Efficient E-Matching for SMT Solvers

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Quantifier reasoning

Motivation

Program verification problems often require dealing with quantified subformulas. [3]

Quantifier reasoning

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Apply Modus Ponens

$$42 < 0$$

Quantifier reasoning

Problem 1

How to figure out which instances are going to be useful?

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Solution

Identify subterms occurring in the quantified formula (*patterns*), and only add instances that make those subterms equal to ground subterms that are currently being considered in the proof.

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Example

In the previous example, one such pattern is $P(f(x))$, which when instantiated with $[x := 42]$, yields its corresponding ground subterm $P(f(42))$.

Quantifier reasoning

Problem 2

Syntactic equality is oftentimes not enough. Consider eg. the following proposition:

$$a = f(42) \wedge P(a) \wedge \forall x.P(f(x))) \Rightarrow x < 0$$

There are no instances of $P(f(x))$ that matches the ground term $P(a)$, ie. there is no substitution for x that make $P(f(x))$ syntactically equal to $P(a)$.

Quantifier reasoning

Problem 2

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There are no instances of $P(f(x))$ that matches the ground term $P(a)$, ie. there is no substitution for x that make $P(f(x))$ syntactically equal to $P(a)$.

Solution

Use the congruence closure of the equality relation induced by the current context.
For example, in the context $P(a), a = f(42)$, the substitution $[x := 42]$ makes $P(f(x))$ equal to $P(a)$, since $P(a) \simeq P(f(42))$.

Quantifier reasoning

Problem 3

How to identify the set of patterns for a given formula?

Quantifier reasoning

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Solution

- Heuristics
- Ask the user to provide the patterns

Quantifier reasoning

Problem 4

How to identify the substitutions that make the pattern equal to some ground term?

Quantifier reasoning

Problem 4

How to identify the substitutions that make the pattern equal to some ground term?

Solution

Apply an E-matching algorithm.

E-Matching algorithm (overview)

Input

- A binary relation E over terms
- A ground term t
- A pattern p

Output

The set of substitutions θ, modulo E, over the variables in p, such that

$$E \models t \simeq \theta(p)$$

E-matching algorithm

Definitions

- Σ is set of function symbols, called the *signature*
- V is the set of variables
- Each function symbol f has an arity, $arity(f)$
- An f-application is a term of the form $f(t_1, \ldots, t_n)$
- $T(\Sigma, V)$ is the set of terms
- $T(\Sigma, \varnothing)$ is the set of ground terms
- The set of non ground terms is called *patterns*
- A substitution β is a mapping from variables to ground terms
- A binary relation E over $T(\Sigma, \varnothing)$

E-graph

- An *E-graph* is a data-structure which maintains the *congruence closure* of a binary relation $E = \{(t_1, t'_1), (t_2, t'_2), \ldots\}$.
- This binary relation is given incrementally with the $union(t_1, t'_1)$ operation.
- Each equivalence class generated by the congruence relation has a *representative*.

E-graph

Operations

For each term t in the *E-graph*:

- $find(t)$ denotes the representative of the equivalence class that contains t
- $class(t)$ denotes the equivalence class that contains t
- $parents(f)$ denotes the set of terms $f(\ldots, t', \ldots)$ in the E-graph such that $t' \simeq t$
- $parents_f(t)$ is a subset of $parents(t)$ which contains only f-applications
- $parents_{f.i}(t)$ is a subset of $parents_f(t)$ which contains only f-applications where the i-th argument t_i is s.t. $t_i \simeq t$
- ...

Quantifier reasoning with E-matching

- Semantically, the formula

$$\forall x_1, \ldots, x_n . F$$

is equivalent to

$$\bigcup_{\beta} \beta(F)$$

where β ranges over all substitutions over the x's.

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  where β ranges over all substitutions over the x's.

- We now want to select from this set, the substitutions that are relevant to the conjecture.

- To do so, we select non ground terms p from F as *patterns*, and consider $\beta(F)$ to be relevant whenever $\beta(p)$ is in the *E-graph.*

E-matching algorithm

$$
\begin{aligned}
match(x, t, S) &= \{\beta \cup \{x \mapsto t\} \mid \beta \in S, x \notin dom(\beta)\} \cup \\
&\quad \{\beta \mid \beta \in S, find(\beta(x)) = find(t)\} \\
match(c, t, S) &= S \ \ if\ c \in class(t) \\
match(c, t, S) &= \emptyset \ \ if\ c \notin class(t) \\
match(f(p_1, \ldots, p_n), t, S) &= \bigcup_{f(t_1,\ldots,t_n) \in class(t)} match(p_n, t_n, \ldots, match(p_1, t_1, S))
\end{aligned}
$$

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\end{aligned}
$$

The abstract matching procedure returns all substitutions that E-match a pattern p with term t. That is, if $\beta \in match(p, t, \varnothing)$ then $E \models \beta(p) = t$.

Efficient E-matching

Complexity

E-matching is in theory NP-hard, and the number of matches can be exponential in the size of the E-graph.

In practice

The cost is dominated by:

- Matching a set of patterns against terms
- Compute sets of patterns that can generate new matches

Solution

Compile patterns into code, and introduce further optimizations.

E-matching abstract machine

Memory

- register pc for storing the current instruction
- an array of registers $reg[]$ for storing ground terms
- a stack $bstack$ for backtracking

E-matching abstract machine

Instructions

- init
- *bind*
- check
- compare
- choose
- *yield*
- *backtrack*
- choose-app

Semantics

init$(f, next)$	assuming $reg[0] = f(t_1, \ldots, t_n)$ $reg[1] := t_1; \ldots; reg[n] := t_n$ $pc := next$
bind$(i, f, o, next)$	$push(bstack,$ choose-app$(o, next, apps_f(reg[i]), 1))$ $pc :=$ backtrack
check$(i, t, next)$	**if** $find(reg[i]) = find(t)$ **then** $pc := next$ **else** $pc :=$ backtrack
compare$(i, j, next)$	**if** $find(reg[i]) = find(reg[j])$ **then** $pc := next$ **else** $pc :=$ backtrack
choose$(alt, next)$	**if** $alt \neq nil$ **then** $push(bstack,\ alt)$ $pc := next$
yield$(i_1, \ldots, i_k)$	yield substitution $\{x_1 \mapsto reg[i_1], \ldots, x_k \mapsto reg[i_k]\}$ $pc :=$ backtrack
backtrack	**if** $bstack$ is not empty **then** $pc := pop(bstack)$ **else stop**
choose-app$(o, next, s, j)$	**if** $\|s\| \geq j$ **then** **let** $f(t_1, \ldots, t_n)$ be the j^{th} term in s. $reg[o] := t_1; \ldots; reg[o + n - 1] := t_n$ $push(bstack,$ choose-app$(o, next, s, j + 1))$ $pc := next$ **else** $pc :=$ backtrack

Figure 1: Semantics of abstract machine instructions

Example

The pattern

$$f(x_1, g(x_1, a), h(x_2), b)$$

is compiled to into the following *labeled instructions*:

Label	Instruction
n_0	init(f, n_1)
n_1	check$(4, b, n_2)$
n_2	bind$(2, g, 5, n_3)$
n_3	compare$(1, 5, n_4)$,
n_4	check$(6, a, n_5)$
n_5	bind$(3, h, 7, n_6)$
n_6	yield$(1, 7)$

E-matching abstract machine

Usage

To compute $match(p, t, \varnothing)$, where p is eg. the pattern on the previous slide:

- Store t in $reg[0]$
- Set $pc = n_0$
- Execute the instruction stored in pc
- Collect the yielded substitutions

Efficient E-matching

To further improve efficiency, the paper introduces two novel techniques:

- E-matching code trees
- Inverted path index

E-matching code trees

Problem

Oftentimes, patterns have some shared structure, and the solver wastes time repeatedly matching against these subpatterns.

Example

- $f(x, g(a, y))$
- $f(x, g(x, y))$
- $f(h(x, y), b)$
- $f(h(x, g(x, y)), b)$

E-matching code trees

Solution

- Minimize the time spent on matching similar patterns by combining their code sequences in a *code tree*
- The matching work common to multiple patterns is *factored out*
- Whenever the DPLL(T) engine asserts a quantified formula, add its patterns to the code tree. Remove them when the engine backtracks.

E-matching code trees

$$P = \{f(x, g(a, y)), f(x, g(x, y)), f(h(x, y), b), f(h(x, g(x, y)), b)\}$$

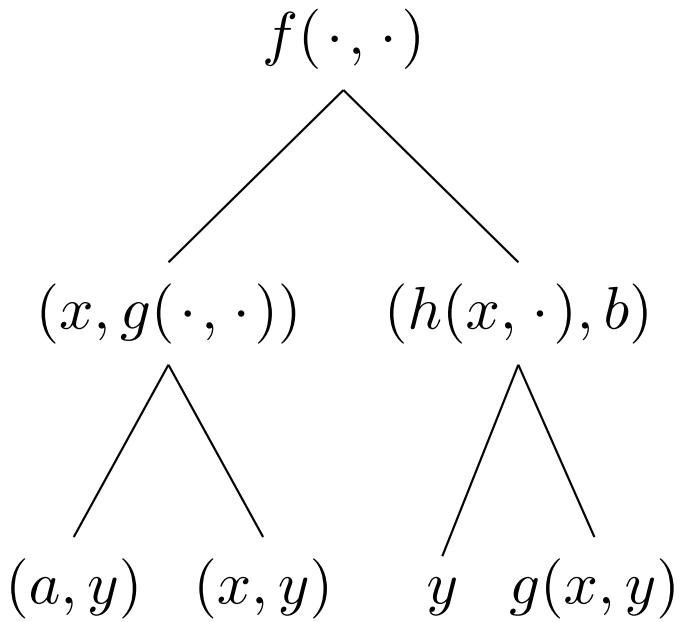

Figure 2: Code tree for P

E-matching code trees

$$P = \{f(x, g(a, y)), f(x, g(x, y)), f(h(x, y), b), f(h(x, g(x, y)), b)\}$$

$\mathsf{init}(f, n_1)$

$n_1 : \mathsf{choose}(n_9, n_2),\ n_2 : \mathsf{bind}(2, g, 3, n_3)$

$n_3 : \mathsf{choose}(n_6, n_4),\ n_4 : \mathsf{check}(3, a, n_5),\ n_5 : \mathsf{yield}(1, 4)$

$n_6 : \mathsf{choose}(nil, n_7),\ n_7 : \mathsf{compare}(1, 3, n_8),\ n_8 : \mathsf{yield}(1, 4)$

$n_9 : \mathsf{choose}(nil, n_{10}),\ n_{10} : \mathsf{check}(2, b, n_{11}),\ n_{11} : \mathsf{bind}(1, h, 5, n_{12})$

$n_{12} : \mathsf{choose}(n_{14}, n_{13}),\ n_{13} : \mathsf{yield}(5, 6)$

$n_{14} : \mathsf{choose}(nil, n_{15}),\ n_{15} : \mathsf{bind}(6, g, 7, n_{16}),\ n_{16} : \mathsf{compare}(5, 7, n_{17}),\ n_{17} : \mathsf{yield}(5, 8)$

Figure 3: Code tree for P

Incrementality

- Say we have a term $f(a, b)$, and a pattern $f(g(x), y)$
- At this stage, there is no way to instantiate $f(g(x), y)$ so that it matches $f(a, b)$
- At some point, the operation $union(a, g(c))$ is executed
- The E-graph now tells us that $f(a, b) \simeq f(g(c), b)$
- It thus becomes possible to instantiate $f(g(x), y)$ with $[x := c, y := b]$ to match $f(g(c), b)$.

Incrementality

Problem

How to identify new terms and patterns that become relevant for matching?

Incrementality

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Solution

Introduce an *inverted path index*.

Inverted path index

pc-pair

A pair of function symbols (f, g) is a *parent-child* pair (*pc-pair*) of a pattern p, if p contains a term of the form:

$$f(\ldots, g(\ldots), \ldots)$$

Inverted path index

Inverted path string

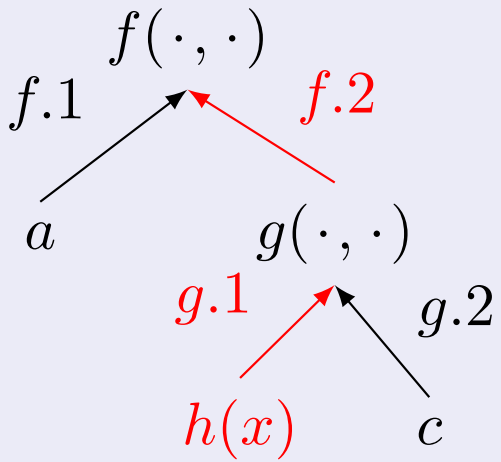
An *inverted path string* over a signature Σ is either:

- the empty string ϵ
- $f.i.\pi$, where π is an inverted path string, $f \in \Sigma$, and $i \in i\mathbb{N}$.

Inverted path index

Inverted path string

The inverted path string $g.1.f.2$ is a path to term $f(a, g(h(x), c))$ from subterm $h(x)$.



Inverted path index

Construction

Given a set of patterns P which share the same set of pc-pairs, one of those pc-pair (f, g), and a pattern $p \in P$:

- $\Pi(p, (f, g))$ denotes the set of inverted path strings from every relevant g-application in p
- The *inverted path index* $\tau(P, (f, g))$ is a trie over $\bigcup_{p \in P} \Pi(p, (f, g))$.

Inverted path index

$$P = \{f(f(g(x), a), x), h(c, f(g(y), x)), f(f(g(x), b), y), f(f(a, g(x)), g(y))\}$$

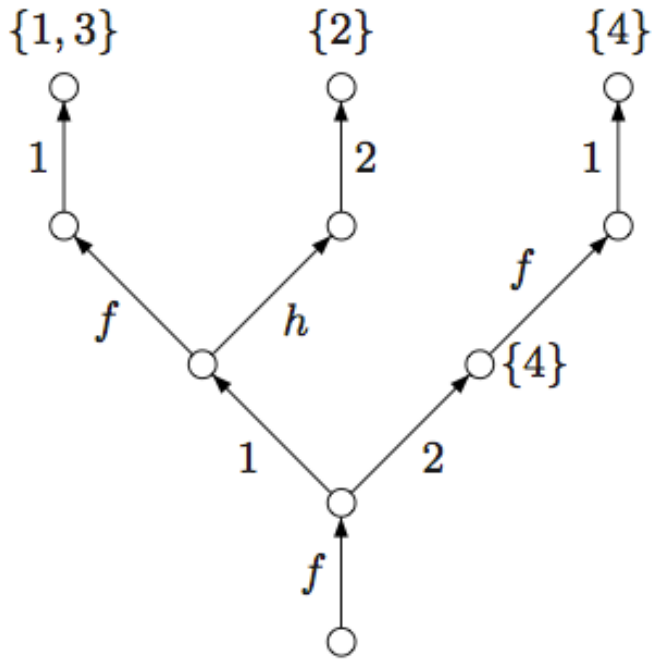


Figure 4: Inverted path index for *pc-pair* (f, g) and patterns P

Inverted path index

$$collect(\langle [f_1.i_1.\tau_1, \ldots f_k.i_k.\tau_k], P\rangle, T) = \{(P,T) \mid P \neq \emptyset\} \cup \bigcup_{j=1}^{k} collect(\tau_j, \{f_j(t_1,\ldots,t_n) \mid f_j(t_1,\ldots,t_n) \in parents_{f_j.i_j}(t), t \in T\})$$

Figure 5: Algorithm to compute U

Reminder

$parents_{f.i}(t)$ is the set of of f-applications where the i-th argument t_i is congruent to t

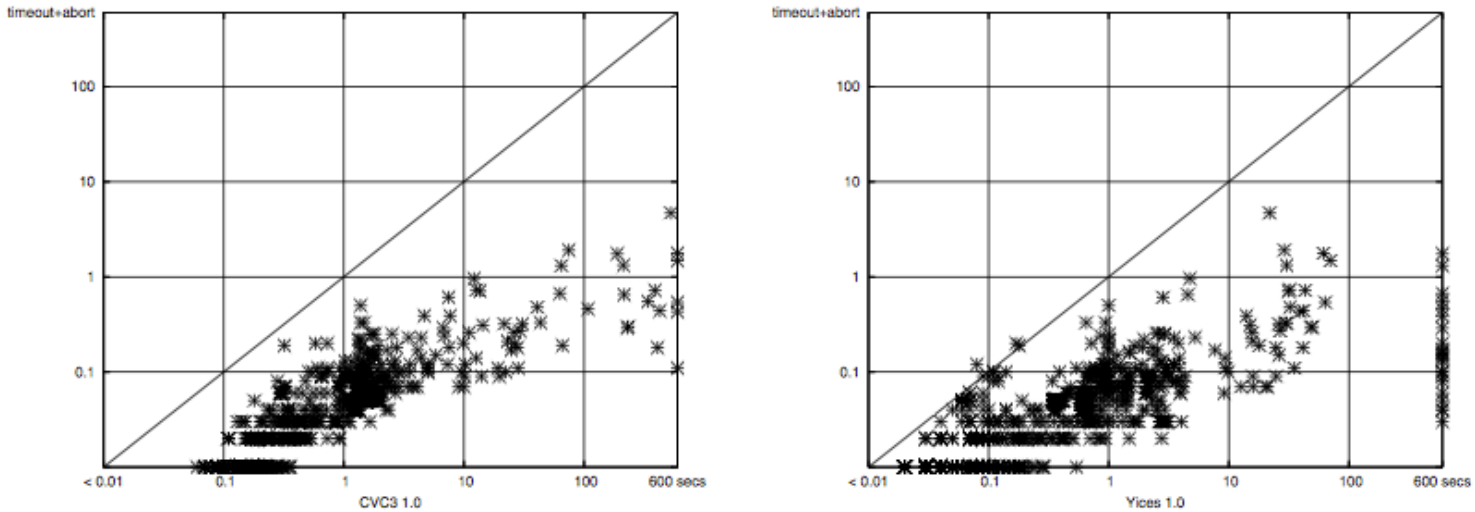
Inverted path index

Usage (?)

Whenever the E-graph is updated, ie. after executing $union(t_1, t_2)$:
For each *pc-pair* (f, g) relevant to t_1 and t_2,

- Define P to be the set of patterns with *pc-pair* (f, g)
- Compute the set $U = collect(\tau(P, (f, g)), \{t_1, t_2\})$ of newly relevant patterns-terms pairs using the inverted path index
- For each $(Q, T) \in U, p \in Q, t \in T$, execute $match(p, t, \varnothing)$, and collect the yielded substitutions

Benchmarks

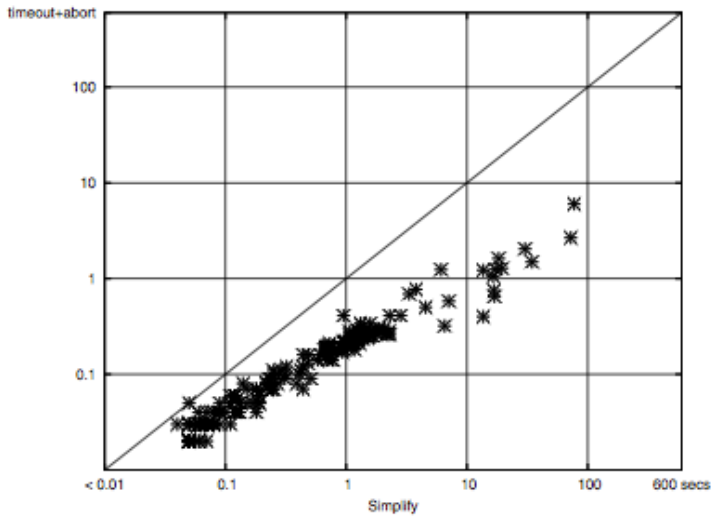


(a) Z3 vs. CVC3 1.0

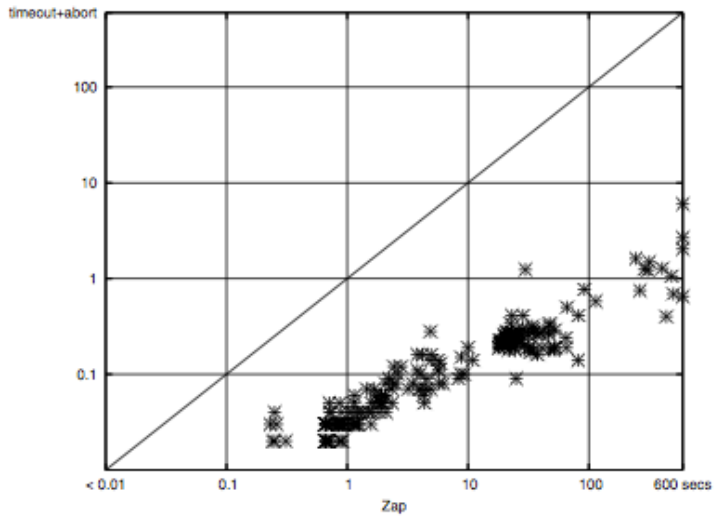
(b) Z3 vs. Yices 1.0

Figure 6: SMT-LIB benchmarks

Benchmarks



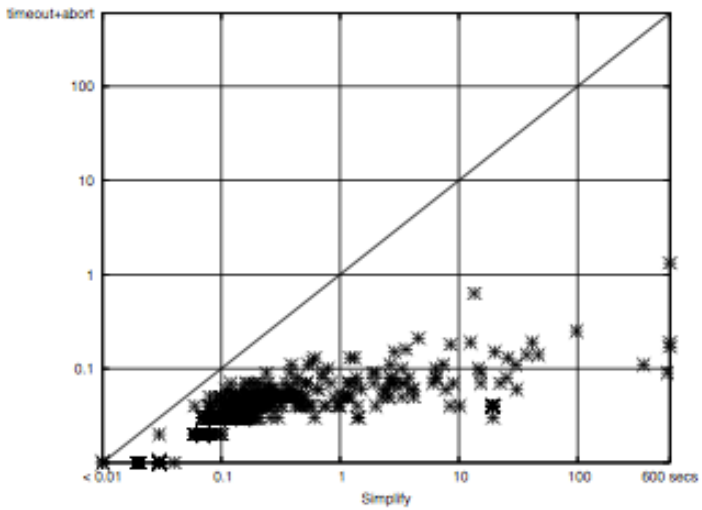
(a) Z3 vs. Simplify



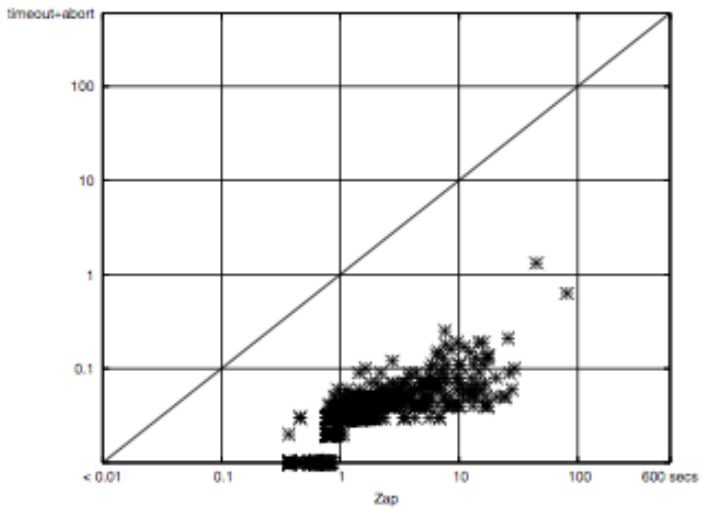
(b) Z3 vs. Zap 2.0

Figure 7: ESC/Java benchmarks

Benchmarks



(a) Z3 vs. Simplify



(b) Z3 vs. Zap 2.0

Figure 8: Boogie benchmarks

Benchmarks

	ESC/Java		Boogie		S-expr Simplifier	
	# valid	time	# valid	time	# valid	time
Simplify	2331	499.03	903	1851.29	18	10985.80
Zap	2222	6297.04	901	2612.64	22	777.78
Z3 (*lazy*)	2331	212.81	907	157.2	32	2904.27
Z3 (*lazy wo. code trees*)	2331	224.14	907	240.44	28	2369.00
Z3 (*eager wo. inc.*)	2331	1495.07	907	229.2	10	2410.52
Z3 (*eager mod-time*)	2331	85.1	907	39.79	32	1341.38
Z3 (*eager wo. code trees*)	2331	48.28	907	26.85	32	654.62
Z3 (*default*)	**2331**	**45.22**	**907**	**18.47**	**32**	**194.54**

Figure 9: Experimental results: summary

Contributions

- An *abstract machine* for E-matching
- *E-matching code trees*, which can efficiently handle matching a term against a large set of patterns simultaneously
- *Inverted path indexing*, which narrowly and efficiently finds a superset of terms that will match a set of patterns

Further work

Use *context trees* for additional sharing

A context tree is a data-structure where not only prefix terms but also common subterms can be shared, even if they occur below different function symbols. [2]

References

[1] Detlefs, D. et al. 2005. Simplify: A theorem prover for program checking. *J. ACM*. 52, 3 (May 2005), 365–473.
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[3] Moskal, M. et al. 2008. E-matching for fun and profit. *Electron. Notes Theor. Comput. Sci.* 198, 2 (May 2008), 19–35.
[4] Moura, L. de and Bjørner, N. 2007. Efficient E-matching for SMT solvers. *Automated deduction – cade-21: 21st international conference on automated deduction bremen, germany, july 17-20, 2007 proceedings*. F. Pfenning, ed. Springer Berlin Heidelberg. 183–198.