Stat 414 Exam #3

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Start Time: 9:52 am/pm Stop time: 11:22 am/pm

You have 1 hour 30 min to complete and 10 minutes to scan/upload. You must show all of your work in order to receive full and/or partial credit. No work=No Credit. Tables/software are only allowed if stated in the problem. 5 pages, 30 points

1. 9 points Let X be a continuous random variable with probability density function (PDF) of

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ c & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) 1 point Find the constant c, such that $f(x)$ is a valid PDF.

$$\int_0^1 x\,dx + \int_1^2 c\,dx \equiv 1$$

$$\left.\frac{x^2}{2}\right]_0^1 + \left.cx\right]_1^2 = \frac{1}{2} + c = 1$$

THEREFORE, $c = \frac{1}{2}$

(b) 2 points Find the cumulative distribution function (CDF) of x, $F(x)$.

$$F(x) = \int_0^x f(t)\,dt$$

WHEN $x \le 1$, $F(x) = \int_0^x t\,dt = \left.\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$

WHEN $x > 1$, $F(x) = \int_0^1 t\,dt + \int_1^x \frac{1}{2}\,dt = \left.\frac{t^2}{2}\right|_0^1 + \left.\frac{1}{2}t\right|_1^x$

$$= \frac{1}{2} + \left(\frac{x}{2} - \frac{1}{2}\right)$$

$$= \frac{x}{2}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ \frac{x}{2} & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

(c) 1 point Find the mean of the random variable X.

$$E(x) = \int_0^2 x f(x)\, dx = \int_0^1 x^2\, dx + \int_1^2 \tfrac{1}{2}x\, dx = \left.\frac{x^3}{3}\right|_0^1 + \left.\frac{x^2}{4}\right|_1^2$$

$$= \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$$

(d) 1 point Find the median of X. Justify.

$\pi_{0.5} = 1$ BECAUSE BY PART (b), $F(1) = \frac{1}{2}$

(e) 2 points Find the 75^{th} percentile of X.

$$F(\pi_{0.75}) = \frac{\pi_{0.75}}{2} = 0.75$$

$$\pi_{0.75} = 1.5$$

(f) 2 points Find $P(X > 1 | X > \frac{1}{4})$.

~~P(X>1 ∩ P(~~ $P(X > 1 \cap X > \frac{1}{4}) = \dfrac{P(X > 1)}{P(X > \frac{1}{4})}$

$$\frac{1 - F(1)}{1 - F(\frac{1}{4})} = \frac{\frac{1}{2}}{0.9688} = 0.5161$$

2. 4 points Let X have an exponential distribution with a mean of θ. Prove that the expected value of X^r is $\theta^r \Gamma(r+1)$, for a real number, r.

$$E(x^r) = \int_0^\infty x^r \frac{1}{\theta} e^{-x/\theta}\, dx$$

$$= \frac{1}{\theta} \int_0^\infty x^r e^{-x/\theta}\, dx$$

I DON'T KNOW HOW TO GET RID OF THAT GUY

3. 3 points The time required to repair a machine follows an exponential distribution with mean 1 hours.

$\lambda = 1 \quad \theta = 1$

(a) 1 point What is the probability that a repair time exceeds 2 hour.

$$P(X > 2) = e^{-2/1} = 0.1353$$

(b) 2 points A man has been working on the machine for 2 hours. What is the probability that the total repair time is less than 4 hours given that information.

$$P(X < 4 \mid X > 2) = \frac{P(X < 4 \cap X > 2)}{P(X > 2)} = \frac{P(2 < X < 4)}{P(X > 2)}$$

$$= \frac{P(X<4) - P(X<2)}{P(X>2)} = \frac{(1 - e^{-4}) - (1 - e^{-2})}{e^{-2}} = \frac{-e^{-4} + e^{-2}}{e^{-2}} = 0.8647$$

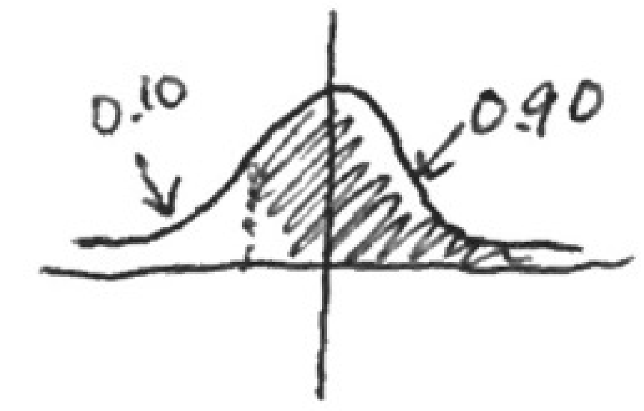
4. 4 points A sugar refinery has three processing plants. The amount of sugar that one plant can process in one day can be modeled as having an exponential distribution with mean of four tons, for each of the three plants. If the plants operate independently, find the probability that exactly two of the three plants each process more than four tons on a given day.

$$P(\text{ONE PLANT PROCESSING MORE 4 TONS}) = e^{-4/4} \approx 0.3679$$

$$P(\text{EXACTLY 2 PLANTS PROCESSING} > 4 \text{ tons}) = \binom{3}{2}(0.3679)^2(1 - 0.3679)^1$$

$$= 0.2567$$

5. 3 points Let X be the life in hours of a radio tube. Assume that X is normally distributed with mean 200 and variance σ^2. If a purchaser of such radio tubes requires that at least 90 percent of the tubes have lives exceeding 150 hours, what is the largest value σ^2 can be and still have the purchaser satisfied? You may used tables/software for this problem. $X \sim N(200, \sigma^2)$



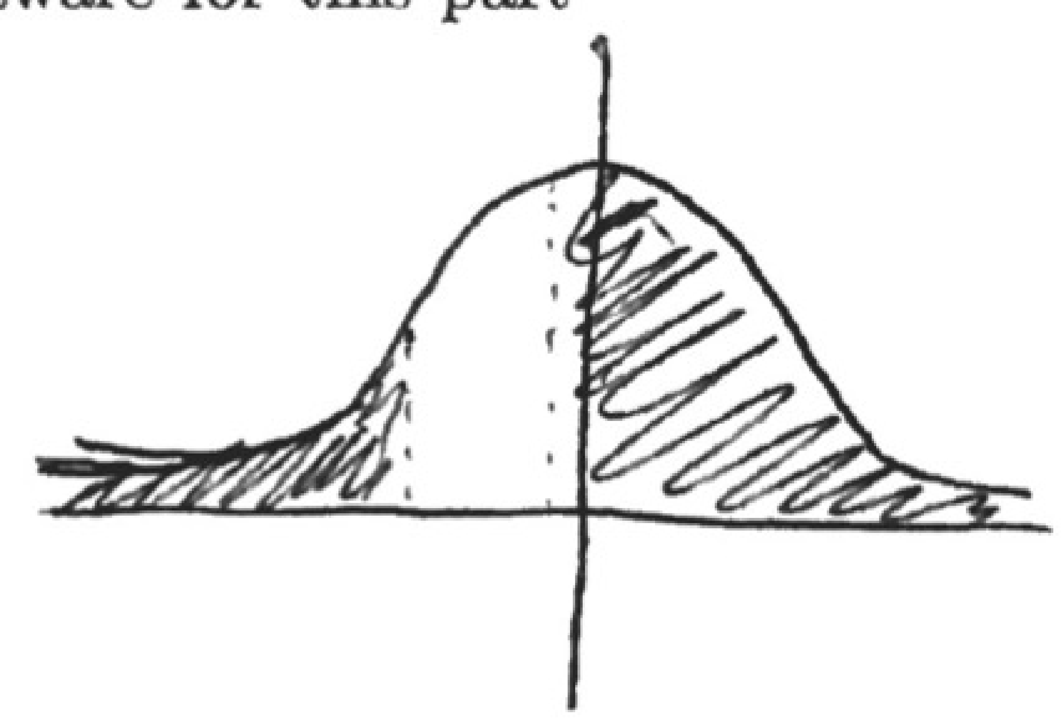
$$P(X > 150) = 0.90$$

$$P\left(\frac{X-\mu}{\sigma} > \frac{150-\mu}{\sigma}\right) = P\left(Z > \frac{-50}{\sigma}\right) = 0.90$$

$$\frac{-50}{\sigma} = -1.28 \Rightarrow \sigma = 39.0625$$

THE LARGEST VALUE THAT σ^2 CAN BE IS AROUND 1526

6. 3 points The random variable W follows a Normal distribution with mean of 4 and variance 9. Find $P(|W| > 1)$. You may use tables/software for this part but you also need to show your steps.



$$P(W < -1 \cup W > 1)$$

$$= P\left(\frac{W-4}{3} < \frac{-1-4}{3} \cup \frac{W-4}{3} > \frac{1-4}{3}\right)$$

$$= P\left(Z < \frac{-5}{3} \cup Z > \frac{-3}{3}\right) = P\left(Z < \frac{-5}{3}\right) + P\left(Z > \frac{-3}{3}\right)$$

$$= 0.04779 + 0.84134$$

$$= 0.88914$$

7. 4 points X has a normal distribution with the pdf $f(x) = c2^{-x^2}$.

(a) 2 points Find c. PDF OF NORMAL DISTRIBUTION $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

IGNORING C FOR A MINUTE, $f(x) = 2^{-x^2} = e^{\ln 2^{-x^2}} = e^{-x^2 \ln 2}$

THIS IS EQUAL TO $e^{\left(-\frac{x^2}{\frac{1}{\ln 2}}\right)}$. So, $\frac{1}{\ln 2} = 2\sigma^2 \Rightarrow \sigma = \sqrt{\frac{1}{2\ln 2}}$

C MUST EQUAL THE COEFFICIENT ON e, $\frac{1}{\sigma\sqrt{2\pi}}$

$$c = \frac{1}{\sqrt{\frac{1}{2\ln 2}}\sqrt{2\pi}} = \frac{1}{\sqrt{\frac{\pi}{\ln 2}}} = \sqrt{\frac{\ln 2}{\pi}}$$

(b) 2 points Find the moment generating function of X.

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

X HAS MEAN $= 0$ AND $\sigma^2 = \frac{1}{2\ln 2}$

So, $M(t) = 1 + \frac{t^2}{4\ln 2}$