Review Exercises I

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1. $P(A) = 0.3, P(B) = 0.5, P(A \cup B) = 0.7$
   (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.3 + 0.5 - 0.7 = 0.1$
   (b) $P(A^C \cup B^C) = P(A^C) + P(B^C) - P(A^C \cap B^C) = 0.7 + 0.5 - 0.3 = 0.9x$
   (c) $P(A^C \cap B) = P(B) - P(A \cap B) = 0.5 - 0.1 = 0.4$
2. $A =$ actress is present
   $D =$ stunt double is present
   $P(A) = 0.4,\ P(D) = 0.3,\ P(A \cap D) = 0.05$
   (a) $P(D \cap A^C) = P(D) - P(A \cap D) = 0.3 - 0.05 = 0.25$
   (b) $P(A^C \cap D^C) = 1 - P(A \cup D) = 1 - (P(A) + P(D) - P(A \cap D)) = 1 - (0.4 + 0.3 - 0.05) = 0.35$
3. First, we show that $f(x) > 0$ for $x = 0, 1, 2, \ldots$.

   We are given that $\lambda > 0$, therefore, the factor $\dfrac{1}{1+\lambda}$ is always positive. The factor $\dfrac{\lambda}{1+\lambda}$ is also always positive and raised to a non-negative power, the product of two positive numbers is also positive.

   Next, we show that $\displaystyle\sum_{x=0}^{\infty} f(x) = 1$

   We start by rewriting the sum,

$$
\begin{aligned}
\sum_{x=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x &= \frac{1}{1+\lambda}\sum_{x=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^x \\
&= \frac{1}{1+\lambda}\left[\left(\frac{\lambda}{1+\lambda}\right)^0 + \left(\frac{\lambda}{1+\lambda}\right)^1 + \left(\frac{\lambda}{1+\lambda}\right)^2 + \cdots\right] \\
&= \frac{1}{1+\lambda}\left[\frac{1}{1 - \frac{\lambda}{1+\lambda}}\right] \\
&= \frac{1}{1+\lambda}(1+\lambda) \\
&= 1
\end{aligned}
$$

The third equality comes from applying the formula for sum of an infinite geometric series since $r = \dfrac{\lambda}{1+\lambda}$ is less than 1.

4. $P(y < 1) = \int_0^1 \frac{1}{9} y^2 dy = \frac{1}{9}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{27}$

5. $P(y > 1.5) = 1 - P(y < 1.5) = 1 - \int_0^{1.5} y e^{-y} dy$

   Applying integration by parts, with $u = y$ and $dv = e^{-y}$, we have

$$\begin{aligned}\int_0^{1.5} y e^{-y} dy &= -ye^{-y}\Big|_0^{1.5} + \int_0^{1.5} e^{-y} dy \\ &= -1.5e^{-1.5} - e^{-y}\Big|_0^{1.5} \\ &= -2.5e^{-1.5} + 1 \\ &= 0.4422\end{aligned}$$

   Therefore, $P(y > 1.5) = 1 - 0.4422 = 0.5578$

6. C = Card is a club
   K = Card is a king
   $P(C|K) = \dfrac{P(C \cap K)}{P(K)} = \dfrac{1/52}{4/52} = 1/4$

7. $P(X \geq 2 | X \geq 1) = \dfrac{P(X \geq 2 \cap X \geq 1)}{P(X \geq 1)} = \dfrac{P(X \geq 2)}{P(X \geq 1)}$

   $P(X \geq 1) = \dfrac{8}{15}\left[\left(\dfrac{1}{2}\right)^1 + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^3\right] = \dfrac{7}{15}$

   $P(X \geq 2) = \dfrac{8}{15}\left[\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^3\right] = \dfrac{1}{5}$

   $P(X \geq 2 | X \geq 1) = \dfrac{1/5}{7/15} = \dfrac{3}{7}$

8. $P(RR - Y - RR - KL) = \dfrac{5}{12}\left(\dfrac{4}{11}\right)\left(\dfrac{4}{10}\right)\left(\dfrac{3}{9}\right) = \dfrac{2}{99}$

9. $R1$ = A red chip is drawn from Urn 1
   $W1$ = A white chip is drawn from Urn 1
   R = A red chip is drawn from Urn 2

   $P(R) = P(R|R1)P(R1) + P(R|W1)P(W1) = \left(\dfrac{4}{5}\right)\left(\dfrac{1}{3}\right) + \left(\dfrac{3}{5}\right)\left(\dfrac{2}{3}\right) = \dfrac{2}{3}$

10. PG = Polygraph says guilty
    G = Actually guilty

    $P(PG|G) = 0.9,\ P(\neg PG|\neg G) = 0.98,\ P(G) = 0.12$

    $$P(\neg G|PG) = \frac{P(PG|\neg G)P(\neg G)}{P(PG|\neg G)P(\neg G) + P(PG|G)P(G)} = \frac{0.02(0.88)}{0.02(0.88) + 0.9(0.12)} = 0.1401$$

11. (a) No, they are not mutually exclusive because $P(A \cap B) \neq 0$
    (b) They are not independent because $P(A \cap B) = 0.2 \neq 0.3 = P(A)P(B)$
    (c) $P(A^C \cup B^C) = P[(A \cap B)^C] = 0.8$

12. $X \sim b(20, 1/78)$

    $$P(X \geq 1) = 1 - P(0) = 1 - \left[\binom{20}{0}\left(\frac{1}{78}\right)^0\left(\frac{77}{78}\right)^{20}\right] = 1 - \left(\frac{77}{78}\right)^{20} \approx 0.2275$$

13. (a) $8! = 40320$
    (b) $4!4! \times 2 = 1152$