

Homework 10

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Section 4.1

1. (a)

$$\begin{aligned}\sum_{x=1}^2 \sum_{y=1}^3 c(x+2y) &= c(1+2(1)) + c(1+2(2)) + c(1+2(3)) + c(2+2(1)) + c(2+2(2)) + c(2+2(3)) \\ &= 33c \stackrel{\text{set}}{=} 1\end{aligned}$$

$$\text{Therefore, } c = \frac{1}{33}$$

(b)

$$\begin{aligned}\sum_{x=1}^3 \sum_{y=1}^x c(x+y) &= c(1+1) + c(2+1) + c(2+2) + c(3+1) + c(3+2) + c(3+3) \\ &= 24c \stackrel{\text{set}}{=} 1\end{aligned}$$

$$\text{Therefore, } c = \frac{1}{24}$$

(c)

$$\sum_{y=0}^5 \sum_{x=6-y}^{8-y} c = 18c \stackrel{\text{set}}{=} 1$$

$$\text{Therefore, } c = \frac{1}{18}$$

(d)

$$\begin{aligned}
\sum_{x=1}^{\infty} \sum_{y=1}^{\infty} c \left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y &= \sum_{x=1}^{\infty} \left\{ c \left(\frac{1}{4}\right)^x \left[\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \cdots \right] \right\} \\
&= \sum_{x=1}^{\infty} \left\{ c \left(\frac{1}{4}\right)^x \left[\frac{1/3}{1-1/3} \right] \right\} \\
&= \sum_{x=1}^{\infty} \frac{c}{2} \left(\frac{1}{4}\right)^x \\
&= \frac{c}{2} \left[\left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \cdots \right] \\
&= \frac{c}{2} \left[\frac{1/4}{1-1/4} \right] \\
&= \frac{c}{6} \stackrel{\text{set}}{=} 1
\end{aligned}$$

Therefore, $c = 6$.

3. (a) $f_x(x) = \sum_{y=1}^4 \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32} = \frac{2x+5}{16}, x = 1, 2$
- (b) $f_y(y) = \sum_{x=1}^2 \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32}, y = 1, 2, 3, 4$
- (c) $P(X > Y) = f(2, 1) = \frac{2+1}{32} = \frac{3}{32}$
- (d) $P(Y = 2X) = f(1, 2) + f(2, 4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$
- (e) $P(X + Y = 3) = f(1, 2) + f(2, 1) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32} = \frac{3}{16}$
- (f) $P(X \leq 3 - Y) = P(X + Y \leq 3) = f(1, 1) + f(1, 2) + f(2, 1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32} = \frac{1}{4}$
- (g) $f(1, 3) = \frac{1+3}{32} = \frac{1}{8} \neq f_x(1)f_y(3) = \left(\frac{7}{16}\right) \left(\frac{9}{32}\right) = \frac{63}{512}$

(h)

$$\begin{aligned}
\mu_x &= \sum_{x=1}^2 \sum_{y=1}^4 x \left(\frac{x+y}{32} \right) \\
&= 1 \left[\frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32} + \frac{1+4}{32} \right] + 2 \left[\frac{2+1}{32} + \frac{2+2}{32} + \frac{2+3}{32} + \frac{2+4}{32} \right] \\
&= \frac{14}{32} + \frac{36}{32} \\
&= \frac{25}{16} = 1.5625
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= \sum_{x=1}^2 \sum_{y=1}^4 \left(x - \frac{25}{16} \right)^2 \left(\frac{x+y}{32} \right) \\
&= \left(1 - \frac{25}{16} \right)^2 \left[\frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32} + \frac{1+4}{32} \right] + \left(2 - \frac{25}{16} \right)^2 \left[\frac{2+1}{32} + \frac{2+2}{32} + \frac{2+3}{32} + \frac{2+4}{32} \right] \\
&= \frac{81}{256} \left(\frac{14}{32} \right) + \frac{49}{256} \left(\frac{18}{32} \right) \\
&= \frac{63}{256} \approx 0.2461
\end{aligned}$$

$$\begin{aligned}
\mu_y &= \sum_{y=1}^4 \sum_{x=1}^2 y \left(\frac{x+y}{32} \right) \\
&= 1 \left[\frac{1+1}{32} + \frac{2+1}{32} \right] + 2 \left[\frac{1+2}{32} + \frac{2+2}{32} \right] + 3 \left[\frac{1+3}{32} + \frac{2+3}{32} \right] + 4 \left[\frac{1+4}{32} + \frac{2+4}{32} \right] \\
&= \frac{5}{32} + \frac{14}{32} + \frac{27}{32} + \frac{44}{32} \\
&= \frac{90}{32} = \frac{45}{16} = 2.8125
\end{aligned}$$

$$\begin{aligned}
\sigma_y^2 &= \sum_{y=1}^4 \sum_{x=1}^2 \left(y - \frac{45}{16} \right)^2 \left(\frac{x+y}{32} \right) \\
&= \left(1 - \frac{45}{16} \right)^2 \left[\frac{1+1}{32} + \frac{2+1}{32} \right] + \left(2 - \frac{45}{16} \right)^2 \left[\frac{1+2}{32} + \frac{2+2}{32} \right] + \left(3 - \frac{45}{16} \right)^2 \left[\frac{1+3}{32} + \frac{2+3}{32} \right] \\
&\quad + \left(4 - \frac{45}{16} \right)^2 \left[\frac{1+4}{32} + \frac{2+4}{32} \right] \\
&= \frac{841}{256} \left(\frac{5}{32} \right) + \frac{169}{256} \left(\frac{7}{32} \right) + \frac{9}{256} \left(\frac{9}{32} \right) + \frac{361}{256} \left(\frac{11}{32} \right) \\
&= \frac{295}{256} \approx 1.1523
\end{aligned}$$

$$8. \quad (a) \quad f(x, y) = \frac{7!}{x!y!(7-x-y)!} (0.78)^x (0.01)^y (0.21)^{7-x-y}, x+y \leq 7$$

$$(b) \quad f_x(x) = \frac{7!}{x!(7-x)!} (0.78)^x (0.22)^{7-x}, x \leq 7$$

Section 4.2

1.

$$\mu_x = \sum_{x=1}^2 \sum_{y=1}^4 x \left(\frac{x+y}{32} \right) = \frac{25}{16} = 1.5625$$

$$\sigma_x^2 = \sum_{x=1}^2 \sum_{y=1}^4 \left(x - \frac{25}{16} \right)^2 \left(\frac{x+y}{32} \right) = \frac{63}{256} \approx 0.2461$$

$$\mu_y = \sum_{y=1}^4 \sum_{x=1}^2 y \left(\frac{x+y}{32} \right) = \frac{90}{32} = \frac{45}{16} = 2.8125$$

$$\sigma_y^2 = \sum_{y=1}^4 \sum_{x=1}^2 \left(y - \frac{45}{16} \right)^2 \left(\frac{x+y}{32} \right) = \frac{295}{256} \approx 1.1523$$

$$\begin{aligned} Cov(X, Y) &= \left[\sum_{x=1}^2 \sum_{y=1}^4 xy \frac{x+y}{32} \right] - \left(\frac{25}{16} \right) \left(\frac{45}{16} \right) \\ &= (1)(1) \frac{2}{32} + (1)(2) \frac{3}{32} + (1)(3) \frac{4}{32} + (1)(4) \frac{5}{32} \\ &\quad + (2)(1) \frac{3}{32} + (2)(2) \frac{4}{32} + (2)(3) \frac{5}{32} + (2)(4) \frac{6}{32} - \frac{1125}{256} \\ &= -\frac{5}{256} \approx -0.01953 \end{aligned}$$

$$\rho = \frac{Cov(X, Y)}{\sigma_x \sigma_y} = \frac{-\frac{5}{256}}{\sqrt{\frac{63}{256} \cdot \frac{295}{256}}} \approx -0.03668$$

$$\begin{aligned}
4. \quad (a) \quad E(X) &= np_x = 3 \left(\frac{1}{6} \right) = \frac{1}{2} \\
(b) \quad E(Y) &= np_y = 3 \left(\frac{1}{2} \right) = \frac{3}{2} \\
(c) \quad Var(X) &= np_x(1 - p_x) = 3 \left(\frac{1}{6} \right) \left(\frac{5}{6} \right) = \frac{5}{12} \\
(d) \quad Var(Y) &= np_y(1 - p_y) = 3 \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = \frac{3}{4} \\
(e) \quad Cov(X, Y) &= \rho \sigma_x \sigma_y = -\sqrt{\left(\frac{1}{5} \right) \left(\frac{5}{12} \right) \left(\frac{3}{4} \right)} = -\frac{1}{4} \\
(f) \quad \rho &= -\sqrt{\frac{p_x p_y}{(1 - p_x)(1 - p_y)}} = -\sqrt{\frac{\left(\frac{1}{6} \right) \left(\frac{1}{2} \right)}{\left(\frac{5}{6} \right) \left(\frac{1}{2} \right)}} = -\sqrt{\frac{1}{5}}
\end{aligned}$$

5. We begin by expanding the expression in the expectation,

$$\begin{aligned}
K(a, b) &= E[(Y - a - bX)^2] \\
&= E[Y^2 - 2aY - 2bXY + a^2 + 2abX + b^2X^2] \\
&= E(Y^2) - 2aE(Y) - 2bE(XY) + a^2 + 2abE(X) + b^2E(X^2) \\
&= E(Y^2) - \mu_y^2 + \mu_y^2 - 2a\mu_y - 2bE(XY) + a^2 + 2ab\mu_x + b^2E(X^2) - b^2\mu_x^2 + b^2\mu_x^2 \\
&= a^2 - 2a\mu_y - 2b(\mu_x\mu_y + \sigma_{xy}) + 2ab\mu_x + b^2(\sigma_x^2 + \mu_x^2) + \sigma_y^2 + \mu_y^2
\end{aligned}$$

Now we will compute the partial derivatives of that expression with respect to a and b ,

$$\frac{\partial K}{\partial a} = 2a - 2\mu_y + 2b\mu_x$$

$$\frac{\partial K}{\partial b} = -2(\mu_x\mu_y + \sigma_{xy}) + 2a\mu_x + 2b(\sigma_x^2 + \mu_x^2)$$

Setting the first of these equal to 0,

$$\begin{aligned}
2a - 2\mu_y + 2b\mu_x &\stackrel{\text{set}}{=} 0 \\
a - \mu_y + b\mu_x &= 0 \\
a &= \mu_y - b\mu_x
\end{aligned}$$

We then substitute this for a in the second partial derivative and set equal to 0,

$$\begin{aligned}
-2(\mu_x\mu_y + \sigma_{xy}) + 2(\mu_y - b\mu_x)\mu_x + 2b(\sigma_x^2 + \mu_x^2) &\stackrel{\text{set}}{=} 0 \\
-\mu_x\mu_y - \sigma_{xy} + \mu_x\mu_y - b\mu_x^2 + b\sigma_x^2 + b\mu_x^2 &= 0 \\
b &= \frac{\sigma_{xy}}{\sigma_x^2}
\end{aligned}$$

Therefore, the line found by method of least squares is,

$$y = \mu_y - \mu_x \left(\frac{\sigma_{xy}}{\sigma_x^2} \right) + \left(\frac{\sigma_{xy}}{\sigma_x^2} \right) x$$

7. (a) X and Y are dependent because the joint probability is not “rectangular”
 (b)

$$\begin{aligned} Cov(X, Y) &= E(XY) - \mu_x \mu_y \\ &= (0)(0) \left(\frac{1}{4} \right) + (1)(1) \left(\frac{1}{4} \right) + (1)(-1) \left(\frac{1}{4} \right) + (2)(0) \left(\frac{1}{4} \right) - (1)(0) \\ &= 0 \end{aligned}$$

$$\rho = \frac{Cov(X, Y)}{\sigma_x \sigma_y} = 0$$