Homework 11

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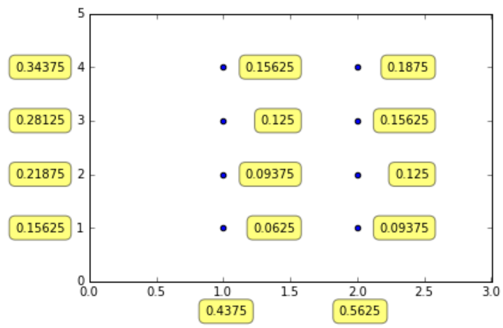
Section 4.3

1. (a) Joint and marginal pmfs:

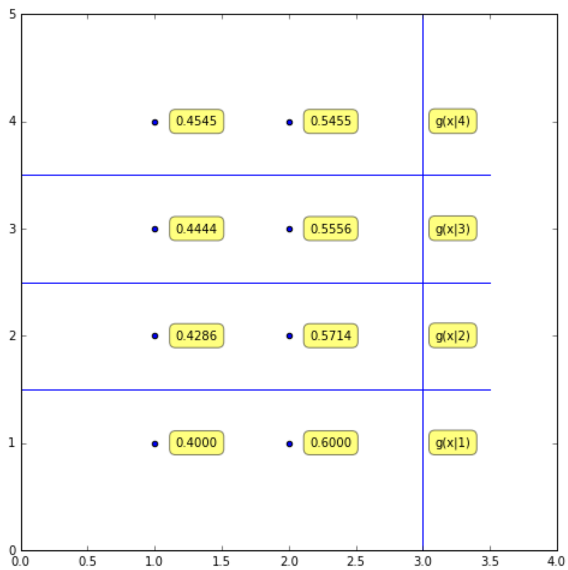


(b) We are given that $f(x,y) = \dfrac{x+y}{32}, x = 1, 2 \quad y = 1, 2, 3, 4$. In order to find $g(x|y)$, we start by finding $f_Y(y)$:

$$\begin{aligned} f_Y(y) &= \sum_{x \in \{1,2\}} \frac{x+y}{32} \\ &= \frac{1+y}{32} + \frac{2+y}{32} \\ &= \frac{2y+3}{32} \end{aligned}$$

Therefore, the conditional pmf of X, given that $Y = y$, is:

$$g(x|y) = \frac{(x+y)/32}{(2y+3)/32} = \frac{x+y}{2y+3}, x = 1, 2, \text{ when } y = 1, 2, 3, \text{ or } 4$$

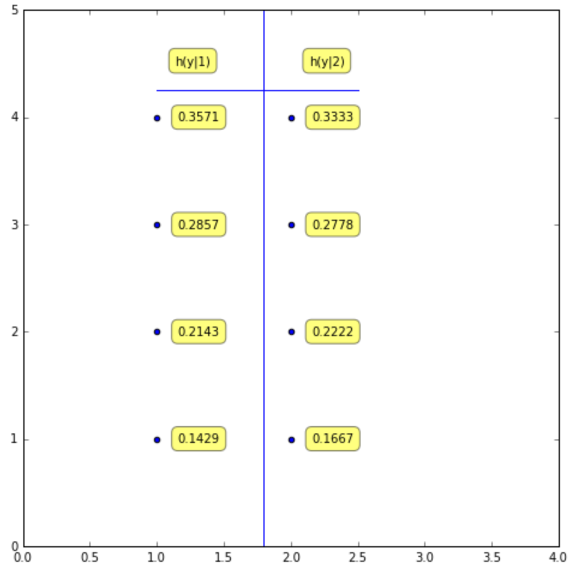


(c) We start by finding $f_X(x)$:

$$\begin{aligned} f_X(x) &= \sum_{y \in \{1,2,3,4\}} \frac{x+y}{32} \\ &= \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} \\ &= \frac{4x+10}{32} \\ &= \frac{2x+5}{16} \end{aligned}$$

Therefore, the conditional pmf of Y, given that $X = x$, is:

$$\begin{aligned} h(y|x) &= \frac{(x+y)/32}{(2x+5)/16} \\ &= \frac{x+y}{4x+10}, y = 1, 2, 3, 4 \text{ when } x = 1 \text{ or } 2. \end{aligned}$$



(d) $P(1 \leq Y \leq 3|X = 1) = \frac{1+1}{4(1)+10} + \frac{1+2}{4(1)+10} + \frac{1+3}{4(1)+10} = \frac{9}{14}$

$P(Y \leq 2|X = 2) = \frac{2+1}{4(2)+10} + \frac{2+2}{4(2)+10} = \frac{7}{18}$

$P(X = 2|Y = 3) = \frac{2+3}{2(3)+3} = \frac{5}{9}$

(e) $E(Y|X = 1) = \sum_{y=1}^{4} y \frac{1+y}{4(1)+10} = (1)\frac{1+1}{14} + (2)\frac{1+2}{14} + (3)\frac{1+3}{14} + (4)\frac{1+4}{14} = \frac{40}{14} = \frac{20}{7}$

$$
\begin{aligned}
Var(Y|X=1) &= E(Y^2|X=1) - [E(Y|x)]^2 \\
&= \left[\sum_{y=1}^{4} y^2 \frac{1+y}{14}\right] - \left(\frac{20}{7}\right)^2 \\
&= (1)\frac{2}{14} + (4)\frac{3}{14} + (9)\frac{4}{14} + (16)\frac{5}{14} - \left(\frac{20}{7}\right)^2 \\
&= \frac{130}{14} - \left(\frac{20}{7}\right)^2 \\
&= \frac{55}{49}
\end{aligned}
$$

2. Reformatting the joint probability mass function table:

		y = 1	y = 2	
x	1	3/8	1/8	1/2
	2	1/8	3/8	1/2
		1/2	1/2	

Then the conditional probability mass function, $g(x|y)$ is given by the following table:

x	$g(x\|y=1)$	$g(x\|y=2)$
1	3/4	1/4
2	1/4	3/4

And the conditional probability mass function, $h(y|x)$ is given by the following table:

y	$h(y\|x=1)$	$h(y\|x=2)$
1	3/4	1/4
2	1/4	3/4

$$\mu_{X|y=1} = \sum_x x g(x|y=1) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$\mu_{X|y=2} = \sum_x x g(x|y=2) = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}$$

$$\mu_{Y|x=1} = \sum_y y h(y|x=1) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$\mu_{Y|x=2} = \sum_y y h(y|x=2) = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}$$

$$\sigma^2_{X|y=1} = \sum_x x^2 g(x|y=1) - \mu^2_{X|y=1} = \frac{3}{4} + 1 - \left(\frac{5}{4}\right)^2 = \frac{3}{16}$$

$$\sigma^2_{X|y=2} = \sum_x x^2 g(x|y=2) - \mu^2_{X|y=2} = \frac{1}{4} + 3 - \left(\frac{7}{4}\right)^2 = \frac{3}{16}$$

$$\sigma^2_{Y|x=1} = \sum_y y^2 h(y|x=1) - \mu^2_{Y|x=1} = \frac{3}{4} + 1 - \left(\frac{5}{4}\right)^2 = \frac{3}{16}$$

$$\sigma^2_{Y|x=2} = \sum_y y^2 h(y|x=2) - \mu^2_{Y|x=2} = \frac{1}{4} + 3 - \left(\frac{7}{4}\right)^2 = \frac{3}{16}$$

3. (a) $f(x,y) = \dfrac{50!}{x!y!(50-x-y)!}(0.02)^x(0.9)^y(0.08)^{(50-x-y)}$

   (b) Y follows a binomial distribution with $n = 50$ and $p = 0.9$.

   (c) $h(y|x=3) = \dfrac{f(3,y)}{f_X(3)} = \dfrac{\frac{50!}{3!y!(47-y)!}(0.02)^3(0.9)^y(0.08)^{(47-y)}}{\frac{50!}{3!47!}(0.02)^3(0.98)^{47}} = \dfrac{47!}{y!(47-y)!}\left(\dfrac{0.9}{0.98}\right)^y\left(\dfrac{0.08}{0.98}\right)^{(47-y)}$

   This shows that $Y \sim binomial\left(47, \dfrac{0.9}{0.98}\right)$.

   (d) $E(Y|X=3) = 47\left(\dfrac{0.9}{0.98}\right) = \dfrac{2115}{49} \approx 43.1633$

   (e) $\rho = -\sqrt{\dfrac{0.02(0.9)}{(1-0.02)(1-0.9)}} = -\sqrt{\dfrac{9}{49}} = -\dfrac{3}{7} \approx -.4286$

Section 4.4

1. (a) $f_X(x) = \displaystyle\int_0^2 \frac{3}{16}xy^2dy = \left[\frac{3}{16}x\frac{y^3}{3}\right]_0^2 = \frac{1}{2}x$ for $0 \le x \le 2$

   $f_Y(y) = \displaystyle\int_0^2 \frac{3}{16}xy^2dx = \left[\left(\frac{3}{16}\right)y^2\left(\frac{x^2}{2}\right)\right]_0^2 = \frac{3}{8}y^2$ for $0 \le y \le 2$

   (b) Yes, they are independent because $f(x,y) = \dfrac{3}{16}xy^2 = \left(\dfrac{x}{2}\right)\left(\dfrac{3y^2}{8}\right) = f_X(x)f_Y(y)$

   (c) $E(X) = \displaystyle\int_0^2 x\frac{x}{2}dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}$

   $Var(X) = \displaystyle\int_0^2 x^2\frac{x}{2}dx - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}$

   $E(Y) = \displaystyle\int_0^2 y\frac{3y^2}{8}dy = \frac{3y^4}{32}\Big|_0^2 = \frac{3}{2}$

   $Var(Y) = \displaystyle\int_0^2 y^2\frac{3y^2}{8}dy - \left(\frac{3}{2}\right)^2 = \frac{12}{5} - \frac{9}{4} = \frac{3}{20}$

(d) $P(X \le Y) = \int_0^2 \int_0^y \frac{3}{16} xy^2 \; dx \; dy = \int_0^2 \frac{3}{16} y^2 \left[\frac{x^2}{2} \right]_0^y \; dy = \int_0^2 \frac{3}{32} y^4 \; dy = \frac{3}{32} \left[\frac{y^5}{5} \right]_0^2 = \frac{3}{5}$

2. (a) $f_X(x) = \int_0^1 (x+y) \; dy = \left[xy + \frac{y^2}{2} \right]_0^1 = x + \frac{1}{2}$ for $0 \le x \le 1$

$f_Y(y) = \int_0^1 (x+y) \; dx = \left[\frac{x^2}{2} + xy \right]_0^1 = y + \frac{1}{2}$ for $0 \le y \le 1$

X and Y are dependent since $x + y \neq \left(x + \frac{1}{2} \right) \left(y + \frac{1}{2} \right)$. For example, when $x = 1$ and $y = 1$,

$$x + y = 1 + 1 = 2 \neq \left(x + \frac{1}{2} \right) \left(y + \frac{1}{2} \right) = \left(1 + \frac{1}{2} \right) \left(1 + \frac{1}{2} \right) = \frac{9}{4}$$

(b) i. $\mu_X = \int_0^1 x \left(x + \frac{1}{2} \right) \; dx = \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \frac{7}{12}$

ii. $\mu_Y = \int_0^1 y \left(y + \frac{1}{2} \right) \; dy = \left[\frac{y^3}{3} + \frac{y^2}{4} \right]_0^1 = \frac{7}{12}$

iii. $\sigma_X^2 = \int_0^1 x^2 \left(x + \frac{1}{2} \right) \; dx - \left(\frac{7}{12} \right)^2 = \left[\frac{x^4}{4} + \frac{x^3}{6} \right]_0^1 - \frac{49}{144} = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$

iv. $\sigma_Y^2 = \int_0^1 y^2 \left(y + \frac{1}{2} \right) \; dy - \left(\frac{7}{12} \right)^2 = \left[\frac{y^4}{4} + \frac{y^3}{6} \right]_0^1 - \frac{49}{144} = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$

3.

$$f_X(x) = \int_x^\infty 2e^{-x-y} \; dy$$

Substitute $u = -x - y$ and $du = -dy$,

$$\begin{aligned} f_X(x) &= 2 \int_{-\infty}^{-2x} e^u \; du \\ &= \lim_{b \to -\infty} 2e^u \Big|_b^{-2x} \\ &= 2e^{-2x} - \lim_{b \to -\infty} 2e^b \\ &= 2e^{-2x}, \; 0 \le x \le \infty \end{aligned}$$

Similarly,

$$f_Y(y) = \int_0^y 2e^{-x-y} \; dx$$

Substitute $u = -x - y$ and $du = -dx$,

$$
\begin{aligned}
f_Y(y) &= 2\int_{-2y}^{-y} e^u \; du \\
&= 2e^u \Big|_{-2y}^{-y} \\
&= 2e^{-y} - 2e^{-2y} \\
&= 2(e^{-y} - e^{-2y}),\; 0 \le y \le \infty
\end{aligned}
$$

X and Y are not independent because $f(x,y) \neq f_X(x) f_Y(y)$. As an example, when $x = 1$ and $y = 1$,

$$f(1,1) = 2e^{-1-1} = 2e^{-2} \neq 2e^{-2x} \cdot 2(e^{-y} - e^{-2y}) = 2e^{-2} \cdot 2(e^{-1} - e^{-2}) = 4e^{-3} - 4e^{-4} = f_X(1) f_Y(1)$$

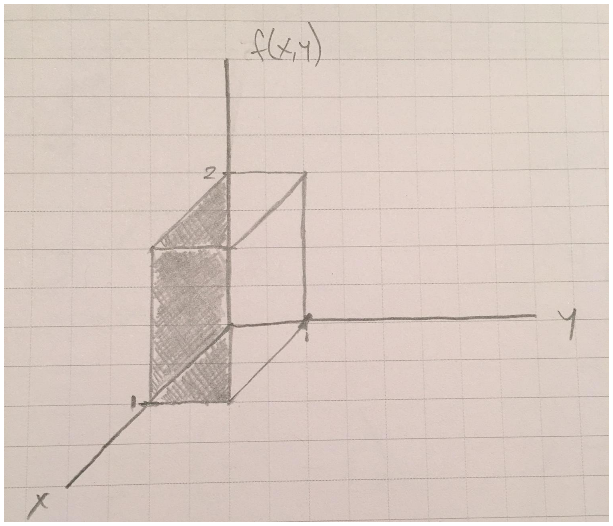
4. (a) $P(0 \le X \le 1/2) = \displaystyle\int_0^{1/2} \int_{x^2}^{1} \frac{3}{2} \; dy \; dx = \int_0^{1/2} \left[\frac{3}{2} y\right]_{x^2}^{1} dx = \int_0^{1/2} \frac{3}{2} - \frac{3x^2}{2} \; dx$
$= \displaystyle\frac{3x}{2} - \frac{3x^3}{6}\Big|_0^{1/2} = \frac{3}{4} - \frac{3}{48} = \frac{11}{16}$

   (b) $P(1/2 \le Y \le 1) = \displaystyle\int_{1/2}^{1} \int_0^{1} \frac{3}{2} \; dx \; dy = \int_{1/2}^{1} \left[\frac{3}{2} x\right]_0^{1} dy = \int_{1/2}^{1} \frac{3}{2} \; dy$
$= \displaystyle\frac{3y}{2}\Big|_{1/2}^{1} = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$

   (c) $P(X \ge 1/2, Y \ge 1/2) = \displaystyle\int_{1/2}^{1} \int_{1/2}^{1} \frac{3}{2} \; dx \; dy = \int_{1/2}^{1} \left[\frac{3x}{2}\right]_{1/2}^{1} dy = \int_{1/2}^{1} \frac{3}{4} dy = \frac{3}{8}$

   (d) No, they are not independent because their support space is not rectangular.

13. Graph of $f(x,y)$



(a) $f_X(x) = \int_0^x 2\ dy = 2y\Big|_0^x = 2x, 0 \leq x \leq 1$

$f_Y(y) = \int_y^1 2\ dx = 2x\Big|_y^1 = 2 - 2y = 2(1-y), 0 \leq y \leq 1$

(b) $\mu_X = \int_0^1 x f_X(x)\ dx = \int_0^1 2x^2\ dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

$\mu_Y = \int_0^1 y f_Y(y)\ dy = \int_0^1 2y - 2y^2\ dy = \left[y^2 - \frac{2y^3}{3}\right]_0^1 = \frac{1}{3}$

$\sigma_X^2 = \int_0^1 x^2 f_X(x) - \mu_X^2 = \int_0^1 2x^3\ dx - \left(\frac{2}{3}\right)^2 = \frac{x^4}{2}\Big|_0^1 - \frac{4}{9} = \frac{1}{18}$

$\sigma_Y^2 = \int_0^1 y^2 f_Y(y) - \mu_Y^2 = \int_0^1 2y^2 - 2y^3\ dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_0^1 = \frac{1}{6}$

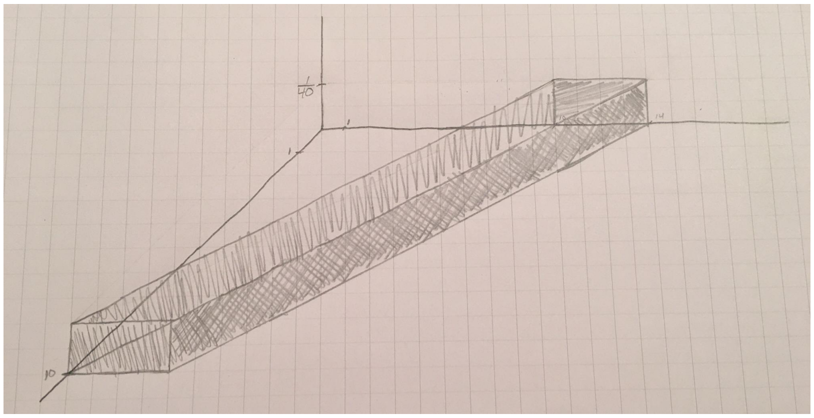
$Cov(X,Y) = E(XY) - \mu_X\mu_Y = \int_0^x \int_y^1 2\ dx\ dy - \frac{2}{9} = \int_0^x [2x]_y^1 dy - \frac{2}{9} = \int_0^x 2 - 2y\ dy - \frac{2}{9}$

$= [2y - y^2]_0^x - \frac{2}{9} = 2x - x^2 - \frac{2}{9}$

$\rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{2x - x^2 - 2/9}{1/108}$

(c) $E(Y|x) = \int_0^1 y\frac{f(x,y)}{f_X(x)}\ dy = \int_0^1 y\left(\frac{2}{2x}\right)\ dy = \frac{y^2}{2x}\Big|_0^1 = \frac{1}{2x}$

17. (a) Region for which $f(x,y) > 0$



(b) $f_X(x) = \displaystyle\int_{10-x}^{14-x} \frac{1}{40}\, dy = \left[\frac{y}{40}\right]_{10-x}^{14-x} = \frac{14-x}{40} - \frac{10-x}{40} = \frac{4}{40} = \frac{1}{10}, 0 \le x \le 10$

(c) $h(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{1/40}{1/10} = \dfrac{1}{4}, 10-x \le y \le 14-x$ for $0 \le x \le 10$

(d) $E(Y|x) = \displaystyle\int_{10-x}^{14-x} y\frac{1}{4}\, dy = \left[\frac{y^2}{8}\right]_{10-x}^{14-x} = \frac{(14-x)^2}{8} - \frac{(10-x)^2}{8} = \frac{196 - 28x + x^2 - 100 + 20x - x^2}{8}$
$= \dfrac{96-8x}{8} = 12 - x$

19. (a) We know that, $h(y|x) = \dfrac{f(x,y)}{f_X(x)}$. We are also given that $h(y|x) = \dfrac{1}{x^2}$ and $f_X(x) = \dfrac{1}{2}$. Therefore,

$$f(x,y) = \frac{1}{x^2}\left(\frac{1}{2}\right) = \frac{1}{2x^2}, 0 < x < 2, 0 < y < x^2$$

(b) $f_Y(y) = \displaystyle\int_{y^{1/2}}^{2} \frac{1}{2x^2}\, dx = -\frac{1}{2x}\Big|_{\sqrt{y}}^{2} = \frac{1}{2\sqrt{y}} - \frac{1}{4}, 0 < y < 4$

(c) $E(X|y) = \displaystyle\int_{\sqrt{y}}^{2} x\left(\frac{2\sqrt{y}}{x^2(2-\sqrt{y})}\right)\, dx = \left(\frac{2\sqrt{y}}{(2-\sqrt{y})}\right)[\log x]_{\sqrt{y}}^{2} = \left(\frac{2\sqrt{y}}{(2-\sqrt{y})}\right)\log\frac{2}{\sqrt{y}}$

(d) $E(Y|x) = \displaystyle\int_0^{x^2} y\left(\frac{1}{x^2}\right)\, dy = \frac{1}{x^2}\left(\frac{y^2}{2}\right)\Big|_0^{x^2} = \frac{x^2}{2}$