Homework 4

Roly Vicaría
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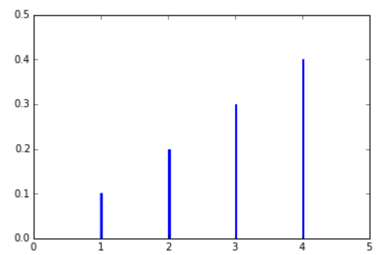
Section 2.1

3. (a) We set the sum of $f(x)$ over all x equal to 1,

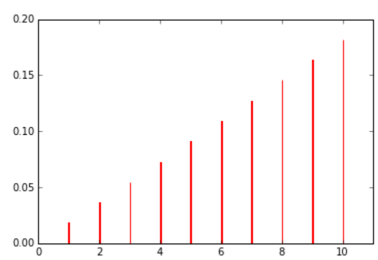
$$\begin{aligned} 1 &= \frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} \\ &= \frac{10}{c} \end{aligned}$$

Therefore, $c = 10$.



(b) $1 = \sum_{x=1}^{10} cx = 1c + 2c + 3c + \cdots + 10c = 55c$

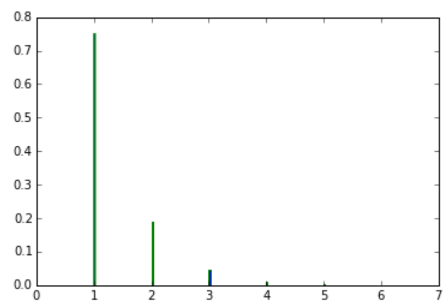
Therefore, $c = \frac{1}{55}$



(c)

$$
\begin{aligned}
1 &= \sum_{x=1}^{\infty} c\left(\frac{1}{4}\right)^x \\
&= c\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x \\
&= c\left[\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right] \\
&= c\left[\frac{\frac{1}{4}}{1-\frac{1}{4}}\right] \\
&= c\left(\frac{1}{3}\right)
\end{aligned}
$$

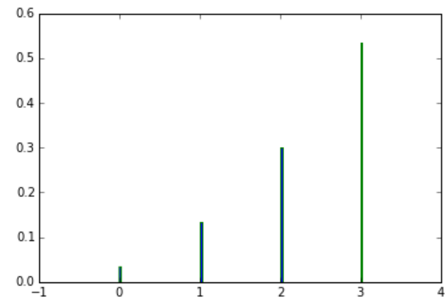
Therefore, $c = 3$.



(d)

$$
\begin{aligned}
1 &= \sum_{x=0}^{3} c(x+1)^2 \\
&= c\sum_{x=0}^{3}(x+1)^2 \\
&= c\left[(0+1)^2+(1+1)^2+(2+1)^2+(3+1)^2\right] \\
&= c\left[1+4+9+16\right] \\
&= 30c
\end{aligned}
$$

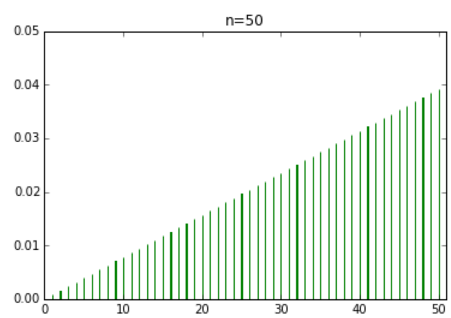
Therefore, $c = \frac{1}{30}$.



(e)

$$
\begin{aligned}
1 &= \sum_{x=1}^{n} \frac{x}{c} \\
&= \frac{1}{c} \sum_{x=1}^{n} x \\
&= \frac{1}{c}(1 + 2 + \cdots + n) \\
&= \frac{1}{c} \cdot \frac{n(n+1)}{2}
\end{aligned}
$$

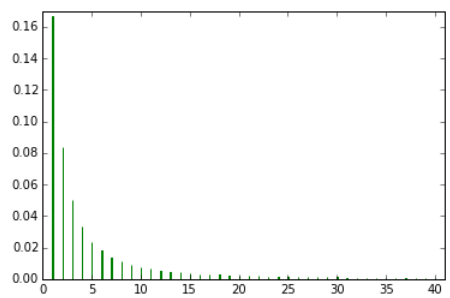
Therefore, $c = \dfrac{n(n+1)}{2}$.



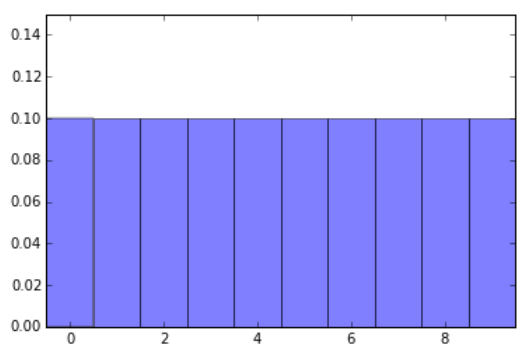
(f)

$$
\begin{aligned}
1 &= \sum_{x=0}^{\infty} \frac{c}{(x+1)(x+2)} \\
&= c\sum_{x=0}^{\infty} \frac{1}{(x+1)(x+2)} \\
&= c\sum_{x=0}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \\
&= c\left[\left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots\right] \\
&= c(1)
\end{aligned}
$$

Therefore, $c = 1$.



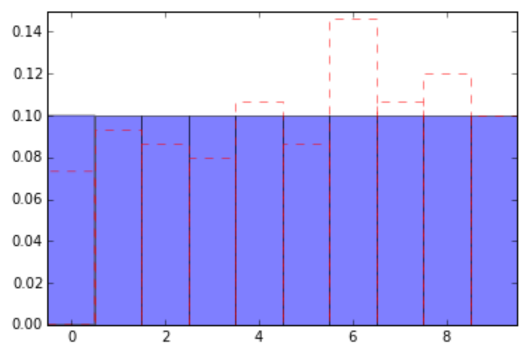
4. (a) The pmf of X for true random numbers is $f(x) = P(X = x) = \dfrac{1}{10}$ for $x = 0, 1, 2, \ldots, 9$.



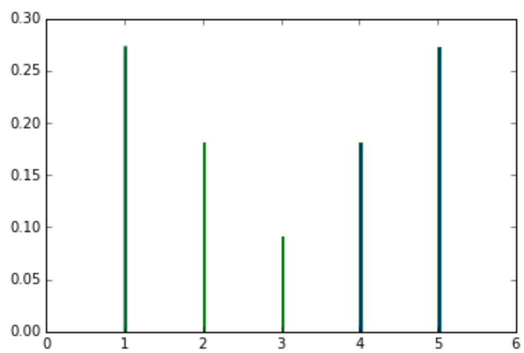
(b) Relative frequencies of integers

Integer	Relative Frequency
0	11/150 (.0733)
1	14/150 (.0933)
2	13/150 (.0867)
3	12/150 (.0800)
4	16/150 (.1067)
5	13/150 (.0867)
6	22/150 (.1467)
7	16/150 (.1067)
8	18/150 (.1200)
9	15/150 (.1000)

(c) Relative frequencies histogram in red dotted line over probability histogram:



9. $f(x) = \dfrac{(1 + |x - 3|)}{11}$



10. (a) $P(X=1) = \dfrac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} = 0.3980$

(b) $P(X \le 1) = P(X=0)+P(X=1) = \dfrac{\binom{3}{0}\binom{47}{10}}{\binom{50}{10}} + \dfrac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} = 0.5041+0.3980 = 0.9021$

11.

$$\begin{aligned} P(X \ge 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}} \\ &= 1 - 0.5838 \\ &= 0.4162 \end{aligned}$$

Section 2.2

1. (a) $E(X) = \sum_{x=1}^{4} x \dfrac{x}{10} = \dfrac{1}{10}[1^2 + 2^2 + 3^2 + 4^2] = 3$

(b) $E(X) = \sum_{x=1}^{10} x \dfrac{x}{55} = \dfrac{1}{55}[1^2 + 2^2 + \cdots + 10^2] = \dfrac{385}{55} = 7$

(c)

$$\begin{aligned} E(X) - \frac{1}{4}E(X) &= \sum_{x=1}^{\infty} x \cdot 3\left(\frac{1}{4}\right)^x - \sum_{x=1}^{\infty} x\left(\frac{1}{4}\right)3\left(\frac{1}{4}\right)^x \\ \frac{3}{4}E(X) &= \left[3(1)\left(\frac{1}{4}\right)^1 + 3(2)\left(\frac{1}{4}\right)^2 + 3(3)\left(\frac{1}{4}\right)^3 + \cdots\right] - \\ &\quad \left[3(1)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^1 + 3(2)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 + 3(3)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^3 + \cdots\right] \\ &= 3\left\{\left[1\left(\frac{1}{4}\right)^1 + 2\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right)^3 + \cdots\right] - \left[1\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^4 + \cdots\right]\right\} \\ &= 3\left[1\left(\frac{1}{4}\right)^1 + 1\left(\frac{1}{4}\right)^2 + 1\left(\frac{1}{4}\right)^3 + \cdots\right] \\ &= 3\left(\frac{1}{3}\right) = 1 \end{aligned}$$

Therefore, $E(X) = \dfrac{4}{3}$

(d)

$$\begin{aligned}
E(X) &= \sum_{x=0}^{3} x \frac{(x+1)^2}{30} \\
&= 0\left(\frac{(0+1)^2}{30}\right) + 1\left(\frac{(1+1)^2}{30}\right) + 2\left(\frac{(2+1)^2}{30}\right) + 3\left(\frac{(3+1)^2}{30}\right) \\
&= 0 + \frac{4}{30} + \frac{18}{30} + \frac{48}{30} \\
&= \frac{70}{30} = \frac{7}{3}
\end{aligned}$$

(e)

$$\begin{aligned}
E(X) &= \sum_{x=1}^{n} x \cdot \frac{x}{\frac{n(n+1)}{2}} = \sum_{x=1}^{n} \frac{2x^2}{n(n+1)} \\
&= \left[\frac{2(1)^2}{n(n+1)} + \frac{2(2)^2}{n(n+1)} + \frac{2(3)^2}{n(n+1)} + \cdots + \frac{2n^2}{n(n+1)}\right] \\
&= 2\left[\frac{n(n+1)(2n+1)}{6n(n+1)}\right] \\
&= \frac{2n+1}{3}
\end{aligned}$$

(f)

$$\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x \cdot \frac{1}{(x+1)(x+2)} \\
&= \sum_{x=0}^{\infty} x \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \\
&= \left[0\left(\frac{1}{1} - \frac{1}{2}\right) + 1\left(\frac{1}{2} - \frac{1}{3}\right) + 2\left(\frac{1}{3} - \frac{1}{4}\right) + 3\left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] \\
&= \left[0 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right] \\
&= \sum_{x=2}^{\infty} \frac{1}{x}
\end{aligned}$$

The sum $\displaystyle\sum_{x=2}^{\infty} \frac{1}{x}$ does not converge to a finite value. Therefore $E(X)$ does not exist.

2.

$$
\begin{aligned}
E(X) &= \sum_{x=-1}^{1} x \cdot \frac{(|x|+1)^2}{9} \\
&= (-1)\left(\frac{(|-1|+1)^2}{9}\right) + (0)\left(\frac{(|0|+1)^2}{9}\right) + (1)\left(\frac{(|1|+1)^2}{9}\right) \\
&= -\frac{4}{9} + 0 + \frac{4}{9} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \sum_{x=-1}^{1} x^2 \cdot \frac{(|x|+1)^2}{9} \\
&= (-1)^2\left(\frac{(|-1|+1)^2}{9}\right) + (0)^2\left(\frac{(|0|+1)^2}{9}\right) + (1)^2\left(\frac{(|1|+1)^2}{9}\right) \\
&= \frac{4}{9} + 0 + \frac{4}{9} \\
&= \frac{8}{9}
\end{aligned}
$$

$$
\begin{aligned}
E(3X^2 - 2X + 4) &= 3E(X^2) - 2E(X) + 4 \\
&= 3\left(\frac{8}{9}\right) - 2(0) + 4 \\
&= \frac{8}{3} + 4 \\
&= \frac{20}{3}
\end{aligned}
$$

4.

$$
\begin{aligned}
0.1 &= \sum_{x=1}^{6} \frac{c}{x} \\
&= \left(\frac{c}{1} + \frac{c}{2} + \frac{c}{3} + \frac{c}{4} + \frac{c}{5} + \frac{c}{6}\right) \\
&= \frac{147c}{60} = \frac{49c}{20}
\end{aligned}
$$

Therefore, $c = \dfrac{2}{49}$

$$
\begin{aligned}
E(X) &= \sum_{x=1}^{6} x \cdot \frac{2}{49x} = \sum_{x=1}^{6} \frac{2}{49} \\
&= 6 \cdot \frac{2}{49} \\
&= \frac{12}{49}
\end{aligned}
$$

Subtracting 1 for the deductible:

$$E(X) - 1 = \frac{12}{49} - \frac{49}{49} = -\frac{37}{49}$$

5. (a) pmf of Z, $h(z) = \dfrac{(4 - \sqrt[3]{z})}{6}$ for $z = 1, 8, 27$

   (b) $E(Z) = (1)\left(\dfrac{3}{6}\right) + (8)\left(\dfrac{2}{6}\right) + (27)\left(\dfrac{1}{6}\right) = \dfrac{46}{6} = \dfrac{23}{3}$

   (c) He can expect to make $10 - \dfrac{23}{3} = \dfrac{7}{3}$ of a dollar on the average each play.

6.

$$
\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x \cdot \frac{6}{\pi^2 x^2} \\
&= \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x}
\end{aligned}
$$

The sum $\sum_{x=1}^{\infty} \frac{1}{x}$ does not converge to a finite value. Therefore $E(X)$ does not exist.

9. (a) $E(X) = (-1)\left(\dfrac{20}{38}\right) + (1)\left(\dfrac{18}{38}\right) = -\dfrac{1}{19}$

   (b) $E(X) = (-1)\left(\dfrac{19}{37}\right) + (1)\left(\dfrac{18}{37}\right) = -\dfrac{1}{37}$

12. (a) Average class size: $\dfrac{16(25) + 3(100) + 1(300)}{20} = \dfrac{1000}{20} = 50$

    (b) pmf of X:

$$f(x) = \begin{cases} \frac{16}{20} & x = 25, \\ \frac{3}{20} & x = 100, \\ \frac{1}{20} & x = 300, \\ 0 & \text{all other x} \end{cases}$$

(c) $E(X) = (25)\left(\frac{16}{20}\right) + (100)\left(\frac{3}{20}\right) + (300)\left(\frac{1}{20}\right) = 50$

No, this answer does not surprise me.