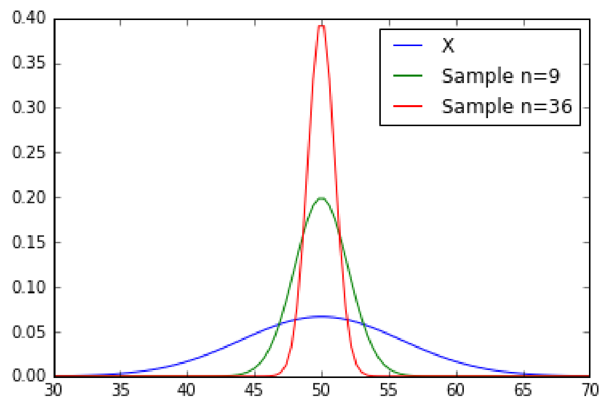
Homework 14

Roly Vicaría
STAT414 Spring 2016

April 24, 2016

Section 5.5

2. $X \sim N(50, 36)$

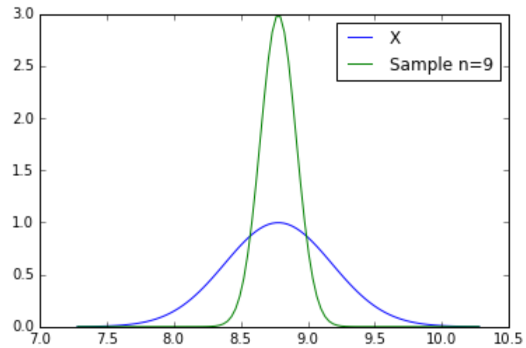


3. (a) $E(\bar{X}) = \mu = 46.58$,
$Var(\bar{X}) = \dfrac{\sigma^2}{n} = \dfrac{40.96}{16} = 2.56$

(b)

$$\begin{aligned} P(44.42 \leq \bar{X} \leq 48.98) &= P\left(\frac{44.42 - 46.58}{\sqrt{2.56}} \leq \frac{\bar{X} - 46.58}{\sqrt{2.56}} \leq \frac{48.98 - 46.58}{\sqrt{2.56}}\right) \\ &= P(-1.35 \leq Z \leq 1.5) \\ &= \Phi(1.5) - \Phi(-1.35) \\ &= 0.9332 - 0.0885 \\ &= 0.8447 \end{aligned}$$

5. (a) $X \sim N(8.78, 0.16)$



(b) To find a and b, such that $P(a \leq S^2 \leq b) = 0.90$, we use the fact that $\dfrac{(n-1)S^2}{\sigma^2}$ is $\chi^2(8)$, therefore, we restate the question as $P\left(\dfrac{8a}{0.16} \leq \dfrac{8S^2}{0.16} \leq \dfrac{8b}{0.16}\right) = 0.90$.

This is the same as $P\left(\dfrac{8S^2}{0.16} \leq \dfrac{8b}{0.16}\right) - P\left(\dfrac{8S^2}{0.16} \leq \dfrac{8a}{0.16}\right) = 0.90$

From Table IV in the book, we can see that $P\left(\dfrac{8S^2}{0.16} \leq 15.51\right) - P\left(\dfrac{8S^2}{0.16} \leq 2.733\right) = 0.90$.

Solving for $a = \dfrac{2.733 \cdot 0.16}{8} = 0.5466$ and $b = \dfrac{15.51 \cdot 0.16}{8} = 0.3102$

7. We start by defining $Y = X_1 + X_2 + X_3$, where $X_i \sim N(1.18, 0.07^2)$. Therefore, $Y \sim N(3.54, 0.0147)$. We are also given $W \sim N(3.22, 0.09^2)$. We can compute that $Y - W \sim N(0.32, 0.0228)$.

   We are asked to find the $P(Y > W) = P(Y - W > 0) = P\left(\dfrac{Y - W - 0.32}{\sqrt{0.0228}} > \dfrac{0 - 0.32}{\sqrt{0.0228}}\right) = P(Z > -2.12) = P(Z < 2.12) = 0.9830$

8. We are given that $X \sim N(184.09, 39.37)$ and $Y \sim N(171.93, 50.88)$. We can compute that $X - Y \sim N(12.16, 90.25)$. Therefore, $P(X > Y) = P(X - Y > 0) = P\left(Z > \dfrac{-12.16}{\sqrt{90.25}}\right) = P(Z > -1.28) = P(Z < 1.28) = 0.8997$

10. We are given that we have n light bulbs, each one follows a normal distribution, $N(800, 100^2)$. We want the sum of the lifetime of the n lightbulbs to be greater than 10,000 hours with probability 0.90. In other words, we are looking for n, such that $P(Y > 10000) = 0.90$), where $Y = X_1 + X_2 + \cdots + X_n$.

We know that $Y \sim N(800n, 100^2 n)$. Therefore,

$$\begin{aligned} P(Y > 10000) &= P\left(\frac{Y - 800n}{\sqrt{100^2 n}} > \frac{10000 - 800n}{\sqrt{100^2 n}}\right) \\ &= P\left(Z > \frac{10000 - 800n}{\sqrt{100^2 n}}\right) = 0.90 \end{aligned}$$

We can rexpress that last equality as $P\left(Z < -\frac{10000 - 800n}{\sqrt{100^2 n}}\right) = 0.90$. From table Va in the book, we find that $P(Z < 1.29) \approx 0.90$.
Solving the following for n,

$$1.29 = -\frac{10000 - 800n}{\sqrt{100^2 n}}$$

We get $n = 13.08$. Rounding up, $n = 14$.

15. (a) $t_{0.01}(17) = 2.567$
    (b) $t_{0.95}(17) = -1.740$
    (c) $P(-1.740 \le T \le 1.740) = 0.90$

16. $T \sim t(8)$
    (a) $P(-t_{0.025} \le T \le t_{0.025}) = 0.95$
    $t_{0.025} = 2.306$
    (b)

$$\begin{aligned} -t_{0.025} &\le T \le t_{0.025} \\ -t_{0.025} &\le \frac{\bar{X} - \mu}{S/\sqrt{n}} \le t_{0.025} \\ -t_{0.025}\frac{S}{\sqrt{n}} &\le \bar{X} - \mu \le t_{0.025}\frac{S}{\sqrt{n}} \\ -\bar{X} - t_{0.025}\frac{S}{\sqrt{n}} &\le -\mu \le -\bar{X} + t_{0.025}\frac{S}{\sqrt{n}} \\ \bar{X} - t_{0.025}\frac{S}{\sqrt{n}} &\le \mu \le \bar{X} + t_{0.025}\frac{S}{\sqrt{n}} \end{aligned}$$

Section 5.6

1. $P(1/2 \le \bar{X} \le 2/3) = P\left(\frac{1/2 - 1/2}{1/12} \le Z \le \frac{2/3 - 1/2}{1/12}\right) = P(0 \le Z \le 2)$
   $= \Phi(2) - \Phi(0) = 0.9772 - 0.5 = 0.4772$

3. $P(2.5 \le \bar{X} \le 4) = P\left(\frac{2.5 - 3}{.5} \le Z \le \frac{4 - 3}{.5}\right) = P(-1 \le Z \le 2)$
   $= \Phi(2) - \Phi(-1) = 0.9772 - 0.1587 = 0.8185$

5. (a) $Y \sim \chi^2(18)$

   (b) To approximate $P(Y \leq 9.390)$ using the central limit theorem, we first observe that Y has a mean $\mu = r = 18$ and a variance $\sigma^2 = 2r = 36$. Therefore,

$$P(Y \leq 9.390) \approx P\left(Z \leq \frac{9.390 - 18}{6/\sqrt{18}}\right) = P(Z \leq -6.088) \approx 0$$

   To approximate $P(Y \leq 34.80)$, we do the same as above and get,

$$P(Y \leq 34.80) \approx P\left(Z \leq \frac{34.80 - 18}{6/\sqrt{18}}\right) = P(Z \leq 11.879) \approx 1$$

   These approximations are fair considering that the exact values are 0.05 and 1. Since χ^2 is a skewed distribution, we require more samples for better approximations.

6. (a)

$$\begin{aligned}
\mu = E(X) &= \int_0^2 x\left(1 - \frac{x}{2}\right) dx \\
&= \int_0^2 x dx - \frac{1}{2}\int_0^2 x^2 dx \\
&= \frac{x^2}{2}\Big|_0^2 - \frac{x^3}{6}\Big|_0^2 \\
&= 2 - \frac{8}{6} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^2 x^2\left(1 - \frac{x}{2}\right) dx \\
&= \int_0^2 x^2 dx - \frac{1}{2}\int_0^2 x^3 dx \\
&= \frac{x^3}{3}\Big|_0^2 - \frac{x^4}{8}\Big|_0^2 \\
&= \frac{8}{3} - 2 \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{2}{3} - \left(\frac{2}{3}\right)^2 \\
&= \frac{2}{9}
\end{aligned}$$

(b)

$$\begin{aligned}
P(2/3 \le \bar{X} \le 5/6) &\approx P\left(\frac{2/3 - 2/3}{\sqrt{2/9}/\sqrt{18}} \le Z \le \frac{5/6 - 2/3}{\sqrt{2/9}/\sqrt{18}}\right) \\
&= P(0 \le Z \le 1.5) \\
&= \Phi(1.5) - \Phi(0) \\
&= 0.9332 - 0.5 \\
&= 0.4332
\end{aligned}$$

7.

$$\begin{aligned}
P(52.761 \le \bar{X} \le 54.453) &\approx P\left(\frac{52.761 - 54.030}{5.8/\sqrt{47}} \le Z \le \frac{54.453 - 54.030}{5.8/\sqrt{47}}\right) \\
&= P(-1.5 \le Z \le 0.5) \\
&= \Phi(0.5) - \Phi(-1.5) \\
&= 0.6915 - 0.0668 \\
&= 0.6247
\end{aligned}$$

12. (a) Assuming independence, we can frame the problem as saying that Y is the sum of the time it takes to sell all 10 tickets. So $Y = X_1 + X_2 + \cdots + X_10$, where $X_i \sim Gamma(\theta = 2, \alpha = 3)$.

We can apply the moment-generating function technique to see that $Y \sim Gamma(\theta = 2, \alpha = 30)$:

$$\begin{aligned}
M_Y(t) &= \prod_{i=1}^{10} M_{X_i}(t) \\
&= \prod_{i=1}^{10} (1-\theta)^{-\alpha} \\
&= (1-\theta)^{-10\alpha}
\end{aligned}$$

Plugging values for θ and α, we get $M_Y(t) = (1-2)^{-30}$, which is the mgf for a Gamma distribution with $\theta = 2$ and $\alpha = 30$.

Therefore, if we wish to find the probability of being sold out within one hour,

$$P(X \le 60) = \int_0^{60} \frac{1}{\Gamma(30)2^{30}} x^{29} e^{-x/2} dx$$

(b) Since X_i follows a Gamma distribution with $\theta = 2$ and $\alpha = 3$, we have $\mu = 6$, and

$\sigma^2 = 12$.

$$\begin{aligned} P(X \le 60) &\approx P\left(\frac{X - 10(6)}{\sqrt{10}\sqrt{12}} \le \frac{60 - 10(6)}{\sqrt{10}\sqrt{12}}\right) \\ &= P(Z \le 0) \\ &= 0.5 \end{aligned}$$

Section 5.7

1. $Y \sim b(25, 1/2)$

   (a) $P(10 < Y \le 12)$
   Table II: $P(Y \le 12) - P(Y \le 10) = 0.5 - 0.2122 = 0.2878$
   Approx: $P\left(\frac{10.5 - 12.5}{5/2} \le Z \le \frac{12.5 - 12.5}{5/2}\right) = P(-0.8 \le Z \le 0)$
   $= \Phi(0) - \Phi(-0.8) = 0.5 - 0.2119 = 0.2881$

   (b) $P(12 \le Y < 15)$
   Table II: $P(Y \le 14) - P(Y \le 11) = 0.7878 - 0.3450 = 0.4428$
   Approx: $P\left(\frac{11.5 - 12.5}{5/2} \le Z \le \frac{14.5 - 12.5}{5/2}\right) = P(-0.4 \le Z \le 0.8)$
   $= \Phi(0.8) - \Phi(-0.4) = 0.7881 - 0.3446 = .4435$

   (c) $P(Y = 12)$
   Table II: $P(Y \le 12) - P(Y \le 11) = 0.5 - 0.3450 = 0.1550$
   Approx: $P\left(\frac{11.5 - 12.5}{5/2} \le Z \le \frac{12.5 - 12.5}{5/2}\right) = P(-0.4 \le Z \le 0)$
   $= \Phi(0) - \Phi(-0.4) = 0.5 - 0.3446 = 0.1554$

2. (a) $P(2 < X < 9) = P(X \le 8) - P(X \le 2) = 0.9532 - 0.0982 = 0.855$

   (b) $P\left(\frac{2.5 - 5}{2} \le Z \le \frac{8.5 - 5}{2}\right) = P(-1.25 \le Z \le 1.75)$
   $= \Phi(1.75) - \Phi(-1.25) = 0.9599 - 0.1056 = 0.8543$

3. $X \sim b(864, 0.6)$, $\mu = 518.4$
   $P(496 \le X \le 548) = P\left(\frac{495.5 - 518.4}{14.4} \le Z \le \frac{548.5 - 518.4}{14.4}\right) = P(-1.59 \le Z \le 2.09)$
   $= \Phi(2.09) - \Phi(-1.59) = 0.9817 - 0.0559 = 0.9258$

7. $X \sim Poisson(49)$, $\mu = 49$
   $P(45 \le X \le 60) = P\left(\frac{45.5 - 49}{7} \le Z \le \frac{59.5 - 49}{7}\right) = P(-0.5 \le Z \le 1.5)$
   $= \Phi(1.5) - \Phi(-0.5) = 0.9332 - 0.3085 = 0.6247$

9. $Y = \sum_{i=1}^{30} X_i \sim Poisson(20)$

   (a) $P(15 < Y \le 22) = P\left(\frac{15.5 - 20}{\sqrt{20}} \le Z \le \frac{22.5 - 20}{\sqrt{20}}\right) = P(-1.01 \le Z \le 0.56)$
   $= \Phi(0.56) - \Phi(-1.01) = 0.7123 - 0.1562 = 0.5561$

(b) $P(21 \leq Y < 27) = P(\left(\dfrac{20.5-20}{\sqrt{20}} \leq Z \leq \dfrac{26.5-20}{\sqrt{20}}\right) = P(0.11 \leq Z \leq 1.45)$
$= \Phi(1.45) - \Phi(0.11) = 0.9265 - 0.5438 = 0.3827$

12. $X \sim b(100, 0.1)$, $\mu = 10$
$P(12 \leq X \leq 14)$

(a) Normal approx:
$P\left(\dfrac{11.5-10}{3} \leq Z \leq \dfrac{14.5-10}{3}\right) = P(0.5 \leq Z \leq 1.5)$
$= \Phi(1.5) - \Phi(0.5) = 0.9332 - 0.6915 = 0.2417$

(b) Poisson approx:
$P(X=12) + P(X=13) + P(X=14) = \dfrac{e^{-10}(10)^{12}}{12!} + \dfrac{e^{-10}(10)^{13}}{13!} + \dfrac{e^{-10}(10)^{14}}{14!}$
$= 0.0948 + 0.0729 + 0.0521 = 0.2198$

(c) Binomial:
$P(X=12) + P(X=13) + P(X=14) = \binom{100}{12}(0.1)^{12}(0.9)^{88} + \binom{100}{13}(0.1)^{13}(0.9)^{87} +$
$\binom{100}{14}(0.1)^{14}(0.9)^{86} = 0.0988 + 0.0743 + 0.0513 = 0.2244$