Homework 12

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Section 4.5

1. (a)

$$
\begin{aligned}
P(-5 < X < 5) &= P\left(\frac{-5-(-3)}{5} < \frac{X-(-3)}{5} < \frac{5-(-3)}{5}\right) \\
&= \Phi\left(\frac{8}{5}\right) - \Phi\left(\frac{-2}{5}\right) \\
&= .9452 - .3446 = 0.6006
\end{aligned}
$$

(b) The conditional pdf of X, given that $Y = 13$, is normal with mean,

$$\mu_X + \rho\left(\frac{\sigma_X}{\sigma_Y}\right)(y - \mu_Y)$$

$$-3 + \left(\frac{3}{5}\right)\left(\frac{5}{3}\right)(13 - 10) = 0$$

and variance,

$$\sigma_X^2(1-\rho^2)$$

$$25\left(1 - \left(\frac{3}{5}\right)^2\right) = 16$$

Therefore,

$$
\begin{aligned}
P(-5 < X < 5 | Y = 13) &= P\left(\frac{-5-0}{4} < \frac{X-0}{4} < \frac{5-0}{4} \,\Big|\, Y = 13\right) \\
&= \Phi\left(\frac{5}{4}\right)\Phi\left(\frac{-5}{4}\right) = 0.8944 - 0.1056 = 0.7888
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(7 < Y < 16) &= P\left(\frac{7-10}{3} < \frac{Y-10}{3} < \frac{16-10}{3}\right) \\
&= \Phi(2) - \Phi(-1) \\
&= 0.9772 - 0.1587 = 0.8185
\end{aligned}
$$

(d) The conditional pdf of Y, given that $X = 2$, is normal with mean,

$$10 + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)(2 - (-3)) = 11.8$$

and variance,

$$9\left(1 - \left(\frac{3}{5}\right)^2\right) = 5.76$$

Therefore,

$$\begin{aligned} P(7 < Y < 16|X = 2) &= P\left(\frac{7 - 11.8}{2.4} < \frac{Y - 11.8}{2.4} < \frac{16 - 11.8}{2.4} \Big| X = 2\right) \\ &= \Phi(1.75) - \Phi(-2) \\ &= 0.9599 - 0.0228 = 0.9371 \end{aligned}$$

2. We have that

$$h(y|x) = \frac{1}{\sigma_Y\sqrt{2\pi}\sqrt{1-\rho^2}} exp\left[-\frac{[y - \mu_Y - \rho(\sigma_Y/\sigma_X)(x - \mu_X)]^2}{2\sigma_Y^2(1-\rho^2)}\right]$$

and that

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right]$$

To add the exponents on e, we start by expanding out the numerator of the first,

$$\begin{aligned} [y - \mu_Y - \rho(\sigma_Y/\sigma_X)(x - \mu_X)]^2 = y^2 - 2y\mu_Y - 2y\rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x - \mu_X) + 2\mu_Y\rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x - \mu_X) \\ + \mu_Y^2 + \rho^2\left(\frac{\sigma_Y}{\sigma_X}\right)^2(x - \mu_X)^2 \end{aligned}$$

TODO: come back to this

We now have to add the exponents on e:

$$-\frac{[y - \mu_Y - \rho(\sigma_Y/\sigma_X)(x - \mu_X)]^2}{2\sigma_Y^2(1-\rho^2)} - \frac{(x-\mu_X)^2}{2\sigma_X^2}$$

4. (a) $E(Y|X = 72) = 80 + \left(\frac{5}{13}\right)\left(\frac{13}{10}\right)(72 - 70) = 81$

   (b) $Var(Y|X = 72) = 169\left(1 - \left(\frac{5}{13}\right)^2\right) = 144$

   (c) $P(Y \leq 84|X = 72) = P\left(\frac{Y - 81}{12} \leq \frac{84 - 81}{12} \Big| X = 72\right) = \Phi\left(\frac{1}{4}\right) = 0.5987$

7. (a)

$$\begin{aligned}P(309.2 < Y < 380.6) &= P\left(\frac{309.2-347}{\sqrt{689}} < \frac{Y-347}{\sqrt{689}} < \frac{380.6-347}{\sqrt{689}}\right)\\ &= \Phi(1.28) - \Phi(-1.44)\\ &= 0.8997 - 0.0749 = 0.8248\end{aligned}$$

(b) $E(Y|x) = 347 + (-0.25)\left(\frac{\sqrt{689}}{\sqrt{611}}\right)(x-415) = 347 + \sqrt{\frac{53}{47}}\left(103.75 - \frac{x}{4}\right)$

(c) $Var(Y|x) = 689(1-(-0.25)^2) = 645.9375$

(d) The mean of Y, given that $X = 385.1$, is

$$347 + \sqrt{\frac{53}{47}}\left(103.75 - \frac{385.1}{4}\right) = 354.9378$$

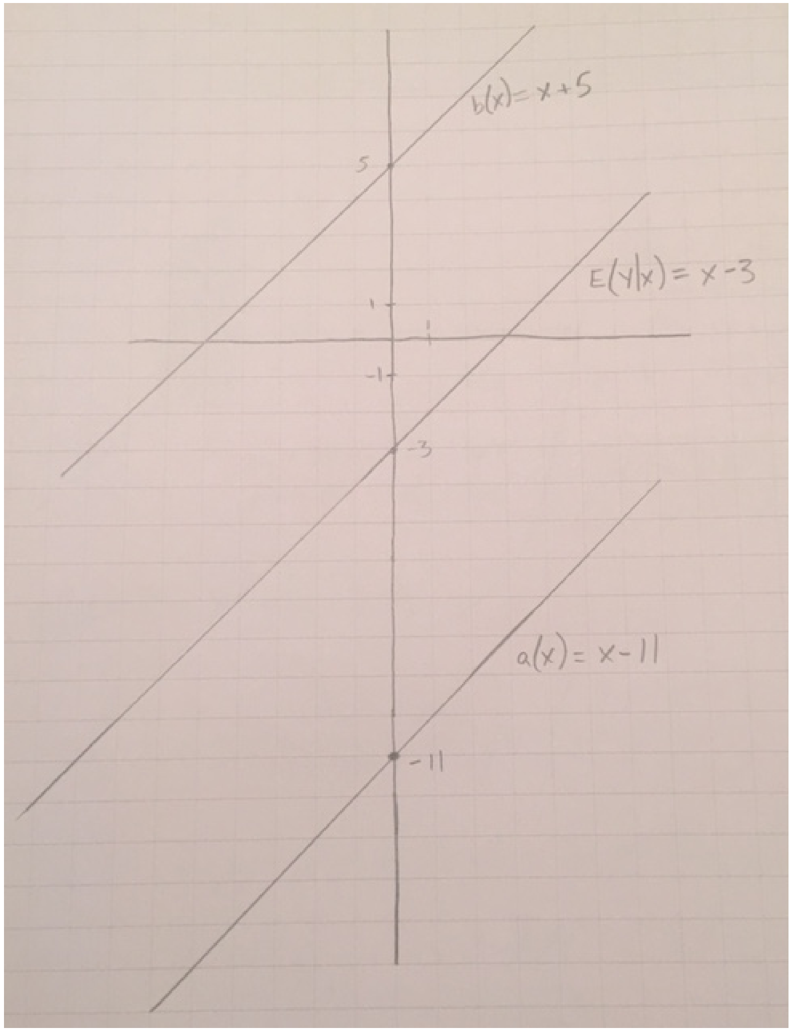
and the variance of Y, given that $X = 385.1$, is

$$645.9375$$

Therefore,

$$\begin{aligned}P(309.2 < Y < 380.6|X = 385.1) &= P\left(\frac{309.2-354.9378}{25.4153} < \frac{Y-354.9378}{25.4153} < \frac{380.6-354.9378}{25.4153}\right)\\ &= \Phi(1.0097) - \Phi(-1.7996)\\ &= 0.8438 - 0.0359 = 0.8079\end{aligned}$$

9. Plot of $E(Y|x) = x - 3$ as well as lines 2 standard deviations above and below: $a(x) = x - 11$ and $b(x) = x + 5$.



Section 5.1

4. (a) $F(x) = \int_0^x 2t\ dt = x^2$ for $0 < x < 1$

   (b) We can simulate observations of X, by first defining $X = F^{-1}(Y) = \sqrt{Y}$, which maps value between 0 and 1 to a value of X. Therefore, we just need to generate random numbers between 0 and 1 and map them back to X values.

   (c) 10 simulated observations of X:

Random number from $U(0,1)$ (Y)	$X = \sqrt{Y}$
0.620355281478075	0.787626359054898
0.146449663756424	0.382687423044479
0.204920874838681	0.45268186935052
0.238088079073957	0.487942700605263
0.433341992859133	0.658287165953532
0.306541892458699	0.553662254861842
0.829008233873843	0.910498892846028
0.84479018798775	0.919124685767796
0.502745995936653	0.709045834863059
0.708640141413394	0.841807662957159

5. We are given that $f(x) = \theta x^{\theta-1}$ for $0 < x < 1, 0 < \theta < \infty$. And $Y = -2\theta \ln X$.

   From this we compute $v(y) = X = e^{-y/2\theta}$.

   Then we compute the derivative, $v'(y) = -\frac{1}{2\theta}e^{-y/2\theta}$

   We then have that

$$\begin{aligned} g(y) &= f[v(y)] \cdot |v'(y)| \\ &= \theta(e^{-y/2\theta})^{\theta-1}\frac{1}{2\theta}e^{-y/2\theta} \\ &= \frac{1}{2}\exp\left[\frac{-\theta y + y}{2\theta} + \frac{-y}{2\theta}\right] \\ &= \frac{1}{2}\exp\left[-\frac{\theta y}{2\theta}\right] \\ &= \frac{1}{2}e^{-y/2} \end{aligned}$$

   Therefore, Y follows an exponential distritbution with mean 2.

6. **TODO: come back to this**

10. We start by breaking up the function into two parts. When $0 < y < 1$, we have, $x_1 = -\sqrt{y}$ for $-1 < x < 0$, and $x_2 = \sqrt{y}$ for $0 < x < 1$.

    We compute the derivatives,

$$\frac{dx_1}{dy} = -\frac{1}{2\sqrt{y}}$$

and

$$\frac{dx_2}{dy} = \frac{1}{2\sqrt{y}}$$

Therefore,

$$g(y) = \begin{cases} \frac{1}{4}\left|\frac{-1}{2\sqrt{y}}\right| + \frac{1}{4}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{4\sqrt{y}} & 0 < y < 1 \\ \frac{1}{4}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{8\sqrt{y}} & 1 < y < 3 \end{cases}$$

14. We break up the function into two parts, $x_1 = -y$ for $-\infty < x_1 < 0$, and $x_2 = y$ for $0 < x_1 < \infty$.

    We compute the derivatives,

$$\frac{dx_1}{dy} = -1$$

and

$$\frac{dx_2}{dy} = 1$$

Therefore,

$$\begin{aligned} g(y) &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(-y)^2}{2}\right] \cdot -1 + \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(y)^2}{2}\right] \cdot 1 \\ &= \frac{1}{\sqrt{2\pi}} \left\{\exp\left[-\frac{y^2}{2}\right] - \exp\left[-\frac{y^2}{2}\right]\right\} \\ &= 0 \end{aligned}$$

Section 5.2

4. (a) $F_{0.05}(9, 24) = 2.30$

   (b) $F_{0.95}(9, 24) = \frac{1}{F_{0.05}(24, 9)} = \frac{1}{2.90} = 0.3448$

   (c) $P(0.277 \le W \le 2.70) = 0.9498$

8. **TODO: come back to this**

   (a)

   (b)

9. $c = \frac{\Gamma(11)}{\Gamma(4)\Gamma(7)} = 840$

11. (a)

$$\begin{aligned}\int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2\ dy &= \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \int_0^{0.4} y^3(1-y)^2\ dy \\ &= \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \int_0^{0.4} y^3 - 2y^4 + y^5\ dy \\ &= \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \left[\frac{y^4}{4} - \frac{2y^5}{5} + \frac{y^6}{6}\right]_0^{0.4} \\ &= \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)}(0.002987) \\ &= 0.1792\end{aligned}$$

(b)

$$\begin{aligned}\sum_{y=4}^{6} \binom{6}{y}(0.4)^y(0.6)^{6-y} &= \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6)^1 + \binom{6}{6}(0.4)^6(0.6)^0 \\ &= 0.13824 + 0.03686 + 0.004096 \\ &= 0.1792\end{aligned}$$