Stat 414 Final Exam

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Start Time: 8:17 am/pm Stop time: 11:17 am/pm

You have 3 hours to complete and 10 minutes to scan/upload (190 min total). You must show all of your work in order to receive full and/or partial credit. No work=No Credit. Tables/software are only allowed if stated in the problem. 10 pages, 76 points

1. 10 points In an experiment, A, B, C and D are events with probabilities $P(A \cup B) = 5/8, P(A) = 3/8, P(C \cap D) = 1/3$ and $P(C) = 1/2$. Furthermore, A and B are disjoint, while C and D are independent.

   (a) 3 points Find $P(B), P(A \cap B')$ and $P(A \cup B')$.

   $P(B) = P(A \cup B) - P(A) + P(A \cap B) = \frac{5}{8} - \frac{3}{8} + 0 = \frac{1}{4}$

   $P(A \cap B') = P(A) = \frac{3}{8}$

   $P(A \cup B') = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$

   (b) 1 points Are A and B independent?

   No, $P(A \cap B) = 0 \neq \frac{3}{8}\left(\frac{1}{4}\right)$

   (c) 4 points Find $P(D), P(C \cap D'), P(C' \cap D')$ and $P(C|D)$.

   $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = 2/3$

   $P(C \cap D') = P(C) - P(C \cap D) = 1/2 - 1/3 = 1/6$

   $P(C' \cap D') = 1 - P(C \cup D) = 1 - [P(C) + P(D) - P(C \cap D)]$
   $= 1 - [1/2 + 2/3 - 1/3] = 1/6$

   $P(C|D) = \frac{P(C \cap D)}{P(D)} = P(C) = 1/2$

(d) 2 points Find $P(C \cup D)$ and $P(C \cup D')$

$$P(C \cup D) = P(C) + P(D) - P(C \cap D) = \left[\tfrac{1}{2} + \tfrac{2}{3} - \tfrac{1}{3}\right] = \underline{\tfrac{5}{6}}$$

$$P(C \cup D') = P(C) + P(D') - P(C \cap D')$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \underline{\frac{2}{3}}$$

2. 3 points A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? (Assume that the probabilities of having a boy and a girl are the same.)

BB
BG
GB
GG

$$P(\text{BOTH BOYS} \mid \text{AT LEAST 1 BOY}) = \frac{P(\text{BOTH BOYS} \cap \text{AT LEAST 1 BOY})}{P(\text{AT LEAST 1 BOY})}$$

$$= \frac{P(\text{BOTH BOYS})}{P(\text{AT LEAST 1})} = \frac{1/4}{3/4} = \underline{1/3}$$

3. 3 points A company has three machines B_1, B_2 and B_3 for making resistors. It has been observed that 80% of resistors produced by B_1 meet a certain specification. 90% of resistors produced by machine B_2 meet the specification and 60% of resistors produced by machine B_3 meet the specification. Machine B_1 produces 30% of all of the company's resistors, B_2 produces 40% and B_3 produces the remaining 40%. What is the probability that a randomly selected resistor manufactured by this company meets the specification?

ASSUMING 30%

M = MEETS SPECIFICATION

$$P(M) = P(M \mid B_1)P(B_1) + P(M \mid B_2)P(B_2) + P(M \mid B_3)P(B_3)$$

$$= .8(.3) + .9(.4) + .6(.3)$$

$$= \underline{0.78}$$

4. 4 points An arrow is fired at random into a circle of radius 8 inches. If it lands within 1 inch of the center, you win \$10. If it lands between 1 and 3 inches from the center, you win \$5. If it lands between 3 and 5 inches from the center, you win \$2. Otherwise, you lose \$4. Find your expected winnings. You may assume the probability of landing any point in the circle is the same.
Hint: Recall that the area of a circle with radius r is given by πr^2.

x	$P(x)$
10	$\pi/64\pi = \frac{1}{64}$
5	$8\pi/64\pi = 1/8$
2	$16\pi/64\pi = 1/4$
-4	$39\pi/64\pi = 39/64$

$$E(X) = 10\left(\tfrac{1}{64}\right) + 5\left(\tfrac{1}{8}\right) + 2\left(\tfrac{1}{4}\right) - 4\left(\tfrac{39}{64}\right)$$

$$= -1.15625$$

5. 8 points X is a random variable with moment generating function (MGF), $M(t)$. Let Y be a function of X defined by $Y = aX + b$, where a and b are constants.

(a) 2 points **Using the definition of a MGF**, show that the MGF for Y would be $e^{tb}M(at)$.

$$M_Y(t) = E\left[e^{Yt}\right] = E\left[e^{(aX+b)t}\right] = E\left[e^{aXt}e^{bt}\right] = e^{tb}E\left[e^{aXt}\right]$$

$$= e^{tb}M_X(at)$$

(b) 3 points **Using the results from (a) and the properties of the MGF**, prove that $E(Y) = aE(X) + b$.

$$M_Y'(t) = e^{tb} \cdot M_X'(at) \cdot a + M_X(at) b e^{tb}$$

$$M_Y'(0) = aM_X'(0) + M_X(0)b$$

$$= aE(X) + b$$

(c) 3 points **Using the results from (a) and (b) and the properties of the MGF**, prove that $\text{Var}(Y) = a^2\text{Var}(X)$.

$$M_Y''(t) = ae^{tb} \cdot M_X''(at) \cdot a + abe^{tb} M_X'(at) + M_X(at) b^2 e^{tb} + M_X'(at) abe^{tb}$$

$$M_Y''(0) = E(Y^2) = a^2 E(X^2) + ab\,E(X) + b^2 + ab\,E(X)$$

$$Var(Y) = M_Y''(0) - [M_Y'(0)]^2 = a^2E(X^2) + abE(X) + b^2 + abE(X) - [aE(X) + b]^2$$

$$= a^2 E(X^2) - a^2[E(X)]^2 = a^2[E(X^2) - [E(X)]^2] = a^2 Var(X)$$

6. 3 points Customers arrive at a checkout counter in a certain store an average of eight per hour.

(a) 1 point For a given hour, find the probability that at least one customer arrives. $\lambda = 8$

$$P(X \geq 1) = 1 - P(0) = 1 - \frac{e^{-8} 8^0}{0!} = 1 - e^{-8} = 0.9997$$

(b) 1 point Find the probability that exactly two customers arrive in one continuous two hour period. $\lambda = 16$

$$P(X = 2) = \frac{e^{-16} 16^2}{2!} = 0.000014$$

(c) 1 point Find the probability that at least two customers arrive in one continuous two hour period.

$$P(X \geq 2) = 1 - [P(0) + P(1) + \cancel{P(2)}] = 1 - \left[\frac{e^{-16} 16^0}{0!} + \frac{e^{-16} 16^1}{1!}\right] + \cancel{\frac{e^{-16}16^2}{}}$$

$$= 1 - [e^{-16} + 16e^{-16}]$$

$$= 0.999998$$

7. 5 points Suppose that X is a random variable for which $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

(a) 2 points Show that $E[X(X-1)] = \mu(\mu-1) + \sigma^2$

$$E[X(X-1)] = E[X^2 - X] = E(X^2) - E(X) = \sigma^2 + \mu^2 - \mu$$
$$= \mu(\mu-1) + \sigma^2$$

(b) 3 points Show that $E[(X-c)^2] = (\mu-c)^2 + \sigma^2$ where c is an arbitrary constant.

$$E[(X-c)^2] = E[X^2 - 2Xc + c^2] = E(X^2) - 2cE(X) + c^2$$
$$= \sigma^2 + \mu^2 - 2c\mu + c^2$$
$$= (\mu - c)^2 + \sigma^2$$

8. 6 points Let X be a continuous random variable with probability density function (PDF) of

$$f(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) 1 point Find the constant c, such that $f(x)$ is a valid PDF.

$$1 \overset{\text{SET}}{=} \int_{-1}^{1} c(1-x^2)\,dx = c\left[\int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx\right] = c\left[2 - \frac{2}{3}\right]$$
$$= c\,\frac{4}{3}$$

THEREFORE, $C = \frac{3}{4}$

(b) 1 point Find the cumulative distribution function (CDF) of x, $F(x)$.

$$F(x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt = \frac{3}{4}\left[\int_{-1}^{x} 1\,dt - \int_{-1}^{x} t^2\,dt\right] = \frac{3}{4}\left[x + 1 - \left(\frac{x^3+1}{3}\right)\right]$$

$$\frac{3}{4}\left[\frac{3x + 3 - x^3 - 1}{3}\right] = \frac{3x - x^3 + 2}{4}, \quad -1 \le x \le 1$$

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{3x - x^3 + 2}{4}, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

(c) 1 points Find $P(X > 0.5)$.

$$P(X > 0.5) = \int_{.5}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[\int_{.5}^{1} 1\,dx - \int_{.5}^{1} x^2\,dx\right]$$

$$= \frac{3}{4}\left[(1-.5) - \left(\frac{1}{3} - \frac{.5^3}{3}\right)\right] = 0.15625$$

(d) 3 points Find the expectation and variance of X.

$$E(x) = \int_{-1}^{1} \frac{3}{4}x(1-x^2)\,dx = \frac{3}{4}\left[\int_{-1}^{1} x\,dx - \int_{-1}^{1} x^3\,dx\right] = \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{4}\right)\right]$$

$$= 0$$

$$E(x^2) = \int_{-1}^{1} \frac{3}{4}x^2(1-x^2)\,dx = \frac{3}{4}\left[\int_{-1}^{1} x^2\,dx - \int_{-1}^{1} x^4\,dx\right] = \frac{3}{4}\left[\left(\frac{1}{3}+\frac{1}{3}\right) - \left(\frac{1}{5}+\frac{1}{5}\right)\right]$$

$$= \frac{1}{5}$$

$$Var(x) = \frac{1}{5} - 0^2 = \frac{1}{5}$$

9. 3 points Suppose X has an exponential distribution. If $P(X \le 1) = P(X > 1)$, solve for the variance of X.

$$\int_0^1 \frac{1}{\theta} e^{-x/\theta}\,dx = \int_1^{\infty} \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$\frac{1}{\theta}\cdot -\theta \cdot e^{-x/\theta}\Big|_0^1 = \frac{1}{\theta}\cdot -\theta \cdot e^{-x/\theta}\Big|_1^{\infty}$$

$$-e^{-1/\theta} + e^0 = \lim_{b\to\infty}\left[-e^{-b/\theta} + e^{-1/\theta}\right]$$

$$-e^{-1/\theta} + 1 = 0 + e^{-1/\theta}$$

$$1 = 2e^{-1/\theta}$$

$$\frac{1}{2} = e^{-1/\theta}$$

$$\ln\frac{1}{2} = -\frac{1}{\theta}$$

$$\theta = -\frac{1}{\ln\frac{1}{2}} = \frac{1}{\ln(2)}$$

$$\sigma^2 = \theta^2 = \frac{1}{[\ln(2)]^2}$$

10. 11 points Let X and Y have the following joint PMF, where the rows are X and the columns are Y.

$f(x,y)$	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

(a) 2 points Find the marginal distributions.

x	$f(x)$
0	4/9
1	4/9
2	1/9

y	$f(y)$
0	4/9
1	4/9
2	1/9

(b) 2 points Find the conditional distribution of X given Y.

$g(x\|y)$		Y: 0	1	2
X	0	1/4	1/2	1
	1	1/2	1/2	0
	2	1/4	0	0

(c) 3 points Find the conditional expectation of X given Y.

$$E(X|Y=0) = \sum_x x\, g(x|y=0) = 0\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{4}\right) = 1$$

$$E(X|Y=1) = 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right) + 2(0) = \tfrac{1}{2}$$

$$E(X|Y=2) = 0(1) + 1(0) + 2(0) = 0$$

(d) 3 points Find the conditional variance of X given Y.

$$\sigma^2_{X|Y=0} = \sum_x (x-1)^2 g(x|y=0) = 1\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right) = \tfrac{1}{2}$$

$$\sigma^2_{X|Y=1} = \sum_x \left(x-\tfrac{1}{2}\right)^2 g(x|y=1) = \tfrac{1}{4}\left(\tfrac{1}{2}\right) + \tfrac{1}{4}\left(\tfrac{1}{2}\right) + \tfrac{9}{4}(0) = \tfrac{1}{4}$$

$$\sigma^2_{X|Y=2} = \sum_x (x-0)^2 g(x|y=2) = 0(1) + 1(0) + 4(0) = 0$$

(e) 4 points Find the correlation of the two variables.

$\mu_X = \sum x f(x,y) = 0(\frac{1}{9}) + 0(\frac{2}{9}) + 0(\frac{1}{9}) + 1(\frac{2}{9}) + 1(\frac{2}{9}) + 1(0) + 2(\frac{1}{9}) + 2(0) + 2(0) = \frac{2}{3}$

$\mu_Y = \sum y f(x,y) = 0(\frac{1}{9}) + 0(\frac{2}{9}) + 0(\frac{1}{9}) + 1(\frac{2}{9}) + 1(\frac{2}{9}) + 1(0) + 2(\frac{1}{9}) + 2(0) + 2(0) = \frac{2}{3}$

$\sigma_X^2 = \sum x^2 f(x,y) - \mu_X^2 = \left[1(\frac{2}{9}) + 1(\frac{2}{9}) + 4(\frac{1}{9})\right] - \frac{4}{9} = \frac{4}{9}$

$\sigma_Y^2 = \sum y^2 f(x,y) - \mu_Y^2 = \frac{4}{9}$

$E(XY) = \sum xy f(x,y) = 1(\frac{2}{9}) = \frac{2}{9}$

$Cov(X,Y) = E(XY) - \mu_X\mu_Y = \frac{2}{9} - \frac{2}{3}(\frac{2}{3}) = \frac{-2}{9}$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-2/9}{\sqrt{4/9 \cdot 4/9}} = \frac{-2}{9} \cdot \frac{9}{4} = -\frac{1}{2}$

11. 4 points Let X have the probability density function given by

$$f(x) = \frac{x+1}{2}, \qquad -1 \le x \le 1$$

and zero otherwise. Find the cumulative distribution function (CDF) and probability density function (PDF) for $W = X^2$. You must also include the support.

$G(w) = P(W \le w) = P(X^2 \le w) = P(-\sqrt{w} \le X \le \sqrt{w})$

$= \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{2}(x+1)dx = \frac{x^2}{4} + \frac{x}{2}\Big|_{-\sqrt{w}}^{\sqrt{w}} = \sqrt{w} \qquad 0 \le w \le 1$

$G'(w) = g(w) = f(\sqrt{w})\left|\frac{1}{2\sqrt{w}}\right| + f(-\sqrt{w})\left|\frac{-1}{2\sqrt{w}}\right|$

$= \frac{\sqrt{w}+1}{2} \cdot \frac{1}{2\sqrt{w}} + \frac{-\sqrt{w}+1}{2} \cdot \frac{1}{2\sqrt{w}} = \frac{2}{4\sqrt{w}} = \frac{1}{2\sqrt{w}}$

$0 \le w \le 1$

12. 4 points The joint probability mass function of X and Y is

$$f(x,y) = \binom{3}{x}\left(\frac{1}{3}\right)^{x+1}\left(\frac{2}{3}\right)^{3-x}$$

for $x = 0,1,2,3$ and $y = 0,1,2$. Are X and Y independent? You must prove mathematically.

$$f_X(x) = \sum_y f(x,y) = 3\left[\binom{3}{x}\left(\frac{1}{3}\right)^{x+1}\left(\frac{2}{3}\right)^{3-x}\right]$$

$$f(y) = \sum_x f(x,y) = \binom{3}{0}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + \binom{3}{1}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 + \binom{3}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + \binom{3}{3}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = \frac{1}{3}$$

$$\cancel{f(0,0)} = \quad f(x,y) = f(x)\,f(y) = 3\left[\binom{3}{x}\left(\frac{1}{3}\right)^{x+1}\left(\frac{2}{3}\right)^{3-x}\right]\cdot\frac{1}{3}$$

$$= \binom{3}{x}\left(\frac{1}{3}\right)^{x+1}\left(\frac{2}{3}\right)^{3-x}$$

13. 3 points X and Y are independent exponential random variables with mean μ_X and μ_Y respectively. Find the MGF of $Z = X - Y$.

$$M_Z(t) = M_X(t)\cdot M_Y(-t)$$

$$= \frac{1}{(1-\mu_X t)}\cdot\frac{1}{(1+\mu_Y t)} = \frac{1}{1+\mu_Y t - \mu_X t - \mu_X\mu_Y t^2}$$

14. 6 points Let X_1, X_2, and X_3 are independent random variables with common PDF,

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = \min(X_1, X_2, X_3)$.

(a) 4 points Find the CDF of Y, $F_Y(y)$.

$$F_Y(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - P(X_1 > y, X_2 > y, X_3 > y)$$

$$= 1 - P(X_1 > y) P(X_2 > y) P(X_3 > y)$$

$$= 1 - \int_y^\infty 2e^{-2x} dx \int_y^\infty 2e^{-2x} dx \int_y^\infty 2e^{-2x} dx$$

$$= 1 - \left[\int_y^\infty 2e^{-2x} dx \right]^3$$

$$= 1 - e^{-6y}$$

(b) 2 points Find the PDF of Y, $f_Y(y)$. What is the distribution of Y?

$$F_Y'(y) = f_Y(y) = 6e^{-6y}$$