An upper bound on the denominator of Eisenstein classes in Bianchi manifolds

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Abstract

A general conjecture of Harder relates the denominator of the Eisenstein cohomology of certain locally symmetric spaces to special values of L-functions. In this paper we consider the locally symmetric space $Y_\Gamma = \mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}_3$ associated to $\mathrm{SL}_2(K)$ where K is an imaginary quadratic field. Berger [Ber08] proves a lower bound on the denominator of the Eisenstein cohomology in certain cases. In this paper, we show how results of Ito [Ito87] and Sczech [Scz84] can be used to prove an upper bound on the denominator in terms of a special value of L-function.

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1 Introduction

Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with ring of integers $\mathscr{O}$. Suppose[1] that $D \equiv 1 \mod 8$. Let $Y_\Gamma = \Gamma\backslash\mathbb{H}_3$ where $\Gamma = \mathrm{SL}_2(\mathscr{O})$ and $\mathbb{H}_3$ is the hyperbolic 3-space. It is a non-compact space and let X_Γ be its Borel-Serre compactification with boundary ∂X_Γ. The boundary has $h = |\mathrm{Cl}(K)|$ connected components, one at every cusp c of Y_Γ, where h is the class number of K.

The cohomology of Y_Γ (or more general arithmetic groups) can be studied through their cuspidal and Eisenstein part; see [Har87; Sch94]. The inclusion $Y_\Gamma \hookrightarrow X_\Gamma$ is a homotopy equivalence, hence $H^1(Y_\Gamma;\mathbb{C})$ is isomorphic to $H^1(X_\Gamma;\mathbb{C})$. By restriction to the boundary we get a map

$$\mathrm{res}\colon H^1(Y_\Gamma;\mathbb{C}) \longrightarrow H^1(\partial X_\Gamma;\mathbb{C}), \tag{1.1}$$

[1] This is required to define the canonical period used in the normalization of the L-function.

whose kernel is the cuspidal (or interior) cohomology that we denote by $H^1_!(Y_\Gamma;\mathbb{C})$. It can be identified with the image of the compactly supported cohomology $H^1_c(Y_\Gamma;\mathbb{C})$ inside $H^1(Y_\Gamma;\mathbb{C})$. The Eisenstein cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$ is the image of $H^1(\partial X_\Gamma;\mathbb{C})$ by Harder's Eisenstein map

$$\mathrm{Eis}\colon H^1(\partial X_\Gamma;\mathbb{C}) \longrightarrow H^1(Y_\Gamma;\mathbb{C}). \tag{1.2}$$

It is a complement to the cuspidal cohomology, so that we have a splitting

$$H^1(Y_\Gamma;\mathbb{C}) = H^1_!(Y_\Gamma;\mathbb{C}) \oplus H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C}). \tag{1.3}$$

Let $H = K(j(\mathscr{O}))$ be the Hilbert class field of K. Let χ be an unramified Hecke character of infinity type $(-2,0)$, and let F/H be the extension of H obtained by adjoining all the values of χ and the h-th roots of unity. By the work of Harder [Har87], we already know that the Eisenstein map is rational, in the sense that it preserves the natural F-structures

$$\mathrm{Eis}\colon H^1(\partial X_\Gamma;F) \longrightarrow H^1(Y_\Gamma;F). \tag{1.4}$$

It is natural to ask how the integral structures behave. Let $\widetilde{H}_1(Y_\Gamma;\mathscr{O}_F)$ be the free part of the homology with coefficients in the ring of integers $\mathscr{O}_F$ of F, and let $\widetilde{H}^1(Y_\Gamma;\mathscr{O}_F)$ be its dual. It is the cohomology of forms whose pairing with any integral homology class is in $\mathscr{O}_F$. It is known that in general that the Eisenstein map is not integral, in the sense that the image of

$$\mathrm{Eis}\colon \widetilde{H}^1(\partial X_\Gamma;\mathscr{O}_F) \longrightarrow H^1(Y_\Gamma;F) \tag{1.5}$$

is not contained in $\widetilde{H}^1(Y_\Gamma;\mathscr{O}_F)$. Hence we have have two integral structures on $H^1(Y_\Gamma;\mathbb{C})$: the canonical one given by

$$\mathcal{L}_0 \coloneqq \widetilde{H}^1(Y_\Gamma;\mathscr{O}_F) \tag{1.6}$$

and the one coming from the integral structure on the boundary, that we denote by

$$\mathcal{L}_{\mathrm{Eis}} \coloneqq \mathrm{Im}(\widetilde{H}^1(\partial X_\Gamma;\mathscr{O}_F)). \tag{1.7}$$

We call the denominator of the Eisenstein cohomology the $\mathscr{O}_F$-ideal

$$\mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}) \coloneqq \{\lambda \in \mathscr{O}_F \,|\, \lambda\mathcal{L}_{\mathrm{Eis}} \subset \mathcal{L}_0\}, \tag{1.8}$$

that relates these two natural integral structures. It was previously studied for Hilbert modular varieties in [Mae93], for the degree 2 cohomology on Bianchi manifolds in [Fel05] and in the special case $K = \mathbb{Q}(i)$ in [Kön91]. See also [Har18] for the latter case. Finally, in the case of Bianchi manifolds, Berger [Ber08] proves a lower bound on the denominator ideal. The main result of this paper is to give an upper bound.

By composing the Hecke character χ with the norm we get a Hecke character $\chi \circ \mathrm{N}_{H/K}$ of the same type on the Hilbert class field. Let $L(\chi \circ \mathrm{N}_{H/K},0)$ be the associated Hecke L-function at $s=0$. Let $L^{\mathrm{int}}(\chi \circ \mathrm{N}_{H/K},0)$ be the normalization by a suitable and canonical complex period to make it an algebraic integer. After replacing F by its localization $F_{\mathfrak{P}}$ at an unramified prime ideal $\mathfrak{P}$ of K, Berger [Ber08] showed that this value is a lower bound on the denominator in the sense of divisiblity, *i.e.*

$$\mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}) \subset L^{\mathrm{int}}(\chi \circ \mathrm{N}_{H/K},0)\mathscr{O}_{F_{\mathfrak{P}}}. \tag{1.9}$$

This means that we can find a class $\mathrm{Eis}(\omega)$ in the Eisenstein cohomology and a cycle C in $H_1(Y_\Gamma;\mathscr{O}_{F_{\mathfrak{P}}})$ such that $L^{\mathrm{int}}(\chi\circ \mathrm{N}_{H/K},0)\int_C \mathrm{Eis}(\omega)$ is a $\mathfrak{P}$-adic unit. In particular, we need to multiply $\mathrm{Eis}(\omega)$ at least by $L^{\mathrm{int}}(\chi\circ \mathrm{N}_{H/K},0)$ to make it an integral class. On the other hand, we prove the following global result.

THEOREM. *(Theorem 5.3) We have the upper bound (in the sense of divisibility) on the denominator*

$$2^{-1}D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi\circ \mathrm{N}_{H/K},0)\mathscr{O}_F\subset \mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}). \tag{1.10}$$

The theorem tells us that given a class $\mathrm{Eis}(\omega)$ in $\mathcal{L}_{\mathrm{Eis}}$, we need to multiply it at most by the value $2^{-1}D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi\circ \mathrm{N}_{H/K},0)$ to make it integral. In particular, if $\mathfrak{P}$ is an unramified prime coprime to 2, then combining with Berger's result we get

$$L^{\mathrm{int}}(\chi\circ \mathrm{N}_{H/K},0)\mathscr{O}_{F_{\mathfrak{P}}}=\mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}), \tag{1.11}$$

since D is a $\mathfrak{P}$-adic unit.

A few words on the proof of the theorem Let χ be an unramified Hecke character of type $(-2,0)$. For every ideal class $\mathfrak{a}$ corresponding to a cusp c of Y_Γ we have a form $\omega_{\chi,\mathfrak{a}}$ on the corresponding boundary component. They are integral forms spanning the cohomology of the boundary $H^1(\partial X_\Gamma;\mathbb{C})$. Their image by the Eisenstein map are forms $E_{\chi,\mathfrak{a}}$ that span the h-dimensional Eisenstein cohomology $H^1_{\mathrm{Eis}}(X_\Gamma;\mathbb{C})$. These forms are not integral, they define classes in $H^1_{\mathrm{Eis}}(X_\Gamma;F)$ but not in $\widetilde{H}^1_{\mathrm{Eis}}(X_\Gamma;\mathscr{O}_F)$.

On the other hand, we have another basis $\widehat{E}_{\chi,\mathfrak{a}}$ of the Eisenstein cohomology. These forms appear in the work of Ito [Ito87]. A more general construction of such Eisenstein classes was done by Bergeron-Charollois-Garcia in [BCG20; BCG21] using the Mathai-Quillen formalism.

For a lattice L in K define the Sczech cocycle $\Phi_L\colon\Gamma\longrightarrow\mathbb{C}$ by

$$\Phi_L\begin{pmatrix}a & b\\ c & d\end{pmatrix}:=\begin{cases} I\left(\frac{a+d}{c}\right)G_2(L)-D(a,c,L) & \text{if}\quad c\neq 0,\\ I\left(\frac{b}{d}\right)G_2(L) & \text{if}\quad c=0,\end{cases}$$

where $G_2(L)$ is an Eisenstein series and $D(a,c,L)$ a Dedekind sum; see Section 2 for the definitions. The forms $\widehat{E}_{\chi,\mathfrak{a}}$ are related to the Sczech cocycle as follows

$$\Phi_{\mathfrak{a}}(\gamma)=\chi(\mathfrak{a})\int_{u_0}^{\gamma u_0}\widehat{E}_{\chi,\mathfrak{a}},$$

where u_0 is any point on $\mathbb{H}_3$ and $\gamma\in\Gamma$. This formula is proved by using the idea of [BCG21] to move the path of integration $[u_0,\gamma u_0]$ to infinity. More precisely, choose a cusp v of Y_Γ and let $[v,\gamma^{-1}v]$ be the modular symbol joining the two cusps v and $\gamma^{-1}v$. There is a homotopy between $[u_0,\gamma u_0]$ and $[v,\gamma^{-1}v]$; see Figure 4. The integral along the modular symbol $[v,\gamma^{-1}v]$ gives the Dedekind sum, whereas the term $I\left(\frac{a+d}{c}\right)G_2(\mathfrak{a})$ is a contribution from the cusps. Note that this formula already appears in the work of Ito [Ito87, Theorem. 3].

After a suitable normalization, the Sczech cocycle takes values in $\mathscr{O}_H$. Hence, contrary to the forms $E_{\chi,\mathfrak{a}}$, the forms $\widehat{E}_{\chi,\mathfrak{a}}$ are integral and define classes in $\widetilde{H}^1(X_\Gamma;\mathscr{O}_F)$. The two bases $E_{\chi,\mathfrak{a}}$ and $\widehat{E}_{\chi,\mathfrak{a}}$ of the Eisenstein cohomology $H^1(X_\Gamma;\mathbb{C})$ are related by a matrix M_χ in $\mathrm{Mat}_h(F)$ whose determinant is $L(\chi\circ\mathrm{N}_{H/K},0)$. This explains the appearance of the L-function in the denominator.

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2 Eisenstein series, L-functions and Sczech's cocycle

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic imaginary field with $D \equiv 1 \pmod 8$. In particular, it's discriminant is $d_K = D$, and since $D \neq -1, -3$ we have $\mathscr{O}^\times = \{\pm 1\}$. Let $\mathscr{O} = \mathbb{Z} + \tau\mathbb{Z}$ be its ring of integers, where $\tau = \frac{1+\sqrt{D}}{2}$. We view $\mathscr{O}$ as a lattice in $\mathbb{C}$ after fixing some embedding $K \subset \mathbb{C}$.

2.1 Canonical periods

We follow [Scz86, Section. 5, p. 103]. Let $\tau = \frac{1+\sqrt{D}}{2}$ so that $\mathscr{O} = \mathbb{Z} + \tau\mathbb{Z}$. Let

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2ni\pi\tau}\right) \tag{2.1}$$

be the Dedekind eta function and define

$$u := -\frac{2^{12}\eta(2\tau)^{24}}{\eta(\tau)^{24}}. \tag{2.2}$$

It is a unit in the Hilbert class field H. Define the elliptic curve

$$E : y^2 = 4x^3 - ax - b \tag{2.3}$$

where

$$a := 12D(u-16), \qquad b := \left(2\sqrt{D}\right)^3 \sqrt{u(j-1728)}, \tag{2.4}$$

j is the j-invariant $j(\tau)$ and the square root is chosen such that b is a positive real number. Then a and b are in $\mathscr{O}_H$, the period lattice is $L_{\mathscr{O}} = \Omega\mathscr{O}$ where

$$\Omega := \frac{\pi}{(144|D|)^{\frac{1}{4}}} \frac{\eta(\tau)^4}{\eta(2\tau)^2}, \tag{2.5}$$

and the discriminant of the elliptic curve is

$$\Delta(L_{\mathscr{O}}) = 12^6 D^3 u. \tag{2.6}$$

The Weierstrass equation is related to the period lattice $L_{\mathscr{O}}$ by $a = g_2(L_{\mathscr{O}})$ and $b = g_3(L_{\mathscr{O}})$, where for any lattice $L \subset K$ we define

$$g_2(L) := 60 \sum_{\omega \in L}{}' \frac{1}{\omega^4}, \qquad g_3(L) := 140 \sum_{\omega \in L}{}' \frac{1}{\omega^6}. \tag{2.7}$$

For any fractional ideal $\mathfrak{a}$ of K let $\sigma = \sigma_{\mathfrak{a}} = (\mathfrak{a}, H/K)$ be the Artin symbol. Let $E_{\mathfrak{a}} = E^{\sigma_{\mathfrak{a}}}$ be the elliptic curve given by the Weierstrass equation

$$E_{\mathfrak{a}}\colon y^2 = 4x^3 - a^{\sigma}x - b^{\sigma}, \tag{2.8}$$

which is also defined over $\mathscr{O}_H$. Since $j(\mathscr{O})^{\sigma_{\mathfrak{a}}} = j(\mathfrak{a}^{-1})$ we have that $E_{\mathfrak{a}} \simeq \mathbb{C}/L_{\mathfrak{a}}$ where $L_{\mathfrak{a}} = \Omega(\mathfrak{a})\mathfrak{a}^{-1}$ for some complex period $\Omega(\mathfrak{a})$. Hence $g_k(L_{\mathfrak{a}}) = g_k(L)^{\sigma}$ is is in $\mathscr{O}_H$ for $k = 2, 3$. Let $\lambda(\mathfrak{a})$ be the complex number

$$\lambda(\mathfrak{a}) := \frac{\Omega(\mathfrak{a})}{\Omega}, \tag{2.9}$$

so that $L_{\mathfrak{a}} = \lambda(\mathfrak{a})\mathfrak{a}^{-1}L_{\mathscr{O}}$. It has the following properties [Rob78, Appendix. D(e) and D(f) on p. 371]

1. $\lambda(\mathfrak{a}) \in H^{\times}$
2. $\lambda(\alpha\mathfrak{a}) = \alpha\lambda(\mathfrak{a})$,
3. $\lambda(\mathfrak{a}\mathfrak{c}) = \lambda(\mathfrak{c})\lambda(\mathfrak{a})^{\sigma_{\mathfrak{c}}}$.

2.2 Kronecker-Eisenstein series

Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be any lattice in $\mathbb{C}$ where ω_1 and ω_2 are complex numbers such that ω_1/ω_2 has positive imaginary part. We define

$$D(L) := I(\omega_1\overline{\omega}_2) = 2i|\omega_2|^2 \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right). \tag{2.10}$$

where $I(z) := z - \bar{z}$. The order of L is the order

$$\mathscr{O}(L) := \{\lambda \in \mathbb{C} |\, \lambda L = L\} \tag{2.11}$$

of $\mathscr{O}$, and the lattice L is homothetic to an ideal in $\mathscr{O}(L)$. In particular, if $L = \mathfrak{a}$ is a fractional ideal then $\mathscr{O}(\mathfrak{a}) = \mathscr{O}$. Consider the character

$$\theta(z) := \exp\left(2i\pi\frac{I(z)}{D(L)}\right). \tag{2.12}$$

For a non-negative integer k and a complex number s we define the Kronecker-Eisenstein series

$$G(s, k, p, q, L) := {\sum_{\omega\in L}}' \theta(w\bar{p}) \frac{\overline{q+\omega}^k}{|q+\omega|^{2s+k}}, \tag{2.13}$$

which converges for $\mathrm{Re}(s) > 1$ and the $'$ means that we remove $\omega = -q$ from the summation if q is in L. This is the series considered by Weil in [Wei76, section VIII]. The function admits a meromorphic continuation to the whole plane with only possible poles at $s = 0$ (if $k = 0$ and q is in L) and at $s = \frac{1}{2}$ (if $k = 0$ and p is in L); see [Wei76, section VIII, p. 80]. Moreover it satisfies the functional equation

$$\mathcal{E}(s, k, p, q, L) = \theta(p\bar{q})\mathcal{E}(1 - s, k, p, q, L) \tag{2.14}$$

where

$$\mathcal{E}(s,k,p,q,L) \coloneqq \left(\frac{2i\pi}{D(L)}\right)^{-s}\Gamma\left(s+\frac{k}{2}\right)G(s,k,p,q,L). \tag{2.15}$$

For positive integers k we set

$$G_k(z,L) \coloneqq G\left(\frac{k}{2},k,0,z,L\right) = \sum_{\omega\in L}{}' \left.\frac{1}{(z+\omega)^k|z+k|^\lambda}\right|_{\lambda=0} \tag{2.16}$$

and

$$G(z,L) \coloneqq \frac{2i\pi}{D(L)}G(0,2,0,z,L). \tag{2.17}$$

They satisfy the following homogeneity properties

$$\begin{aligned} G_k(\alpha z,\alpha L) &= \alpha^{-k}G_k(z,L)\\ G(\alpha z,\alpha L) &= \frac{\bar{\alpha}}{\alpha}G(z,L). \end{aligned} \tag{2.18}$$

When $z=0$ we set

$$\begin{aligned} G(L) &\coloneqq G(0,L),\\ G_k(L) &\coloneqq G_k(0,L). \end{aligned} \tag{2.19}$$

Let $L_{\mathfrak{a}}$ be the lattices of Section 2.1. The following result is well known and was proved by Damerell [Dam71]. See also [Rob78, Appendix D(c)].

Proposition 2.1. *The value $2\sqrt{D}G_2(L_{\mathfrak{a}})$ is an algebraic integer in $\mathscr{O}_H$.*

2.3 Sczech cocycle

For a and c in $\mathscr{O}$, and c nonzero, we define the Dedekind sum

$$D(a,c,L) \coloneqq \frac{1}{c}\sum_{r\in L/cL} G_1\left(\frac{ar}{c},L\right)G_1\left(\frac{r}{c},L\right). \tag{2.20}$$

In [Scz84], Sczech shows that the map $\Phi_L\colon\Gamma\longrightarrow\mathbb{C}$ defined by

$$\Phi_L\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{cases} I\left(\frac{a+d}{c}\right)G_2(L) - D(a,c,L) & \text{if} \quad c\neq 0,\\ I\left(\frac{b}{d}\right)G_2(L) & \text{if} \quad c=0\end{cases}$$

is a cocycle.

Remark 2.1. We will show in Theorem 4.9 that $\Phi_{\mathfrak{a}}(\gamma) = \chi(\mathfrak{a}) \int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}}$ for a certain closed form $\widehat{E}_{\chi,\mathfrak{a}}$ in $\Omega^1(Y_\Gamma)$. Since the form is closed, the integral is independant of the basepoint u_0 and we have

$$\begin{aligned}\Phi_{\mathfrak{a}}(\gamma_1\gamma_2) &= \int_{u_0}^{\gamma_1\gamma_2 u_0} \widehat{E}_{\chi,\mathfrak{a}} \\ &= \int_{\gamma_2 u_0}^{\gamma_1\gamma_2 u_0} \widehat{E}_{\chi,\mathfrak{a}} + \int_{u_0}^{\gamma_2 u_0} \widehat{E}_{\chi,\mathfrak{a}} \\ &= \int_{u_0}^{\gamma_1 u_0} \widehat{E}_{\chi,\mathfrak{a}} + \int_{u_0}^{\gamma_2 u_0} \widehat{E}_{\chi,\mathfrak{a}} \\ &= \Phi_{\mathfrak{a}}(\gamma_1) + \Phi_{\mathfrak{a}}(\gamma_2). \end{aligned} \tag{2.21}$$

This gives an alternative proof of the cocycle property of the Sczech cocycle.

Furthermore, Sczech proved that the cocycle is integral. The following is proved in [Scz84, Satz. 4].

PROPOSITION 2.2 (Sczech)**.** *For any γ in Γ, the value $2\Phi_{L_{\mathfrak{a}}}(\gamma)$ is an algebraic integer in $\mathscr{O}_H$.*

2.4 Hecke characters and L-functions

Let $\chi\colon \mathcal{I} \longrightarrow \mathbb{C}^\times$ be an unramified algebraic Hecke character of infinity type (k, j). Let F_χ be the finite extension of H obtained by adjoining the image of χ. So χ is a character

$$\chi\colon \mathcal{I} \longrightarrow F_\chi^\times \subset \mathbb{C}^\times \tag{2.22}$$

on the group $\mathcal{I}$ of fractional ideals of K such that on principal ideals (α) in K we have

$$\chi((\alpha)) = \alpha^k \bar{\alpha}^j. \tag{2.23}$$

Let F be the field obtained by adjoining the h-th roots of unity to F_χ. If χ and $\widetilde{\chi}$ are two Hecke characters of the same type, then there is a character $\varphi \in \widehat{\mathrm{Cl}(K)}$ on the class group such that $\widetilde{\chi} = \varphi\chi$. Thus F is the smallest field containing all the fields F_χ.

PROPOSITION 2.3. *For any fractional ideal $\mathfrak{a}$ and any Hecke character χ of infinity type $(-2, 0)$, the value $\chi(\mathfrak{a})\lambda(\mathfrak{a})^2$ is a unit in $\mathscr{O}_F$.*

Proof. For $\sigma \in \mathrm{Gal}(H/K)$ we have $\Delta(L_{\mathscr{O}})^\sigma = 12^6 D^3 u^\sigma$ where $u^\sigma \in \mathscr{O}_H^\times$ is a unit. Hence

$$c(\sigma) := \frac{\Delta(L_{\mathscr{O}})}{\Delta(L_{\mathscr{O}})^\sigma} = \frac{u}{u^\sigma} \tag{2.24}$$

is a unit. When $\sigma = \sigma_{\mathfrak{a}}$ then $c(\sigma_{\mathfrak{a}}) = \Delta(L_{\mathscr{O}})/\Delta(L_{\mathfrak{a}})$. Since $\Delta(\alpha L_{\mathscr{O}}) = \alpha^{-12}\Delta(L_{\mathscr{O}})$ we have

$$c(\sigma_{\mathfrak{a}}) = \Delta(L_{\mathscr{O}})/\Delta(L_{\mathfrak{a}}) = \frac{\Omega^{-12}\Delta(\mathscr{O})}{\Omega(\mathfrak{a})^{-12}\Delta(\mathfrak{a}^{-1})} = \lambda(\mathfrak{a})^{12}\frac{\Delta(\mathscr{O})}{\Delta(\mathfrak{a}^{-1})}. \tag{2.25}$$

Moreover, by [Sha87, p.49] or[2] [Lan78, Theorem. 5 p.165] we have

$$\frac{\Delta(\mathfrak{a}^{-1})}{\Delta(\mathscr{O})}\mathscr{O}_H = \mathfrak{a}^{12}\mathscr{O}_H. \tag{2.26}$$

Combining the two gives $\lambda(\mathfrak{a})^{12}\mathscr{O}_H = \mathfrak{a}^{12}\mathscr{O}_H$, and by comparing the prime factorizations we get

$$\lambda(\mathfrak{a})\mathscr{O}_H = \mathfrak{a}\mathscr{O}_H. \tag{2.27}$$

On the other hand, since $\mathfrak{a}^h = \alpha\mathscr{O}$ for some $\alpha \in K^\times$ and h the class number, we have $\chi(\mathfrak{a})^h\mathscr{O}_F = \mathfrak{a}^{-2h}\mathscr{O}_F$. By comparing the prime decomposition we then also get

$$\chi(\mathfrak{a})\mathscr{O}_F = \mathfrak{a}^{-2}\mathscr{O}_F. \tag{2.28}$$

Combining (2.27) and (2.28) we then get

$$\chi(\mathfrak{a})\lambda(\mathfrak{a})^2\mathscr{O}_F = \mathscr{O}_F. \tag{2.29}$$

□

Let χ be a Hecke character of infinity type (k, j). The Hecke L-function

$$L(\chi, s) \coloneqq \sum_{0\neq\mathfrak{a}\subseteq\mathscr{O}} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s} \tag{2.30}$$

converges for $\mathrm{Re}(s) > 1 + \frac{k+j}{2}$, admits a meromorphic continuation to the whole plane and a functional equation.

Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ be integral ideal representatives of the class group of K. Every integral ideal $\mathfrak{a}$ can be written $\mathfrak{a} = (\alpha)\mathfrak{a}_i$ where $(\alpha) \subset \mathfrak{a}_i^{-1}$. Hence we can write the L-function as

$$\begin{aligned} L(\chi, s) &= \sum_{i=1}^h \frac{\chi(\mathfrak{a}_i)}{\mathrm{N}(\mathfrak{a}_i)^s}\frac{1}{w(\mathfrak{a}_i^{-1})} \sum_{\alpha\in\mathfrak{a}_i^{-1}-\{0\}} \frac{\chi((\alpha))}{(\alpha\bar{\alpha})^s} \\ &= \sum_{i=1}^h \frac{\chi(\mathfrak{a}_i)}{\mathrm{N}(\mathfrak{a}_i)^s}\frac{1}{w(\mathfrak{a}_i^{-1})} \sum_{\alpha\in\mathfrak{a}_i^{-1}-\{0\}} \frac{\alpha^k\bar{\alpha}^j}{|\alpha|^{2s}}, \\ &= \sum_{i=1}^h \frac{\chi(\mathfrak{a}_i)}{\mathrm{N}(\mathfrak{a}_i)^s}\frac{1}{w(\mathfrak{a}_i^{-1})} G\left(s - \frac{k+j}{2}, j-k, 0, 0; \mathfrak{a}_i^{-1}\right) \end{aligned} \tag{2.31}$$

where $w(\mathfrak{a}_i) = 2^{|\mathscr{O}^\times\cap\mathfrak{a}_i|} \in \{1, 2\}$ is the number of units in $\mathfrak{a}_i$.

From now on, let χ be of infinity type $(-2, 0)$. At $s = 0$ we then have

$$L(\chi, 0) = \sum_{i=1}^h \frac{\chi(\mathfrak{a}_i)}{w(\mathfrak{a}_i)} G_2(\mathfrak{a}_i^{-1}). \tag{2.32}$$

[2] Note that in [Lan78] the result is proved for a split prime ideal $\mathfrak{p}$. However, any class in $\mathrm{Cl}(K)$ contains such an ideal and equality (2.25) only depends on the ideal class of $\mathfrak{a}$.

Note that (2.32) does not depend on the choice of representatives. We define the algebraic L-function

$$L^{\mathrm{alg}}(\chi, s) \coloneqq \Omega^{-2} L(\chi, s), \tag{2.33}$$

where Ω is our the canonical period (2.5). We also define the integral L-function

$$L^{\mathrm{int}}(\chi, s) \coloneqq 4\sqrt{D} L^{\mathrm{alg}}(\chi, s). \tag{2.34}$$

The normalizations are chosen such that we have the following.

Proposition 2.4. *The value $L^{\mathrm{alg}}(\chi, 0)$ is in F and nonzero. More precisely, the value $L^{\mathrm{int}}(\chi, 0)$ is in $\mathscr{O}_F$.*

Proof. By the homogeneity of G_2 we have $G_2(\mathfrak{a}_i^{-1}) = \Omega(\mathfrak{a}_i)^2 G_2(L_{\mathfrak{a}_i})$ and hence

$$4\sqrt{D}\Omega^{-2} L(\chi, 0) = \sum_{i=1}^{h} \chi(\mathfrak{a}_i)\lambda(\mathfrak{a}_i)^2 \frac{2}{w(\mathfrak{a}_i^{-1})} 2\sqrt{D} G_2(L_{\mathfrak{a}_i}). \tag{2.35}$$

and all the terms are algebraic. Furthermore, by Proposition 2.3 we have that $\chi(\mathfrak{a}_i)\lambda(\mathfrak{a}_i)^2$ is a unit in $\mathscr{O}_F$. The integrality of $L^{\mathrm{int}}(\chi, 0)$ follows since $2\sqrt{D} G_2(L_{\mathfrak{a}_i})$ is in $\mathscr{O}_H$ by Proposition 2.1, and $w(\mathfrak{a}_i^{-1}) = 1$ or 2.

The character χ has infinity type $(-2, 0)$. Hence the Hecke character $\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^2$ has type $(0, 2)$, and $\mathcal{C}(\mathfrak{a}) \coloneqq \overline{\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^2}$ has type $(2, 0)$. Then we have

$$L(\chi, 0) = L(\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^2, 2) = \frac{2\pi}{\sqrt{|D|}} L(\mathcal{C}, 1), \tag{2.36}$$

where the second equality is the functional equation; see [Gre85]. By arguing as in [Gre85] (in the case $n = 2$), we have that $L(\mathcal{C}, 1)$ is nonzero. $\square$

Composing with the norm, one gets a Hecke character on H

$$\chi \circ \mathrm{N}_{H/K} \colon \mathrm{Cl}(H) \longrightarrow F^\times \subset \mathbb{C}^\times \tag{2.37}$$

of infinity type $(-2, 0)$, where for any fractional ideal $\mathfrak{b}$ of H

$$\mathrm{N}_{H/K}(\mathfrak{b}) = K \cap \prod_{\sigma \in \mathrm{Gal}(H/K)} \sigma(\mathfrak{b}) \tag{2.38}$$

is the relative ideal norm. We can then also consider consider the L-function

$$L(\chi \circ \mathrm{N}_{H/K}, s) = \sum_{0 \neq \mathfrak{b} \subseteq \mathscr{O}_H} \frac{\chi(\mathrm{N}_{H/K}(\mathfrak{b}))}{\mathrm{N}(\mathfrak{b})^s} = \prod_{\varphi \in \widehat{\mathrm{Cl}(K)}} L(\varphi\chi, s). \tag{2.39}$$

It follows from Proposition 2.4 that

$$L^{\mathrm{alg}}(\chi \circ \mathrm{N}_{H/K}, 0) \coloneqq \Omega^{-2h} L(\chi \circ \mathrm{N}_{H/K}, 0) \tag{2.40}$$

is a nonzero algebraic number in F and that

$$L^{\mathrm{int}}(\chi \circ \mathrm{N}_{H/K}, 0) \coloneqq (4D)^{\frac{h}{2}} L^{\mathrm{alg}}(\chi \circ \mathrm{N}_{H/K}, 0) \tag{2.41}$$

is an algebraic integer in $\mathscr{O}_F$.

3 Eisenstein cohomology

Let $\mathbb{H}_3$ be the hyperbolic 3-space

$$\mathbb{H}_3 \coloneqq \{u = z + jv \mid z \in \mathbb{C}, v \in \mathbb{R}_{>0}\}, \tag{3.1}$$

where $ij = -ji$ and $i^2 = j^2 = -1$. For $u = z + jv$ let $\overline{u} = \overline{z} - jv$ and $|u| = u\overline{u} = |z|^2 + v^2$. The group $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathbb{H}_3$ by

$$u \longmapsto (au+b)(cu+d)^{-1} = \frac{(au+b)(\overline{cu+d})}{|cz+d|^2+v^2}, \tag{3.2}$$

and the stabilizer of j is $\mathrm{SU}(2)$. Hence the symmetric space $\mathrm{SL}_2(\mathbb{C})/\,\mathrm{SU}(2)$ is isomorphic to $\mathbb{H}_3$. For a fractional ideal $\mathfrak{a}$ of K let

$$\Gamma(\mathfrak{a}) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \,\middle|\, a, d \in \mathscr{O}, b \in \mathfrak{a}, c \in \mathfrak{a}^{-1} \right\} \tag{3.3}$$

be the subgroup of $\mathrm{SL}_2(K)$ preserving $\mathfrak{a} \oplus \mathscr{O}$. Let $\Gamma \coloneqq \Gamma(\mathscr{O}) = \mathrm{SL}_2(\mathscr{O})$ and

$$Y_\Gamma \coloneqq \Gamma \backslash \mathbb{H}_3. \tag{3.4}$$

This space is a non-compact 3-dimensional orbifold and can be compactified in several ways, one of them is the Borel-Serre compactification.

3.1 Borel-Serre compactification

We describe the Borel-Serre compactification of Y_Γ; see [BJ06] or [JM02] for more on compactifications of locally symmetric spaces. Define the space

$$\mathbb{H}_3^* \coloneqq \mathbb{H}_3 \cup \bigsqcup_{r \in \mathbb{P}^1(K)} \mathcal{H}_r \tag{3.5}$$

where $\mathcal{H}_r = \mathbb{P}^1(\mathbb{C}) - \{r\}$. We have a canonical map

$$\begin{aligned} \mathcal{H}_r &\longrightarrow \mathbb{C} \\ (x:y) &\longmapsto \frac{mx}{my - nx} \end{aligned} \tag{3.6}$$

where $r = (m : n)$. Hence we can view the space $\mathbb{H}_3^*$ as adding a copy of $\mathbb{C}$ at every cusp $\frac{m}{n}$. The topology on $\mathbb{H}_3^*$ is defined as follows: let $\mathcal{H}_\infty$ be the boundary component at ∞ corresponding to $r = (1 : 0)$. A sequence $u_n = z_n + jv_n$ converges to $(z_0 : 1) \in \mathcal{H}_\infty$ if $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} z_n = z_0$. If γ maps ∞ to r then u_n converges to $(x : y) \in \mathcal{H}_r$ if $\gamma^{-1}u_n$ converges to $\gamma^{-1}(x : y) \in \mathcal{H}_\infty$. The action of Γ extends to $\mathbb{H}_3^*$ by sending $z \in \mathcal{H}_r$ to $\gamma z \in \mathcal{H}_{\gamma r}$, where $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x : y) = (ax + by : cx + dy). \tag{3.7}$$

We define $X_\Gamma := \Gamma\backslash\mathbb{H}_3^*$. Let $C_\Gamma := \Gamma\backslash\mathbb{P}^1(K)$ be the set of equivalence class of cusps of Y_Γ. We can represent cusps by fractional ideals in K since we have a bijection [Fre90, Lemma. 3.5]

$$\begin{aligned} C_\Gamma &\longrightarrow \mathrm{Cl}(K) \\ c = \Gamma(m:n) &\longmapsto [\mathfrak{a}_c] \end{aligned} \tag{3.8}$$

where $\mathfrak{a}_c := m\mathscr{O} + n\mathscr{O}$ and $\mathrm{Cl}(K)$ is the class group. We have a bijection

$$\begin{aligned} \Gamma\backslash \bigsqcup_{r\in\mathbb{P}^1(K)} \mathcal{H}_r &\longrightarrow \bigsqcup_{c=[r]\in C_\Gamma} \Gamma_r\backslash\mathcal{H}_r \\ (x:y) &\longmapsto \gamma(x:y) \end{aligned} \tag{3.9}$$

where $(x:y)\in\mathcal{H}_r$, the element γ maps $(1:0)$ to some $[r]\in C_\Gamma$ and Γ_r is the unipotent radical of the stabilizer $\{\gamma\in\Gamma \mid \gamma c = c\}$ of r in Γ. The Borel-Serre compactification is

$$X_\Gamma := Y_\Gamma \cup \bigsqcup_{c=[r]\in C_\Gamma} \Gamma_r\backslash\mathcal{H}_r, \tag{3.10}$$

and the boundary of the Borel-Serre compactification is

$$\partial X_\Gamma = \bigsqcup_{c=[r]\in C_\Gamma} \Gamma_r\backslash\mathcal{H}_r. \tag{3.11}$$

Note that

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b\in\mathscr{O} \right\} \simeq \mathscr{O}, \tag{3.12}$$

hence

$$\begin{aligned} \Gamma_\infty\backslash\mathcal{H}_\infty &\longrightarrow \mathbb{C}/\mathscr{O} \\ \Gamma_\infty(z:1) &\longmapsto z+\mathscr{O}. \end{aligned} \tag{3.13}$$

For any $M = \begin{pmatrix} y & -x \\ -n & m \end{pmatrix}$ in $\mathrm{SL}_2(K)$ that maps $r=(m:n)$ to ∞ we have

$$M\Gamma_r M^{-1} = (M\Gamma M^{-1})_\infty. \tag{3.14}$$

Let $\mathfrak{a}_M = m\mathscr{O}+n\mathscr{O}$ be the ideal generated by the bottom row of M. Then

$$M\Gamma M^{-1} = \Gamma(\mathfrak{a}_M^{-2});$$

see for example [EGM98, Lemma. 2.2, Chapter 8]. In particular

$$(M\Gamma M^{-1})_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b\in\mathfrak{a}_M^{-2} \right\}.$$

Thus we have a map

$$\Gamma_r\backslash\mathcal{H}_r \longrightarrow (M\Gamma M^{-1})_\infty\backslash\mathcal{H}_\infty \simeq \mathbb{C}/\mathfrak{a}_M^{-2}, \tag{3.15}$$

where the first map is $(z:1)\mapsto M(z:1)$ and the second is (3.13).

3.2 Equivariant homology and cohomology

Since Y_Γ is not a manifold but rather an orbifold, we need to work with equivariant (co)homology.

Restriction of differential forms At ∞ the boundary component of $\mathbb{H}_3^*$ can be canonically embedded inside $\mathbb{H}_3$ as a horocycle by the map

$$\begin{aligned}\iota_v \colon \mathcal{H}_\infty \simeq \mathbb{C} &\longhookrightarrow \mathbb{H}_3 \\ z &\longmapsto z + jv.\end{aligned} \tag{3.16}$$

For a differential form ω in $\Omega^k(\mathbb{H}_3)$ we define the restriction map

$$\begin{aligned}\mathrm{res}_\infty \colon \Omega^k(\mathbb{H}_3) &\longrightarrow \Omega^k(\mathcal{H}_\infty) \\ \omega &\longmapsto \mathrm{res}_\infty(\omega) \coloneqq \lim_{v\to\infty} i_v^*\omega\end{aligned} \tag{3.17}$$

to be the restriction of ω to the boundary component at the cusp ∞. Note that ι_v is $\mathscr{O}$-equivariant in the sense that for every $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ in Γ_∞ we have

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \iota_v(z) = \iota_v(z+b). \tag{3.18}$$

Hence it descends to a map $\iota_v \colon \Gamma_\infty\backslash\mathcal{H}_\infty \to Y_\Gamma$, and so does the restriction map:

$$\mathrm{res}_\infty \colon \Omega^k(Y_\Gamma) \longrightarrow \Omega^k(\Gamma_\infty\backslash\mathcal{H}_\infty). \tag{3.19}$$

The other boundary components $\mathcal{H}_r$ can be embedded as horosphere at the other cusps. Let M be any matrix in $\mathrm{SL}_2(K)$ sending r to ∞. We define the embedding

$$\iota_{v,M}(z) \coloneqq M^{-1} \circ \iota_v \circ M \colon \mathcal{H}_r \longrightarrow \mathbb{H}_3. \tag{3.20}$$

Its image is a horosphere, see figure 1. If N is another matrix sending r to ∞, then

$$M = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} N, \tag{3.21}$$

and replacing M by N gives

$$\iota_{N,r} = M^{-1} \circ \iota_{v/a^2} \circ M. \tag{3.22}$$

Hence the embedding depends on the choice of M, but taking the limits gives a well-defined map

$$\begin{aligned}\mathrm{res}_r \colon \Omega^k(\mathbb{H}_3) &\longrightarrow \Omega^k(\mathcal{H}_r) \\ \omega &\longmapsto \mathrm{res}_r(\omega) \coloneqq \lim_{v\to\infty} i_{M,v}^*\omega,\end{aligned} \tag{3.23}$$

which also descends to the quotient

$$\mathrm{res}_r \colon \Omega^k(Y_\Gamma) \longrightarrow \Omega^k(\Gamma_r\backslash\mathcal{H}_r). \tag{3.24}$$

Finally, note that for any matrix A in $\mathrm{SL}_2(K)$ and any form ω on $\mathbb{H}_3$ we have

$$A^* \,\mathrm{res}_{Ar}(\omega) = \mathrm{res}_r(A^*\omega). \tag{3.25}$$

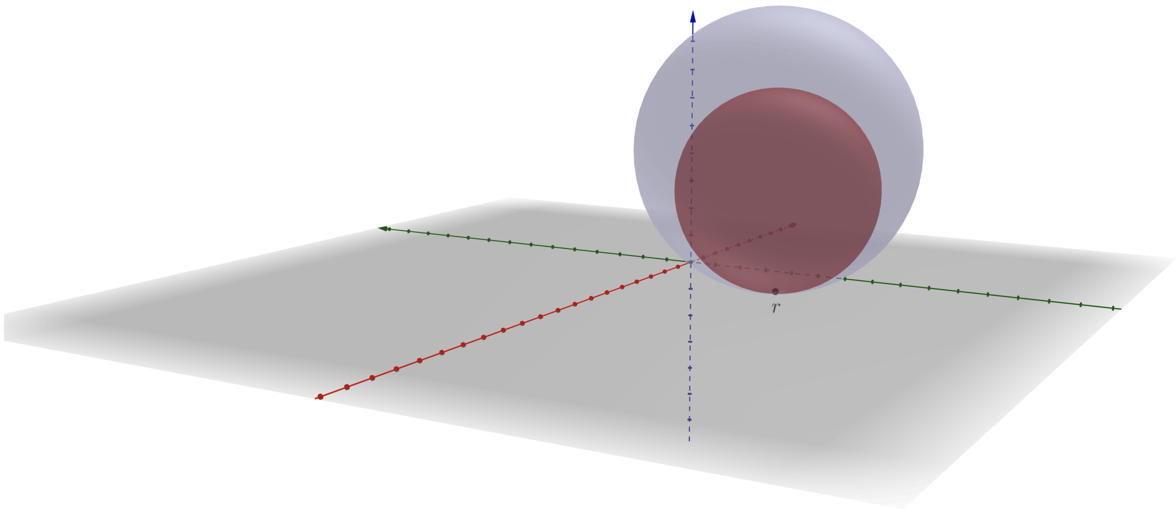

Figure 1: The embedding of $\mathcal{H}_r$ by $\iota_{r,v}$ is a horosphere in $\mathbb{H}_3$, tangent to the plane $v = 0$ at the cusp r. As v increases, the radius of the sphere decreases. Hence we can see the boundary components as horospheres at infinity.

Cohomology Following [Ste89, Section. 2] we define a k-form ω on $\mathbb{H}_3^*$ to be a k-form ω_0 on $\mathbb{H}_3$ and a family k-forms ω_r on $\mathcal{H}_r$ such that $\mathrm{res}_r(\omega_0) = \omega_r$. We denote by $\Omega^k(\mathbb{H}_3^*)$ the space of such forms. Let

$$\Omega^k(\mathbb{H}_3^*; \mathbb{C}) := \Omega^k(\mathbb{H}_3^*) \otimes_{\mathbb{R}} \mathbb{C} \tag{3.26}$$

be the differential forms valued in $\mathbb{C}$. Let $\Omega^k(\mathbb{H}_3^*; \mathbb{C})^\Gamma$ be the complex of Γ-invariant forms, consisting of forms that satisfy $\gamma^*\omega_0 = \omega_0$ and $\gamma^*\omega_r = \omega_{\gamma^{-1}r}$. Let $H^k(X_\Gamma; \mathbb{C}) = H^k(\Omega^\bullet(\mathbb{H}_3^*; \mathbb{C})^\Gamma)$ be the cohomology of this complex. Similarly we have

$$H^k(\partial X_\Gamma; \mathbb{C}) = \bigoplus_{c=[r]\in C_\Gamma} H^k(\Omega^\bullet(\mathcal{H}_r; \mathbb{C})^{\Gamma_r}) \simeq H^k(\Omega^\bullet(\partial X_\Gamma; \mathbb{C})^\Gamma). \tag{3.27}$$

It follows from (3.25) that the restriction map induces a map

$$\begin{aligned} \mathrm{res}\colon H^k(Y_\Gamma; \mathbb{C}) &\longrightarrow H^k(\partial X_\Gamma; \mathbb{C}) \\ \omega &\longmapsto (\mathrm{res}_r(\omega))_{c=[r]} \end{aligned} \tag{3.28}$$

Relative cohomology Let $\Omega^k(\mathbb{H}_3^*, \partial\mathbb{H}_3^*; \mathbb{C})^\Gamma := \Omega^k(\mathbb{H}_3^*; \mathbb{C})^\Gamma \oplus \Omega^{k-1}(\partial\mathbb{H}_3^*; \mathbb{C})^\Gamma$ be the complex with the coboundary operator

$$\begin{aligned} \delta\colon \Omega^k(\mathbb{H}_3^*, \partial\mathbb{H}_3^*; \mathbb{C})^\Gamma &\longrightarrow \Omega^{k+1}(\mathbb{H}_3^*, \partial\mathbb{H}_3^*; \mathbb{C})^\Gamma \\ (\omega, \theta) &\longmapsto (d\omega, \iota^*\omega - d\theta). \end{aligned} \tag{3.29}$$

The cohomology $H^k(X_\Gamma, \partial X_\Gamma; \mathbb{C})$ of X_Γ relative to ∂X_Γ is the cohomology associated to this complex. We have an exact sequence

$$0 \longrightarrow \Omega^{k-1}(\partial\mathbb{H}_3^*; \mathbb{C})^\Gamma \xhookrightarrow{\alpha} \Omega^k(\mathbb{H}_3^*, \partial\mathbb{H}_3^*; \mathbb{C})^\Gamma \xrightarrow{\beta} \Omega^k(\mathbb{H}_3^*; \mathbb{C})^\Gamma \longrightarrow 0 \tag{3.30}$$

where the first map is given by $\alpha(\theta) = (0, \theta)$ and the second by $\beta(\omega, \theta) = \omega$. This induces a long exact sequence in cohomology

$$\begin{tikzcd} H^{k-1}(\partial X_\Gamma;\mathbb{C}) \arrow[r, "\alpha^*"] & H^k(X_\Gamma, \partial X_\Gamma;\mathbb{C}) \arrow[r, "\beta^*"] & H^k(X_\Gamma;\mathbb{C}) \arrow[dll, "\mathrm{res}"] \\ H^k(\partial X_\Gamma;\mathbb{C}) \arrow[r, "\alpha^*"] & H^{k+1}(X_\Gamma, \partial X_\Gamma;\mathbb{C}) \arrow[r, "\beta^*"] & H^{k+1}(X_\Gamma;\mathbb{C}) \end{tikzcd} \tag{3.31}$$

and the boundary map res is the restriction to the boundary.

Homology For an abelian group A let $C_n(\mathbb{H}_3; A)$ be the $\mathbb{Z}$-module of singular n-chains valued in A. The action of Γ on $\mathbb{H}_3$ endows $C_n(\mathbb{H}_3; A)$ with a Γ-module structure and we define the complex of coinvariant chains

$$C_n(\mathbb{H}_3; A)_\Gamma \coloneqq C_n(\mathbb{H}_3; A)/\langle \sigma - \gamma\sigma \rangle, \tag{3.32}$$

where we quotient by the submodule generated by all $\sigma - \gamma\sigma$ with $\sigma \in C_n(\mathbb{H}_3; A)$. Let $H_n(Y_\Gamma; A)$ be the homology of this complex. We define $H_n(X_\Gamma; A)$ similarly.

Let $u_0 \in \mathbb{H}_3$ be any basepoint and $[u_0, \gamma u_0]$ be a path segment joining u_0 and γu_0 for $\gamma \in \Gamma$. The boundary of $[u_0, \gamma u_0]$ is $\gamma u_0 - u_0$ hence it represents a class in $H_1(Y_\Gamma; \mathbb{C})$.

PROPOSITION 3.1. *The map $\Gamma \longrightarrow H_1(Y_\Gamma; \mathbb{Z})$ sending γ to $[u_0, \gamma u_0]$ is a surjective morphism and independant of u_0.*

Proof. If Y_Γ were a manifold (for example if Γ were some congruence subgroup of $\mathrm{SL}_2(\mathscr{O})$) we could work with singular homology. Then the map would be the Hurewicz homomorphism, which is surjective since Y_Γ is path connected; see [Hat01, Theorem. 2A.1]. In the case of equivariant homology it works almost in the same way and we follow the proof of [Hat01].

We write $[a, a'] \sim [b, b']$ for two homologous paths joining points a, a', b and b' in $\mathbb{H}_3$. We have the following relations:

(1) $[a, a] \sim 0,$

(2) $[a, b] + [b, c] \sim [a, c]$

(3) $[a, b] \sim -[b, a]$

(4) $[a, \gamma a] \sim [b, \gamma b],$

(5) $[a, b] \sim [\gamma a, \gamma b].$

The first two ones are clear since $[a, a]$ is the boundary of the constant 2-simplex $C = \{a\}$ and $[a, b] - [a, c] + [b, c]$ is the boundary of the simplex joining a, b and c. The third one follows from (1) and (2). For (4) consider the two simplices C_1 and C_2 as in Figure 2. The boundaries are given by

$$\begin{aligned} \partial C_1 &= [\gamma a, \gamma b] - [a, \gamma b] + [a, \gamma a] \\ \partial C_2 &= [b, \gamma b] - [a, \gamma b] + [a, b] \end{aligned} \tag{3.33}$$

so that

$$\partial(C_1 - C_2) = [a, \gamma a] - [b; \gamma b] + [\gamma a, \gamma b] - [a, b] = [a, \gamma a] - [b, \gamma b] \tag{3.34}$$

where $[\gamma a, \gamma b] - [a, b] = 0$ since we are in the complex of coinvariants. For (5), consider the simplices

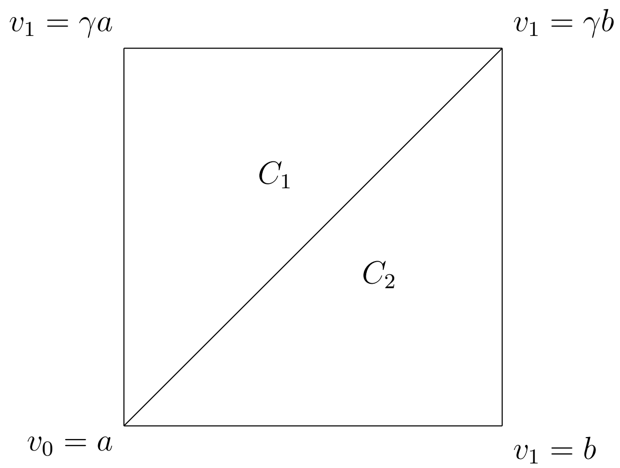

Figure 2: The equivalence $[a, \gamma a] \sim [b, \gamma b]$.

C_1 and C_2 as in Figure 3. The boundaries are given by

$$\partial C_1 = [b, \gamma a] - [\gamma b, \gamma a] + [\gamma b, b] \tag{3.35}$$
$$\partial C_2 = [b, \gamma a] - [a, \gamma a] + [a, b] \tag{3.36}$$

so that

$$\partial(C_1 - C_2) = [\gamma a, \gamma b] - [b, \gamma b] + [a, \gamma a] - [a, b] = [\gamma a, \gamma b] - [a, b] \tag{3.37}$$

Let us now prove the statement of the proposition. The fact that it is independant of the basepoint follows from (4), and the fact that it is a homorphism follows from (2) and (4) since

$$[u_0, \gamma_1 u_0] + [u_0, \gamma_2 u_0] \sim [\gamma_2 u_0, \gamma_1 \gamma_2 u_0] + [u_0, \gamma_2 u_0] \sim [u_0, \gamma_1 \gamma_2 u_0]. \tag{3.38}$$

For surjectivity suppose we have a class represented by a cycle

$$\sigma = \sum_{i=1}^{m} n_i [a_i, b_i] \in C_1(\mathbb{H}_3)_\Gamma. \tag{3.39}$$

After relabeling the paths we can suppose that $n_i = \pm 1$ and using (3) we can suppose that $n_i = 1$. Since the boundary is

$$\partial\sigma = \sum_i (b_i - a_i) = 0 \tag{3.40}$$

we necessarily have that every b_j is equal to $\gamma_{ij} a_i$ for a unique a_i and some $\gamma_{ij} \in \Gamma$. We can see it as a permutation on the set $\{1, \ldots, m\}$, where we send i to j if a_i is Γ-equivalent to b_j.

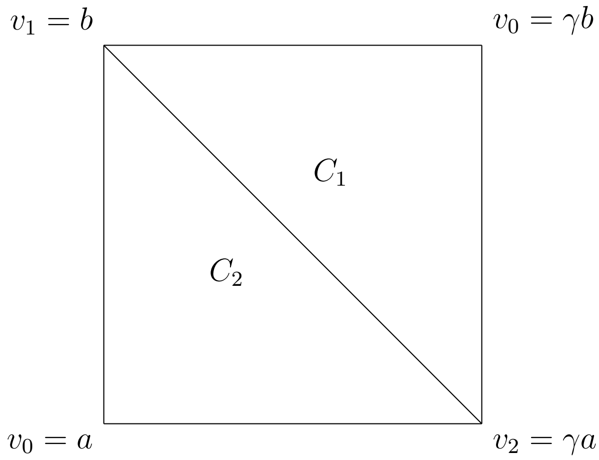

Figure 3: The equivalence $[a, b] \sim [\gamma a, \gamma b]$.

First suppose that the corresponding permutation is the identity $i.e$ $b_i = \gamma_i a_i$ for every i . Then using (4) and the fact that the map is a morphism we get

$$\sigma = \sum_{i=1}^{m} [a_i, \gamma_i a_i] \sim \sum_{i=1}^{m} [u_0, \gamma_i u_0] \sim [u_0, \gamma_1 \cdots \gamma_m u_0]. \tag{3.41}$$

Now suppose that the permutation contains some cycle of order n, which means that σ contains a cycle

$$\sigma' = [a_1, \gamma_n a_n] + [a_2, \gamma_1 a_1] + [a_3, \gamma_2 a_2] + \cdots + [a_n, \gamma_{n-1} a_{n-1}]. \tag{3.42}$$

Using (5) and (2) we can sum the first two terms

$$[a_1, \gamma_n a_n] + [a_2, \gamma_1 a_1] \sim [\gamma_1 a_1, \gamma_1 \gamma_n a_n] + [a_2, \gamma_1 a_1] \sim [a_2, \gamma_1 \gamma_n a_n]. \tag{3.43}$$

By induction we then get $\sigma' \sim [a_n, \gamma_{n-1}\gamma_{n-2} \cdots \gamma_1 \gamma_n a_n]$, so that we are reduced to the first case. $\square$

Integral structure on the cohomology Let $\mathcal{R} \subset \mathbb{C}$ be a torsion free $\mathscr{O}$-submodule. We will later take $\mathcal{R} = \mathscr{O}_F$. We have a pairing

$$\begin{aligned} \langle \quad , \quad \rangle \colon H_1(Y_\Gamma; \mathbb{C}) \otimes H^1(Y_\Gamma; \mathbb{C}) &\longrightarrow \mathbb{C} \\ ([\sigma], [\omega]) &\longmapsto \int_\sigma \omega \end{aligned} \tag{3.44}$$

where $\omega \in \Omega^1(\mathbb{H}_3; \mathbb{C})^\Gamma$ and $\sigma \in C_1(\mathbb{H}_3)_\Gamma$; see [Fel05, Satz. 3]. Note that $\mathcal{R} \otimes_{\mathcal{R}} \mathbb{C} = \mathbb{C}$, so that we have a map

$$H_1(Y_\Gamma; \mathcal{R}) \longrightarrow H_1(Y_\Gamma; \mathcal{R}) \otimes_{\mathcal{R}} \mathbb{C} = H_1(Y_\Gamma; \mathbb{C}). \tag{3.45}$$

Let $\widetilde{H}_1(Y_\Gamma; \mathcal{R})$ be the image of this map. The kernel is the torsion part of $H_1(Y_\Gamma; \mathcal{R})$, so that we can identify $\widetilde{H}_1(Y_\Gamma; \mathcal{R})$ with the free part of $H_1(Y_\Gamma; \mathcal{R})$. We use the pairing to define the cohomology groups

$$\widetilde{H}^1(Y_\Gamma; \mathcal{R}) := \left\{ [\omega] \in H^1(Y_\Gamma; \mathbb{C}) \,\middle|\, \langle [\omega], [\sigma] \rangle \in \mathcal{R} \quad \text{for all } [\sigma] \in \widetilde{H}_1(Y_\Gamma; \mathcal{R}) \right\}. \tag{3.46}$$

Since $\mathcal{R}$ is torsion-free we can identify $H_1(Y_\Gamma;\mathcal{R})$ with $H_1(Y_\Gamma;\mathbb{Z})\otimes_\mathbb{Z}\mathcal{R}$. Hence, by Proposition 3.1 a class $[\omega]$ is in $\widetilde{H}^1(Y_\Gamma;\mathcal{R})$ if and only if

$$\int_{u_0}^{\gamma u_0}\omega\in\mathcal{R} \tag{3.47}$$

for all $\gamma\in\Gamma$.

Remark 3.1. The $\mathcal{R}$-module $\widetilde{H}_1(Y_\Gamma;\mathcal{R})$ is the torsion free part of the sheaf cohomology $H^1(Y_\Gamma;\mathcal{R})$, that we identify with the image

$$\operatorname{Im}\left(H^1(Y_\Gamma;\mathcal{R})\longrightarrow H^1(Y_\Gamma;\mathbb{C})\right). \tag{3.48}$$

4 Cohomology of the boundary

Recall that for any matrix M in $\mathrm{SL}_2(K)$ sending r to ∞ we had a map (3.15)

$$\begin{aligned}\phi_{M,r}\colon \Gamma_r\backslash\mathcal{H}_r &\longrightarrow \mathbb{C}/\mathfrak{a}_M^{-2},\\ \Gamma_r(z:1)&\longmapsto Mz\end{aligned}$$

where $\mathfrak{a}_M=m\mathscr{O}+n\mathscr{O}$ is the ideal generated by the bottom row of M. Note that $[\mathfrak{a}_M]=[\mathfrak{a}_c]$ in the bijection (3.8) between C_Γ and $\mathrm{Cl}(K)$.

Lemma 4.1. *Let χ be an unramified Hecke character of infinity type $(-2,0)$. The forms*

$$\begin{aligned}\omega_{\chi,r}&\coloneqq\chi(\mathfrak{a}_M)^{-1}\phi_{M,r}^*dz\\ \bar\omega_{\chi,r}&\coloneqq\chi(\mathfrak{a}_M)^{-1}\phi_{M,r}^*d\bar z\end{aligned}$$

lie in $\widetilde{H}^1(\Gamma_r\backslash\mathcal{H}_r,\mathscr{O}_F)$ and do not depend on the choice of M. Furthermore, we have $\gamma^*\omega_{\chi,r}=\omega_{\chi,\gamma^{-1}r}$ and $\gamma^*\bar\omega_{\chi,r}=\bar\omega_{\chi,\gamma^{-1}r}$.

Proof. We prove the statements for $\omega_{\chi,r}$, they are similar for $\bar\omega_{\chi,r}$. Since $\mathfrak{a}_M^{-2}$ is the period lattice of the elliptic curve $\mathbb{C}/\mathfrak{a}_M^{-2}$ we have

$$\int_\gamma dz\in\mathfrak{a}_M^{-2}$$

for any $\gamma\in H_1(\mathbb{C}/\mathfrak{a}_M^{-2},\mathbb{Z})$. After tensoring with $\mathscr{O}_F$ and recalling from the proof of Proposition 2.3 that $\chi(\mathfrak{a}_M)\mathscr{O}_F=\mathfrak{a}_M^{-2}\mathscr{O}_F$, we get that

$$\int_\gamma\chi(\mathfrak{a}_M)^{-1}dz\in\chi(\mathfrak{a}_M)^{-1}\mathfrak{a}_M^{-2}\mathscr{O}_F\subset\mathscr{O}_F. \tag{4.1}$$

It follows that

$$\chi(\mathfrak{a})^{-1}dz\in\widetilde{H}^1(\mathbb{C}/\mathfrak{a}_M^{-2},\mathscr{O}_F). \tag{4.2}$$

Hence the pullback $\omega_{\chi,r}$ is also integral. Now suppose that N is another matrix in $\mathrm{SL}_2(K)$ sending r to ∞. Then $M = PN$ where $P = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. We have $\mathfrak{a}_N = a^{-1}\mathfrak{a}_M$ and the following diagram commutes

$$\begin{tikzcd} \Gamma_r\backslash\mathcal{H}_r \arrow[r, "\Phi_{M,r}"] \arrow[dr, "\Phi_{N,r}"'] & \mathbb{C}/\mathfrak{a}_M^{-2} \arrow[d] \\ & \mathbb{C}/a^2\mathfrak{a}_M^{-2}, \end{tikzcd} \tag{4.3}$$

where the vertical map sends z to $Nz = a^2z + ab$. Hence $N^*dz = a^2dz$

$$\chi(\mathfrak{a}_N^{-2})\phi_{N,r}^*dz = a^{-2}\chi(\mathfrak{a}_M^{-2})\phi_{M,r}^*N^*dz = \chi(\mathfrak{a}_M^{-2})\phi_{M,r}^*dz. \tag{4.4}$$

Finally, let γ be in Γ. We have seen that the form does not depend on the choice of M, so we can take $M\gamma$ to be the matrix in $\mathrm{SL}_2(K)$ sending $\gamma^{-1}r$ to ∞. We have $\mathfrak{a}_{M\gamma^{-1}} = \mathfrak{a}_M$ and the following diagram commutes

$$\begin{tikzcd} \Gamma_r\backslash\mathcal{H}_r \arrow[r, "\Phi_{M,r}"] \arrow[d] & \mathbb{C}/\mathfrak{a}_M^{-2} \arrow[d] \\ \Gamma_{\gamma^{-1}r}\backslash\mathcal{H}_{\gamma^{-1}r} \arrow[r, "\Phi_{M\gamma,r}"] & \mathbb{C}/\mathfrak{a}_M^{-2}. \end{tikzcd} \tag{4.5}$$

Hence

$$\omega_{\chi,\gamma^{-1}r} = \chi(\mathfrak{a}_{M\gamma})^{-1}\phi_{M\gamma,r}^*dz = \chi(\mathfrak{a}_M)^{-1}\gamma^*\phi_{M,r}^*dz = \omega_{\chi,r}. \tag{4.6}$$

□

The complex conjugation on $\mathbb{C}$ induces an involution on $\mathbb{H}_3$

$$\begin{aligned} \iota \colon \mathbb{H}_3 &\longrightarrow \mathbb{H}_3 \\ z + jv &\longmapsto \bar{z} + jv. \end{aligned} \tag{4.7}$$

It extends canonically to $\partial\mathbb{H}_3$ by sending $z \in \mathcal{H}_r$ to $\bar{z} \in \mathcal{H}_{\bar{r}}$. Consider the involution $I(\gamma) = \bar{\gamma}$ on Γ. One can check that

$$\iota \circ \gamma(u) = I(\gamma) \circ \iota(u). \tag{4.8}$$

Hence the involution ι descents to an involution on X_Γ, and restricts to an involution on ∂X_Γ. The pullback of differential forms by this involution induces an involution on $H^1(X_\Gamma;\mathbb{C})$ and $H^1(\partial X_\Gamma;\mathbb{C})$. At the level of the boundary forms we have

$$\iota^*\omega_{\chi,r} = \bar{\omega}_{\chi,\bar{r}}. \tag{4.9}$$

Let $H^1(\partial X_\Gamma;\mathbb{C})^-$ be the (-1)-eigenspace of this involution.

PROPOSITION 4.2. *We have*

$$\dim \mathrm{Im}(\mathrm{res}) = \frac{1}{2}\dim_{\mathbb{C}} H^1(\partial X_\Gamma;\mathbb{C}) = h.$$

More precisely, the map

$$H^1(X_\Gamma, \mathbb{C}) \longrightarrow H^1(\partial X_\Gamma, \mathbb{C})^-$$

is surjective.

Proof. The result follows from a theorem of Serre, see [Ber09, Proposition. 24, Corollary. 26] for a proof. However let us prove the statement about the dimension. Let

$$\alpha^*\colon H^1(\partial X_\Gamma;\mathbb{C}) \longrightarrow H^2(X_\Gamma, \partial X_\Gamma;\mathbb{C}) \tag{4.10}$$

be the map from the long exact sequence, so that $\ker(\alpha^*) = \mathrm{Im}(\mathrm{res})$. By Poincaré duality we have

$$H^1(\partial X_\Gamma;\mathbb{C})^\vee \simeq H^1(\partial X_\Gamma;\mathbb{C}) \tag{4.11}$$

and

$$H^2(X_\Gamma, \partial X_\Gamma;\mathbb{C}) \simeq H^2_c(Y_\Gamma;\mathbb{C}) \simeq H^1(Y_\Gamma;\mathbb{C})^\vee \tag{4.12}$$

so that we can see α^* as a map

$$\alpha^*\colon H^1(\partial X_\Gamma;\mathbb{C})^\vee \longrightarrow H^1(Y_\Gamma;\mathbb{C})^\vee \tag{4.13}$$

Since for $\theta \in \Omega^1(\partial X_\Gamma)$ and $\omega \in H^1(X_\Gamma, \partial X_\Gamma)$

$$\int_{X_\Gamma} \alpha(\theta) \wedge \omega = \int_{\partial X_\Gamma} \theta \wedge \mathrm{res}(\omega) \tag{4.14}$$

we have that $\alpha^* = \mathrm{res}^\vee$ is adjoint to res. It follows from

$$\mathrm{Im}(\mathrm{res}) = \ker(\alpha^*) = \ker(\mathrm{res}^\vee) = \mathrm{Im}(\mathrm{res})^\perp, \tag{4.15}$$

that $\mathrm{Im}(\mathrm{res})$ is an isotropic subspace, and thus it must be of half the dimension of the total space. □

4.1 Eisenstein map

In (3.28) we defined a restriction map

$$\mathrm{res}\colon H^1(Y_\Gamma;\mathbb{C}) \longrightarrow H^1(\partial X_\Gamma;\mathbb{C}). \tag{4.16}$$

The kernel is the interior cohomology $H^1_!(Y_\Gamma;\mathbb{C})$ and can be identified with the image of the compactly supported cohomology inside $H^1(Y_\Gamma;\mathbb{C})$. We will define an Eisenstein map

$$\mathrm{Eis}\colon H^1(\partial X_\Gamma;\mathbb{C}) \longrightarrow H^1(Y_\Gamma;\mathbb{C}) \tag{4.17}$$

whose image will be the Eisenstein cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$.

Remark 4.1. As it will follow from Proposition 4.7, the map Eis is not a section of res, *i.e.* we do not have $\mathrm{res} \circ \mathrm{Eis} = \mathbf{1}$.

We begin by defining a map

$$\mathrm{Eis}\colon H^1(\Gamma_\infty\backslash\mathcal{H}_\infty;\mathbb{C}) \longrightarrow H^1(Y_\Gamma;\mathbb{C}) \tag{4.18}$$

at the cusp ∞. We have a Γ_∞-equivariant map

$$\begin{aligned} p_\infty \colon \mathbb{H}_3 &\longrightarrow \mathcal{H}_\infty, \\ z + jv &\longmapsto (z:1), \end{aligned} \tag{4.19}$$

that we can use to pull back a form ω_∞ in $\Omega^1(\mathcal{H}_\infty)^{\Gamma_\infty}$ to a form

$$p_\infty^* \omega_\infty \in \Omega^1(\mathbb{H}_3)^{\Gamma_\infty}. \tag{4.20}$$

To obtain a form on Y_Γ we define

$$\operatorname{Eis}(\omega_\infty) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \gamma^* p^* \omega \in \Omega^1(\mathbb{H}_3)^\Gamma. \tag{4.21}$$

Similarly, at the other cusps r we can define the Γ_r-equivariant map

$$p_r \colon \mathbb{H}_r \longrightarrow \mathcal{H}_r \tag{4.22}$$

to be the composition $p_r = M^{-1} \circ p_\infty \circ M$:

$$\begin{array}{ccc} \mathbb{H}_3 & \overset{p_r}{\dashrightarrow} & \mathcal{H}_r \\ \big\downarrow M & & M^{-1} \big\uparrow \\ \mathbb{H}_3 & \xrightarrow{p_\infty} & \mathcal{H}_\infty, \end{array} \tag{4.23}$$

where M is any matrix in $\mathrm{SL}_2(K)$ sending r to ∞. Note that p_r does not depend on the choice of M since

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}^{-1} \circ p_\infty \circ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = p_\infty. \tag{4.24}$$

For a form ω_r in $\Omega^1(\mathcal{H}_r)^{\Gamma_r}$ we define

$$\operatorname{Eis}(\omega_r) = \sum_{\gamma \in \Gamma_r \backslash \Gamma} \gamma^* p_r^* \omega_r \in \Omega^1(\mathbb{H}_3)^\Gamma. \tag{4.25}$$

However, the sums (4.21) and (4.25) are not convergent, and need to be regularized.

4.2 Regularization of the Eisenstein series

Since $\omega_{\chi,r}$ and $\bar{\omega}_{\chi,r} = \iota^* \omega_{\chi,\bar{r}}$ span the space of forms on $\Gamma_r \backslash \mathcal{H}_r$, it is enough to regularize

$$E_{\chi,c} := \operatorname{Eis}(\omega_{\chi,r}) \in \Omega^1(\mathbb{H}_3)^\Gamma, \tag{4.26}$$

where $c = [r]$ is a cusp. Note that the left hand side only depends on c since for γ in Γ we have

$$\operatorname{Eis}(\omega_{\chi,\gamma^{-1}r}) = \operatorname{Eis}(\gamma^* \omega_{\chi,r}) = \operatorname{Eis}(\omega_{\chi,r}). \tag{4.27}$$

In particular, we will frequently denote this form by $E_{\chi,\mathfrak{a}}$ where $\mathfrak{a}$ is in the ideal class of $\mathfrak{a}_c$, corresponding to the cusp c.

First note that for any matrix M in $\mathrm{SL}_2(K)$ sending r to ∞ we have:

$$\begin{aligned} E_{\chi,c} &= \chi(\mathfrak{a}_M)^{-1} \sum_{\gamma \in \Gamma_r \backslash \Gamma} \gamma^* M^* dz \\ &= \chi(\mathfrak{a}_M)^{-1} \sum_{\gamma \in (M\Gamma_r M^{-1}) \backslash M\Gamma} \gamma^* dz \\ &= \chi(\mathfrak{a}_M)^{-1} \sum_{\gamma \in \Gamma(\mathfrak{a}_M^{-2}) \backslash M\Gamma} \alpha^* dz. \end{aligned} \tag{4.28}$$

Since $[\mathfrak{a}_M] = [\mathfrak{a}_c]$ we have

$$E_{\chi,\mathfrak{a}} = \chi(\mathfrak{a})^{-1} \sum_{\gamma \in \Gamma(\mathfrak{a}^{-2}) \backslash M\Gamma} \alpha^* dz. \tag{4.29}$$

LEMMA 4.3. *The map*

$$\begin{aligned} \Gamma(\mathfrak{a}^{-2})_\infty \backslash M\Gamma &\longrightarrow \{(c,d) \in \mathfrak{a} \times \mathfrak{a} |\, \gcd(c,d) = \mathfrak{a}\} / \mathscr{O}^\times \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (c,d) \end{aligned} \tag{4.30}$$

is a bijection.

Proof. First we have a bijection

$$\begin{aligned} (\pm\Gamma_\infty)\backslash\Gamma &\longrightarrow \Gamma(1:0) \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \gamma^{-1}(1:0). \end{aligned} \tag{4.31}$$

Hence after moving the orbit by $M = \begin{pmatrix} y & -x \\ -n & m \end{pmatrix}$ this becomes a bijection

$$\begin{aligned} (\pm\Gamma(\mathfrak{a}^{-2})_\infty)\backslash M\Gamma &\longrightarrow \Gamma M^{-1}(1:0) = \Gamma(m:n) \\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \alpha^{-1}(1:0) = (d:-c) \end{aligned} \tag{4.32}$$

where α is in $M\Gamma$. Moreover, the orbit $\Gamma(m:n)$ is in bijection with the set

$$\{(c,d) \in \mathfrak{a} \times \mathfrak{a} |\, \gcd(c,d) = \mathfrak{a}\} / \mathscr{O}^\times \tag{4.33}$$

where $\mathscr{O}^\times = \{\pm 1\}$ acts diagonally on $\mathfrak{a} \times \mathfrak{a}$. Hence we get the bijection

$$\begin{aligned} (\pm\Gamma(\mathfrak{a}^{-2})_\infty)\backslash M\Gamma &\longrightarrow \{(c,d) \in \mathfrak{a} \times \mathfrak{a} |\, \gcd(c,d) = \mathfrak{a}\} / \mathscr{O}^\times \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (c,d). \end{aligned} \tag{4.34}$$

The lemma follows from the observation that the action of ± 1 on the left handside correspond to the action of $\mathscr{O}^\times$ on the right handside. $\square$

For $u = z + jv \in \mathbb{H}_3$ let $z(u) = z$, $\bar{z}(u) = \bar{z}$ and $v(u) = v$ be the coordinate functions. Let $\alpha \in \mathrm{SL}_2(\mathbb{C})$ and

$$\eta(u, c, d) := \alpha^*(dz) \tag{4.35}$$

where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It follows from (4.8) that

$$\iota^*\eta(u, \overline{c}, \overline{d}) = \iota^* I(\alpha)^*(dz) = \alpha^*(\iota^* dz) = \alpha^*(d\bar{z}). \tag{4.36}$$

The following calculations will indeed show that η and $\overline{\eta}$ depend only on c and d. We have

$$z(\alpha u) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}v^2}{|cz+d|^2 + |cv|^2}, \tag{4.37}$$

and

$$v(\alpha u) = \frac{v}{|cz+d|^2 + |cv|^2}. \tag{4.38}$$

We view dz as the differential of the coordinate map $z(u)$, hence

$$\eta(u, c, d) = \eta(u, c, d)_z dz + \eta(u, c, d)_{\bar{z}} d\bar{z} + \eta(u, c, d)_v dv \tag{4.39}$$

where

$$\begin{aligned} \eta(u, c, d)_z &= \frac{\partial z(\alpha u)}{\partial z} = \frac{\left(\overline{cz+d}\right)^2}{\left(|cz+d|^2 + |cv|^2\right)^2}, \\ \eta(u, c, d)_{\bar{z}} &= \frac{\partial z(\alpha u)}{\partial \bar{z}} = \frac{-(\bar{c}v)^2}{\left(|cz+d|^2 + |cv|^2\right)^2}, \\ \eta(u, c, d)_v &= \frac{\partial z(\alpha u)}{\partial v} = 2\frac{\left(\overline{cz+d}\right)\bar{c}v}{\left(|cz+d|^2 + |cv|^2\right)^2}. \end{aligned} \tag{4.40}$$

If $J((c, d), u)$ is the Jacobian

$$J((c, d), u) := \begin{pmatrix} \frac{\partial z(\alpha u)}{\partial z} & \frac{\partial z(\alpha u)}{\partial \bar{z}} & \frac{\partial z(\alpha u)}{\partial v} \\ \frac{\partial \bar{z}(\alpha u)}{\partial z} & \frac{\partial \bar{z}(\alpha u)}{\partial \bar{z}} & \frac{\partial \bar{z}(\alpha u)}{\partial v} \\ \frac{\partial v(\alpha u)}{\partial z} & \frac{\partial v(\alpha u)}{\partial \bar{z}} & \frac{\partial v(\alpha u)}{\partial v} \end{pmatrix}, \tag{4.41}$$

then

$$\eta(u, c, d) = (1, 0, 0) J((c, d), u) \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}. \tag{4.42}$$

It follows from the previous lemma

$$E_{\chi, \mathfrak{a}} = \chi(\mathfrak{a})^{-1} \sum_{\substack{c, d \in \mathfrak{a} \\ \gcd(c, d) = \mathfrak{a}}} \eta(u, c, d). \tag{4.43}$$

This sum does not converge, and in order to regularize it we define for a complex number s:

$$\eta(u, c, d, s) := (1, 0, 0) J((c, d), u) v((c, d), u)^s \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}, \tag{4.44}$$

where we write $v((c, d), u) = v(\alpha u)$. More precisely we have

$$\eta(u, c, d, s) = \eta(u, c, d, s)_z dz + \eta(u, c, d, s)_{\bar{z}} d\bar{z} + \eta(u, c, d, s)_v dv \tag{4.45}$$

with

$$\begin{aligned} \eta(u, c, d, s)_z &:= v^s \frac{\left(\overline{cz + d}\right)^2}{(|cz + d|^2 + |cv|^2)^{2+s}}, \\ \eta(u, c, d, s)_{\bar{z}} &:= v^s \frac{-(\bar{c}v)^2}{(|cz + d|^2 + |cv|^2)^{2+s}}, \\ \eta(u, c, d, s)_v &:= 2v^s \frac{\left(\overline{cz + d}\right) \bar{c}v}{(|cz + d|^2 + |cv|^2)^{2+s}}. \end{aligned} \tag{4.46}$$

For an ideal $\mathfrak{a}$ and an unramified Hecke character χ of infinity type $(-2, 0)$ we define

$$E_{\chi,\mathfrak{a}}(u, s) := \chi(\mathfrak{a})^{-1} \, \mathrm{N}(\mathfrak{a})^s \sum_{\substack{c,d \in \mathfrak{a} \\ \gcd(c,d) = \mathfrak{a}}} \eta(u, c, d, s). \tag{4.47}$$

Let us also define the forms

$$\widehat{E}_{\chi,\mathfrak{a}}(u, s) := \chi(\mathfrak{a})^{-1} \, \mathrm{N}(\mathfrak{a})^s \sum_{c,d \in \mathfrak{a}} \eta(u, c, d, s), \tag{4.48}$$

that appear in [Ito87; BCG21]. Note that here we sum over $\mathfrak{a} \times \mathfrak{a}$ instead of the subset with $\gcd(c, d) = \mathfrak{a}$ as is (4.47). We have

$$\widehat{E}_{\chi,\mathfrak{a}}(u, s) = \widehat{E}_{\chi,\mathfrak{a},z}(u, s) dz + \widehat{E}_{\chi,\mathfrak{a},\bar{z}}(u, s) d\bar{z} + \widehat{E}_{\chi,\mathfrak{a},v}(u, s) dv \tag{4.49}$$

where

$$\widehat{E}_{\chi,\mathfrak{a},z}(u, s) := \chi(\mathfrak{a})^{-1} \, \mathrm{N}(\mathfrak{a})^s \sum_{c,d \in \mathfrak{a}} \eta(u, c, d, s)_z, \tag{4.50}$$

and similarly for $\widehat{E}_{\chi,\mathfrak{a},\bar{z}}(u, s)$ and $\widehat{E}_{\chi,\mathfrak{a},v}(u, s)$. By (4.42) we have

$$\left(\widehat{E}_{\chi,\mathfrak{a},z}(u, s), \widehat{E}_{\chi,\mathfrak{a},\bar{z}}(u, s), \widehat{E}_{\chi,\mathfrak{a},v}(u, s)\right) = \chi(\mathfrak{a})^{-1} \, \mathrm{N}(\mathfrak{a})^s \sum_{c,d \in \mathfrak{a}} (1, 0, 0) J((m, n), u) v((m, n), u)^s. \tag{4.51}$$

Since $\eta(u, \kappa c, \kappa d, s) = \kappa^{-2} |\kappa|^{-2s} \eta(u, c, d, s)$ and $\chi((\kappa))^{-1} \, \mathrm{N}(\kappa)^s = \kappa^2 |\kappa|^{2s}$, the forms $E_{\chi,\mathfrak{a}}(u, s)$ and $\widehat{E}_{\chi,\mathfrak{a}}(u, s)$ do not depend on the choice of the set of the representative $\mathfrak{a}$ of the class $[\mathfrak{a}]$.

PROPOSITION 4.4. *We have*

$$\widehat{E}_{\chi,\mathfrak{a}}(u,s) = \sum_{i=1}^{h} \frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{w(\mathfrak{a}_i^{-1}\mathfrak{a})\,\mathrm{N}(\mathfrak{a}_i\mathfrak{a}^{-1})^s} G(1+s,2,0,0;\mathfrak{a}_i^{-1}\mathfrak{a}) E_{\chi,\mathfrak{a}_i}(u,s).$$

Proof. First we have

$$\begin{aligned}\widehat{E}_{\chi,\mathfrak{a}}(u,s) &= \frac{\mathrm{N}(\mathfrak{a})^s}{\chi(\mathfrak{a})} \sum_{c,d\in\mathfrak{a}} \eta(u,c,d,s) \\ &= \sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \frac{\chi(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^s} \frac{\mathrm{N}(\mathfrak{a}\mathfrak{b})^s}{\chi(\mathfrak{a}\mathfrak{b})} \sum_{\substack{c,d\in\mathfrak{a}\\ \gcd(c,d)=\mathfrak{a}\mathfrak{b}}} \eta(u,c,d,s). \end{aligned} \tag{4.52}$$

Note that if $c,d\in\mathfrak{a}$ with $\gcd(c,d)=\mathfrak{a}\mathfrak{b}$ then c and d are in $\mathfrak{a}\mathfrak{b}$. Hence

$$\widehat{E}_{\chi,\mathfrak{a}}(u,s) = \sum_{0\neq\mathfrak{b}\subseteq\mathscr{O}} \frac{\chi(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^s} E_{\chi,\mathfrak{b}\mathfrak{a}}(u,s). \tag{4.53}$$

After writing $\mathfrak{b}\mathfrak{a} = \mathfrak{a}_i(\alpha)$ for some representant $\mathfrak{a}_i$, we get

$$\widehat{E}_{\chi,\mathfrak{a}}(u,s) = \sum_{i=1}^{h} \frac{1}{w(\mathfrak{a}_i^{-1}\mathfrak{a})} \frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{\mathrm{N}(\mathfrak{a}_i\mathfrak{a}^{-1})^s} \sum_{0\neq\alpha\subseteq\mathfrak{a}_i^{-1}\mathfrak{a}} \frac{1}{\alpha^2|\alpha|^{2s}} E_{\chi,\mathfrak{a}_i}(u,s). \tag{4.54}$$

□

PROPOSITION 4.5. *The series $\widehat{E}_{\chi,\mathfrak{a}}(u,s)$ and $E_{\chi,\mathfrak{a}}(u,s)$ converge for $\mathrm{Re}(s)\gg 0$ and admit an analytic continuation to the whole plane. Moreover, the forms by $\widehat{E}_{\chi,\mathfrak{a}}(u)$ and $E_{\chi,\mathfrak{a}}(u)$ at $s=0$ are closed and are related by*

$$\widehat{E}_{\chi,\mathfrak{a}}(u) = \sum_{i=1}^{h} \frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{w(\mathfrak{a}_i^{-1}\mathfrak{a})} G_2(\mathfrak{a}_i^{-1}\mathfrak{a}) E_{\chi,\mathfrak{a}_i}(u).$$

Proof. The analytic continuation of $\widehat{E}_{\chi,\mathfrak{a}}(u,s)$ can be done by Poisson summation, see for example [BCG21, page. 18]. The fact that the forms $\widehat{E}_{\chi,\mathfrak{a}}(u,s)$ are closed is the content of [BCG21, Proposition. 3.3]. The same results holds for $E_{\chi,\mathfrak{a}}$ by the previous proposition. □

The Eisenstein operator It follows from (4.8) and (4.9) that

$$\mathrm{Eis}(\bar{\omega}_{\chi,r}) = \mathrm{Eis}(\iota^*\omega_{\chi,\bar{r}}) = \iota^*\,\mathrm{Eis}(\omega_{\chi,\bar{r}}) = \iota^* E_{\chi,c}, \tag{4.55}$$

where ι is the involution induced by complex conjugation. The cohomology $H^1(\Gamma_r\backslash\mathcal{H}_r;\mathbb{C})$ is spanned by $\omega_{\chi,r}$ and $\bar{\omega}_{\chi,r}$. Hence if $\epsilon_r = \alpha\omega_{\chi,r} + \beta\bar{\omega}_{\chi,r}$ we have

$$\mathrm{Eis}(\epsilon_r) = \alpha E_{\chi,c} + \beta\iota^* E_{\chi,c}. \tag{4.56}$$

Since $H^1(\partial X_\Gamma;\mathbb{C}) = \bigoplus_{c=[r]\in C_\Gamma} H^1(\Gamma_r\backslash\mathcal{H}_r;\mathbb{C})$ we then have a map

$$\begin{aligned} \mathrm{Eis}\colon H^1(\partial X_\Gamma;\mathbb{C}) &\longrightarrow H^1(Y_\Gamma;\mathbb{C}) \\ \sum_{c=[r]\in C_\Gamma} \lambda_c\epsilon_r &\longmapsto \sum_{c=[r]\in C_\Gamma} \lambda_c(\alpha E_{\chi,c} + \beta\iota^* E_{\chi,c}). \end{aligned} \tag{4.57}$$

4.3 Fourier expansions and constant terms

Let M be a matrix in $\mathrm{SL}_2(K)$ such that $Mr = \infty$. In this section let N denote the inverse of M Hence

$$N = M^{-1} = \begin{pmatrix} m & x \\ n & y \end{pmatrix} \in \begin{pmatrix} \mathfrak{a}_M & \mathfrak{a}_M^{-1} \\ \mathfrak{a}_M & \mathfrak{a}_M^{-1} \end{pmatrix} \tag{4.58}$$

where $\mathfrak{a}_M = m\mathscr{O} + n\mathscr{O}$. Since

$$\widehat{E}_{\chi,\mathfrak{a}}(u) = (\widehat{E}_{\chi,\mathfrak{a},z}(u,s), \widehat{E}_{\chi,\mathfrak{a},\bar{z}}(u), \widehat{E}_{\chi,\mathfrak{a},v}(u)) \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}, \tag{4.59}$$

we have

$$(N^*\widehat{E}_{\chi,\mathfrak{a}}(u) = (\widehat{E}_{\chi,\mathfrak{a},z}(Nu), \widehat{E}_{\chi,\mathfrak{a},\bar{z}}(Nu), \widehat{E}_{\chi,\mathfrak{a},v}(Nu))J(N,u) \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}, \tag{4.60}$$

where $J(N,u)$ is the Jacobian as in (4.41) with α replaced by N. On the other hand, since

$$(\widehat{E}_{\chi,\mathfrak{a},z}(Nu), \widehat{E}_{\chi,\mathfrak{a},\bar{z}}(Nu), \widehat{E}_{\chi,\mathfrak{a},v}(Nu)) = \chi(\mathfrak{a})^{-1} \sum_{c,d\in\mathfrak{a}} (1,0,0)J((c,d),u) \tag{4.61}$$

and $J((c,d),Nu)J(M,u) = J((c,d)N,u)$, we have

$$N^*\widehat{E}_{\chi,\mathfrak{a}}(u) = \chi(\mathfrak{a})^{-1} \sum_{c,d\in\mathfrak{a}} (1,0,0)J((c,d)N,u) \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}. \tag{4.62}$$

Let us define

$$(\widehat{E}_z^{(N)}(u,s), \widehat{E}_{\bar{z}}^{(N)}(u,s), \widehat{E}_v^{(N)}(u,s)) \coloneqq \chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^s \sum_{c,d\in\mathfrak{a}} (1,0,0)J((c,d)N,u)v((c,d)N,u)^s, \tag{4.63}$$

so that at $s = 0$

$$N^*\widehat{E}_{\chi,\mathfrak{a}}(u) = (\widehat{E}_z^{(N)}(u), \widehat{E}_{\bar{z}}^{(N)}(u), \widehat{E}_v^{(N)}(u)) \begin{pmatrix} dz \\ d\bar{z} \\ dv \end{pmatrix}. \tag{4.64}$$

By [Scz84, p. 536] and [Ito87, p. 162], for each $c^* \in \mathfrak{a}\mathfrak{a}_M$ there exists a complex number $\rho(c^*)$ such that

$$\begin{aligned}\{yc - dx |\, c, d \in \mathfrak{a}, dm - cn = c^*\} &= \frac{(\mathfrak{a} + c^* x)}{n} \cap \frac{(\mathfrak{a} + c^* y)}{m} \\ &= \rho(c^*) + \mathfrak{a}\mathfrak{a}_M^{-1}. \end{aligned} \tag{4.65}$$

Hence, as in [Ito87] we have

$$\begin{aligned}(\widehat{E}_z^{(N)}(u,s), \widehat{E}_{\bar{z}}^{(N)}(u,s), \widehat{E}_v^{(N)}(u,s)) =&\chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^s \sum_{c \in \mathfrak{a}\mathfrak{a}_M} \sum_{d \in \rho(c)+\mathfrak{a}\mathfrak{a}_M^{-1}} (1,0,0) J((c,d),u) v(c,d),u)^s, \\ &= \chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^s \sum_{c \in \mathfrak{a}\mathfrak{a}_M} \sum_{d \in \rho(c)+\mathfrak{a}\mathfrak{a}_M^{-1}} \eta(u,c,d,s). \end{aligned} \tag{4.66}$$

LEMMA 4.6. *If $\mathfrak{b}$ is a fractional ideal then $D(\mathfrak{b}) = -\sqrt{D}\,\mathrm{N}(\mathfrak{b})$.*

Proof. Recall that if $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\mathrm{Im}(\omega_1/\omega_2) > 0$ then

$$D(L) = \omega_1\overline{\omega_2} - \overline{\omega_1}\omega_2 = \begin{vmatrix} \omega_1 & \omega_2 \\ \overline{\omega_1} & \overline{\omega_2} \end{vmatrix}.$$

Let $\mathfrak{b} = (a + b\tau)\mathbb{Z} + (c + d\tau)\mathbb{Z}$ a basis of $\mathfrak{b}$ such that

$$\mathrm{Im}\left(\frac{a+b\tau}{c+d\tau}\right) = \frac{(ad-bc)\sqrt{|D|}}{2|c+d\tau|} > 0,$$

and where $\tau = \frac{1+\sqrt{D}}{2}$. Then

$$D(\mathfrak{b}) = \begin{vmatrix} a+b\tau & c+d\tau \\ \overline{a+b\tau} & \overline{c+d\tau} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} 1 & 1 \\ \tau & \overline{\tau} \end{vmatrix} = -\sqrt{D}(ad - bc).$$

Since the determinant $ad - bc$ is positive we have

$$ad - bc = |ad - bc| = \mathrm{N}(\mathfrak{b}).$$

$\square$

PROPOSITION 4.7. *(a) The restriction of the Eisenstein forms $\widehat{E}_{\chi,\mathfrak{a}}$ to the boundary components are*

$$\mathrm{res}_r(\widehat{E}_{\chi,\mathfrak{a}}) = \chi(\mathfrak{a}^{-1}\mathfrak{a}_c)\left(G_2(\mathfrak{a}\mathfrak{a}_c^{-1})\omega_{\chi,r} - G_2(\mathfrak{a}\bar{\mathfrak{a}}_c^{-1})\bar{\omega}_{\chi,r}\right),$$

where $c = [r] \in C_\Gamma$.

(b) For the Eisenstein series $E_{\chi,\mathfrak{a}}$ the restriction is

$$\mathrm{res}_r(E_{\chi,\mathfrak{a}}) = \delta_{\mathfrak{a},\mathfrak{a}_c}\omega_{\chi,r} - \delta_{\mathfrak{a},\bar{\mathfrak{a}}_c}\bar{\omega}_{\chi,r},$$

where $\delta_{\mathfrak{a},\mathfrak{a}_c}$ is defined by

$$\delta_{\mathfrak{a},\mathfrak{b}} = \begin{cases} 1 & \text{if} \quad [\mathfrak{a}] = [\mathfrak{b}] \\ 0 & \text{otherwise} \end{cases}.$$

In particular the restriction of $E_{\chi,\mathfrak{a}}$ to the boundary is integral.

Proof. (*a*). Let M be any matrix in $\mathrm{SL}_2(K)$ sending r to ∞, and $N = M^{-1}$. By (3.25) we have

$$\mathrm{res}_r(\widehat{E}_{\chi,\mathfrak{a}}) = M^* \,\mathrm{res}_\infty(N^* \widehat{E}_{\chi,\mathfrak{a}}). \tag{4.67}$$

where $\mathrm{res}_\infty(\omega) = \lim_{v\to\infty} \iota_v^* \omega$. After pulling back $N^*\widehat{E}_{\chi,\mathfrak{a}}(u)$ by the map $\iota_v(z) = z + jv$ we get

$$\iota_v^* N^* \widehat{E}_{\chi,\mathfrak{a}}(u) = \widehat{E}_z^{(N)}(u)dz + \widehat{E}_{\bar{z}}^{(N)}(u)d\bar{z}. \tag{4.68}$$

The limit of $\widehat{E}_z^{(N)}(u)$ as $v \to \infty$ is the constant term in the Fourier expansion. Hence by [Ito87, Section. 5, p. 162], we have[3]

$$\lim_{v\to\infty} \widehat{E}_z^{(N)}(u) = \chi(\mathfrak{a})^{-1} G_2(\mathfrak{a}\mathfrak{a}_M^{-1}). \tag{4.72}$$

Similarly

$$\lim_{v\to\infty} \widehat{E}_{\bar{z}}^{(N)}(u) = -\chi(\mathfrak{a})^{-1} \frac{D(\mathfrak{a}\mathfrak{a}_M)}{D(\mathfrak{a}\mathfrak{a}_M^{-1})} G(\mathfrak{a}\mathfrak{a}_M). \tag{4.73}$$

By the functional equation (2.14) we have

$$G_2(\mathfrak{a}\mathfrak{a}_M) = G(\mathfrak{a}\mathfrak{a}_M). \tag{4.74}$$

[3]Note that the constant term of $\widehat{E}_z^{(N)}(u,s)$ in *loc. cit.* is $v^s G(s+1,2,0,0;\mathfrak{a}\mathfrak{a}_M)$, which is $G_2(\mathfrak{a}\mathfrak{a}_M)$ at $s=0$. We believe it is as mistake and should be $v^s G(s+1,2,0,0;\mathfrak{a}\mathfrak{a}_M^{-1})$. Indeed, the form is

$$\widehat{E}_z^{(N)}(u) = \chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^s v^s \sum_{c\in\mathfrak{a}\mathfrak{a}_M} \sum_{d\in\rho(c)+\mathfrak{a}\mathfrak{a}_M^{-1}} \frac{\left(\overline{cz+d}\right)^2}{(|cz+d|^2+|cv|^2)^{2+s}}$$

where c and d are not both zero. The constant term is coming from the terms where $c = 0$:

$$\chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^s v^s \sum_{d\in\rho(0)+\mathfrak{a}\mathfrak{a}_M^{-1}} \frac{1}{d^2|d|^{2s}} = \chi(\mathfrak{a})^{-1}\,\mathrm{N}(\mathfrak{a})^{-s} G(1+s,2,0,0;\mathfrak{a}\mathfrak{a}_M^{-1}). \tag{4.69}$$

We used that $\rho(0) = 0$. To see this, note that for any c^* we have

$$\{cy - dx|\, c,d \in \mathfrak{a}, dm - cn = c^*\} \subset \mathfrak{a}\mathfrak{a}_M^{-1}, \tag{4.70}$$

since $x, y \in \mathfrak{a}_M^{-1}$. On the other hand, when $c^* = 0$ we have

$$\mathfrak{a}\mathfrak{a}_M^{-1} \subset \frac{\mathfrak{a}}{n} \cap \frac{\mathfrak{a}}{m} = \{cy - dx|\, c,d\in\mathfrak{a}, dm - cn = 0\} \subset \mathfrak{a}\mathfrak{a}_M^{-1}. \tag{4.71}$$

Furthermore, we have

$$\frac{D(\mathfrak{a}\mathfrak{a}_M)}{D(\mathfrak{a}\mathfrak{a}_M^{-1})} = \mathrm{N}(\mathfrak{a}_M)^2$$

and $\mathfrak{a}_M = \mathrm{N}(\mathfrak{a}_M)\bar{\mathfrak{a}}_M^{-1}$. Using the homogeneity of G_2, we get from (3.25)

$$\lim_{v\to\infty} \widehat{E}_{\bar{z}}^{(N)}(u) = -\chi(\mathfrak{a})^{-1}G_2(\mathfrak{a}\bar{\mathfrak{a}}_M^{-1}). \tag{4.75}$$

and

$$\operatorname{res}_\infty N^*\widehat{E}_{\chi,\mathfrak{a}} = \chi(\mathfrak{a})^{-1}\left(G_2(\mathfrak{a}\mathfrak{a}_M^{-1})dz - G_2(\mathfrak{a}\bar{\mathfrak{a}}_M^{-1})d\bar{z}\right). \tag{4.76}$$

After pulling back by M, and using that $[\mathfrak{a}_M] = [\mathfrak{a}_c]$ we get

$$\operatorname{res}_r \widehat{E}_{\chi,\mathfrak{a}} = \chi(\mathfrak{a}_c\mathfrak{a}^{-1})\left(G_2(\mathfrak{a}\mathfrak{a}_c^{-1})\omega_{\chi,r} - G_2(\mathfrak{a}\bar{\mathfrak{a}}_c^{-1})\bar{\omega}_{\chi,r}\right). \tag{4.77}$$

Note that the right hand side of (4.77) does not depend on the ideal representatives.

(b). Let $\operatorname{res}_r(E_{\chi,\mathfrak{a}}) = \alpha(\mathfrak{a})\omega_{\chi,r} + \beta(\mathfrak{a})\bar{\omega}_{\chi,r}$ be the restriction of $E_{\chi,\mathfrak{a}}$ to the boundary component r. Since the restriction map is linear, it follows from Lemma 4.4 that

$$\begin{aligned}\operatorname{res}_r(\widehat{E}_{\chi,\mathfrak{a}}) &= \sum_{i=1}^{h}\frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{w(\mathfrak{a}_i^{-1}\mathfrak{a})}G_2(\mathfrak{a}_i^{-1}\mathfrak{a})\operatorname{res}_r(E_{\chi,\mathfrak{a}_i})\\ &= \left(\sum_{i=1}^{h}\frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{w(\mathfrak{a}_i^{-1}\mathfrak{a})}G_2(\mathfrak{a}_i^{-1}\mathfrak{a})\alpha(\mathfrak{a}_i)\right)\omega_{\chi,r} + \left(\sum_{i=1}^{h}\frac{\chi(\mathfrak{a}_i\mathfrak{a}^{-1})}{w(\mathfrak{a}_i^{-1}\mathfrak{a})}G_2(\mathfrak{a}_i^{-1}\mathfrak{a})\beta(\mathfrak{a}_i)\right)\bar{\omega}_{\chi,r}.\end{aligned} \tag{4.78}$$

Comparing with (4.77) we see that

$$\alpha(\mathfrak{a}_i) = \begin{cases}1 & \text{if} \quad [\mathfrak{a}_i] = [\mathfrak{a}_c]\\ 0 & \text{otherwise}\end{cases} \tag{4.79}$$

and

$$\beta(\mathfrak{a}_i) = \begin{cases}-1 & \text{if} \quad [\mathfrak{a}_i] = [\bar{\mathfrak{a}}_c]\\ 0 & \text{otherwise}\end{cases}. \tag{4.80}$$

Thus $\alpha(\mathfrak{a}) = \delta_{\mathfrak{a},\mathfrak{a}_c}$ and $\beta(\mathfrak{a}) = \delta_{\mathfrak{a},\bar{\mathfrak{a}}_c}$.

□

PROPOSITION 4.8. *The Eisenstein cohomology $H^1_{\mathrm{Eis}}(Y_\Gamma;\mathbb{C})$ is spanned by the forms $E_{\chi,\mathfrak{a}}$ as $\mathfrak{a}$ ranges over the class group.*

Proof. Let V be the subspace of the Eisenstein cohomology spanned by the forms $E_{\chi,\mathfrak{a}}$. We know that the Eisenstein cohomology is h-dimensional, so $\dim(V) \le h$. A class in $H^1(\partial X_\Gamma;\mathbb{C})$ is represented by a collection of forms $\alpha_r\omega_{\chi,r} + \beta_r\bar{\omega}_{\chi,r}$ with α_r, β_r two complex numbers such that $\alpha_{\gamma r} = \alpha_r$ and $\beta_{\gamma r} = \beta_r$. By Proposition 4.2, the image of the restriction is the -1 eigenspace by the involution ι. Hence if the class lies in Im(Res), then we have in addition $\beta_r = -\alpha_{\bar{r}}$, by (4.9). It follows that the space Im(Res) is spanned by the forms $\omega_{\chi,r} - \bar{\omega}_{\chi,\bar{r}}$. By the previous proposition, we have

$$\mathrm{Res}(E_{\chi,\mathfrak{a}_c}) = \omega_{\chi,r} - \bar{\omega}_{\chi,\bar{r}}, \tag{4.81}$$

Hence the restriction map Res: $V \longrightarrow \mathrm{Im}(\mathrm{Res})$ is surjective and $\dim(V) = h$. □

Remark 4.2. We will not discuss this further in the rest of the paper, but let us mention that the fact that the Eisenstein classes have an integral restriction to the boundary is an important property to find congruences. See [Ber09] for more on the relation between the denominator ideal, congruences and the Selmer group.

4.4 Relation to the Sczech cocycle

The forms $\widehat{E}_{\chi,\mathfrak{a}}$ in $\Omega^1(Y_\Gamma;\mathbb{C})$ define a cocycle in $H^1(\Gamma;\mathbb{C})$

$$\gamma \longmapsto \int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}}. \tag{4.82}$$

Since the form is closed, the integrals do not depend on the path from u_0 to γu_0.

Theorem 4.9. *We have*

$$\Phi_{\mathfrak{a}}(\gamma) = \chi(\mathfrak{a}) \int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}}, \tag{4.83}$$

where $\Phi_{\mathfrak{a}}$ is the Sczech cocycle.

Proof. Let $\widehat{E}^*_{\chi,\mathfrak{a}}$ be the extension of $\widehat{E}_{\chi,\mathfrak{a}}$ to $\mathbb{H}_3^*$ with boundary components $\widehat{E}^{(r)}_{\chi,\mathfrak{a}} \coloneqq \mathrm{res}_r(\widehat{E}^*_{\chi,\mathfrak{a}})$. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

1. Suppose that $c \neq 0$. Let $P(\gamma)$ be the closed path in $\mathbb{H}_3^*$ pictured in Figure 4. Since the path and the form are closed we have

$$\int_{P(\gamma)} \widehat{E}^*_{\chi,\mathfrak{a}} = 0. \tag{4.84}$$

Note that E is an arc in the horocycle $\mathcal{H}_{\gamma^{-1}\infty}$ and C is a segment in $\mathcal{H}_\infty$. Let $\widetilde{D} \coloneqq \gamma D$, $\widetilde{E} \coloneqq \gamma E$, $\widetilde{F} \coloneqq \gamma F$ and $\widetilde{P}(\gamma)$ be the paths in Figure 5. Since $\widehat{E}_{\chi,\mathfrak{a}}$ is Γ-invariant we have

$$\int_{\widetilde{P}(\gamma)} \widehat{E}^*_{\chi,\mathfrak{a}} = \int_{P(\gamma)} \widehat{E}^*_{\chi,\mathfrak{a}} = 0. \tag{4.85}$$

Moreover, since the integrals along B and $\widetilde{F}$ cancel we get

$$\int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}} = -\int_{C+\widetilde{E}} \widehat{E}^{(\infty)}_{\chi,\mathfrak{a}} - \int_{\widetilde{D}} \widehat{E}_{\chi,\mathfrak{a}}. \tag{4.86}$$

We will compute these two integrals separately.

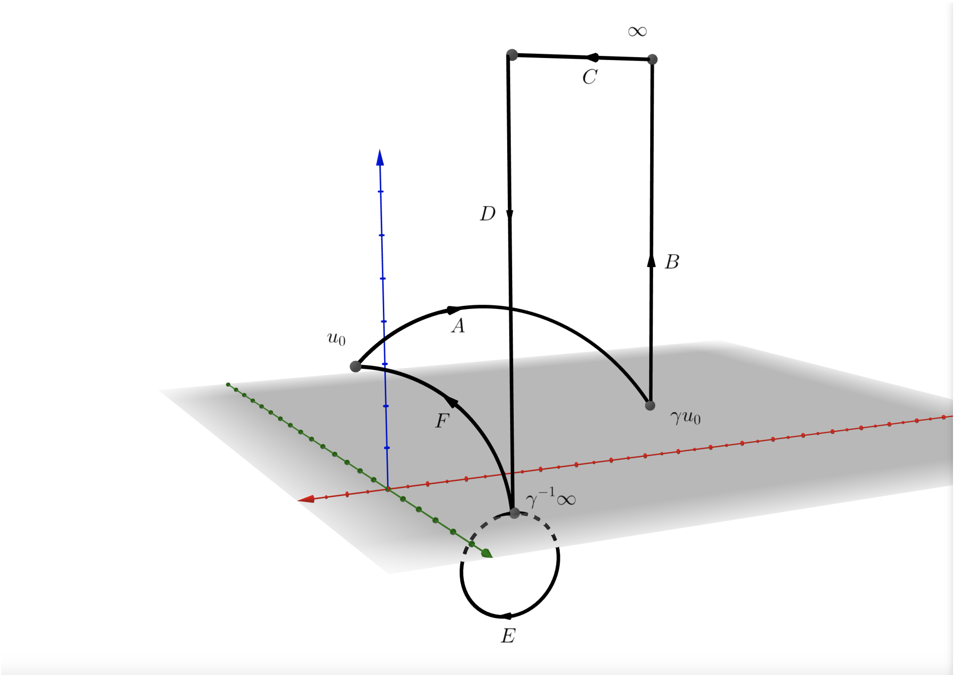

Figure 4: The path $P(\gamma)$ when $c \neq 0$.

1a) We begin with the integral along $C + \widetilde{E}$. Let $z(t) = \frac{a}{c} - t\frac{a+d}{c}$ with $0 \leq t \leq 1$ be the straight line joining the endpoints of $\widetilde{E} + C$, where $\gamma\infty = \frac{a}{c}$ and $\gamma^{-1}\infty = -\frac{d}{c}$. Note that the three segments $\widetilde{E}$,C and $z(t)$ form a closed triangle (with suitable orientations of the edges). Since

$$\widehat{E}^{(\infty)}_{\chi,\mathfrak{a}} = \chi(\mathfrak{a}^{-1})G_2(\mathfrak{a})(dz - d\bar{z}) \tag{4.87}$$

is closed, the integral along $\widetilde{E} + C$ is the same as along the line $z(t)$. Then

$$\begin{aligned}\chi(\mathfrak{a}) \int_{C+\widetilde{E}} \widehat{E}^{(\infty)}_{\chi,\mathfrak{a}} &= G_2(\mathfrak{a}) \int_0^1 (z(t) - \bar{z}(t))\, dt \\ &= -G_2(\mathfrak{a}) I\left(\frac{a+d}{c}\right).\end{aligned} \tag{4.88}$$

1b) Let us now compute the integral along $\widetilde{D}$. The path $\widetilde{D}$ is parametrized by $u(t) = \frac{a}{c} + jt$ with $0 < t < \infty$. We have

$$\begin{aligned}\chi(\mathfrak{a}) \int_{\widetilde{D}} \widehat{E}_{\chi,\mathfrak{a}}(u,s) &= \chi(\mathfrak{a}) \int_{\widetilde{D}} \widehat{E}_{\chi,\mathfrak{a},v}(u,s) \\ &= 2\,\mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \int_0^\infty \frac{\left(\overline{m\frac{a}{c} + n}\right) \overline{m} t^{1+s}}{\left(|m\frac{a}{c} + n|^2 + |m|^2 t^2\right)^{2+s}} dt \\ &= 2\,\mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c} + n}\right) \overline{m}}{|m\frac{a}{c} + n|^{2(2+s)}} \int_0^\infty \frac{t^{1+s}}{\left(1 + \frac{|m|^2}{|m\frac{a}{c}+n|^2} t^2\right)^{2+s}} dt.\end{aligned} \tag{4.89}$$

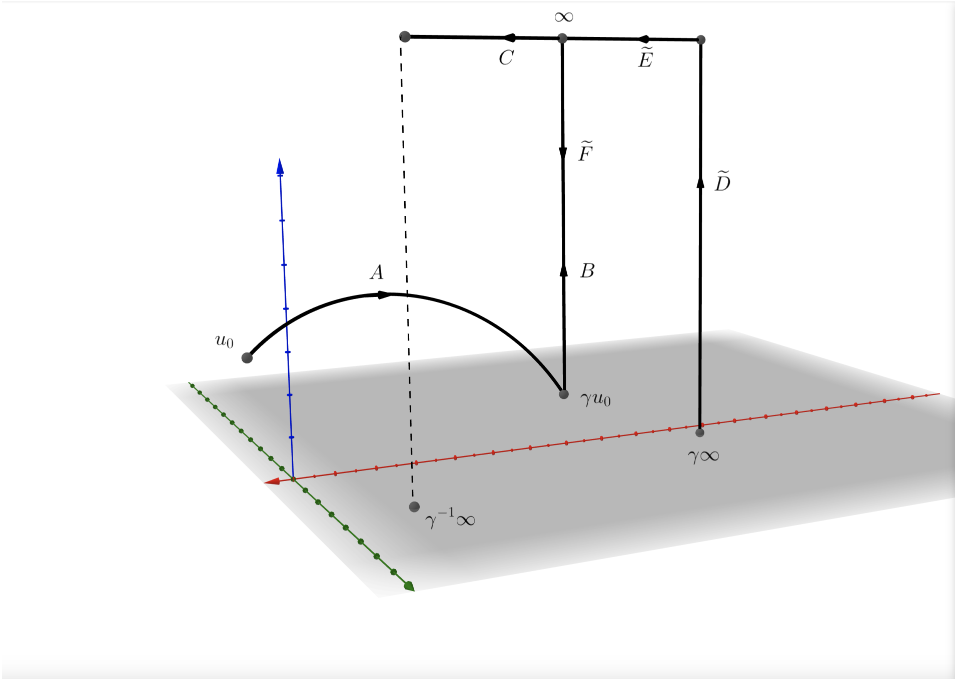

Figure 5: The path $\widetilde{P}(\gamma)$ is Γ-equivalent to the path $P(\gamma)$ in Figure 4.

By substituting $\alpha = \frac{|m|}{|m\frac{a}{c}+n|}t$ this becomes

$$\begin{aligned} &2\,\mathrm{N}(\mathfrak{a})^s \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c}+n}\right)}{|m\frac{a}{c}+n|^{2+s}} \frac{\overline{m}}{|m|^{2+s}} \int_0^\infty \frac{\alpha^{1+s}}{(1+\alpha^2)^{2+s}} d\alpha \\ &= \mathrm{N}(\mathfrak{a})^s B\left(1+\frac{s}{2}, 1+\frac{s}{2}\right) \sum_{(m,n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{m\frac{a}{c}+n}\right)}{|m\frac{a}{c}+n|^{2+s}} \frac{\overline{m}}{|m|^{2+s}}, \end{aligned} \tag{4.90}$$

where

$$B\left(x,y\right) = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}} dt = \frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma(x+y)} \tag{4.91}$$

is the Beta function. By writing $m = c\widetilde{m} + r$ and summing over $\widetilde{m}$ and r we rewrite the previous inner sum as

$$\sum_{r\in\mathfrak{a}/c\mathfrak{a}} \sum_{(\widetilde{m},n)\in\mathfrak{a}\times\mathfrak{a}} \frac{\left(\overline{r\frac{a}{c}+a\widetilde{m}+n}\right)}{|r\frac{a}{c}+a\widetilde{m}+n|^{2+s}} \frac{\overline{c\widetilde{m}+r}}{|c\widetilde{m}+r|^{2+s}}, \tag{4.92}$$

where the inner sum is restricted to $(\widetilde{m}, n) \neq (-r/c, 0)$. By summing over $(\widetilde{m}, a\widetilde{m}+n)$ instead of $(\widetilde{m}, n)$, we rewrite the sum as

$$\sum_{r\in\mathfrak{a}/c\mathfrak{a}} \frac{1}{c|c|^s} G\left(\frac{1+s}{2}, 1, \frac{ar}{c}, 0, \mathfrak{a}\right) G\left(\frac{1+s}{2}, 1, \frac{r}{c}, 0, \mathfrak{a}\right), \tag{4.93}$$

where the inner sum in the first line is restricted to $(\widetilde{m}, \widetilde{n}) \neq (-r/c, -ar/c)$. Hence at $s = 0$ we get

$$\chi(\mathfrak{a}) \int_{\widetilde{D}} \widehat{E}_{\chi,\mathfrak{a}} = B(1,1)\frac{1}{c} \sum_{r \in \mathfrak{a}/c\mathfrak{a}} G_1\left(\frac{ar}{c}, \mathfrak{a}\right) G_1\left(\frac{r}{c}, \mathfrak{a}\right) = D(a, c, \mathfrak{a}). \tag{4.94}$$

2. Suppose that $c = 0$. Let $P(\gamma)$ the closed path pictured in Figure 6. The integrals along the paths B and D in Figure 6 cancel, and we have

$$\int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}} = -\int_C \widehat{E}_{\chi,\mathfrak{a}}^{(\infty)}. \tag{4.95}$$

Since K has no non-trivial units we have $a, d = \pm 1$ and $a/d = 1$. If $u_0 = z_0 + jv_0$ then $\gamma u_0 = z_0 + \frac{b}{d} + jv_0$. Hence the path C is parametrized by $z(t) = z_0 + \frac{b}{d}(1-t)$ for $0 \leq t \leq 1$. At the boundary the form is

$$\widehat{E}_{\chi,\mathfrak{a}}^{(\infty)} = \chi(\mathfrak{a})^{-1} G_2(\mathfrak{a})\,(dz - d\bar{z}) \tag{4.96}$$

and thus

$$\chi(\mathfrak{a}) \int_C \widehat{E}_{\chi,\mathfrak{a}}^{(\infty)} = G_2(\mathfrak{a}) \int_0^1 (z(t) - \bar{z}(t))dt = -G_2(\mathfrak{a}) I\left(\frac{b}{d}\right). \tag{4.97}$$

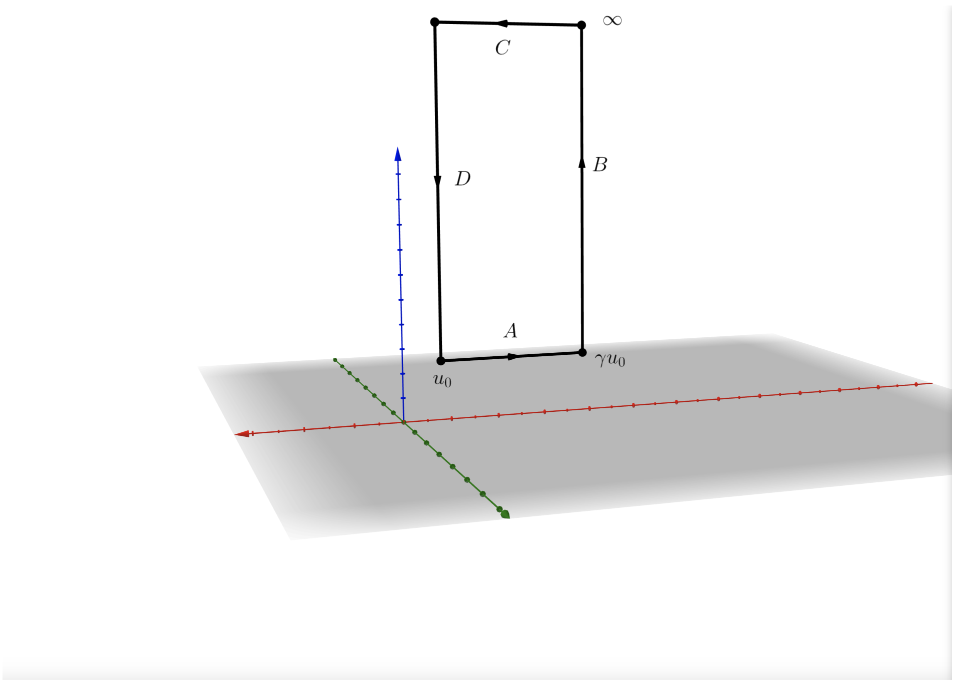

Figure 6: The path $P(\gamma)$ for $c = 0$.

$\square$

Corollary 4.10. *The forms $2\widehat{E}_{\chi,\mathfrak{a}}^{\mathrm{alg}} := 2\Omega^{-2}\widehat{E}_{\chi,\mathfrak{a}}$ are in $\widetilde{H}^1(X_\Gamma, \mathscr{O}_F)$.*

Proof. We have

$$\int_{u_0}^{\gamma u_0} 2\widehat{E}^{\mathrm{alg}}_{\chi,\mathfrak{a}} = 2\Omega^{-2}\int_{u_0}^{\gamma u_0} \widehat{E}_{\chi,\mathfrak{a}} = \Omega^{-2}\chi(\mathfrak{a}^{-1})2\Phi_{\mathfrak{a}}(\gamma) = \chi(\mathfrak{a}^{-1})\lambda(\mathfrak{a}^{-1})^2 2\Phi_{L_{\mathfrak{a}^{-1}}}(\gamma). \tag{4.98}$$

By Proposition (2.3) the factor $\chi(\mathfrak{a})\lambda(\mathfrak{a})^2$ is a unit in $\mathscr{O}_F$ for any fractional ideal $\mathfrak{a}$, and the result follows from the integrality of the Sczech cocycle. □

5 Denominators of the Eisenstein cohomology

We have seen that $H^1_{\mathrm{Eis}}(X_\Gamma;\mathbb{C})$ is an h-dimensional complex vector spaces spanned by the forms $E_{\chi,\mathfrak{a}}$. The $\mathscr{O}_F$-lattice $\mathcal{L}_{\mathrm{Eis}}$ of $H^1_{\mathrm{Eis}}(X_\Gamma,\mathbb{C})$ defined by

$$\mathcal{L}_{\mathrm{Eis}} := \mathrm{Eis}(\widetilde{H}^1(\partial X_\Gamma;\mathscr{O}_F)) = \bigoplus_{[\mathfrak{a}]\in \mathrm{Cl}(K)} E_{\chi,\mathfrak{a}}\mathscr{O}_F \tag{5.1}$$

gives us an integral structure on $H^1_{\mathrm{Eis}}(X_\Gamma,\mathbb{C})$, *i.e* $\mathcal{L}_{\mathrm{Eis}}\otimes_{\mathscr{O}_F}\mathbb{C}\simeq H^1_{\mathrm{Eis}}(X_\Gamma,\mathbb{C})$. Another $\mathscr{O}_F$-lattice is given by the integral Eisenstein classes

$$\mathcal{L}_0 := \widetilde{H}^1(X_\Gamma;\mathscr{O}_F)\cap H^1_{\mathrm{Eis}}(X_\Gamma;\mathbb{C}) \subset H^1_{\mathrm{Eis}}(X_\Gamma;\mathbb{C}). \tag{5.2}$$

We define the denominator of the Eisenstein cohomology to be the $\mathscr{O}_F$ ideal

$$\mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}) := \{\lambda\in\mathscr{O}_F \mid \lambda\mathcal{L}_{\mathrm{Eis}}\subset\mathcal{L}_0\}. \tag{5.3}$$

Lemma 5.1. *The submodule $\mathcal{L}_{\mathrm{Eis}}$ does not depend on the choice of the Hecke character χ.*

Proof. Let us temporarily denote by $\mathcal{L}_{\chi,\mathrm{Eis}}$ and $\mathcal{L}_{\widetilde{\chi},\mathrm{Eis}}$ the lattice that we obtain for two different Hecke characters $\widetilde{\chi}$ and χ of infinity type $(-2,0)$. Then $\varphi=\chi/\widetilde{\chi}$ is a character on the class group, valued in F (since χ and $\widetilde{\chi}$ are). Since $E_{\widetilde{\chi},\mathfrak{a}}=\varphi(\mathfrak{a})^{-1}E_{\chi,\mathfrak{a}}$, we have $\mathcal{L}_{\chi,\mathrm{Eis}}=M\mathcal{L}_{\widetilde{\chi},\mathrm{Eis}}$, where

$$M=\mathrm{diag}(\varphi(\mathfrak{a}_1)^{-1},\dots,\varphi(\mathfrak{a}_h)^{-1}).$$

Since $\varphi(\mathfrak{a})^h=1$, the value $\varphi(\mathfrak{a})$ is a unit in $\mathscr{O}_F$ and $M\in\mathrm{Mat}_h(\mathscr{O}_F)$. Furthermore, the determinant of M is

$$\det(M)=\prod_{\mathfrak{a}\in\mathrm{Cl}(K)}\varphi(\mathfrak{a})^{-1}.$$

The substitution $\mathfrak{a}\to\mathfrak{a}^{-1}$ shows that $\det(M)=\det(M)^{-1}$. Hence $\det(M)=\pm1$ and M is in $\mathrm{GL}_h(\mathscr{O}_F)$. □

We will need the following result on Dedekind determinants, of which a proof can be found in [Lan90, Chapter. 3, Theorem. 6.1].

Lemma 5.2. *Let f any complex valued function on a finite abelian group G. Then*

$$\det(f(a^{-1}b))_{a,b\in G}=\prod_{\varphi\in\widehat{G}}\left(\sum_{a\in G}\varphi(a)f(a^{-1})\right).$$

THEOREM 5.3. *We have*

$$D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi\circ\mathrm{N}_{H/K},0)E_{\chi,\mathfrak{a}}\in\mathscr{O}_F.$$

In particular, we get the following bound on the denominator

$$D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi\circ\mathrm{N}_{H/K},0)\mathscr{O}_F\subset\mathrm{Den}(\mathcal{L}_{\mathrm{Eis}}).$$

Proof. Let $M_\chi\in\mathrm{GL}_h(F)$ be the matrix such that

$$\begin{pmatrix}\widehat{E}^{\mathrm{alg}}_{\chi,\mathfrak{a}_1}\\ \vdots\\ \widehat{E}^{\mathrm{alg}}_{\chi,\mathfrak{a}_h}\end{pmatrix}=M_\chi\begin{pmatrix}E_{\chi,\mathfrak{a}_1}\\ \vdots\\ E_{\chi,\mathfrak{a}_h}\end{pmatrix}. \tag{5.4}$$

By Proposition 4.4 we have

$$M_\chi=\left[\Omega^{-2}\frac{\chi(\mathfrak{a}\mathfrak{b}^{-1})}{w(\mathfrak{a}^{-1}\mathfrak{b})}G_2(\mathfrak{a}^{-1}\mathfrak{b})\right]_{[\mathfrak{a}],[\mathfrak{b}]\in\mathrm{Cl}(K)}.$$

Applying the previous lemma to the function

$$f(\mathfrak{a})=\Omega^{-2}\frac{\chi(\mathfrak{a}^{-1})}{w(\mathfrak{a})}G_2(\mathfrak{a}),$$

we then have

$$\begin{aligned}\det(M_\chi)&=\prod_{\varphi\in\widehat{\mathrm{Cl}(K)}}\left(\Omega^{-2}\sum_{\mathfrak{a}\in\mathrm{Cl}(K)}\varphi(\mathfrak{a})\frac{\chi(\mathfrak{a})}{w(\mathfrak{a}^{-1})}G_2(\mathfrak{a}^{-1})\right)\\&=\prod_{\varphi\in\widehat{\mathrm{Cl}(K)}}\Omega^{-2}L(\varphi\chi,0)\\&=\prod_{\varphi\in\widehat{\mathrm{Cl}(K)}}L^{\mathrm{alg}}(\varphi\chi,0)\\&=L^{\mathrm{alg}}(\chi\circ\mathrm{N}_{H/K},0).\end{aligned}\tag{5.5}$$

We have already seen in the proof of Proposition 2.4 that for any fractional ideal $\mathfrak{a}$

$$4\sqrt{D}f(\mathfrak{a}^{-1})=4\sqrt{D}\frac{\chi(\mathfrak{a})}{w(\mathfrak{a}^{-1})}\Lambda(\mathfrak{a})^2G_2(L_\mathfrak{a})\in\mathscr{O}_F \tag{5.6}$$

is integral. Hence the matrix $\widetilde{M}_\chi:=4\sqrt{D}M_\chi$ is in $\mathrm{GL}_h(F)\cap\mathrm{Mat}_h(\mathscr{O}_F)$. Furthermore, the adjoint matrix $N_\chi:=\det(\widetilde{M}_\chi)\widetilde{M}_\chi^{-1}$ also has coefficients in $\mathscr{O}_F$ and

$$\begin{aligned}N_\chi M_\chi&=4^{-1}D^{-\frac{1}{2}}N_\chi\widetilde{M}_\chi\\&=4^{-1}D^{-\frac{1}{2}}\det(\widetilde{M}_\chi)\\&=4^{h-1}D^{\frac{h-1}{2}}\det(M_\chi)\\&=4^{h-1}D^{\frac{h-1}{2}}L^{\mathrm{alg}}(\chi\circ\mathrm{N}_{H/K},0)\\&=4^{-1}D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi\circ\mathrm{N}_{H/K},0).\end{aligned}\tag{5.7}$$

Applying $2N_\chi$ to both sides of (5.4) we find that

$$2N_\chi \begin{pmatrix} \widehat{E}^{\mathrm{alg}}_{\chi,\mathfrak{a}_1} \\ \vdots \\ \widehat{E}^{\mathrm{alg}}_{\chi,\mathfrak{a}_h} \end{pmatrix} = 2^{-1}D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi \circ \mathrm{N}_{H/K}, 0) \begin{pmatrix} E_{\chi,\mathfrak{a}_1} \\ \vdots \\ E_{\chi,\mathfrak{a}_h} \end{pmatrix}. \tag{5.8}$$

Since the left hand side is integral, it follows that $2^{-1}D^{-\frac{1}{2}}L^{\mathrm{int}}(\chi \circ \mathrm{N}_{H/K}, 0)E_{\chi,\mathfrak{a}_i}$ is integral for any i. □

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