}} \frac{1}{w(s,\rho_i)}} \\
&= \frac{1}{\sum_j \frac{\rho_i}{\rho_j}\frac{w(s,\rho_j)}{w(s,\rho_i)}} \\
&= \frac{\frac{w(s,\rho_i)}{\rho_i}}{\sum_j \frac{w(s,\rho_j)}{\rho_j}} \\
&= \frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}.
\end{aligned}$$

Now calculate the desired derivative.

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} = \frac{1}{e(s)^2}\left[e(s)\frac{dw(s,\rho_i)}{de(s)} - w(s,\rho_i)\right].$$

Using the result obtained for $\frac{dw(s,\rho_i)}{de(s)}$ in this equation

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} = \frac{w(s,\rho_i)}{e(s)^2}\left(\frac{\rho_H(s)}{\rho_i} - 1\right).$$

2. Calculation of $\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)}$: Notice that $V(s,\rho)$ is a smooth function of arguments in the CDF over p, i.e. of $\mu(s)$ and the threshold θ_s^*. Hence, for risk-averse agents, wealth fractions will be smooth functions as well.

Noticing that $\hat{\lambda}$ depends on $\mu(s)$, the implicit function can be written as $F(w(s,\rho_i), \mu(s))$. Again, use the Implicit Function Theorem.

$$\frac{dw(s,\rho_i)}{d\mu(s)} = -\frac{\frac{\partial F}{\partial \mu(s)}}{\frac{\partial F}{\partial w(s,\rho_i)}} = -\frac{\sum_j \frac{1}{\rho_j}\frac{\hat{\lambda}_i}{\hat{\lambda}_j}(\frac{\hat{\lambda}_j}{\hat{\lambda}_i})^{\frac{1}{\rho_j}} w(s,\rho_i)^{\frac{\rho_i}{\rho_j}} \frac{\lambda_j}{\lambda_i}\frac{\partial}{\partial \mu(s)}(\frac{V(s,\rho_j)}{V(s,\rho_i)})}{\sum_j \frac{\rho_i}{\rho_j}(\frac{\hat{\lambda}_j}{\hat{\lambda}_i})^{\frac{1}{\rho_j}} w(s,\rho_i)^{\frac{\rho_i}{\rho_j}} \frac{1}{w(s,\rho_i)}}$$

$$= -\frac{\sum_j \frac{1}{\rho_j} w(s,\rho_j)\frac{V(s,\rho_i)}{V(s,\rho_j)}\frac{\partial}{\partial \mu(s)}(\frac{V(s,\rho_j)}{V(s,\rho_i)})}{\sum_j \frac{\rho_i}{\rho_j}\frac{w(s,\rho_j)}{w(s,\rho_i)}}$$

$$= -\frac{\frac{w(s,\rho_i)}{\rho_i}}{\sum_j \frac{w(s,\rho_j)}{\rho_j}}\left(\sum_j \frac{1}{\rho_j} w(s,\rho_j)\frac{V(s,\rho_i)}{V(s,\rho_j)}\frac{\partial}{\partial \mu(s)}\left(\frac{V(s,\rho_j)}{V(s,\rho_i)}\right)\right)$$

$$= -\frac{\frac{w(s,\rho_i)}{\rho_i}}{\sum_j \frac{w(s,\rho_j)}{\rho_j}}\left(\sum_j \frac{1}{\rho_j} w(s,\rho_j)\frac{\partial}{\partial \mu(s)}\left[\log V(s,\rho_j) - \log V(s,\rho_i)\right]\right)$$

$$= -\frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}\left(\sum_j \frac{1}{\rho_j} w(s,\rho_j)\frac{\partial}{\partial \mu(s)}\left[\log V(s,\rho_j) - \log V(s,\rho_i)\right]\right).$$

Realizing that

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} = \frac{1}{e(s)}\frac{dw(s,\rho_i)}{d\mu(s)}$$

define

$$J(\rho_i) = \sum_j \frac{1}{\rho_j}\frac{w(s,\rho_j)}{e(s)}\frac{\partial}{\partial \mu(s)}\left[\log V(s,\rho_j) - \log V(s,\rho_i)\right].$$

Then

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} = -\frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}J(\rho_i).$$

3. Calculation of $\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}$: The result

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*} = -\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} = \frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}J(\rho_i)$$

immediately follows from

$$\frac{\partial \log V(s,\rho_j)}{\partial \theta_s^*} = -\frac{\partial \log V(s,\rho_j)}{\partial \mu(s)}.$$

To show that this relationship holds, calculate

$$\frac{\partial \log V(s,\rho_j)}{\partial \mu(s)} = -\frac{1}{V(s,\rho_j)}\int_0^1 \frac{\partial \zeta(p,\rho_j)}{\partial p}\frac{\partial \Gamma_{\mu(s)}(p \mid \theta_s^*, \theta_s^*)}{\partial \mu(s)}\,dp$$

and

$$\frac{\partial \log V(s,\rho_j)}{\partial \theta_s^*} = -\frac{1}{V(s,\rho_j)}\int_0^1 \frac{\partial \zeta(p,\rho_j)}{\partial p}\frac{\partial \Gamma_{\mu(s)}(p \mid \theta_s^*, \theta_s^*)}{\partial \theta_s^*}\,dp$$

where

$$\Gamma_\mu\big(p \mid \theta_s^*, \theta_s^*\big) = \Phi\left(\frac{\sigma_\varepsilon}{\sigma_\theta^2}\big(\theta_s^* - \mu\big) - \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right).$$

Inspection of the functional form of $\Gamma_{\mu(s)}(p \mid \theta_s^*, \theta_s^*)$ reveals $\frac{\partial \Gamma_{\mu(s)}(p|\theta_s^*,\theta_s^*)}{\partial \mu(s)} = -\frac{\partial \Gamma_{\mu(s)}(p|\theta_s^*,\theta_s^*)}{\partial \theta_s^*}$, so that

$$\frac{\partial \log V(s,\rho_j)}{\partial \theta_s^*} = -\frac{1}{V(s,\rho_j)}\int_0^1 \frac{\partial \zeta(p,\rho_j)}{\partial p}\left(-\frac{\partial \Gamma_{\mu(s)}(p \mid \theta_s^*, \theta_s^*)}{\partial \mu(s)}\right)dp = -\frac{\partial \log V(s,\rho_j)}{\partial \mu(s)}.$$

Finally, with this result, $J(\rho_i)$ can also be written in the following form.

$$J(\rho_i) = \sum_j \frac{1}{\rho_j}\frac{w(s,\rho_j)}{e(s)}\frac{\partial}{\partial \theta_s^*}\big[\log V(s,\rho_i) - \log V(s,\rho_j)\big]. \qquad \square$$

Proof of Theorem 1. Define

$$M = \frac{1}{1 - (\sum_i \frac{w(s,\rho_i)}{e(s)}\frac{\partial A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)}{\partial \theta_s^*} + \sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}A(\mu(s),\rho_i,\theta_s^*,\theta_s^*))}.$$

Notice that at an extremal or unique threshold equilibrium the denominator is between 0 and 1. Therefore, $M > 1$. Use the definition of M in (31) to obtain

$$\begin{aligned}\frac{d\theta_s^*}{ds} = M\Bigg[&\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)}A\big(\mu(s),\rho_i,\theta_s^*,\theta_s^*\big)\frac{de(s)}{ds}\\ &+\Bigg(\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)}A\big(\mu(s),\rho_i,\theta_s^*,\theta_s^*\big)\\ &+\sum_i \frac{w(s,\rho_i)}{e(s)}\frac{\partial A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)}{\partial \mu(s)}\Bigg)\frac{d\mu(s)}{ds}\Bigg].\end{aligned}$$

From its definition, the dependence of $A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)$ on both $\mu(s)$ and θ_s^* is only through the conditional distribution on beliefs. Consider what would happen if $\mu(s)$ and θ_s^* by the same infinitesimal amount. Consider the functional form of

$$\Gamma_\mu\big(p \mid \theta_s^*, \theta_s^*\big) = \Phi\left(\frac{\sigma_\varepsilon}{\sigma_\theta^2}\big(\theta_s^* - \mu\big) - \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right).$$

A simultaneous increase of $\mu(s)$ and θ_s^* leaves the distribution over beliefs at the threshold unchanged. This implies that

$$\frac{\partial A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)}{\partial \mu(s)} + \frac{\partial A(\mu(s),\rho_i,\theta_s^*,\theta_s^*)}{\partial \theta_s^*} = 0.$$

Also, by Lemma 4,

$$\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} = -\frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*}$$

so that M can be written in terms of the derivatives with respect to $\mu(s)$

$$M = \left[1 + \left(\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big) + \sum_i \frac{w(s,\rho_i)}{e(s)} \frac{\partial A(\mu(s), \rho_i, \theta_s^*, \theta_s^*)}{\partial \mu(s)}\right)\right]^{-1}$$

which can be rearranged to obtain

$$\left(\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\mu(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big) + \sum_i \frac{w(s,\rho_i)}{e(s)} \frac{\partial A(\mu(s), \rho_i, \theta_s^*, \theta_s^*)}{\partial \mu(s)}\right) = \frac{1-M}{M}.$$

Hence,

$$\begin{aligned}\frac{d\theta_s^*}{ds} &= M\left[\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)\frac{de(s)}{ds} + \frac{1-M}{M}\frac{d\mu(s)}{ds}\right] \\ &= M\sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{de(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)\frac{de(s)}{ds} + (1-M)\frac{d\mu(s)}{ds}.\end{aligned}$$

Substituting it into Eq. (36) permits to solve for T_1 and T_2 and yields

$$\begin{aligned}&T_1 = 1 - M, \\ &T_2 = M\sum_i \frac{w(s,\rho_i)}{e(s)}\frac{1}{e(s)}\left(\frac{\rho_H(s)}{\rho_i} - 1\right)A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big),\end{aligned}$$

T_1 is negative because $M > 1$. If the population is homogeneous (consists of a single type ρ), $\rho_H(s) = \rho$ and $T_2 = 0$. Otherwise, the sign of T_2 is determined by the sign of

$$\begin{aligned}&\left[\sum_i \frac{w(s,\rho_i)}{e(s)}\left(\frac{\rho_H(s)}{\rho_i} - 1\right)A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)\right] \\ &\quad = \left[\sum_i \frac{\rho_H(s)}{\rho_i}\frac{w(s,\rho_i)}{e(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)\right] - \left[\sum_i \frac{w(s,\rho_i)}{e(s)} A\big(\mu(s), \rho_i, \theta_s^*, \theta_s^*\big)\right].\end{aligned}$$

Also, notice that

$$\sum_i \frac{\rho_H(s)}{\rho_i}\frac{w(s,\rho_i)}{e(s)} = \sum_i \frac{w(s,\rho_i)}{e(s)} = 1$$

so that the weights in both summations can be interpreted as probabilities. Write them as probability measures ν_1 and ν_2, respectively. Then, the above difference can be written as the difference of two expectations over the variable ρ.

$$\sum_\rho \nu_1(\rho) A\big(\mu(s), \rho, \theta_s^*, \theta_s^*\big) - \sum_\rho \nu_2(\rho) A\big(\mu(s), \rho, \theta_s^*, \theta_s^*\big).$$

The probability measure ν_1 is first-order stochastically dominated by ν_2 (because $\frac{\rho_H(s)}{\rho_i}$ is decreasing in ρ_i). Since $A(\mu(s), \rho, \theta_s^*, \theta_s^*)$ is increasing in ρ, this implies that the second expectation is greater than the first. Therefore,

$$\left[\sum_i \frac{w(s,\rho_i)}{e(s)} \frac{1}{e(s)} \left(\frac{\rho_H(s)}{\rho_i} - 1\right) A\left(\mu(s), \rho_i, \theta_s^*, \theta_s^*\right)\right] < 0$$

and therefore $T_2 < 0$. □

Proof of Theorem 2. Denote the limit of expression X when $\sigma_\varepsilon \to 0$ as $\tilde{X}$ and use the superindex h for the restriction to a homogeneous population. As $\sigma_\varepsilon \to 0$, the CDF over beliefs $\Gamma(p \mid \theta_s^*, \theta_s^*)$ converges in distribution to the uniform distribution $\tilde{\Gamma} = p$.

$$\lim_{\sigma_\varepsilon \to 0} \Gamma\left(p \mid \theta_s^*, \theta_s^*\right) = \Phi\left(-\Phi^{-1}(1-p)\right) = 1 - \Phi\left(\Phi^{-1}(1-p)\right) = p.$$

In consequence, $A(\mu(s), \rho_i, \theta_s^*, \theta_s^*) \to \tilde{A}(\rho_i)$, which is independent of θ_s^* and $\mu(s)$. This implies that in the limit

$$M \to \tilde{M} \equiv \left[1 - \sum_i \frac{d\frac{w(s,\rho_i)}{e(s)}}{d\theta_s^*} \tilde{A}(\rho_i)\right]^{-1} > 1.$$

With a homogeneous population with risk-aversion ρ, $w(s,\rho) = e(s)$. Therefore, $\frac{d\frac{w(s,\rho)}{e(s)}}{d\theta_s^*} = 0$ and $\tilde{M}^h = 1$. In consequence,

$$\tilde{T}_1^h = 1 - \tilde{M}^h = 0, \qquad \tilde{T}_1 = 1 - \tilde{M} < 0,$$
$$\tilde{T}_2^h = 0, \qquad \tilde{T}_2 = \tilde{M} \sum_i \frac{w(s,\rho_i)}{e(s)} \frac{1}{e(s)} \left(\frac{\rho_H(s)}{\rho_i} - 1\right) \tilde{A}(\rho_i) < 0.$$

Using these results in the expressions derived in Theorem 1, this implies

$$\frac{d\tilde{\psi}(s)^h}{ds} = -\frac{1}{\sigma_\theta} \phi(\cdot) \frac{d\mu(s)}{ds} < 0$$

and

$$\frac{d\tilde{\psi}(s)}{ds} = \frac{d\tilde{\psi}(s)^h}{ds} + \frac{1}{\sigma_\theta} \phi(\cdot) \left[\tilde{T}_1 \frac{d\mu(s)}{ds} + \tilde{T}_2 \frac{de(s)}{ds}\right]. \qquad \square$$

Proof of Theorem 3. The theorem follows from two results. The first result is that, for the specified parameter values, the equilibrium is unique among, in principle, possibly multiple threshold equilibria. The second result states that there exist no equilibria that are not of the threshold kind. This second result is taken directly from [13]. The first result, however, is new and requires a proof.

A. Uniqueness among threshold equilibria. Uniqueness among threshold equilibria obtains if

$$LHS \equiv \sum_i \frac{\partial}{\partial \theta_s^*} \left(\frac{w(s,\rho_i)}{e(s)}\right) \tilde{A}(\rho_i) < 1.$$

The proof proceeds by obtaining an upper bound for the left hand side of this equation which depends only on parameters of the problem. Since $0 \leqslant \tilde{A}(\rho_i) \leqslant 1$,

$$LHS \equiv \sum_i \frac{\partial}{\partial \theta_s^*} \left(\frac{w(s,\rho_i)}{e(s)}\right) \tilde{A}(\rho_i) \leqslant \sum_i \left|\frac{\partial}{\partial \theta_s^*} \left(\frac{w(s,\rho_i)}{e(s)}\right) \tilde{A}(\rho_i)\right| \leqslant \sum_i \left|\frac{\partial}{\partial \theta_s^*} \left(\frac{w(s,\rho_i)}{e(s)}\right)\right|.$$

The derivative $\frac{\partial}{\partial \theta_s^*}(\frac{w(s,\rho_i)}{e(s)})$ was calculated in the proof of Lemma 4:

$$\frac{\partial}{\partial \theta_s^*}\left(\frac{w(s,\rho_i)}{e(s)}\right) = \frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}\sum_j \frac{w(s,\rho_j)}{e(s)}\frac{1}{\rho_j}\frac{\partial}{\partial \theta_s^*}\left[\log V(s,\rho_i) - \log V(s,\rho_j)\right].$$

Substituting this expression in the right hand side of the previous equation:

$$\begin{aligned}&\sum_i \left|\frac{\partial}{\partial \theta_s^*}\left(\frac{w(s,\rho_i)}{e(s)}\right)\right| \\ &\quad = \sum_i \frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i}\sum_j \frac{w(s,\rho_j)}{e(s)}\frac{1}{\rho_j}\left|\frac{\partial}{\partial \theta_s^*}\left[\log V(s,\rho_i) - \log V(s,\rho_j)\right]\right|.\end{aligned}$$

By the definition of the harmonic mean ρ_H,

$$\sum_j \frac{w(s,\rho_j)}{e(s)}\frac{1}{\rho_j} = \frac{1}{\rho_H} \quad \text{and} \quad \sum_i \frac{w(s,\rho_i)}{e(s)}\frac{\rho_H(s)}{\rho_i} = 1.$$

Therefore,

$$\begin{aligned} LHS \leqslant \sum_i \left|\frac{\partial}{\partial \theta_s^*}\left(\frac{w(s,\rho_i)}{e(s)}\right)\right| &\leqslant \frac{1}{\rho_H}\max_{\rho_i,\rho_j}\left|\frac{\partial}{\partial \theta_s^*}\left[\log V(s,\rho_i) - \log V(s,\rho_j)\right]\right| \\ &\leqslant \frac{1}{\rho_H}\left|\max_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho) - \min_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho)\right|.\end{aligned}$$

Extreme values of $\log V(s,\rho)$ **when** $\sigma_\varepsilon \to 0$**:** By definition, from (40),

$$\begin{aligned} V(s,\rho) = \int_\Theta \Bigg[\int_0^1 \big\{p\big[R-(R-1)\alpha(p,\rho)\big]^{1-\rho} \\ + (1-p)\big[f + (1-f)\alpha(p,\rho)\big]^{1-\rho}\big\} d\Gamma_{\mu(s)}\big(p \mid \theta_s^*, \theta\big)\Bigg] d\Phi\left(\frac{\theta-\mu(s)}{\sigma_\theta}\right),\end{aligned}$$

where

$$\Gamma_\mu\big(p \mid \theta_s^*, \theta\big) = \Phi\left(\frac{\theta_s^* - \theta}{\sigma_\varepsilon} + \frac{\sigma_\varepsilon}{\sigma_\theta^2}\big(\theta_s^* - \mu\big) - \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\Phi^{-1}(1-p)\right).$$

This last expression for $\Gamma_\mu(p \mid \theta_s^*, \theta)$ is derived in the proof of Lemma 2.

Take the limit as $\sigma_\varepsilon \to 0$. The CDF over p converges to a degenerate distribution. If $\theta < \theta_s^*$, then $\Gamma_\mu(p \mid \theta_s^*, \theta)$ puts all mass on $p = 0$. When θ is bigger than the threshold then it puts all mass on $p = 1$. If $\sigma_\varepsilon \to 0$, then $\Gamma_\mu(p \mid \theta_s^*, \theta) \to \Phi(-\infty) = 0$ if $\theta_s^* < \theta$ and $\Gamma_\mu(p \mid \theta_s^*, \theta) \to \Phi(+\infty) = 1$ if $\theta_s^* > \theta$. From $\alpha(0,\rho) = 1$ and $\alpha(1,\rho) = 0$, it then follows that

$$\begin{aligned}\tilde{V}(s,\rho) \equiv \lim_{\sigma_\varepsilon \to 0} V(s,\rho) &= \int_{\theta < \theta_s^*} d\Phi\left(\frac{\theta-\mu(s)}{\sigma_\theta}\right) + \int_{\theta > \theta_s^*} R^{1-\rho}\, d\Phi\left(\frac{\theta-\mu(s)}{\sigma_\theta}\right) \\ &= \Phi\left(\frac{\theta_s^*-\mu(s)}{\sigma_\theta}\right) + \left[1 - \Phi\left(\frac{\theta_s^*-\mu(s)}{\sigma_\theta}\right)\right] R^{1-\rho}.\end{aligned}$$

Take the log and calculate the derivative with respect to θ_s^*

$$\frac{\partial \log \tilde{V}(s,\rho)}{\partial \theta_s^*} = \frac{1}{\sigma_\theta}\phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right)\frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)}.$$

The expression $\max_\rho \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)}$ **has an upper bound of** 1**:** If only levels of risk-aversion that are less than one are allowed, then the numerator is negative. Since the denominator is positive, the expression is bounded by zero in this case. For levels of risk-aversion greater than one the numerator is positive. In this case, the numerator has an upper bound of 1 and the denominator a lower bound of 1. Therefore, the whole expression can be at most 1. The previous cases exhaust all possibilities.

$$\forall \rho\colon \quad \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)} \leqslant 1 \quad \Longrightarrow \quad \max_\rho \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)} \leqslant 1.$$

The expression $\min_\rho \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)}$ **has a lower bound of** $1-R$**:** If only levels of risk-aversion that are greater than one are allowed, then both the numerator and the denominator are positive, meaning that zero is a lower bound. If levels of risk-aversion less than one are admitted, then the numerator can be negative while the denominator is positive. In this case, a lower bound for the denominator is $1-R$. The lower bound for the denominator is again 1 (here the lower bound is relevant since the whole expression is negative). The previous cases exhaust all possibilities.

$$\forall \rho\colon \quad \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)} \geqslant 1-R \quad \Longrightarrow \quad \min_\rho \frac{1-R^{1-\rho}}{\tilde{V}(s,\rho)} \geqslant 1-R.$$

A bound for the expression $\frac{1}{\rho_H}|\max_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho) - \min_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho)|$ **(and hence LHS) is obtained by combining the previous findings:**

$$\begin{aligned} LHS &\leqslant \frac{1}{\rho_H}\left|\max_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho) - \min_\rho \frac{\partial}{\partial \theta_s^*}\log V(s,\rho)\right| \\ &\leqslant \frac{1}{\rho_H}\frac{1}{\sigma_\theta}\phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right)\bigl|1-(1-R)\bigr| \\ &= \frac{1}{\rho_H}\frac{1}{\sigma_\theta}\phi\left(\frac{\theta_s^* - \mu(s)}{\sigma_\theta}\right)R. \end{aligned}$$

The CDF of a Normal distribution takes at most value $\frac{1}{2\pi}$ and $\frac{1}{\rho_H}$ is at least $\frac{1}{\underline{\rho}}$. Therefore,

$$LHS \equiv \sum_i \frac{\partial}{\partial \theta_s^*}\left(\frac{w(s,\rho_i)}{e(s)}\right)\tilde{A}(\rho_i) \leqslant \frac{1}{\underline{\rho}}\frac{1}{\sigma_\theta}\frac{1}{2\pi}R.$$

This implies that if the condition stated in the theorem holds, then

$$\sum_i \frac{\partial}{\partial \theta_s^*}\left(\frac{w(s,\rho_i)}{e(s)}\right)\tilde{A}(\rho_i) < 1.$$

There is a single threshold equilibrium for the parameters assumed.

B. There are no other (non-threshold) equilibria (Guimarães and Morris). The proof is a standard argument in the Global Games literature. It consists of showing that the threshold equilibrium found above is the only strategy profile that survives iterated elimination of strategies that are never best responses. The proof for the particular case of risk-averse agents with a Normal prior and Normal signals is contained in [13] and [12]. □

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