Homework 5

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Section 2.3

1. (a) Mean:

$$E(X) = \sum_{x \in S} x \cdot f(x) = 5\left(\frac{1}{5}\right) + 10\left(\frac{1}{5}\right) + 15\left(\frac{1}{5}\right) + 20\left(\frac{1}{5}\right) + 25\left(\frac{1}{5}\right)$$
$$= 1 + 2 + 3 + 4 + 5$$
$$= 15$$

Variance:
We first have to compute $E(X^2)$

$$E(X^2) = \sum_{x^2 \in S} x^2 \cdot f(x) = 5^2\left(\frac{1}{5}\right) + 10^2\left(\frac{1}{5}\right) + 15^2\left(\frac{1}{5}\right) + 20^2\left(\frac{1}{5}\right) + 25^2\left(\frac{1}{5}\right)$$
$$= 5 + 20 + 45 + 80 + 125$$
$$= 275$$

Therefore, $\sigma^2 = E(X^2) - [E(X)]^2 = 275 - 15^2 = 50$

(b) Mean: $E(X) = \sum_{x} x \cdot f(x) = 5(1) = 5$

Variance:
We first compute $E(X^2)$: $E(X^2) = \sum_{x} x^2 \cdot f(x) = 25(1) = 25$

Therefore, $\sigma^2 = E(X^2) - [E(X)]^2 = 25 - 25 = 0$

(c) Mean: $E(X) = \sum_{x \in S} x \cdot f(x) = 1\left(\frac{3}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{1}{6}\right) = \frac{10}{6} = \frac{5}{3}$

Variance:
We first compute $E(X^2)$:
$E(X^2) = \sum_{x \in S} x^2 \cdot f(x) = 1^2\left(\frac{3}{6}\right) + 2^2\left(\frac{2}{6}\right) + 3^2\left(\frac{1}{6}\right) = \frac{20}{6} = \frac{10}{3}$

Therefore, $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{10}{3} - \left(\frac{5}{3}\right)^2 = \frac{5}{9}$

3. (a) $Var(X+4) = E[(X+4)^2] - [E(X+4)]^2 = 116 - 10^2 = 16$

   (b) We are given that $E(X+4) = 10$. That means that $E(X) + 4 = 10$. Therefore, $\mu = E(X) = 6$

   (c) We are given that $E[(X+4)^2] = 116$. That means that $E(X^2 + 8X + 16) = E(X^2) + 8E(X) + 16 = 116$. Therefore, $E(X^2) = 116 - 16 - 8(6) = 52$. So $\sigma^2 = E(X^2) - [E(X)]^2 = 52 - 36 = 16$

4.

$$\begin{aligned} E\left[\frac{(X-\mu)}{\sigma}\right] &= E\left(\frac{X}{\sigma}\right) - E\left(\frac{\mu}{\sigma}\right) \\ &= \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} \\ &= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \\ &= 0 \end{aligned}$$

$$\begin{aligned} E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] &= E\left[\frac{X^2 - 2\mu X + \mu^2}{\sigma^2}\right] \\ &= E\left(\frac{X^2}{\sigma^2}\right) - E\left(\frac{2\mu X}{\sigma^2}\right) + E\left(\frac{\mu^2}{\sigma^2}\right) \\ &= \frac{1}{\sigma^2}[E(X^2) - 2\mu E(X) + \mu^2] \\ &= \frac{1}{\sigma^2}[E(X^2) - \mu^2] \\ &= \frac{1}{\sigma^2}\sigma^2 \\ &= 1 \end{aligned}$$

11. $M'(t) = \frac{2}{5}e^t + (2)\frac{1}{5}e^{2t} + (3)\frac{2}{5}e^{3t}$.

    Evaluating this when $t = 0$ gives us: $\mu = \frac{2}{5} + (2)\frac{1}{5} + (3)\frac{2}{5} = \frac{10}{5} = 2$.

    $M''(t) = \frac{2}{5}e^t + (4)\frac{1}{5}e^{2t} + (9)\frac{2}{5}e^{3t}$.

    Evaluting this when $t = 0$ gives us $E(X^2) = \frac{2}{5} + (4)\frac{1}{5} + (9)\frac{2}{5} = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} = \frac{24}{5}$

    Therefore, $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{24}{5} - 4 = \frac{4}{5}$

    pmf of X: $f(x) = \frac{|2-x|+1}{5}$ for $x = 1, 2, 3$.

15. (a)

$$\begin{aligned} P(X \geq 20) &= 1 - P(X \leq 19) \\ &= 1 - (1 - .96^{19}) \\ &= 0.4604 \end{aligned}$$

(b)

$$\begin{aligned} P(X \leq 20) &= P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = 20) \\ &= (.96^0)(.04^1) + (.96^1)(.04^1) + \cdots + (.96^{19})(.04^1) \\ &= .04 \cdot (.96^0 + .96^1 + \cdots + .96^{19}) \\ &= .04 \cdot \sum_{k=0}^{19} .96^k = .04 \cdot \frac{1 - .96^{20}}{1 - .96} \\ &= 0.5580 \end{aligned}$$

(c) $P(X = 20) = .96^{19}(.04) = 0.0184$

18.

$$\begin{aligned} P(X > k + j | X > k) &= \frac{P(X > k + j \cap X > k)}{P(X > k)} \\ &= \frac{P(X > k + j)}{P(X > k)} \\ &= \frac{1 - P(X \leq k + j)}{1 - P(X \leq k)} \\ &= \frac{1 - (1 - p^{k+j})}{1 - (1 - p^k)} \\ &= \frac{p^{k+j}}{p^k} \\ &= p^j \\ &= 1 - (1 - p^j) \\ &= 1 - P(X \leq j) \\ &= P(X > j) \end{aligned}$$

Section 2.4

4. (a) The distribution of X is $b(7, 0.15)$.

(b) i. $P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.7166 = 0.2834$

ii. $P(X = 1) = P(X \leq 1) - P(X \leq 0) = 0.7166 - 0.3206 = 0.3960$

iii. $P(X \leq 3) = 0.9879$

5. $X \sim b(25, 0.2)$

   (a) $P(X \leq 4) = 0.4207$

   (b) $P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.4207 = 0.5793$

   (c) $P(X = 6) = P(X \leq 6) - P(X \leq 5) = 0.7800 - 0.6167 = 0.1633$

   (d) $\mu = np = 25(0.2) = 5$
   $\sigma^2 = np(1-p) = 25(0.2)(0.8) = 4$
   $\sigma = \sqrt{np(1-p)} = \sqrt{4} = 2$

6. (a) $X \sim b(15, 0.75)$, $Y = 15 - X \sim b(15, 0.25)$

   (b) $P(X \geq 10) = 1 - P(X \leq 9) = 1 - P(Y \geq 6) = 1 - (1 - P(Y \leq 5)) = 0.8516$

   (c) $P(X \leq 10) = P(Y \geq 5) = 1 - P(Y \leq 4) = 1 - 0.6865 = 0.3135$

   (d) $\mu = np = 15(0.75) = 11.25$
   $\sigma^2 = np(1-p) = 15(0.75)(0.25) = 2.8125$
   $\sigma = \sqrt{np(1-p)} = \sqrt{2.125} = 1.6771$

8. (a) $P(X \geq 1) = 1 - P(X = 0) = 1 - \left[\binom{4}{0}(0.99^0)(0.01^4)\right] = 1 - 0.00000001 = 0.99999999 \approx 1$

   (b) $P(X = 4) = \binom{4}{4}(0.99^4)(0.01)^0 = 0.9606$

9. (a) $X \sim b(20, 0.8)$

   (b) $\mu = np = 20(0.8) = 16$
   $\sigma^2 = np(1-p) = 20(0.8)(0.2) = 3.2$
   $\sigma = \sqrt{np(1-p)} = \sqrt{3.2} = 1.789$

   (c) i. $P(X = 15) = \binom{20}{15}(0.8^{15})(0.2^5) = 0.1746$

   ii. $P(X > 15) = 1 - P(X \leq 15) = 1 - 0.3704 = 0.6296$

   iii. $P(X \leq 15) = 0.3704$

10. (a) $X \sim b(8, 0.9)$

   (b) i. $P(X = 8) = \binom{8}{8}(0.9^8)(0.1^0) = 0.4305$

   ii. $P(X \leq 6) = 0.1869$

   iii. $P(X \geq 6) = 1 - P(X \leq 5) = 1 - 0.0381 = 0.9619$

11. $\mu = np = 6, \sigma^2 = np(1-p) = 3.6$
   $1 - p = \dfrac{3.6}{6} = 0.6 \rightarrow p = 0.4$
   $n = \dfrac{6}{0.4} = 15$
   $P(X = 4) = \binom{15}{4}(0.4^4)(0.6^{11}) = 0.1268$

13. (a) $X \sim b(10, 0.1)$

$P(X \geq 1) = 1 - P(X = 0) = 1 - \left[\binom{10}{0}(0.1^0)(0.9^{10})\right] = 1 - 0.3487 = 0.6513$

(b) $X \sim b(15, 0.1)$

$P(X \geq 1) = 1 - P(X = 0) = 1 - \left[\binom{15}{0}(0.1^0)(0.9^{15})\right] = 1 - 0.2059 = 0.7941$

15. $P(X = 0|C) = \binom{5}{0}(.05^0)(.95^5) = 0.7738$

$P(X = 0|B) = \binom{5}{0}(.02^0)(.98^5) = 0.9039$

$P(X = 0|A) = \binom{5}{0}(.03^0)(.97^5) = 0.8587$

$$\begin{aligned} P(C|X \geq 1) &= \frac{P(X \geq 1|C)P(C)}{P(X \geq 1|C)P(C) + P(X \geq 1|B)P(B) + P(X \geq 1|A)P(A)} \\ &= \frac{(1 - P(X = 0|C))P(C)}{(1 - P(X = 0|C))P(C) + (1 - P(X = 0|B))P(B) + (1 - P(X = 0|A))P(A)} \\ &= \frac{(1 - 0.7738)(0.1)}{(1 - 0.7738)(0.1) + (1 - 0.9039)(0.5) + (1 - 0.8587)(0.4)} \\ &= 0.1778 \end{aligned}$$

19. (a) $f(x) = \binom{1}{x}(2/3)^x(1/3)^{1-x}$

$\mu = np = 1(2/3) = 2/3$

$\sigma^2 = np(1-p) = 1(2/3)(1/3) = 2/9$

$\sigma = \sqrt{np(1-p)} = \sqrt{2/9} \approx 0.4714$

(b) $f(x) = \binom{12}{x}(.75)^x(.25)^{12-x}$

$\mu = np = 12(.75) = 9$

$\sigma^2 = np(1-p) = 12(.75)(.25) = 2.25$

$\sigma = \sqrt{2.25} = 1.5$

20. (a) i. Binomial with $n = 5$ and $p = 0.7$

ii. $M'(t) = 5(0.7e^t)(0.3 + 0.7e^t)^4$

$\mu = M'(0) = 5(0.7)(1)^4 = 3.5$

$M''(t) = 3.5e^t(4)(0.3 + 0.7e^t)^3(0.7e^t) + 3.5e^t(0.3 + 0.7e^t)^4$

$E(X^2) = M''(0) = 9.8 + 3.5 = 13.3$

$\sigma^2 = E(X^2) - E(X)^2 = 13.3 - (3.5)^2 = 1.05$

iii. $P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = \binom{5}{1}(0.7)^1(0.3)^4 + \binom{5}{2}(0.7)^2(0.3)^3 = 0.16065$

(b) i. Geometric distribution with $p = 0.3$

ii. $M'(t) = \dfrac{0.3e^t(1-0.7e^t) - 0.3e^t(-0.7e^t)}{(1-0.7e^t)^2} = \dfrac{0.3e^t - 0.21e^{2t} + 0.21e^{2t}}{(1-0.7e^t)^2} = \dfrac{0.3e^t}{(1-0.7e^t)^2}$

$\mu = M'(0) = \dfrac{0.3}{0.3^2} = 3.333$

$M''(t) = \dfrac{0.3e^t(1-0.7e^t)^2 - 0.3e^t(2)(-0.7e^t)(1-0.7e^t)}{(1-0.7e^t)^4}$

$E(X^2) = M''(0) = \dfrac{0.3(0.3)^2 - 0.3(2)(-0.7)(0.3)}{0.3^4} = 18.8889$

$\sigma^2 = E(X^2) - E(X)^2 = 18.8889 - (3.333)^2 = 7.78$

iii. $P(1 \le X \le 2) = P(X=1) + P(X=2) = (0.7)^0(0.3)^1 + (0.7)^1(0.3)^1 = 0.51$

(c) i. Bernoulli distribution with $p = 0.55$

ii. $M'(t) = 0.55e^t$

$\mu = M'(0) = 0.55$

$M''(t) = 0.55e^t$

$E(X^2) = M''(0) = 0.55$

$\sigma^2 = E(X^2) - E(X)^2 = 0.55 - 0.3025 = 0.2475$

iii. $P(1 \le X \le 2) = P(X=1) + P(X=2) = 0.55$

(d) i. I don't think this distribution has a name.

ii. $M'(t) = 0.3e^t + 0.8e^{2t} + 0.6e^{3t} + 0.4e^{4t}$

$\mu = M'(0) = 0.3 + 0.8 + 0.6 + 0.4 = 2.1$

$M''(t) = 0.3e^t + 1.6e^{2t} + 1.8e^{3t} + 1.6e^{4t}$

$E(X^2) = M''(0) = 0.3 + 1.6 + 1.8 + 1.6 = 5.3$

$\sigma^2 = E(X^2) - E(X)^2 = 5.3 - 2.1^2 = 0.89$

iii. $P(1 \le X \le 2) = P(X=1) + P(X=2) = 0.3 + 0.4 = 0.7$

(e) i. Uniform distribution where $f(x) = 1/10$ for $x = 1, 2, ..., 10$

ii. $M'(t) = \displaystyle\sum_{x=1}^{10} (0.1x)e^{tx}$

$\mu = M'(0) = \displaystyle\sum_{x=1}^{10} 0.1x = 0.1(1 + 2 + \cdots + 10) = 5.5$

$M''(t) = \displaystyle\sum_{x=1}^{10} (0.1x^2)e^{tx}$

$E(X^2) = M''(0) = \displaystyle\sum_{x=1}^{10} 0.1x^2 = 0.1(1 + 4 + 9 + \cdots + 100) = (0.1)\frac{10(11)(21)}{6} =$ $(0.1)(385) = 38.5$

$\sigma^2 = E(X^2) - E(X)^2 = 38.5 - (5.5)^2 = 8.25$

iii. $P(1 \le X \le 2) = P(X=1) + P(X=2) = 1/10 + 1/10 = 1/5$