Homework 8

Roly Vicaría
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Section 3.2

1. (a) $f(x) = \frac{1}{3}e^{-x/3}$
   $\mu = 3$
   $\sigma^2 = 3^2 = 9$

   (b) $f(x) = 3e^{-3x}$
   $\mu = \frac{1}{3}$
   $\sigma^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

2. (a) $f(X) = \frac{3}{2}e^{-3x/2}$

   (b) $P(X > 2) = e^{-3(2)/2} = e^{-3} \approx 0.04979$

3.

$$\begin{aligned}
P(X > x + y | X > x) &= \frac{P((X > x + y) \cap (X > x))}{P(X > x)} \\
&= \frac{P(X > x + y)}{P(X > x)} \\
&= \frac{e^{-(x+y)/\theta}}{e^{-x/\theta}} \\
&= e^{-y/\theta} \\
&= P(X > y)
\end{aligned}$$

4. We define $g(x) = 1 - F(x)$.

$$\begin{aligned}
P(X > x + y | X > x) &= \frac{P((X > x + y) \cap (X > x))}{P(X > x)} \\
&= \frac{P(X > x + y)}{P(X > x)} \\
&= \frac{1 - F(x + y)}{1 - F(x)} \qquad (1)
\end{aligned}$$

By assumption, we know that $P(X > x + y | X > x) = P(X > y)$. We can express $P(X > y)$ as

$$P(X > y) = 1 - F(y) \tag{2}$$

We can see by the equality of line (1) and line (2), that

$$1 - F(x + y) = (1 - F(x))(1 - F(y))$$

which shows that $g(x)$ satisfies the functional equation $g(x + y) = g(x)g(y)$.

This is as far as I got...I throw in the towel...I'm not sure how to get to $e^{-\lambda x}$ from here. Mercy

6. (a) $P(no\ flaws\ in\ first\ 40\ feet) = e^{-3(40)/100} = 0.3012$
   (b) I assumed that the flaws in the sheets of aluminum follow a Poisson distribution and that the mean number of occurrences in an interval of length w is poprtional to λ.

7.
$$\begin{aligned}
M(t) = E(e^{tX}) &= \int_0^\infty e^{tx} \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} dx \\
&= \int_0^\infty \frac{x^{\alpha-1}e^{-(1-\theta t)x/\theta}}{\Gamma(\alpha)\theta^\alpha} dx \\
&= \int_0^\infty \frac{(\frac{\theta y}{1-\theta t})^{\alpha-1}e^{-y}}{\Gamma(\alpha)\theta^\alpha} \frac{\theta}{1-\theta t} dy \\
&= \int_0^\infty \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)(1-\theta t)^\alpha} dy \\
&= \frac{1}{\Gamma(\alpha)(1-\theta t)^\alpha} \int_0^\infty y^{\alpha-1}e^{-y} dy \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha)(1-\theta t)^\alpha} \\
&= \frac{1}{(1-\theta t)^\alpha}
\end{aligned}$$

8. $P(X < 5; \alpha = 2, \theta = 4) = \int_0^5 \frac{1}{\Gamma(2)4^2} x^1 e^{-x/4} dx = \frac{1}{16} \int_0^5 x e^{-x/4} dx$

   Doing integration by parts with $u = x$ and $dv = e^{-x/4}dx$, we get $du = dx$ and $v = -4e^{-x/4}$:

$$\begin{aligned}
\frac{1}{16}\left\{ -4xe^{-x/4}\Big|_0^5 + 4\int_0^5 e^{-x/4}dx \right\} &= \frac{1}{16}\left\{ -20e^{-5/4} + 4\left[-4e^{-x/4} \right]_0^5 \right\} \\
&= \frac{1}{16}\left\{ -20e^{-5/4} + 4\left[-4e^{-5/4} + 4 \right] \right\} \\
&= \frac{1}{16}\left[-36e^{-5/4} + 16 \right] \\
&= 0.3554
\end{aligned}$$

10. From 3.2-7, we found that $M(t) = (1-\theta t)^{-\alpha}$. We start by computing $M'(t)$:
$$M'(t) = \alpha\theta(1-\theta t)^{-\alpha-1}$$
Evaluating this when $t = 0$: $M'(0) = E(X) = \alpha\theta$
To find $Var(X)$, we compute $M''(t)$:
$$M''(t) = \alpha(\alpha+1)\theta^2(1-\theta t)^{-\alpha-2}$$
Evaluating this when $t = 0$: $M''(0) = \alpha^2\theta^2 + \alpha\theta^2$
Therefore,
$$Var(X) = M''(0) - [M'(0)]^2 = \alpha^2\theta^2 + \alpha\theta^2 - (\alpha\theta)^2 = \alpha\theta^2$$

11. $X \sim \chi^2(17)$
   (a) $P(X < 7.564) = 0.025$
   (b) $P(X > 27.59) = 0.05$
   (c) $P(6.408 < X < 27.59) = 0.95 - 0.01 = 0.94$
   (d) $\chi^2_{0.95}(17) = 8.672$
   (e) $\chi^2_{0.025}(17) = 30.19$

12. (a) $W \sim Gamma(7, 1/16)$
   (b) $P(W \leq 0.5) = 1 - \sum_{k=0}^{6} \frac{8^k e^{-8}}{k!} = 0.6866$

13. $X \sim \chi^2(23)$
   (a) $P(14.85 < X < 32.01) = 0.90 - 0.10 = 0.80$
   (b) $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$
   $a = 11.69, b = 38.08$
   (c) $\mu = 23$
   $\sigma^2 = 46$
   (d) $\chi^2_{0.05}(23) = 35.17$
   $\chi^2_{0.95}(23) = 13.09$

14. $X \sim \chi^2(12)$
   $P(a < X < b) = 0.90$ and $P(X < a) = 0.05$
   $a = 5.226, b = 21.03$

16. $X \sim Gamma(8, 2)$
$$\begin{aligned} P(X > 26.3) &= \sum_{k=0}^{7} \frac{(26.3/2)^k e^{-26.3/2}}{k!} \\ &= e^{-13.15}\left[\frac{13.15^0}{0!} + \frac{13.15^1}{1!} + \frac{13.15^2}{2!} + \cdots + \frac{13.15^7}{7!}\right] \\ &= 25675.093e^{-13.15} \\ &= 0.04995 \end{aligned}$$

20.

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\begin{aligned}
E[v(t)] &= \int_0^3 v(t)f(t)dt + \int_3^\infty 0f(t)dt \\
&= \int_0^3 100(2^{3-t}-1)\cdot\frac{1}{5}e^{-t/5}dt \\
&= 20\int_0^3 (2^{3-t}-1)e^{-t/5}dt \\
&= 20\left\{\int_0^3 2^{3-t}e^{-t/5}dt - \int_0^3 e^{-t/5}dt\right\} \\
&= 20\left\{\int_0^3 e^{(3-t)\ln 2}e^{-t/5}dt - \int_0^3 e^{-t/5}dt\right\} \\
&= 20\left\{\int_0^3 e^{3\ln 2 - t\ln 2 - t/5}dt - \int_0^3 e^{-t/5}dt\right\} \\
&= 20\left\{\int_0^3 e^{3\ln 2 - (\ln 2 + 1/5)t}dt - \int_0^3 e^{-t/5}dt\right\} \\
&= 20\left\{\left[\frac{e^{3\ln 2 - (\ln 2+1/5)t}}{-(\ln 2 + 1/5)}\right]_0^3 - \left[-5e^{-t/5}\right]_0^3\right\} \\
&= 20\,(8.3426 - 2.2559) \\
&= 121.734
\end{aligned}
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