Homework 10

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Section 4.1

1. (a)

$$\sum_{x=1}^{2}\sum_{y=1}^{3} c(x+2y) = c(1+2(1)) + c(1+2(2)) + c(1+2(3)) + c(2+2(1)) + c(2+2(2)) + c(2+2(3))$$
$$= 33c \overset{\text{set}}{=} 1$$

Therefore, $c = \dfrac{1}{33}$

(b)

$$\sum_{x=1}^{3}\sum_{y=1}^{x} c(x+y) = c(1+1) + c(2+1) + c(2+2) + c(3+1) + c(3+2) + c(3+3)$$
$$= 24c \overset{\text{set}}{=} 1$$

Therefore, $c = \dfrac{1}{24}$

(c)

$$\sum_{y=0}^{5}\sum_{x=6-y}^{8-y} c = 18c \overset{\text{set}}{=} 1$$

Therefore, $c = \dfrac{1}{18}$

(d)

$$\begin{aligned}
\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} c\left(\frac{1}{4}\right)^{x}\left(\frac{1}{3}\right)^{y} &= \sum_{x=1}^{\infty}\left\{c\left(\frac{1}{4}\right)^{x}\left[\left(\frac{1}{3}\right)^{1}+\left(\frac{1}{3}\right)^{2}+\cdots\right]\right\} \\
&= \sum_{x=1}^{\infty}\left\{c\left(\frac{1}{4}\right)^{x}\left[\frac{1/3}{1-1/3}\right]\right\} \\
&= \sum_{x=1}^{\infty}\frac{c}{2}\left(\frac{1}{4}\right)^{x} \\
&= \frac{c}{2}\left[\left(\frac{1}{4}\right)^{1}+\left(\frac{1}{4}\right)^{2}+\cdots\right] \\
&= \frac{c}{2}\left[\frac{1/4}{1-1/4}\right] \\
&= \frac{c}{6} \overset{\text{set}}{=} 1
\end{aligned}$$

Therefore, $c = 6$.

3. (a) $f_x(x) = \sum_{y=1}^{4}\frac{x+y}{32} = \frac{x+1}{32}+\frac{x+2}{32}+\frac{x+3}{32}+\frac{x+4}{32} = \frac{4x+10}{32} = \frac{2x+5}{16}, x = 1, 2$

(b) $f_y(y) = \sum_{x=1}^{2}\frac{x+y}{32} = \frac{1+y}{32}+\frac{2+y}{32} = \frac{3+2y}{32}, y = 1, 2, 3, 4$

(c) $P(X > Y) = f(2,1) = \frac{2+1}{32} = \frac{3}{32}$

(d) $P(Y = 2X) = f(1,2) + f(2,4) = \frac{1+2}{32}+\frac{2+4}{32} = \frac{9}{32}$

(e) $P(X + Y = 3) = f(1,2) + f(2,1) = \frac{1+2}{32}+\frac{2+1}{32} = \frac{6}{32} = \frac{3}{16}$

(f) $P(X \leq 3 - Y) = P(X + Y \leq 3) = f(1,1) + f(1,2) + f(2,1) = \frac{1+1}{32}+\frac{1+2}{32}+\frac{2+1}{32} = \frac{8}{32} = \frac{1}{4}$

(g) $f(1,3) = \frac{1+3}{32} = \frac{1}{8} \neq f_x(1)f_y(3) = \left(\frac{7}{16}\right)\left(\frac{9}{32}\right) = \frac{63}{512}$

(h)

$$\begin{aligned}
\mu_x &= \sum_{x=1}^{2}\sum_{y=1}^{4} x\left(\frac{x+y}{32}\right) \\
&= 1\left[\frac{1+1}{32}+\frac{1+2}{32}+\frac{1+3}{32}+\frac{1+4}{32}\right]+2\left[\frac{2+1}{32}+\frac{2+2}{32}+\frac{2+3}{32}+\frac{2+4}{32}\right] \\
&= \frac{14}{32}+\frac{36}{32} \\
&= \frac{25}{16} = 1.5625
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= \sum_{x=1}^{2}\sum_{y=1}^{4}\left(x-\frac{25}{16}\right)^2\left(\frac{x+y}{32}\right) \\
&= \left(1-\frac{25}{16}\right)^2\left[\frac{1+1}{32}+\frac{1+2}{32}+\frac{1+3}{32}+\frac{1+4}{32}\right]+\left(2-\frac{25}{16}\right)^2\left[\frac{2+1}{32}+\frac{2+2}{32}+\frac{2+3}{32}+\frac{2+4}{32}\right] \\
&= \frac{81}{256}\left(\frac{14}{32}\right)+\frac{49}{256}\left(\frac{18}{32}\right) \\
&= \frac{63}{256} \approx 0.2461
\end{aligned}$$

$$\begin{aligned}
\mu_y &= \sum_{y=1}^{4}\sum_{x=1}^{2} y\left(\frac{x+y}{32}\right) \\
&= 1\left[\frac{1+1}{32}+\frac{2+1}{32}\right]+2\left[\frac{1+2}{32}+\frac{2+2}{32}\right]+3\left[\frac{1+3}{32}+\frac{2+3}{32}\right]+4\left[\frac{1+4}{32}+\frac{2+4}{32}\right] \\
&= \frac{5}{32}+\frac{14}{32}+\frac{27}{32}+\frac{44}{32} \\
&= \frac{90}{32} = \frac{45}{16} = 2.8125
\end{aligned}$$

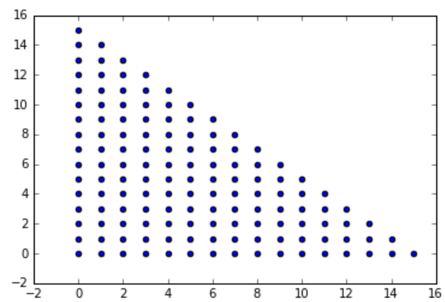
$$\begin{aligned}
\sigma_y^2 &= \sum_{y=1}^{4}\sum_{x=1}^{2}\left(y-\frac{45}{16}\right)^2\left(\frac{x+y}{32}\right) \\
&= \left(1-\frac{45}{16}\right)^2\left[\frac{1+1}{32}+\frac{2+1}{32}\right]+\left(2-\frac{45}{16}\right)^2\left[\frac{1+2}{32}+\frac{2+2}{32}\right]+\left(3-\frac{45}{16}\right)^2\left[\frac{1+3}{32}+\frac{2+3}{32}\right] \\
&\quad+\left(4-\frac{45}{16}\right)^2\left[\frac{1+4}{32}+\frac{2+4}{32}\right] \\
&= \frac{841}{256}\left(\frac{5}{32}\right)+\frac{169}{256}\left(\frac{7}{32}\right)+\frac{9}{256}\left(\frac{9}{32}\right)+\frac{361}{256}\left(\frac{11}{32}\right) \\
&= \frac{295}{256} \approx 1.1523
\end{aligned}$$

8. (a) $f(x,y) = \dfrac{7!}{x!y!(7-x-y)!}(0.78)^x(0.01)^y(0.21)^{7-x-y}, x+y \leq 7$

   (b) $f_x(x) = \dfrac{7!}{x!(7-x)!}(0.78)^x(0.22)^{7-x}, x \leq 7$

9. (a) $f(x,y) = \left(\dfrac{15!}{x!y!(15-x-y)!}\right)\left(\dfrac{6}{10}\right)^x\left(\dfrac{3}{10}\right)^y\left(\dfrac{1}{10}\right)^{15-x-y}, 0 \leq x+y \leq 15$

   (b) Based on the shape of the region, they cannot be independent because the region is not rectangular.



   (c) $f(10,4) = \left(\dfrac{15!}{10!4!1!}\right)\left(\dfrac{6}{10}\right)^{10}\left(\dfrac{3}{10}\right)^4\left(\dfrac{1}{10}\right)^1 \approx 0.07354$

   (d) $X \sim b(15, 0.6)$

   (e)

$$\begin{aligned}
P(X \leq 11) &= 1 - P(X > 11) \\
&= 1 - [P(X=12) + P(X=13) + P(X=14) + P(X=15)] \\
&= 1 - \left[\frac{15!}{12!3!}(.6)^{12}(.4)^3 + \frac{15!}{13!2!}(.6)^{13}(.4)^2 + \frac{15!}{14!1!}(.6)^{14}(.4)^1 + \frac{15!}{15!0!}(.6)^{15}(.4)^0\right] \\
&= 1 - 0.0905 \\
&= 0.9095
\end{aligned}$$

Section 4.2

1.

$$\mu_x = \sum_{x=1}^{2}\sum_{y=1}^{4} x\left(\frac{x+y}{32}\right) = \frac{25}{16} = 1.5625$$

$$\sigma_x^2 = \sum_{x=1}^{2}\sum_{y=1}^{4}\left(x - \frac{25}{16}\right)^2\left(\frac{x+y}{32}\right) = \frac{63}{256} \approx 0.2461$$

$$\mu_y = \sum_{y=1}^{4}\sum_{x=1}^{2} y\left(\frac{x+y}{32}\right) = \frac{90}{32} = \frac{45}{16} = 2.8125$$

$$\sigma_y^2 = \sum_{y=1}^{4}\sum_{x=1}^{2}\left(y - \frac{45}{16}\right)^2\left(\frac{x+y}{32}\right) = \frac{295}{256} \approx 1.1523$$

$$\begin{aligned} Cov(X,Y) &= \left[\sum_{x=1}^{2}\sum_{y=1}^{4} xy\frac{x+y}{32}\right] - \left(\frac{25}{16}\right)\left(\frac{45}{16}\right) \\ &= (1)(1)\frac{2}{32} + (1)(2)\frac{3}{32} + (1)(3)\frac{4}{32} + (1)(4)\frac{5}{32} \\ &\quad + (2)(1)\frac{3}{32} + (2)(2)\frac{4}{32} + (2)(3)\frac{5}{32} + (2)(4)\frac{6}{32} - \frac{1125}{256} \\ &= -\frac{5}{256} \approx -0.01953 \end{aligned}$$

$$\rho = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = \frac{-\frac{5}{256}}{\sqrt{\frac{63}{256}\cdot\frac{295}{256}}} \approx -0.03668$$

4. (a) $E(X) = np_x = 3\left(\frac{1}{6}\right) = \frac{1}{2}$

   (b) $E(Y) = np_y = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

   (c) $Var(X) = np_x(1-p_x) = 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{5}{12}$

   (d) $Var(Y) = np_y(1-p_y) = 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$

   (e) $Cov(X,Y) = \rho\sigma_x\sigma_y = -\sqrt{\left(\frac{1}{5}\right)\left(\frac{5}{12}\right)\left(\frac{3}{4}\right)} = -\frac{1}{4}$

   (f) $\rho = -\sqrt{\frac{p_x p_y}{(1-p_x)(1-p_y)}} = -\sqrt{\frac{\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)}{\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)}} = -\sqrt{\frac{1}{5}}$

5. We begin by expanding the expression in the expectation,

$$\begin{aligned}
K(a,b) &= E[(Y-a-bX)^2] \\
&= E[Y^2 - 2aY - 2bXY + a^2 + 2abX + b^2X^2] \\
&= E(Y^2) - 2aE(Y) - 2bE(XY) + a^2 + 2abE(X) + b^2E(X^2) \\
&= E(Y^2) - \mu_y^2 + \mu_y^2 - 2a\mu_y - 2bE(XY) + a^2 + 2ab\mu_x + b^2E(X^2) - b^2\mu_x^2 + b^2\mu_x^2 \\
&= a^2 - 2a\mu_y - 2b(\mu_x\mu_y + \sigma_{xy}) + 2ab\mu_x + b^2(\sigma_x^2 + \mu_x^2) + \sigma_y^2 + \mu_y^2
\end{aligned}$$

Now we will compute the partial derivatives of that expression with respect to a and b,

$$\frac{\partial K}{\partial a} = 2a - 2\mu_y + 2b\mu_x$$

$$\frac{\partial K}{\partial b} = -2(\mu_x\mu_y + \sigma_{xy}) + 2a\mu_x + 2b(\sigma_x^2 + \mu_x^2)$$

Setting the first of these equal to 0,

$$\begin{aligned}
2a - 2\mu_y + 2b\mu_x &\overset{\text{set}}{=} 0 \\
a - \mu_y + b\mu_x &= 0 \\
a &= \mu_y - b\mu_x
\end{aligned}$$

We then substitute this for a in the second partial derivative and set equal to 0,

$$\begin{aligned}
-2(\mu_x\mu_y + \sigma_{xy}) + 2(\mu_y - b\mu_x)\mu_x + 2b(\sigma_x^2 + \mu_x^2) &\overset{\text{set}}{=} 0 \\
-\mu_x\mu_y - \sigma_{xy} + \mu_x\mu_y - b\mu_x^2 + b\sigma_x^2 + b\mu_x^2 &= 0 \\
b &= \frac{\sigma_{xy}}{\sigma_x^2}
\end{aligned}$$

Therefore, the line found by method of least squares is,

$$y = \mu_y - \mu_x \left(\frac{\sigma_{xy}}{\sigma_x^2}\right) + \left(\frac{\sigma_{xy}}{\sigma_x^2}\right) x$$

7. (a) X and Y are dependent because the joint probability is not "rectangular"

   (b)

$$\begin{aligned} Cov(X,Y) &= E(XY) - \mu_x\mu_y \\ &= (0)(0)\left(\frac{1}{4}\right) + (1)(1)\left(\frac{1}{4}\right) + (1)(-1)\left(\frac{1}{4}\right) + (2)(0)\left(\frac{1}{4}\right) - (1)(0) \\ &= 0 \end{aligned}$$

$$\rho = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = 0$$