Homework 9

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Section 3.3

1. (a) $P(0.53 < Z \leq 2.06) = \Phi(2.06) - \Phi(0.53) = 0.9803 - 0.7019 = 0.2784$
   (b) $P(-0.79 \leq Z < 1.52) = \Phi(1.52) - \Phi(-0.79) = 0.9357 - 0.2148 = 0.7209$
   (c) $P(Z > -1.77) = \Phi(1.77) = 0.9616$
   (d) $P(Z > 2.89) = 0.0019$
   (e) $P(|Z| < 1.96) = P(-1.96 < Z < 1.96) = \Phi(1.96) - P(Z > 1.96) = 0.9750 - 0.0250 = 0.95$
   (f) $P(|Z| < 1) = P(-1 < Z < 1) = \Phi(1) - P(Z > 1) = 0.8413 - 0.1587 = 0.6826$
   (g) $P(|Z| < 2) = P(-2 < Z < 2) = \Phi(2) - P(Z > 2) = 0.9772 - 0.0228 = 0.9544$
   (h) $P(|Z| < 3) = P(-3 < Z < 3) = \Phi(3) - P(Z > 3) = 0.9987 - 0.0013 = 0.9974$

3. (a) $P(Z \geq c) = 0.025$
   $c = 1.96$
   (b) $P(|Z| \leq c) = 0.95$
   $c = 1.96$
   (c) $P(Z > c) = 0.05)$
   $c = 1.645$
   (d) $P(|Z| \leq c) = 0.90$
   $c = 1.645$

4. (a) $z_{0.10} = 1.282$
   (b) $-z_{0.05} = -1.645$
   (c) $-z_{0.0485} = -1.66$
   (d) $z_{0.9656} = -1.82$

5. $X \sim N(6, 25)$
   (a) $P(6 \leq X \leq 12) = P(0 \leq Z \leq 1.2) = \Phi(1.2) - \Phi(0) = 0.8849 - 0.5 = 0.3849$
   (b) $P(0 \leq X \leq 8) = P(-1.2 \leq Z \leq 0.4) = \Phi(0.4) - \Phi(-1.2) = 0.6554 - 0.1151 = 0.5403$
   (c) $P(-2 < X \leq 0) = P(-1.6 < Z \leq -1.2) = \Phi(-1.2) - \Phi(-1.6) = 0.1151 - 0.0548 = 0.0603$

(d) $P(X > 21) = P(Z > 3) = 0.0013$

(e) $P(|X - 6| < 5) = P(-1 < Z < 1) = 0.6826$

(f) $P(|X - 6| < 10) = P(-2 < Z < 2) = 0.9544$

(g) $P(|X - 6| < 15) = P(-3 < Z < 3) = 0.9974$

(h) $P(|X - 6| < 12.41) = P(-2.482 < Z < 2.482) = \Phi(2.482) - \Phi(-2.482) = 0.9934 - 0.0066 = 0.9868$

6. $M(t) = exp(166t + 200t^2)$
   $X \sim N(166, 400)$

   (a) $\mu = 166$

   (b) $\sigma^2 = 400$

   (c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = \Phi(1.7) - \Phi(0.2) = 0.9554 - 0.5793 = 0.3761$

   (d) $P(148 \leq X \leq 172) = P(-0.9 \leq Z \leq 0.3) = \Phi(0.3) - \Phi(-0.9) = 0.6179 - 0.1841 = 0.4338$

7. $X \sim N(650, 625)$

   (a) $P(600 \leq X < 660) = P(-2 \leq Z < 0.4) = \Phi(0.4) - \Phi(-2) = 0.6554 - 0.0228 = 0.6326$

   (b) $P(|X - 650| \leq c) = 0.9544$
   $c = 50$

8. Since $X \sim N(\mu, \sigma^2)$, then the PDF of X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

In order to find the points of inflection, we have to find the values of x where the second derivative of the PDF is equal to 0. We start by first computing the first derivative of the PDF:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) &= \frac{1}{\sigma\sqrt{2\pi}}\frac{d}{dx}\left(exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\frac{d}{dx}\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\left(-\frac{x-\mu}{\sigma^2}\right) \\
&= -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]
\end{aligned}$$

We then use that result to calculate the second derivative of the PDF:

$$\begin{aligned}
&\frac{d^2}{dx^2}\left(\frac{1}{\sigma\sqrt{2\pi}}exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) \\
&= \frac{d}{dx}\left(-\frac{x-\mu}{\sigma^3\sqrt{2\pi}}exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) \\
&= -\frac{x-\mu}{\sigma^3\sqrt{2\pi}}\frac{d}{dx}\left(exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\right) + exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\frac{d}{dx}\left(-\frac{x-\mu}{\sigma^3\sqrt{2\pi}}\right) \\
&= \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}}exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] - \frac{1}{\sigma^3\sqrt{2\pi}}exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{(x-\mu)^2-\sigma^2}{\sigma^5\sqrt{2\pi}}exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]
\end{aligned}$$

We can see that this expression can only be equal to 0 when the first factor is 0, since the $exp[...]$ factor is always positive. Therefore, we set the first factor equal to 0 and solve for x:

$$\begin{aligned}
\frac{(x-\mu)^2-\sigma^2}{\sigma^5\sqrt{2\pi}} &\overset{\text{set}}{=} 0 \\
(x-\mu)^2-\sigma^2 &= 0 \\
(x-\mu)^2 &= \sigma^2 \\
x-\mu &= \pm\sigma \\
x &= \mu\pm\sigma
\end{aligned}$$

9. $W = X^2$

   (a) $X \sim N(0,4)$
   Cumulative distribution function of W:

$$G(w) = P(W \le w) = P(X^2 \le w) = P(-\sqrt{w} \le X \le \sqrt{w})$$

   Let's integrate the PDF of X, a normal random variable with $\mu = 0$ and $\sigma = 2$:

$$G(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{2\sqrt{2\pi}}exp\left(-\frac{x^2}{8}\right)dx$$

   We do the following change of variables: Let $x = \sqrt{y} = y^{1/2}$, so $dx = \frac{1}{2}y^{-1/2}dy = \frac{1}{2\sqrt{y}}dy$. Therefore, $x^2 = y$ and $x = 0 \implies y = 0$ and $x = \sqrt{w} \implies y = w$.
   This gives us:

$$G(w) = 2\int_0^w \frac{1}{2\sqrt{2\pi}}exp\left(-\frac{y}{8}\right)\left(\frac{1}{2\sqrt{y}}\right)dy$$

$$G(w) = \int_0^w \frac{1}{\sqrt{8}\sqrt{\pi}}y^{\frac{1}{2}-1}exp\left(-\frac{y}{8}\right)dy$$

We take the derivative of $G(w)$ to get the probability density function $g(w)$:

$$g(w) = \frac{1}{\sqrt{8}\sqrt{\pi}} y^{\frac{1}{2}-1} exp\left(-\frac{y}{8}\right)$$

which is the gamma distribution PDF with $\alpha = \frac{1}{2}$ and $\theta = 8$.

(b) $X \sim N(0, \sigma^2)$
Cumulative distribution function of W:

$$G(w) = P(W \leq w) = P(X^2 \leq w) = P(-\sqrt{w} \leq X \leq \sqrt{w})$$

Let's integrate the PDF of X, a normal random variable with $\mu = 0$ and variance σ^2:

$$G(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

We do the following change of variables: Let $x = \sqrt{y} = y^{1/2}$, so $dx = \frac{1}{2}y^{-1/2}dy = \frac{1}{2\sqrt{y}}dy$. Therefore, $x^2 = y$ and $x = 0 \implies y = 0$ and $x = \sqrt{w} \implies y = w$.
This gives us:

$$G(w) = 2\int_{0}^{w} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{y}{2\sigma^2}\right)\left(\frac{1}{2\sqrt{y}}\right) dy$$

$$G(w) = \int_{0}^{w} \frac{1}{\sqrt{2\sigma^2}\sqrt{\pi}} y^{\frac{1}{2}-1} exp\left(-\frac{y}{2\sigma^2}\right) dy$$

We take the derivative of $G(w)$ to get the probability density function $g(w)$:

$$g(w) = \frac{1}{\sqrt{2\sigma^2}\sqrt{\pi}} y^{\frac{1}{2}-1} exp\left(-\frac{y}{2\sigma^2}\right)$$

which is the gamma distribution PDF with $\alpha = \frac{1}{2}$ and $\theta = 2\sigma^2$.

11. $X \sim N(21.37, 0.16)$

(a) $P(X > 22.07) = P(Z > 1.75) = 0.0401$

(b) First we compute the probability of a mint weighing less than 20.857 grams,

$$P(X < 20.857) = P(Z < -1.2825) = 0.1003$$

The distribution of Y is binomial with $n = 15$ and $p = 0.1003$. We wish to find $P(Y \leq 2)$:

$$\begin{aligned} P(Y \leq 2) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\ &= (1 - 0.1003)^{15} + 15(0.1003)(1 - 0.1003)^14 + 105(0.1003^2)(1 - 0.1003)^13 \\ &= 0.8148 \end{aligned}$$

15. (a) If $X \sim N(12.1, \sigma^2)$, then $Z = \frac{X - 12.1}{\sigma}$. Therefore, we can rewrite the problem so that we're looking for c such that $P(Z < c) = 0.01$. From Table Va, we know that this is the value $-z_{0.01} = -2.326$. Now we can solve for σ:

$$\begin{aligned} -2.326 &= \frac{12 - 12.1}{\sigma} \\ \sigma &= \frac{12 - 12.1}{-2.326} \\ &= 0.04299 \end{aligned}$$

(b)

$$\begin{aligned} -2.326 &= \frac{12 - \mu}{0.05} \\ 12 - \mu &= -0.1163 \\ \mu &= 12.1163 \end{aligned}$$