Stat 414 Exam #4

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Start Time: 10:08 am/pm Stop time: 10:36 am/pm

You have 1 hour 30 min to complete and 10 minutes to scan/upload. You must show all of your work in order to receive full and/or partial credit. No work=No Credit. Tables/software are only allowed if stated in the problem. 5 pages, 30 points

1. 4 points Suppose that X and Y have a discrete joint distribution for which the joint pdf is defined below.

$$f(x,y) = \begin{cases} c|x+y| & \text{for } x=-2,-1,0,1,2 \text{ and } y=-2,-1,0,1,2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) 1 point Find the value of the constant c.

$$\boxed{c = \frac{1}{40}}$$

$$1 = \sum_x \sum_y c|x+y| = c \sum_x \sum_y |x+y| = c[(-2+-2)+(-2+-1)+(-2+0)$$
$$+(-2+1)+(-2+2)+(-1+-2)+(-1+-1)+(-1+0)+(-1+1)+(-1+2)$$
$$+(0+-2)+(0+-1)+(0+0)+(0+1)+(0+2)+(1+-2)+(1+-1)+(1+0)$$
$$+(1+1)+(1+2)+(2+-2)+(2+-1)+(2+0)+(2+1)+(2+2)] = c40$$

(b) 1 point Find $P(X=1)$.

$$f_X(x) = \sum_y \frac{1}{40}|x+y| = \frac{1}{40}[|x+-2|+|x+-1|+|x+0|+|x+1|+|x+2|]$$

$$f_X(1) = \frac{1}{40}[7] = \boxed{\frac{7}{40}}$$

(c) 2 points Find $P(|X-Y| \le 1)$.

~~P(-1 ≤ X~~ $P(-1 \le X - Y \le 1) = \frac{13}{25}$ (BY COUNTING)

$(1,2)\ (1,3)\ (2,3)\ (2,1)\ (3,1)\ (3,2)$

2. 1 points A box contains 3 slips of paper numbered 1, 2, 3. You make two draws without replacement from the box. Let X denote the first number drawn and let Y be the second number drawn. Find $Cov(X, Y)$.

$$\mu_X = \sum x\, f(x,y) = 1\left(\tfrac{1}{6}\right) + 1\left(\tfrac{1}{6}\right) + 2\left(\tfrac{1}{6}\right) + 2\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right)$$

$$= 2$$

$$\mu_Y = \sum y\, f(x,y) = 2\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right) + 1\left(\tfrac{1}{6}\right) + 1\left(\tfrac{1}{6}\right) + 2\left(\tfrac{1}{6}\right)$$

$$= 2$$

$$E(XY) = \sum xy\, f(x,y) = 1(2)\left(\tfrac{1}{6}\right) + 1(3)\left(\tfrac{1}{6}\right) + 2(3)\left(\tfrac{1}{6}\right) + 2(1)\left(\tfrac{1}{6}\right) + 3(1)\left(\tfrac{1}{6}\right) + 3(2)\left(\tfrac{1}{6}\right)$$

$$= \frac{22}{6}$$

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y = \frac{22}{6} - 4 = -\frac{1}{3}$$

3. 7 points Let (X, Y) have the following joint probability density function

$$f(x,y) = \begin{cases} 4xy & \text{if } 0 \le x \le 1,\ 0 \le y \le 1,\ x \ge y \\ 6x^2 & \text{if } 0 \le x \le 1,\ 0 \le y \le 1,\ x < y \\ 0 & \text{otherwise} \end{cases}.$$

(a) 2 points Find the marginal probability density function for X.

$$\cancel{f_X(x) = \int_0^x 4xy\,dy + \int_x^1 6x^2\,dy = \frac{4xy^2}{2}\Big|_0^x + 6x^2y\Big|_x^1}$$

$$\cancel{= \frac{4x^3}{2} + 6x^2 - 6x^3}$$

$$f_X(x) = \begin{cases} 4x^3/2 & 0 < y \le x \le 1 \\ 6x^2 - 6x^3 & 0 \le x < y \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

(b) 2 points Find the marginal probability density function for Y.

$$f_Y(y) = \int_y^1 4xy\,dx = 2x^2y\Big|_y^1 = 2y - 2y^3$$

$$f_Y(y) = \int_0^y 6x^2\,dx = 2x^3\Big|_0^y = 2y^3$$

$$f_Y(y) = \begin{cases} 2y - 2y^3 & 0 < y \le x \le 1 \\ 2y^3 & 0 \le x < y \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

(c) 3 points If $A = \{X \le \frac{1}{2}\}$ and $B = \{Y \le \frac{1}{2}\}$, find $P(A \cup B)$.

$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \int_0^{1/2}\int_0^{1/2} \frac{1}{2\pi \frac{1}{\sqrt{2}} \frac{1}{2} \sqrt{1-\frac{1}{2}}} \exp\left[-(2x^2 - 4xy + 4y^2)\right] dx\,dy$$

4. 5 points Let X and Y are continuous random variables having the following joint density.

$$f(x,y) = \begin{cases} cx^2y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) 1 point Find the constant c.

$$1 = \int_0^1\int_0^1 cx^2y\,dx\,dy = \int_0^1 cy\left[\frac{x^3}{3}\right]_0^1 dy = \frac{c}{3}\int_0^1 y\,dy$$

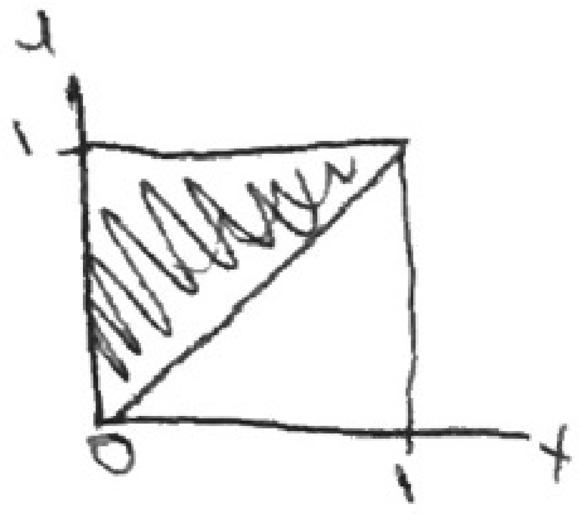
$$= \frac{c}{3}\left[\frac{y^2}{2}\right]_0^1 = \frac{c}{6} \Rightarrow c = 6$$

(b) 2 points Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^1 6x^2y\,dy = 6x^2\left[\frac{y^2}{2}\right]_0^1 = 3x^2 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

$$f_Y(y) = \int_0^1 6x^2y\,dx = 6y\left[\frac{x^3}{3}\right]_0^1 = 2y \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

$0<x<y<1$



(c) 2 points Find $P(X < Y)$.

$P(X<Y) = \cancel{\int_0^y \int_0^1 6x^2y\,dy\,dx = \int_0^y 6x^2\left[\frac{y^2}{2}\right]_0^1 dx}$

$\cancel{= \int_0^y 3x^2\,dx = x^3\Big|_0^y = y^3}$

$\int_0^1\int_0^y 6x^2y\,dx\,dy = \int_0^1 2x^3y\Big|_0^y dy \qquad \int_0^1 2y^4\,dy = \frac{2y^5}{5}\Big|_0^1 = \boxed{\frac{2}{5}}$

5. 4 points Random variables X and Y have a bivariate normal joint pdf

$$f(x,y) = ce^{-(2x^2-4xy+4y^2)}$$

where $-\infty < x < \infty, -\infty < y < \infty$.

(a) 3 points What are $E(X)$ and $E(Y)$?

$e^{-(2x^2-4xy+4y^2)} = e^{-\frac{q(x,y)}{2}}$

$q(x,y) = 4x^2 - 8xy + 8y^2 = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$

$\cancel{4(x^2-2xy+2y^2)}$

$2(2x^2-4xy+4y^2) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$

$2 = \frac{1}{1-\rho^2} \Rightarrow \rho = \frac{1}{\sqrt{2}}$

$\left(\frac{x-\mu_x}{\sigma_x}\right)^2 = 2x^2 \Rightarrow \boxed{\mu_x = 0} \quad \sigma_x = \frac{1}{\sqrt{2}}, \quad \left(\frac{y-\mu_y}{\sigma_y}\right)^2 = 4y^2 \Rightarrow \boxed{\mu_y = 0} \quad \sigma_y = \frac{1}{2}$

(b) 1 point Find ρ, the correlation coefficient of X and Y.

$\rho = \frac{1}{\sqrt{2}}$

6. 3 points The continuous random variable X has a PDF, $f_X(x)$, and a CDF, $F_X(x)$. Let U be a random variable such that $U = X^2$. Prove the PDF of U in terms of $f_X(x)$.

~~ASSUMING STRICTLY INCREASING FOR SIMPLICITY~~

$\cancel{v(u) = x = \sqrt{u}}$

$\cancel{g(u) = f_x(-\sqrt{u}) \cdot \frac{1}{2\sqrt{u}}}$

$$
\begin{aligned}
F(U \leq u) &= F(X^2 \leq u) \\
&= F(-\sqrt{u} \leq X \leq \sqrt{u}) \\
&= F(X \leq \sqrt{u}) - (1 - F(X \leq -\sqrt{u})) \\
&= F(X \leq \sqrt{u}) + F(X \leq -\sqrt{u}) - 1
\end{aligned}
$$

7. 3 points As you know, uniform random variables can be used to "simulate" random variables from other, more complicated distributions. For example, suppose X is uniformly distributed on $(0, 1)$. Let $Y = -\theta \ln X$. Find the PDF of Y and also provide the name of the distribution.

$f(x) = 1$

$$v(y) = x = e^{-\frac{y}{\theta}}$$

$$v'(y) = -\frac{1}{\theta} e^{-\frac{y}{\theta}}$$

$$g(y) = f[v(y)] \cdot |v'(y)| = 1 \cdot \frac{1}{\theta} e^{-\frac{y}{\theta}} = \frac{1}{\theta} e^{-y/\theta}$$

Y FOLLOWS THE EXPONENTIAL DISTRIBUTION WITH MEAN OF θ.