Homework 7

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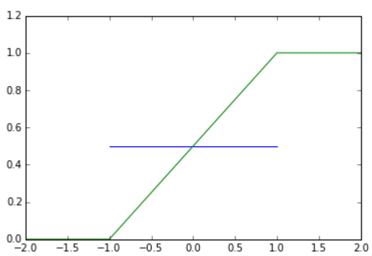
Section 3.1

2. $\mu = \int_{-1}^{1} x \frac{1}{2} dx = \frac{x^2}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

$\sigma^2 = E(X^2) - \mu^2 = \int_{-1}^{1} x^2 \frac{1}{2} dx - 0^2 = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$



3. (a) $f(x) = \frac{1}{10}$ for $0 \leq x \leq 10$

(b) $P(X \geq 8) = \int_{8}^{10} \frac{1}{10} dx = \frac{x}{10}\Big|_{8}^{10} = \frac{10}{10} - \frac{8}{10} = \frac{1}{5}$

(c) $P(2 \leq X < 8) = \int_{2}^{8} \frac{1}{10} dx = \frac{x}{10}\Big|_{2}^{8} = \frac{8}{10} - \frac{2}{10} = \frac{6}{10} = \frac{3}{5}$

(d) $E(X) = \int_{0}^{10} x \frac{1}{10} dx = \frac{x^2}{20}\Big|_{0}^{10} = \frac{100}{20} - \frac{0}{20} = 5$

(e) $Var(X) = E(X^2) - [E(X)]^2 = \int_{0}^{10} x^2 \frac{1}{10} dx - 5^2 = \left[\frac{x^3}{30}\right]_{0}^{10} - 25 = \frac{100}{3} - \frac{75}{3} = \frac{25}{3}$

4. (a) $E(X) = \frac{5+4}{2} = 4.5$

(b) $Var(X) = \frac{(5-4)^2}{12} = \frac{1}{12}$

(c) $P(4.2 < X \leq 4.7) = \int_{4.2}^{4.7} 1 dx = x\Big|_{4.2}^{4.7} = 4.7 - 4.2 = 0.5$

5. $F(y) = \int_0^y 1 dt = t\Big|_0^y = y$

   (a)

$$\begin{aligned} G(W) &= P(W \leq w) \\ &= P(a + (b-a)Y \leq w) \\ &= P\left(Y \leq \frac{w-a}{b-a}\right) \\ &= F\left(\frac{w-a}{b-a}\right) \\ &= \frac{w-a}{b-a}, a \leq w \leq b \end{aligned}$$

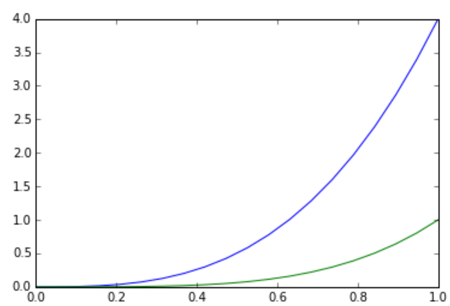
   (b) $U(a, b)$

7. (a) i. $1 = \int_0^1 4x^c dx = 4\left[\frac{x^{c+1}}{c+1}\right]_0^1 = 4\left[\frac{1}{c+1} - 0\right] = \frac{4}{c+1}$

   Therefore, $c = 3$.

   ii. $F(X) = \int_0^x 4t^3 dt = t^4\Big|_0^x = x^4$ for $0 \leq x \leq 1$.

   iii. Pdf (blue) and Cdf (green):



   iv. $\mu = \int_0^1 x4x^3 dx = \frac{4x^5}{5}\Big|_0^1 = \frac{4}{5}$

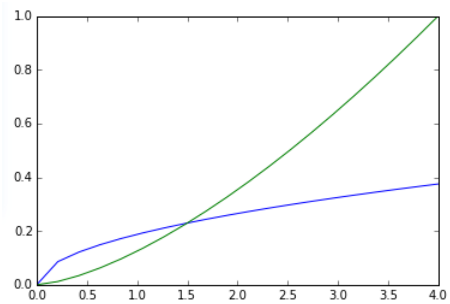
   $\sigma^2 = \int_0^1 x^2 4x^3 dx - \left[\frac{4}{5}\right]^2 = \left[\frac{2x^6}{3}\right]_0^1 - \frac{16}{25} = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$

(b) i. $1 = \int_0^4 c\sqrt{x}dx = c\left[\frac{2x^{3/2}}{3}\right]_0^4 = c\left[\frac{16}{3} - 0\right] = \frac{16c}{3}$

Therefore, $c = \frac{3}{16}$.

ii. $F(X) = \int_0^x \frac{3\sqrt{t}}{16}dt = \frac{3}{16}\left[\frac{2t^{3/2}}{3}\right]_0^x = \frac{3}{16}\left[\frac{2x^{3/2}}{3}\right] = \frac{x^{3/2}}{8}$ for $0 \le x \le 4$

iii. Pdf and Cdf:



iv. $\mu = \int_0^4 x\frac{3}{16}\sqrt{x}dx = \frac{3}{16}\int_0^4 x^{3/2}dx = \frac{3}{16}\left[\frac{2x^{5/2}}{5}\right]_0^4 = \frac{3}{16}\left(\frac{64}{5}\right) = \frac{12}{5}$

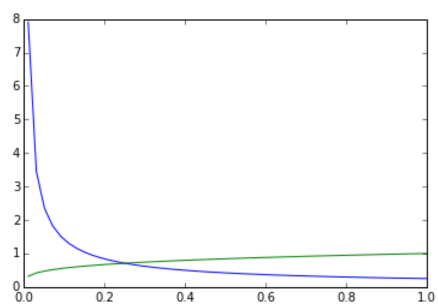
$\sigma^2 = \int_0^4 x^2\frac{3}{16}\sqrt{x}dx - \left[\frac{12}{5}\right]^2 = \frac{3}{16}\left[\frac{2x^{7/2}}{7}\right]_0^4 - \frac{144}{25} = \frac{3}{16}\left(\frac{256}{7}\right) - \frac{144}{25} = \frac{192}{175}$

(c) i. $1 = \int_0^1 \frac{c}{x^{3/4}}dx = c\left[4x^{1/4}\right]_0^1 = c\left[4(1^{1/4}) - 0\right] = 4c$

Therefore, $c = 1/4$.

ii. $F(X) = \int_0^x \frac{1}{4t^{3/4}}dt = \frac{1}{4}\int_0^x t^{-3/4}dt = \frac{1}{4}[4t^{1/4}]_0^x = t^{1/4}\Big|_0^x = x^{1/4}$ for $0 \le x \le 1$

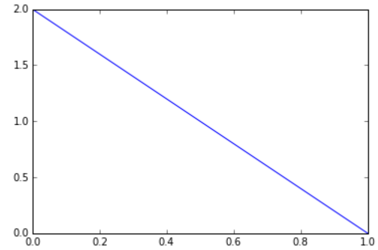
iii. Pdf and Cdf:



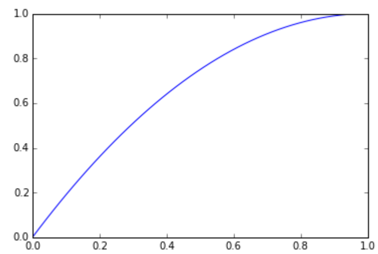
iv. $\mu = \int_0^1 x\frac{1}{4x^{3/4}}dx = \frac{1}{4}\int_0^1 x^{1/4}dx = \frac{1}{4}\left[\frac{4x^{5/4}}{5}\right]_0^1 = \frac{1}{4}\left[\frac{4}{5}\right] = \frac{1}{5}$

$\sigma^2 = \int_0^1 x^2\frac{1}{4x^{3/4}}dx - \left[\frac{1}{5}\right]^2 = \frac{1}{4}\left[\frac{4x^{9/4}}{9}\right]_0^1 - \frac{1}{25} = \frac{1}{4}\left[\frac{4}{9}\right] - \frac{1}{25} = \frac{16}{225}$

9. (a) PDF:



(b) $F(X) = \int_0^x 2(1-t)dt = 2\int_0^x (1-t)dt = 2\left[t - \frac{t^2}{2}\right]_0^x = 2x - x^2$



(c) i. $P(0 \leq X \leq 1/2) = F(1/2) - F(0) = [2(1/2) - (1/2)^2] - 0 = 3/4$

ii. $P(1/4 \leq X \leq 3/4) = F(3/4) - F(1/4) = [2(3/4) - (3/4)^2] - [2(1/4) - (1/4)^2] = 1/2$

iii. $P(X = 3/4) = 0$

iv. $P(X \geq 3/4) = 1 - P(X \leq 3/4) = 1 - F(3/4) = 1 - [2(3/4) - (3/4)^2] = 1/16$

10. (a) $\int_1^\infty \frac{c}{x^2}dx = \lim_{b\to\infty}\left[-\frac{c}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{c}{b} + c\right] = c$

Setting the value above equal to 1, we get that $c = 1$.

(b) $\int_1^\infty x\frac{1}{x^2}dx = \lim_{b\to\infty}[\ln|x|]_1^b = \lim_{b\to\infty}[\ln b - \ln 1] = \lim_{b\to\infty}\ln b = \infty$

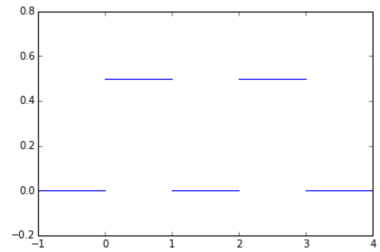
11. (a) $\int_1^\infty \frac{d}{y^3}dy = \lim_{b\to\infty}\left[-\frac{d}{2y^2}\right]_1^b = \lim_{b\to\infty}\left[-\frac{d}{2b^2} + \frac{d}{2}\right] = \frac{d}{2}$

Setting the value above equal to 1, we get that $d = 2$.

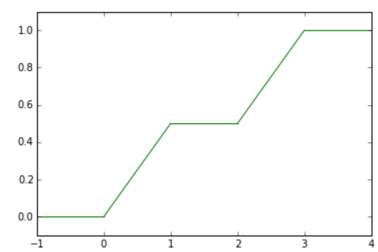
(b) $E(Y) = \int_1^\infty y\frac{2}{y^3}dy = \lim_{b\to\infty}\left[-\frac{2}{y}\right]_1^b = \lim_{b\to\infty}\left[-\frac{2}{b} + 2\right] = 2$

(c) $Var(Y) = \int_1^\infty y^2\frac{2}{y^3}dy = \lim_{b\to\infty}\int_1^b \frac{2}{y}dy = \lim_{b\to\infty}[2\ln y]_1^b = \lim_{b\to\infty} 2\ln b = \infty$

14. (a) PDF:



(b) $F(x) = \begin{cases} 0 & x \leq 0 \\ x/2 & 0 < x < 1 \\ 1/2 & 1 \leq x \leq 2 \\ (x-1)/2 & 2 < x < 3 \\ 1 & x \geq 3 \end{cases}$



(c) $F(\pi_{0.25}) = \pi_{0.25}/2 = 0.25$
Therefore, $\pi_{0.25} = 1/2$.

(d) $F(\pi_{0.5}) = 1/2$
The cdf evaluates to 1/2 for $1 \leq x \leq 2$. So, no, it's not unique.

(e) $F(\pi_{0.75}) = (\pi_{0.75} - 1)/2 = 0.75$
Therefore, $\pi_{0.75} = 2.5$.

15. (a)

$$\begin{aligned} P(X \geq 7) &= 1 - P(X < 7) \\ &= 1 - \int_0^7 \frac{3x^2}{7^3} e^{-(x/7)^3}\, dx \\ &= 1 + \left[e^{-(x/7)^3} \right]_0^7 \\ &= 1 + \left[e^{-1} - e^{-0} \right] \\ &= 1 + \frac{1}{e} - 1 \\ &= \frac{1}{e} \approx 0.3679 \end{aligned}$$

(b)

$$\begin{aligned}
P(X \geq 7 + 3.5 | X \geq 7) &= \frac{P((X \geq 10.5) \cap (X \geq 7))}{P(X \geq 7)} \\
&= \frac{P(X \geq 10.5)}{P(X \geq 7)} \\
&= \frac{e^{-(10.5/7)^3}}{1/e} \\
&= e^{-2.375} \approx 0.0930
\end{aligned}$$

16. $F(x) = \displaystyle\int_{-1}^{x} \frac{t+1}{2} dt = \frac{t^2+2t}{4}\Big|_{-1}^{x} = \frac{x^2+2x}{4} - \frac{(-1)^2+2(-1)}{4} = \frac{x^2+2x}{4} + \frac{1}{4} = \left(\frac{x+1}{2}\right)^2$

(a) $F(\pi_{0.64}) = \left(\dfrac{\pi_{0.64}+1}{2}\right)^2 = 0.64$
$\pi_{0.64} = 2\sqrt{0.64} - 1 = 2(\pm 0.8) - 1 = -2.6$ or 0.6
Since -2.6 is out of range, $\pi_{0.64} = 0.6$.

(b) $F(\pi_{0.25}) = \left(\dfrac{\pi_{0.25}+1}{2}\right)^2 = 0.25$
$\pi_{0.25} = 2\sqrt{0.25} - 1 = 2(\pm 0.5) - 1 = -2$ or 0
Since -2 is out of range, $\pi_{0.25} = 0$.

(c) $F(\pi_{0.81} = \left(\dfrac{\pi_{0.81}+1}{2}\right)^2 = 0.81$
$\pi_{0.81} = 2\sqrt{0.81} - 1 = 2(\pm 0.9) - 1 = -2.8$ or 0.8
Since -2.8 is out of range, $\pi_{0.81} = 0.8$