Homework 2

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Section 1.2

1. $8 \cdot 8 \cdot 8 \cdot 8 = 8^4 = 4096$

3. (a) $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6760000$
   (b) $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$

4. (a) $\binom{4}{1} \cdot \binom{6}{3} = \frac{4!}{1!3!} \cdot \frac{6!}{3!3!} = 80$
   (b)
$$\begin{aligned}
&\binom{4}{1} \cdot \left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}\right] \\
&= 4 \cdot (1 + 6 + 15 + 20 + 15 + 6 + 1) \\
&= 4 \cdot (54) \\
&= 216
\end{aligned}$$
   (c) $\binom{4-1+3}{3} = \frac{(4-1+3)!}{3!(4-1)!} = 20$

5. (a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
   (b) $4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256$

6. A player can win in 3, 4 or 5 sets, therefore:
$$\begin{aligned}
&\binom{2}{1} \cdot \left[\binom{3}{3} + \binom{4}{3} + \binom{5}{3}\right] \\
&= 2 \cdot (1 + 4 + 10) \\
&= 30
\end{aligned}$$

7. $N(S) = 10^4 = 10000$
   (a) $N(A) = 4! = 24$
   $P(A) = \frac{24}{10000} = 0.0024$

(b) $N(A) = \dfrac{4!}{1!1!2!} = \dfrac{24}{2} = 12$

$P(A) = \dfrac{12}{10000} = 0.0012$

(c) $N(A) = \dfrac{4!}{2!2!} = \dfrac{24}{4} = 6$

$P(A) = \dfrac{6}{10000} = 0.0006$

(d) $N(A) = \dfrac{4!}{1!3!} = \dfrac{24}{6} = 4$

$P(A) = \dfrac{4}{10000} = 0.0004$

8.

$$3 \cdot 3 \cdot \sum_{i=0}^{12} \binom{12}{i} = 9 \cdot 4096 = 36864$$

9. (a) $\binom{2}{1} \cdot \binom{4}{4} = 2$

(b) 3 wins out of first 4 games: $\binom{2}{1} \cdot \binom{4}{3} = 8$

(c) 3 wins out of first 5 games: $\binom{2}{1} \cdot \binom{5}{3} = 20$

(d) 3 wins out of first 6 games: $\binom{2}{1} \cdot \binom{6}{3} = 40$

10. Starting from the sum and working backwards:

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!}$$

In order to add up the terms, we first multiply the first term by $\dfrac{n-r}{n-r}$ and the second term by $\dfrac{r}{r}$:

$$\begin{aligned}
&\frac{(n-r)(n-1)!}{r!(n-r)(n-r-1)!} + \frac{r(n-1)!}{r(r-1)!(n-r)!} \\
&= \frac{(n-r)(n-1)! + r(n-1)!}{r!(n-r)!} \\
&= \frac{(n-1)!(n-r+r)}{r!(n-r)!} = \frac{(n-1)! \cdot n}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

11. (a) $9! = 362880$

    (b) $\dfrac{9!}{3!6!} = 84$

    (c) $2^9 = 512$

12. We can show the first part by restarting 0 as $(1-1)^n$ and applying binomial expansion (equation 1.2-1):

$$(1-1)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r (1)^{n-r} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$$

We can show the second part by induction and using Pascal's equation

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

We start with the base case $n = 0$:

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{0}{0} = 1 = 2^0$$

Assuming the statement is true for $n-1$, then for all $n > 0$ and $r \in 0, \dots, n$:

$$\begin{aligned}
\sum_{r} \binom{n}{r} &= \sum_{r} \left[\binom{n-1}{r-1} + \binom{n-1}{r} \right] \\
&= \sum_{r} \binom{n-1}{r-1} + \sum_{r} \binom{n-1}{r} \\
&= 2^{n-1} + 2^{n-1} \\
&= 2^n
\end{aligned}$$

13. $N(S) = \binom{52}{13} = 635013559600$

    (a) $N(A) = \binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1} = 3421322190$

    $P(A) = \dfrac{3421322190}{635013559600} = 0.00539$

    (b) $N(A) = \binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2} = 5598527220$

    $P(A) = \dfrac{5598527220}{635013559600} = 0.00882$

    (c) $N(A) = \binom{13}{5}\binom{13}{4}\binom{13}{1}\binom{13}{3} = 3421322190$

    $P(A) = \dfrac{3421322190}{635013559600} = 0.00539$

(d) No, by the results above, there is a higher probability for hands where the other suits are split 2 and 2.

14. $\binom{10+36-1}{36} = \frac{45!}{36!9!} = 886163135$

15. Equation 1.2-2 states that number of distinguishable permutations of n objects when n_1 are similar, n_2 are similar, ..., n_s are similar (where $n_1 + n_2 + \cdots + n_s = n$) is equal to:

$$\binom{n}{n_1, n_2, \cdots, n_s} = \frac{n!}{n_1!n_2!\cdots n_s!}$$

We can arrive at this equation by applying the multiplication rule for each of the n_s subsets:

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n-n_1-\cdots-n_{s-1}}{n_s}$$

Expanding this out:

$$\frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-n_2-\cdots-n_{s-1})!}{n_s!(n-n_1-n_2-\cdots-n_s)!}$$

We can see that the numerator of every factor after the first cancels out with part of the denominator of the factor before it. Also, the denominator of the last factor reduces to $n_s!(0!) = n_s!$ since by definition $n_1 + n_2 + \cdots + n_s = n$.

Therefore, the previous expression reduces to:

$$\frac{n!}{n_1!n_2!\cdots n_s!}$$

16. $N(S) = \binom{52}{9} = 3679075400$

(a) $N(A) = \binom{19}{3} \cdot \binom{33}{6} = 1073233392$

$P(A) = \frac{1073233392}{3679075400} = 0.2917$

(b) $N(A) = \binom{19}{3} \cdot \binom{10}{2} \cdot \binom{7}{1} \cdot \binom{5}{1} \cdot \binom{6}{2} = 22892625$

$P(A) = \frac{22892625}{1073233392} = 0.02133$

17. $N(S) = \binom{52}{5} = 2598960$

(a) $N(A) = \binom{13}{1}\binom{12}{1}\binom{4}{1} = 624$

$P(A) = \frac{624}{2598960} = 0.00024$

(b) $N(A) = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$

$P(A) = \dfrac{3744}{2598960} = 0.00144$

(c) $N(A) = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54912$

$P(A) = \dfrac{54912}{2598960} = 0.02113$

(d) $N(A) = \binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123552$

$P(A) = \dfrac{123552}{2598960} = 0.04754$

(e) $N(A) = \binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1098240$

$P(A) = \dfrac{1098240}{2598960} = 0.4226$

Section 1.3

1. (a) $P(B_1) = \dfrac{5000}{1000000} = 0.005$

   (b) $P(A_1) = \dfrac{78515}{1000000} = 0.078515$

   (c) $P(A_1|B_2) = \dfrac{73630}{995000} = 0.074$

   (d) $P(B_1|A_1) = \dfrac{4885}{78515} = 0.0622$

   (e) Part (c) says that there is a 7.4% probability of a positive test result given that the person is not carrying the AIDS virus. Part (d) says there is a 6.22probability of a carrying the AIDS virus given that the test result was positive.

3. (a) $P(A_1 \cap B_1) = \dfrac{5}{35} = 0.1429$

   (b) $P(A_1 \cup B_1) = \dfrac{12 + 19 - 5}{35} = \dfrac{26/35}{=}0.7429$

   (c) $P(A_1|B_1) = \dfrac{5}{19} = 0.2632$

   (d) $P(B_2|A_2) = \dfrac{9}{23} = 0.3913$

   (e) The event of a student being right-eye-dominant is A_2. So if all we can observe is which thumb is on top, left (B_1) or right (B_2), we choose the one which gives us the highest probability of A_2. $P(A_2|B_1) = \dfrac{14}{19} = 0.7368$

   $P(A_2|B_2) = \dfrac{9}{16} = 0.5625$

   Based on the conditional probabilities, we have a better chance of selecting a right-eye-dominant student given that their left thumb is on top.

4. (a) $P(H_1) \cdot P(H_2|H_1) = \frac{13}{52} \cdot \frac{12}{51} = 0.0588$

   (b) $P(H_1) \cdot P(C_2|H_1) = \frac{13}{52} \cdot \frac{13}{51} = 0.0637$

   (c) We denote drawing a heart first as H_1, drawing an ace first

$$\begin{aligned} P(H_1 \cap A_2) &= P(H_1 \cap A_1 \cap A_2) + P(H_1 \cap A_1' \cap A_2) \\ &= P(H_1) \cdot P(A_1|H_1) \cdot P(A_2|H_1 \cap A_1) + P(H_1) \cdot P(A_1'|H_1) \cdot P(A_2|H_1 \cap A_1') \\ &= \frac{13}{52} \cdot \frac{1}{13} \cdot \frac{3}{51} + \frac{13}{52} \cdot \frac{12}{13} \cdot \frac{4}{51} \\ &= 0.01923 \end{aligned}$$

6. Let H = event that at least one parent had some heart disease
   Let D = event that randomly selected man died of some heart disease

   We are looking for $P(D|H') = \frac{P(D \cap H')}{P(H')} = \frac{\frac{110}{982}}{\frac{648}{982}} = \frac{110}{648} = 0.1698$

11. (a) 365^r

    (b) ${}_{365}P_r = \frac{365!}{(365-r)!}$

    (c) It would be 1 minus the probability of everyone having different birthdays which is part (b) divided by part (a): $1 - \frac{{}_{365}P_r}{365^r}$

    (d) $1 - \frac{{}_{365}P_r}{365^r} = .5 \to r \approx 23$

    Yes, this number is suprisingly small. In a small room there is already a 50% chance of two people with the same birthday.

12. (a) If you draw first:
    $P(B_1) = \frac{1}{18} = 0.0556$
    If you draw fifth:

$$\begin{aligned} &P(R_1 \cap R_2 \cap R_3 \cap R_4 \cap B_5) \\ &= P(R_1) \cdot P(R_2|R_1) \cdot P(R_3|R_1 \cap R_2) P(R_4|R_1 \cap R_2 \cap R_3) \cdot P(B_5|R_1 \cap R_2 \cap R_3 \cap R_4) \\ &= \frac{17}{18} \cdot \frac{16}{17} \cdot \frac{15}{16} \cdot \frac{14}{15} \cdot \frac{1}{14} \\ &= 0.0556 \end{aligned}$$

If you draw last:

$$\begin{aligned}
&P(R_1 \cap R_2 \cap \cdots \cap R_{17} \cap B_{18}) \\
&= P(R_1)P(R_2|R_1)\cdots P(R_{17}|R_1 \cap R_2 \cap \cdots \cap R_{16})P(B_{18}|R_1 \cap R_2 \cap \cdots \cap R_{17}) \\
&= \frac{17}{18} \cdot \frac{16}{17} \cdots \frac{1}{2} \cdot \frac{1}{1} \\
&= 0.0556
\end{aligned}$$

You have the same chance of drawing the blue pebble regardless which position we choose.

(b) If you draw first:
$P(B_1) = \dfrac{2}{18} = 0.1111$
If you draw fifth:

$$\begin{aligned}
&P(R_1 \cap R_2 \cap R_3 \cap R_4 \cap B_5) \\
&+ P(B_1 \cap R_2 \cap R_3 \cap R_4 \cap B_5) \\
&+ P(R_1 \cap B_2 \cap R_3 \cap R_4 \cap B_5) \\
&+ P(R_1 \cap R_2 \cap B_3 \cap R_4 \cap B_5) \\
&+ P(R_1 \cap R_2 \cap R_3 \cap B_4 \cap B_5) \\
&= \frac{16}{18} \cdot \frac{15}{17} \cdot \frac{14}{16} \cdot \frac{13}{15} \cdot \frac{2}{14} \\
&+ 4 \cdot \left(\frac{16}{18} \cdot \frac{15}{17} \cdot \frac{14}{16} \cdot \frac{2}{15} \cdot \frac{1}{14}\right) \\
&= 0.1765
\end{aligned}$$

If you draw last, the probability reduces to the probability of drawing 1 blue pebble out of the 18 pebbles: $\dfrac{1}{18} = 0.0556$.
Therefore, in the case of 2 blue pebbles, I choose to draw 5th.

14. (a) $P(A_1) = \dfrac{7 + 11 + 12}{100} = 0.3$

(b) $P(A_3 \cap B_2) = \dfrac{9}{100} = 0.09$

(c) $P(A_2 \cup B_3) = \dfrac{11 + 21 + 9 + 12 + 7}{100} = 0.6$

(d) $P(A_1|B_2) = \dfrac{11}{11 + 21 + 9} = 0.2683$

(e) $P(B_1|A_3) = \dfrac{13}{13 + 9 + 7} = 0.4483$

16. We define:
RA as drawing a red chip from bowl A

RB as drawing a red chip from bowl B
WA as drawing a white chip from bowl A

$$\begin{aligned}P(RB) &= P(RA) \cdot P(RB|RA) + P(WA) \cdot P(RB|WA) \\ &= \frac{3}{5} \cdot \frac{5}{8} + \frac{2}{5} \cdot \frac{4}{8} \\ &= 0.575\end{aligned}$$