Homework 13

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Section 5.3

1. (a) $P(X_1 = 3, X_2 = 5) = P(X_1 = 3)P(X_2 = 5) = \left(\frac{2^3 e^{-2}}{3!}\right)\left(\frac{3^5 e^{-3}}{5!}\right) = 0.01819$

   (b) $P(X_1 + X_2 = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 1) =$
   $P(X_1 = 1)P(X_2 = 0) + P(X_1 = 0)P(X_2 = 1) = \left(\frac{2^1 e^{-2}}{1!}\right)\left(\frac{3^0 e^{-3}}{0!}\right) + \left(\frac{2^0 e^{-2}}{0!}\right)\left(\frac{3^1 e^{-3}}{1!}\right)$
   $= 0.03369$

3. (a) $P(0.5 < X_1 < 1 \text{ and } 0.4 < X_2 < 0.8) = P(0.5 < X_1 < 1)P(0.4 < X_2 < 0.8)$
   $= \left[\int_{0.5}^{1} 2x_1 dx_1\right]\left[\int_{0.4}^{0.8} 4x_2^3 dx_2\right] = x_1^2\Big|_{0.5}^{1} \cdot x_2^4\Big|_{0.4}^{0.8} = (1 - 0.25)(0.4096 - 0.0256) = 0.288$

   (b) $E(X_1^2 X_2^3) = E(X_1^2)E(X_2^3) = \left[\int_0^1 x_1^2 2x_1 dx\right]\left[\int_0^1 x_2^3 4x_2^3 dx\right] = \frac{x_1^4}{2}\Big|_0^1 \cdot \frac{4x_2^7}{7}\Big|_0^1 = \frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$

5. pmf of Y:

$y = x_1 + x_2$	2	3	4	5	6
$g(y)$	1/36	4/36	10/36	12/36	9/36

Mean and variance #1:

$E(Y) = \sum_y y g(y) = 2\left(\frac{1}{36}\right) + 3\left(\frac{4}{36}\right) + 4\left(\frac{10}{36}\right) + 5\left(\frac{12}{36}\right) + 6\left(\frac{9}{36}\right) = \frac{14}{3}$

$E(Y^2) = \sum_y y^2 g(y) = 4\left(\frac{1}{36}\right) + 9\left(\frac{4}{36}\right) + 16\left(\frac{10}{36}\right) + 25\left(\frac{12}{36}\right) + 36\left(\frac{9}{36}\right) = \frac{206}{9}$

$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{10}{9}$

Mean and variance #2:

$E(Y) = E(X_1 + X_2) = E(X_1) + E(X_2) = 2E(X) = 2\left[1\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{3}{6}\right)\right]$

$= 2\left(\frac{14}{6}\right) = \frac{14}{3}$

$$Var(Y) = Var(X_1) + Var(X_2) = 2Var(X) = 2\left[\sum_x (x - 14/6)^2 f(x)\right]$$
$$= 2\left[\left(1 - \frac{14}{6}\right)^2 \left(\frac{1}{6}\right) + \left(2 - \frac{14}{6}\right)^2 \left(\frac{2}{6}\right) + \left(3 - \frac{14}{6}\right)^2 \left(\frac{3}{6}\right)\right] = \frac{10}{9}$$

6. The mean of X_1 and X_2,

$$E(X_1) = E(X_2) = E(X) = \int_0^1 x \cdot 6x(1-x)dx = \frac{1}{2}$$

Therefore,

$$E(Y) = E(X_1 + X_2) = E(X_1) + E(X_2) = 2E(X) = 2(1) = 2$$

The variance of X_1 and X_2,

$$Var(X_1) = Var(X_2) = Var(X) = \int_0^1 x^2 \cdot 6x(1-x)dx = \frac{6}{20}$$

Therefore,

$$Var(Y) = Var(X_1 + X_2) = Var(X_1) + Var(X_2) = 2Var(X) = 2\left(\frac{6}{20}\right) = \frac{6}{10}$$

11. (a)

$$\begin{aligned} P(X_1 = 2, X_2 = 2, X_3 = 5) &= P(X_1 = 2)P(X_2 = 2)P(X_3 = 5) \\ &= \left[\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right]\left[\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4\right]\left[\binom{12}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^7\right] \\ &= 0.00351 \end{aligned}$$

(b)

$$\begin{aligned} E(X_1X_2X_3) &= E(X_1)E(X_2)E(X_3) \\ &= 4\left(\frac{1}{2}\right) \cdot 6\left(\frac{1}{3}\right) \cdot 12\left(\frac{1}{6}\right) \\ &= 8 \end{aligned}$$

(c)

$$\begin{aligned} E(Y) &= E(X_1 + X_2 + X_3) \\ &= E(X_1) + E(X_2) + E(X_3) \\ &= 3(2) \\ &= 6 \end{aligned}$$

$$
\begin{aligned}
Var(Y) &= Var(X_1 + X_2 + X_3) \\
&= Var(X_1) + Var(X_2) + Var(X_3) \\
&= 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + 12\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) \\
&= 4
\end{aligned}
$$

12.

$$
\begin{aligned}
P(1 < \min X_i) &= P(1 < X_1, 1 < X_2, 1 < X_3) \\
&= P(1 < X_1)P(1 < X_2)P(1 < X_3) \\
&= [1 - P(X_1 \leq 1)][1 - P(X_2 \leq 1)][1 - P(X_3 \leq 1)] \\
&= (e^{-1})^3 \\
&= e^{-3} \approx 0.04979
\end{aligned}
$$

17. We're given that,

$$
\sigma_X^2 = 8100, \quad \sigma_Y^2 = 10000, \quad \sigma_{X+Y}^2 = 20000
$$

Since, $Var(X+Y)$ is not equal to the sum of $Var(X)$ and $Var(Y)$, we know they must be correlated. We can find the correlation coefficient, by solving the following for rho_{XY},

$$
\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y
$$

$$
\rho_{XY} = \frac{20000 - 8100 - 10000}{90(100)} = 0.2111
$$

We can define $W = X + 500$, and $T = 1.08Y$, which are linear combinations of X and Y, respectively. Thus, to compute, $Var(W)$ and $Var(T)$,

$$
Var(W) = Var(X + 500) = Var(X) = 8100
$$

$$
Var(T) = Var(1.08Y) = 1.08^2 Var(Y) = 11664
$$

Therefore, we can define $Z = W + T$, and compute its variance as follows,

$$
\begin{aligned}
Var(Z) &= Var(W) + Var(T) + \rho_{XY}\sigma_W\sigma_T \\
&= 8100 + 11664 + 0.2111(90)(108) \\
&= 21816
\end{aligned}
$$

18. Since these are random samples of a gamma distribution with $\alpha = \theta = 2$, then

$$
E(X_1) = E(X_2) = E(X_3) = E(X) = \alpha\theta
$$

and,

$$Var(X_1) = Var(X_2) = Var(X_3) = Var(X) = \alpha\theta^2$$

Therefore,

$$\begin{aligned} E(X_1 + X_2 + X_3) &= E(X_1) + E(X_2) + E(X_3) \\ &= 3(\alpha\theta) \\ &= 3(2)(2) \\ &= 12 \end{aligned}$$

$$\begin{aligned} Var(X_1 + X_2 + X_3) &= Var(X_1) + Var(X_2) + Var(X_3) \\ &= 3(\alpha\theta^2) \\ &= 3(2)(2)^2 \\ &= 24 \end{aligned}$$

Section 5.4

2. Binomial mgf: $(q + pe^t)^n$

   We are given that $Y = X_1 + X_2$, where $X_1 \sim b(n_1, p)$ and $X_2 \sim b(n_2, p)$. Therefore, $M_{X_1} = (q + pe^t)^{n_1}$ and $M_{X_2} = (q + pe^t)^{n_2}$.

   The mgf of Y is

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) M_{X_2}(t) \\ &= (q + pe^t)^{n_1}(q + pe^t)^{n_2} \\ &= (q + pe^t)^{(n_1+n_2)} \end{aligned}$$

   This last line is the mgf of a binomial distribution, $b(n_1 + n_2, p)$.

3. (a) $M_{X_1}(t) = e^{2(e^t-1)}$, $M_{X_2}(t) = e^{(e^t-1)}$, $M_{X_3}(t) = e^{4(e^t-1)}$
   Therefore,

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) M_{X_2}(t) M_{X_3}(t) \\ &= e^{2(e^t-1)} e^{(e^t-1)} e^{4(e^t-1)} \\ &= e^{7(e^t-1)} \end{aligned}$$

   (b) Y has a Poisson distribution with mean 7.

   (c) $P(3 \leq Y \leq 9) = P(Y \leq 9) - P(Y \leq 2) = 0.830 - 0.030 = 0.800$

4. If we have a random variable, Y, equal to the sum of n Poisson random variables with means $\mu_1, \mu_2, \cdots, \mu_n$, with each having mgf,

$$M_{X_i}(t) = e^{\mu_i(e^t-1)}$$

then by theorem 5.4-1, the moment-generating function,

$$\begin{aligned} M_Y(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= (e^{\mu_1(e^t-1)})(e^{\mu_2(e^t-1)})\cdots(e^{\mu_n(e^t-1)}) \\ &= e^{(e^t-1)\sum_{i=1}^{n}\mu_i} \end{aligned}$$

This last line is the mgf of a Poisson random variable with mean $\sum_{i=1}^{n}\mu_i$

5. By Corollary 5.4-2, W has a distribution that is $\chi^2(7)$. Therefore,

$$\begin{aligned} P(1.69 < W < 14.07) &= P(W < 14.07) - P(W < 1.69) \\ &= 0.95 - 0.025 = 0.925 \end{aligned}$$

8. We have h mutually independent and identically exponential random variables with mean θ. Each of these h random variables has mgf $M_{X_i}(t) = (1-\theta)^{-1}$. By theorem, 5.4-1, the moment-generating function of W is,

$$\begin{aligned} M_W(t) &= \prod_{i=1}^{h} M_{X_i}(t) \\ &= \prod_{i=1}^{h} (1-\theta)^{-1} \\ &= ((1-\theta)^{-1})^h \\ &= (1-\theta)^{-h} \end{aligned}$$

This last line is the mgf of a Gamma distribution with parameters $\alpha = h$ and θ.

14. $Y = X_1 + X_2 + X_3 \sim Poisson(6)$
$P(Y = 7) = \dfrac{6^7 e^{-6}}{7!} = 0.1377$

16. $Y = X_1 + X_2 + X_3 + X_4 \sim Poission(8)$
$P(Y > 10) = 1 - P(Y \le 10) = 1 - 0.816 = 0.184$

19. The sum of the three exponential random variables equates to a gamma random variable with $\alpha = 3$ and $\theta = 2$. Therefore,

$$\begin{aligned} P(X \le 6) &= \int_0^6 \frac{1}{\Gamma(3)2^3} x^2 e^{-x/2} dx \\ &= \frac{1}{\Gamma(3)2^3} \int_0^6 x^2 e^{-x/2} dx \end{aligned}$$

Doing integration by parts with $u = x^2$ and $dv = e^{-x/2}dx$. Therefore, $du = 2xdx$ and $v = -2e^{-x/2}$

$$\frac{1}{\Gamma(2)2^3}\int_0^6 x^2 e^{-x/2}dx = \frac{1}{\Gamma(3)2^3}\left[-2x^2e^{-x/2}\Big|_0^6 - \int_0^6 -4xe^{-x/2}dx\right]$$
$$= \frac{1}{\Gamma(3)2^3}\left[-72e^{-3} + 4\int_0^6 xe^{-x/2}dx\right]$$

Again doing integration by parts with $u = x$ and $dv = e^{-x/2}dx$. Therefore, $du = dx$ and $v = -2e^{-x/2}$

$$\frac{1}{\Gamma(3)2^3}\left[-72e^{-3} + 4\int_0^6 xe^{-x/2}dx\right] = \frac{1}{\Gamma(3)2^3}\left\{-72e^{-3} + 4\left[-2xe^{-x/2}\Big|_0^6 - \int_0^6 -2e^{-x/2}dx\right]\right\}$$
$$= \frac{1}{\Gamma(3)2^3}\left\{-72e^{-3} + 4\left[-12e^{-3} + 2\int_0^6 e^{-x/2}dx\right]\right\}$$
$$= \frac{1}{\Gamma(3)2^3}\left[-120e^{-3} + 8\int_0^6 e^{-x/2}dx\right]$$
$$= \frac{1}{\Gamma(3)2^3}\left[-120e^{-3} - 16(e^{-x/2}\Big|_0^6)\right]$$
$$= \frac{1}{\Gamma(3)2^3}\left[-120e^{-3} - 16(e^{-3} - 1)\right]$$
$$= \frac{1}{16}\left(-136e^{-3} + 16\right)$$
$$= -8.5e^{-3} + 1 \approx 0.5768$$