Theta correspondence and the Borisov-Gunnells relations

Romain Branchereau

Abstract

We consider a geometric theta correspondence from the first homology of a modular curve, to modular forms of weight 2. Using Stevens' description of the homology, we find that this map sends modular symbols to product of weight one Eisenstein series, modular caps to weight 2 Eisenstein series, and hyperbolic cycles to diagonal restrictions of Hilbert-Eisenstein series. We use it to revisit work of Borisov and Gunnells, and explain its connection to a theorem of Li. In particular, we give a geometric proof of certain relations between Eisenstein series.

Contents

1. Introduction

Let $Y_1(N)$ be the modular curve with $N \geq 4$. In previous work [Bra25], we combined ideas due to Bergeron-Charollois-Garcia [BCG20; BCG23a] and Kudla-Millson [KM86; KM87; KM90] to construct a closed differential form

$$\mathscr{E}(z,\tau) \in \Omega^1(Y_1(N)) \otimes C^\infty(\mathbb{H})$$

that transforms in the variable τ as a modular form of weight 2 for $\Gamma_1(N)$. The key feature of this construction is that it induces a theta lift

$$\mathscr{E}\colon H_1(Y_1(N);\mathbb{Z}) \longrightarrow M_2(\Gamma_1(N)), \qquad \mathcal{Z} \longrightarrow \int_{\mathcal{Z}} \mathscr{E}(z,\tau) \tag{1.1}$$

to holomorphic modular forms, whose Fourier coefficients are Poincaré duals of linear combinations of modular symbols. More precisely, we have the following.

Theorem 1.1. *Let $T_n\{0,\infty\}$ be the Hecke translate of the modular symbol $\{0,\infty\}$, and $\mathrm{PD}(T_n\{0,\infty\})$ its Poincaré dual in $H^1(Y_1(N);\mathbb{Z})$. In cohomology, the differential form $\mathscr{E}$ has a Fourier expansion*

$$[\mathscr{E}] = -\frac{1}{2i\pi} d\log(g_{0,1}) - \sum_{n=1}^{\infty} \mathrm{PD}(T_n\{0,\infty\})q^n \in H^1(Y_1(N);\mathbb{Q}) \otimes M_2(\Gamma_1(N)), \qquad q = e^{2i\pi\tau},$$

where $g_{0,1}$ is a Siegel unit.

For a residue $r \in \mathbb{Z}$, we define the Eisenstein series of weight k

$$G_r^{(k)}(\tau) := N\frac{(k-1)!}{(-2i\pi)^k} \lim_{s\to 0} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(mN\tau+n)^k|mN\tau+n|^{2s}} e\left(-\frac{rn}{N}\right)$$

by analytic continuation. We will also need the Eisenstein series

$$\widehat{G}_r^{(k)}(\tau) := \frac{(k-1)!}{(-2i\pi)^k} \lim_{s\to 0} \sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv r \mod N}}{}' \frac{1}{(m\tau+n)^k|m\tau+n|^{2s}}.$$

Both define holomorphic Eisenstein series in $E_k(\Gamma_1(N))$, except in some cases when weight $k=2$. To get a holomorphic Eisenstein series of weight 2, we set for $p,q \in \mathbb{Z}$

$$H_{p,q}^{(2)} := G_q^{(2)} - \delta_{q0}\widehat{G}_p^{(2)} \in E_2(\Gamma_1(N)).$$

By passing from homology to cohomology, the map (1.1) restricts to the map considered by Borisov-Gunnells in [BG01, Theorem. 4.11]. Let $S_{2,\mathrm{rk}=0}^{\mathrm{new}}(\Gamma) \subseteq S_2(\Gamma)$ be the subspace spanned by rank 0 newforms, and $\mathcal{H}^{(2)} \subset E_2(\Gamma_1(N))$ be the subspace spanned by the forms $G_q^{(2)}(\tau)$ with $q \not\equiv 0 \mod N$, and the forms $H_{p,0}^{(2)}$ with $p \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Corollary 1.1.1. *We have the inclusion*

$$\mathcal{H}^{(2)} \oplus S_{2,\mathrm{rk}=0}^{new}(\Gamma_1(N)) \subseteq \mathrm{Im}(\mathscr{E}),$$

which is an equality when N is prime.

Two complementary results in the literature describe spanning sets of $S_{2,\mathrm{rk}=0}^{\mathrm{new}}(\Gamma)$. For $\Gamma = \Gamma_0(N)$, Li proves in [Li17] that this space is spanned by diagonal restrictions of certain Hilbert-Eisenstein series. On the other hand, Borisov-Gunnells show in [BG01] that it can also be spanned by Eisenstein series of weight 2, and products of weight 1 Eisenstein series.

The first goal of this paper is to explain how both results arise naturally from the above theta lift, by evaluating it on appropriate generators of the homology. The second goal is to use this theta lift to obtain relations among Eisenstein series. The main novelty lies in the use of the theta correspondence, which moreover admits a natural generalization to a correspondence from the $(n-1)$-st homology of a symmetric space of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ to modular forms of weight n, leading to corresponding generalizations of the results obtained here.

1.1. Evaluation on hyperbolic matrices and Li's result. Every class in $H_1(Y_1(N);\mathbb{Z})$ is represented by a cycle

$$\mathcal{Z}_\gamma := \{z_0, \gamma z_0\}, \qquad \gamma \in \Gamma_1(N).$$

As explained in [Bra25] in the case of the modular curve $Y_0(p)$, we can recover Li's result by evaluating the theta lift on cycles $\mathcal{Z}_\gamma$, with γ a hyperbolic matrix. This argument extends naturally to $\Gamma_1(N)$.

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with ring of integers $\mathscr{O}_F$. Let $\mathfrak{f} \subset F$ be a lattice of rank 2, and $\mathscr{O} \subset \mathscr{O}_F$ be the order that preserves this lattice. The subgroup of units $U := \mathscr{O}^{\times,+} \cap (1 + N\mathfrak{f}) \subset \mathscr{O}^{\times,+}$ preserves the coset $1 + N\mathfrak{f}$. We define the Hilbert-Eisenstein series

$$E^{(k)}_{1,\mathfrak{f}}(\tau,\tau') := (-1)^k \frac{D^{k-\frac{1}{2}}}{(2i\pi)^{2k}} \lim_{s\to 0} \sideset{}{'}\sum_{(m,n)\in(1+N\mathfrak{f})\times N\mathfrak{f}/U} \frac{(yy')^s}{\mathrm{N}(m\tau+n)^k|\mathrm{N}(m\tau+n)|^{2s}},$$

of parallel weight k, for the congruence subgroup $\Gamma_1(N\mathfrak{f}) \subset \mathrm{SL}_2(\mathscr{O})$. The second part of the following theorem result can be compared to [Li17, Theorem. 1.2].

Theorem 1.2. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a primitive hyperbolic matrix in $\Gamma_1(N)$, let $D := \mathrm{tr}(\gamma)^2 - 4$, and $\mathfrak{f} = \mathbb{Z} + \nu\mathbb{Z}$ with $\nu = \frac{1}{2c}(a - d - \sqrt{D})$. Then*

$$\mathscr{E}(\mathcal{Z}_\gamma) = E^{(1)}_{1,\mathfrak{f}}(\tau,\tau).$$

Hence, every form in $\mathcal{H}^{(2)} \oplus S^{new}_{2,\mathrm{rk}=0}(\Gamma_1(N))$ is a linear combination of such modular forms.

1.2. Modular caps, modular symbols and the result of Borisov-Gunnells. The main result of this paper is to recover the results of Borisov-Gunnells from the theta lift.

We use Stevens' [Ste89] description of the homology of the modular curve. Let $\overline{Y_1(N)}^{\mathrm{BS}}$ be the Borel-Serre compactification of $Y_1(N)$. From the long exact sequence in homology, one can write

$$H_1(Y_1(N);\mathbb{Z}) \simeq H_1(\overline{Y_1(N)}^{\mathrm{BS}};\mathbb{Z}) \simeq \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z}), \tag{1.2}$$

where the space $\mathcal{C}(\mathbb{Z})$ is spanned by modular caps $\mathcal{C}_r$ (a loop around the cusp r), and $\mathcal{MS}_0(\mathbb{Z})$ is spanned by modular symbols of degree 0. Thus, it suffices to evaluate the periods of $\mathscr{E}(z,\tau)$ over modular caps and modular symbols.

Theorem 1.3. 1. *Let $\mathcal{C}_r$ be the modular cap around the cusp $r = \gamma_r\infty$ for some matrix $\gamma_r = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The period of $\mathscr{E}(z,\tau)$ over $\mathcal{C}_r$ is $H^{(2)}_{d,-c}(\tau)$.*

2. *Let $\mathcal{M} = \gamma\{0,\infty\}$ be the unimodular symbol with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. The period of $\mathscr{E}(z,\tau)$ over $\mathcal{M}$ is $-G^{(1)}_d(\tau)G^{(1)}_c(\tau)$.*

Remark 1.1. The theta lift (1.1) is similar to a cocycle constructed in [BCG20, Theorem. 5] or [BCG23a, Théorème p. 2.10], without the geometric interpretation of the Fourier coefficients. The construction in *loc. cit.* also uses the Mathai-Quillen formalism [MQ86], as well as previous work of Charollois-Sczech [CS16]. The cocycle is given by a product of weight one Eisenstein series, so that the case $N = 2$ of [BCG23a, Théorème p. 2.10] is analogous to the second part of Theorem 1.3. See also [BCG23a, Section. 2.3.4] or [BCG23b] in the setting of $\mathrm{SL}_N(\mathscr{O}_K)$ over an imaginary quadratic field. The connection to the work of Borisov-Gunnells was already suggested in [BCG20, Example p. 8] and [BCG23a, Remark p. 30].

Next, we would like to write any cycle $\mathcal{Z}_\gamma$ as a linear combination of modular caps and unimodular symbols, with respect to the splitting (1.2). First, if $\gamma = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, then it is easy to

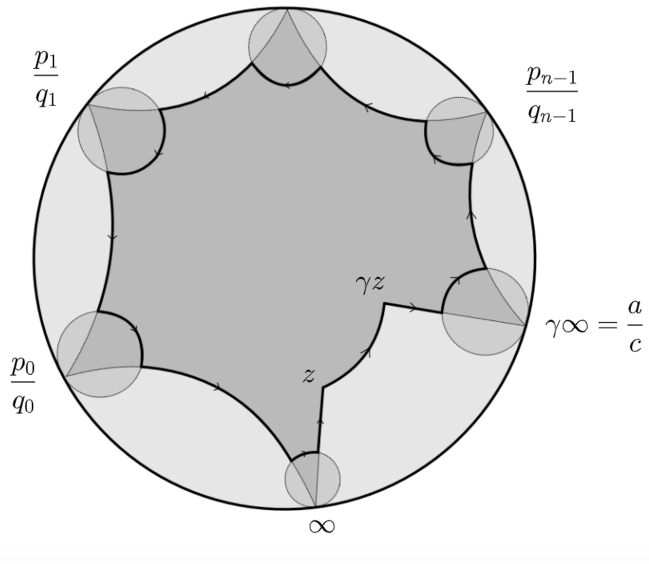

Figure 1: We represent $\mathbb{H}$ in the Poincaré disc model. The polygon $\mathcal{P}$ is closed in the Borel-Serre compactification. The circles around the cusps are the horocycles at infinity containing the modular caps. The theta lift sends the modular symbols on each side to a product of two weight 1 Eisenstein series and each modular cap to a weight 2 Eisenstein series. The two segments between the interior and the cusp cancel out.

see that $\mathcal{Z}_\gamma$ can be pushed to the cusp and it is equal to $n\mathcal{C}_\infty$. The same can be done for any parabolic matrix, after translation.

For hyperbolic matrices, this can be achieved by adapting the continued fraction algorithm explained in [Ste89]. Suppose that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$. Consider the continued fraction expansion of $\frac{a}{c} = \gamma\infty$

$$\frac{a}{c} = b_0 - \cfrac{1}{b_1 - \cfrac{1}{\cdots - \cfrac{1}{b_{n-1} - \cfrac{1}{b_n}}}}$$

with $b_k \in \mathbb{Z}$ and convergents $\frac{p_0}{q_0} = \frac{b_0}{1}, \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} = \frac{a}{c}$. Set $(p_{-1}, q_{-1}) \coloneqq (1, 0)$ and

$$\gamma_k \coloneqq \begin{pmatrix} -p_k & p_{k-1} \\ -q_k & q_{k-1} \end{pmatrix}, \qquad k = 0, \dots, n.$$

As shown in Figure 1, we can find a polygon $\mathcal{P}$ whose boundary is $\mathcal{Z}_\gamma$ plus a linear combination of modular caps and unimodular symbols. It follows that we can write the cycle in homology as

$$\mathcal{Z}_\gamma = \left[(b_0 + bq_{n-1} - p_{n-1}d)\mathcal{C}_\infty + \sum_{k=0}^{n-1} b_{k+1}\mathcal{C}_{\gamma_k\infty}\right] \oplus \sum_{k=0}^{n} \gamma_k\{0, \infty\} \in \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z}).$$

By combining this with the previous result, we deduce the following theorem.

Theorem 1.4. *If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a hyperbolic matrix in $\Gamma_1(N)$, then*

$$\mathscr{E}(\mathcal{Z}_\gamma) = (b_0 - p_{n-1}d + bq_{n-1})H^{(2)}_{1,0} + \sum_{k=0}^{n-1} b_{k+1} H^{(2)}_{q_{k-1},q_k} - \sum_{k=0}^{n} G^{(1)}_{q_k} G^{(1)}_{q_{k-1}}.$$

If we denote by $\mathcal{H}^{(1,1)}$ the span of the products $G^{(1)}_a G^{(1)}_b$, then it follows from Corollary 1.1.1 that

$$\mathcal{H}^{(2)} \oplus S^{new}_{2,\mathrm{rk}=0}(\Gamma_1(N)) \subseteq \mathrm{span}\{\mathcal{H}^{(2)}, \mathcal{H}^{(1,1)}\}$$

Remark 1.2. The second part of this theorem can be compared with [BG01, Theorem. 4.11]. Note that the result in *loc. cit.* is stronger, as it proves an equality.

Remark 1.3. There have been several results on spanning modular forms by Eisenstein series, extending and generalizing the work of Borisov-Gunnells; e.g. by Dickson-Neururer [DN18], Xue [Xue23], or Raum-Xia [RX20].

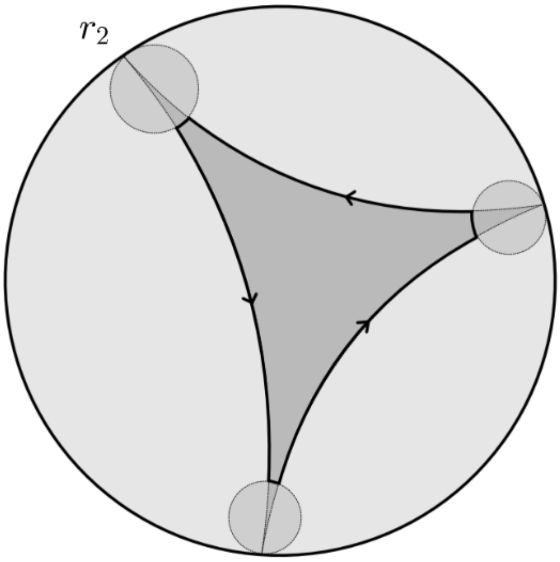

Figure 2: The hyperbolic triangle closes in the Borel–Serre compactification. Its vertices r_1, r_2, r_3 are cusps connected by unimodular sides and completed by modular caps. Each side is mapped to a product of two weight 1 Eisenstein series, while each modular cap is mapped to a weight 2 Eisenstein series.

1.3. Relation between Eisenstein series. Finally, we can use the theta lift to deduce relations between Eisenstein series. Let $\mathcal{T} \subset \mathbb{H}$ be a hyperbolic triangle whose sides are three unimodular symbols and closed by three modular caps, as in Figure 2. The image of the boundary $\partial\mathcal{T}$ is trivial in $H_1(Y_1(N);\mathbb{Z})$, so that $\mathscr{E}(\partial\mathcal{T}) = 0$. The boundary $\partial\mathcal{T}$ is a linear combination of unimodular symbols and modular caps, whose image have been computed in Theorem 1.3. We deduce the following result, similar to [BG01, Proposition. 3.7].

Theorem 1.5. *Let a, b, c be three coprime integers that satisfy $a+b+c \equiv 0 \mod N$ and such that $a, b, c \neq 0 \mod N$. We have*

$$G^{(1)}_a G^{(1)}_b + G^{(1)}_b G^{(1)}_c + G^{(1)}_c G^{(1)}_a = G^{(2)}_a + G^{(2)}_b + G^{(2)}_c.$$

Remark 1.4. Note that the proof of Theorem 1.5 generalizes to the case of d Eisenstein series of weight 2, and d products of two Eisenstein series of weight 1; see Corollary 4.7.1. We conclude by mentioning several further results in the literature concerning relations among Eisenstein series, such as the recent work of Brunault [Bru25], Brunault-Zudilin [BZ23], Khuri-Makdisi–Raji [KMR17], or Zhang [Zha20].

2. Preliminaries on the modular curve

2.1. Homology classes. Let $Y := \Gamma\backslash\mathbb{H}$ be the modular curve for some congruence subgroup Γ that will be $\Gamma(N)$ or $\Gamma_1(N)$ with $N \geq 4$. By Hurewicz's theorem, the group homorphism $\Gamma \longrightarrow H_1(Y;\mathbb{Z})$ that sends γ to the cycle $\mathcal{Z}_\gamma := \{z_0, \gamma z_0\}$ is independent of the choice of the basepoint z_0 and surjective. The trace of a matrix γ in $\Gamma_1(N)$ is congruent to 2 modulo N. In particular, if $N \geq 4$, then $|\mathrm{tr}(\gamma)| \geq 2$.

Definition 2.1. *The matrix is* parabolic *if* $|\mathrm{tr}(\gamma)| = 2$*, in which case it stabilizes a cusp in* $\mathbb{P}^1(\mathbb{Q})$*. The matrix is* hyperbolic *if* $|\mathrm{tr}(\gamma)| > 2$*, and in that case it stabilizes two real quadratic points in* $\mathbb{P}^1(\mathbb{R}) \smallsetminus \mathbb{P}^1(\mathbb{Q})$*. Moreover, it is* primitive *in* $\Gamma_1(N)$ *if it cannot be written as a nontrivial power* $\gamma = \gamma_1^m$ *of another hyperbolic matrix* $\gamma_1 \in \Gamma_1(N)$*.*

2.2. Borel-Serre compactification. We follow Stevens [Ste89]. Let $\mathbb{P}^1(\mathbb{Q})$ be the boundary of $\mathbb{H}$ and let

$$\overline{\mathbb{H}} = \mathbb{H} \sqcup \bigsqcup_{r \in \mathbb{P}^1(\mathbb{Q})} B_r$$

be the Borel-Serre completion, obtained by gluing horocycles $B_r = \mathbb{P}^1(\mathbb{R}) \smallsetminus \{r\}$ at the cusp r. At ∞, the gluing is done such that a sequence $z_n = x_n + iy_n$ converges to $\alpha \in \mathbb{P}^1(\mathbb{R}) \smallsetminus \{\infty\} \simeq \mathbb{R}$ if $\lim_{n\to\infty} x_n = \alpha$ and $\lim_{n\to\infty} y_n = \infty$. At a cusp $r = \gamma^{-1}\infty$, a sequence z_n converges to $\alpha \in \mathbb{P}^1(\mathbb{R}) \smallsetminus \{r\}$ if γz_n converges to $\gamma\alpha$.

The group Γ acts on $\bigsqcup_{r\in\mathbb{P}^1(\mathbb{Q})} B_r$ by sending $z \in B_r$ to $\gamma z \in B_{\gamma z}$. Let

$$\overline{Y}^{\mathrm{BS}} := \Gamma\backslash\overline{\mathbb{H}}$$

be the Borel-Serre compactification of $Y := Y$. If we write $C_N := \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ for the set of cusps of Y, then

$$\overline{Y}^{\mathrm{BS}} = Y \cup \bigsqcup_{r \in C_N} \mathcal{C}_r$$

is the union of Y with a circle $\mathcal{C}_r := \Gamma_r\backslash B_r \simeq S^1$ at each cusp, where Γ_r is the stabilizer of the cusp.

For $r \in \mathbb{P}^1(\mathbb{Q})$ and $x \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$ with $x \neq r$, let $\pi_r(x)$ be the endpoint in the horocycle B_r of the geodesic joining x to r. Note that this maps is Γ-equivariant in the sense that $\gamma(\pi_r(x)) = \pi_{\gamma r}(\gamma x)$, and continuous: if $\lim_{n\to\infty} z_n = \alpha \in B_r$, then

$$\lim_{n\to\infty} \pi_r(z_n) = \pi_r(\alpha).$$

Given two distinct points $r_1, r_2 \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$, let $\{r_1, r_2\} \in Z_1(\overline{\mathbb{H}})$ be the geodesic oriented from r_1 to r_2. If r_1, r_2 are in $\mathbb{P}^1(\mathbb{Q})$, we call it a *modular symbol*. Its boundary in $\overline{\mathbb{H}}$ is

$$\partial\{r_1, r_2\} = \pi_{r_2}(r_1) - \pi_{r_1}(r_2).$$

By abuse of notation, we will also denote by $\{r_1, r_2\}$ its image in Y, which represents a 1-cycle

$$\{r_1, r_2\} \in H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}).$$

Definition 2.2. *A modular symbol* $\{r_1, r_2\}$ *is* unimodular *if there is a matrix* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $\{\alpha, \beta\} = \gamma\{0, \infty\}$.

For a cusp r and $x, y \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$ with $x \neq r$, $y \neq r$, we define the *modular cap* $[x, y]_r$ to be the segment in B_r from $\pi_r(y)$ to $\pi_r(x)$. We have

$$\partial[x, y]_r = \pi_r(y) - \pi_r(x).$$

Let $C_k(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) := C_k(\overline{Y}^{\mathrm{BS}}; \mathbb{Z})/C_k(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$ be the complex of relative cycles, whose homology is the relative homology $H_k(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$. The short exact sequence

$$0 \longrightarrow C_k(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \longrightarrow C_k(\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \longrightarrow C_k(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \longrightarrow 0$$

induces a long exact sequence

$$\begin{tikzcd}
H_2(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_2(\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_2(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[dll] \\
H_1(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_1(\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[dll] \\
H_0(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_0(\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[r] & H_0(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}),
\end{tikzcd} \tag{2.1}$$

where the connecting morphism are the boundary operators. Since the inclusion of Y in $\overline{Y}^{\mathrm{BS}}$ is a homotopy equivalence, we have $H_1(Y; \mathbb{Z}) \simeq H_1(\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$ and $H_2(\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \simeq H_2(Y; \mathbb{Z}) \simeq H_c^0(Y; \mathbb{Z}) = 0$ (since Y is noncompact). Moreover, note $\partial\overline{Y}^{\mathrm{BS}}$ is a union of circles (one $\mathcal{C}_r$ at each cusp r). Thus, we have $H_2(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) = 0$ and

$$H_0(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \simeq H_1(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \simeq \mathbb{Z}[C_N],$$

where $\mathbb{Z}[C_N]$ is the $\mathbb{Z}$-module generated by the set of cusps C_N. At each cusp, a generator is given by a loop that goes once around the cusp, oriented such that the boundary map sends the fundamental class $1 \in H_2(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$ to 1 at each cusp. We get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[dll] \\
\mathbb{Z}[C_N] \arrow[r] & H_1(Y; \mathbb{Z}) \arrow[r] & H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \arrow[dll] \\
\mathbb{Z}[C_N] \arrow[r] & \mathbb{Z} \arrow[r] & H_0(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}).
\end{tikzcd}$$

The first connecting map $\mathbb{Z} \longrightarrow \mathbb{Z}[C_N]$ sends 1 to $\sum_{r \in C_N} \mathcal{C}_r$. Let $\mathcal{C}(\mathbb{Z})$ be the cokernel of this map, which consists of linear combinations of closed modular caps, modulo $\sum_{r \in C_N} \mathcal{C}_r$. Ash and Rudolph [AR79] showed (in much greater generality for congruence subgroups of SL_n)

that the homology $H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$ is generated by modular symbols $\{\alpha, \beta\}$. Hence, the kernel of the second connecting map $H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}) \longrightarrow \mathbb{Z}[C_N]$ consists of classes represented by linear combinations of modular symbols of degree 0, *i.e.* such that $\partial c = 0 \in \mathbb{Z}[C_N] \simeq H_0(\partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z})$. Let us denote by $\mathcal{MS}_0(\mathbb{Z})$ this kernel, so that we get a short exact sequence

$$0 \longrightarrow \mathcal{C}(\mathbb{Z}) \longrightarrow H_1(Y;\mathbb{Z}) \longrightarrow \mathcal{MS}_0(\mathbb{Z}) \longrightarrow 0$$

At each cusp $r \in C_N$, the maps π_r define a projection on the corresponding boundary component, which induces a map $\pi\colon H_1(Y;\mathbb{Z}) \longrightarrow \mathbb{Z}[C_N] \longrightarrow \mathcal{C}(\mathbb{Z})$. The inclusion $i_1\colon \mathcal{C}(\mathbb{Z}) \longrightarrow H_1(Y;\mathbb{Z})$ satisfies $\pi \circ i_1 = \mathbf{1}$, so that the sequence splits and that we have an isomorphism

$$H_1(Y;\mathbb{Z}) \longrightarrow \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z}).$$

2.3. Continued fraction. We will now explain how a cycle decomposes with respect to the decomposition $H_1(Y;\mathbb{Z}) \simeq \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z})$. We will write $\mathcal{Z}_\gamma = \{z, \gamma z\}$ as a linear combination of unimodular symbols and closed modular caps.

If γ is a parabolic matrix that stabilizes the cusp $r \in \mathbb{P}^1(\mathbb{Q})$, then

$$\mathcal{Z}_\gamma = \{z_0, \gamma z_0\} = [z_0, \gamma z_0]_r\,;$$

see Figure 3. Let $\gamma_r \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma_r\infty = r$. The matrix $\gamma_r^{-1}\gamma\gamma_r \in \mathrm{SL}_2(\mathbb{Z})$ preserves ∞ and

$$\gamma_r^{-1}\gamma\gamma_r = \begin{pmatrix} 1 & b(\gamma) \\ 0 & 1 \end{pmatrix}$$

for some $b(\gamma) \in \mathbb{Z}$.

Remark 2.1. If w_r denotes the width of the cusp, then $w(\gamma) := \frac{b(\gamma)}{w_r}$ is the winding number of the loop around the cusp.

If we set $z_0' := \gamma_r^{-1} z_0$, we find that

$$[z_0, \gamma z_0]_r = \gamma_r[\gamma_r^{-1}z_0, \gamma_r^{-1}\gamma z_0]_\infty = \gamma_r[z_0', \gamma_r^{-1}\gamma\gamma_r z_0']_\infty = b(\gamma)\gamma_r[0,1]_\infty = b(\gamma)\mathcal{C}_r.$$

Let $\gamma \in \Gamma$ be a matrix such that $c \neq 0$. Consider the continued fraction of $\frac{a}{c} = \gamma\infty$

$$\frac{a}{c} = b_0 - \cfrac{1}{b_1 - \cfrac{1}{\cdots - \cfrac{1}{b_{n-1} - \cfrac{1}{b_n}}}} \tag{2.2}$$

with $b_0 \in \mathbb{Z}$, $b_k \in \mathrm{N}$ for $k \geq 1$, and convergents $\frac{p_0}{q_0} = \frac{b_0}{1}, \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} = \frac{a}{c}$. For $-1 \leq k \leq n$ let $\gamma_k \in \mathrm{SL}_2(\mathbb{Z})$ be the matrices $\gamma_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\gamma_k := \begin{pmatrix} -p_k & p_{k-1} \\ -q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} -b_0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -b_k & 1 \\ -1 & 0 \end{pmatrix}.$$

In particular, we have $b_0 = \frac{p_0}{q_0}$ and $\frac{a}{c} = \frac{p_n}{q_n}$, as well as the recursion

$$p_{k+1} := p_{k-1} - b_{k+1}p_k, \quad q_{k+1} := q_{k-1} - b_{k+1}q_k.$$

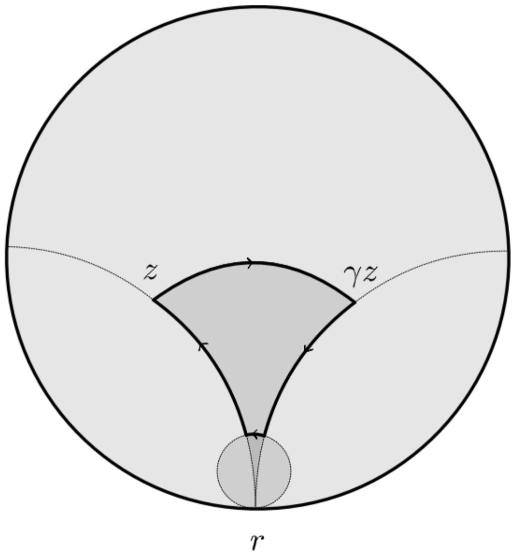

Figure 3: We visualize the hyperbolic 2*-space in the disk model. The cycle* $\mathcal{Z}_\gamma$ *is moved to the boundary component at the cusp* r*. The two sides are* Γ*-translates and cancel out.*

For $k \geq 0$ we deduce the following action of γ_k on the cusps

$$\gamma_k b_{k+1} = \frac{p_{k+1}}{q_{k+1}}, \quad \gamma_k \infty = \frac{p_k}{q_k}, \quad \gamma_k 0 = \frac{p_{k-1}}{q_{k-1}}.$$

Let $\mathcal{P}$ be the closure of the hyperbolic polygon with endpoints the cusps $p_0/q_0, \ldots, p_n/q_n$ as well as the interior points z and γz, which is closed by adding modular caps at each cusp; see Figure 4.

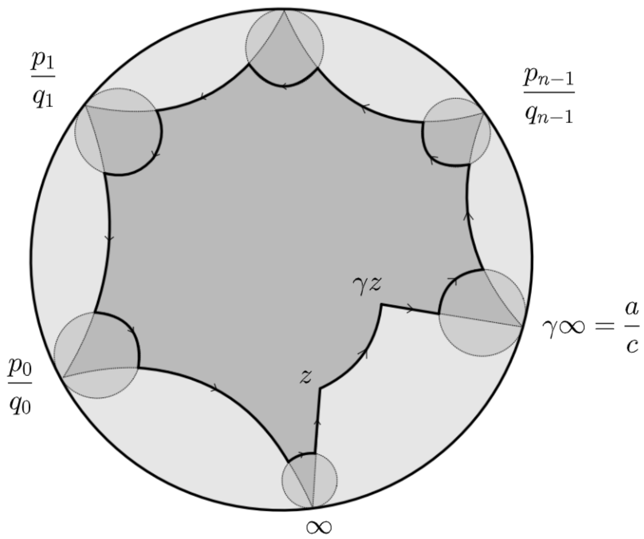

Figure 4: The polygon $\mathcal{P}$*.*

Its boundary consists of modular caps, modular symbols and some geodesic segments with endpoints in the interior of $\mathbb{H}$:

$$\begin{aligned}
\partial\mathcal{P} = & \left[\frac{p_0}{q_0}, z\right]_\infty + \left[\gamma z, \frac{p_{n-1}}{q_{n-1}}\right]_{\gamma\infty} + \sum_{k=1}^{n-1}\left[\frac{p_{k+1}}{q_{k+1}}, \frac{p_{k-1}}{q_{k-1}}\right]_{\frac{p_k}{q_k}} + \left[\frac{p_1}{q_1}, \infty\right]_{\frac{p_0}{q_0}} && \textit{(modular caps)} \\
& + \sum_{k=1}^{n}\left\{\frac{p_k}{q_k}, \frac{p_{k-1}}{q_{k-1}}\right\} + \left\{\frac{p_0}{q_0}, \infty\right\} && \textit{(modular symbols)} \\
& + \{z_0, \gamma z_0\} + \{\infty, z\} + \{\gamma z, \gamma\infty\}. && \textit{(remaining geodesic segments)}
\end{aligned}$$

It follows from the previous relations that

$$\begin{aligned}
\left[\frac{p_{k+1}}{q_{k+1}}, \frac{p_{k-1}}{q_{k-1}}\right]_{\frac{p_k}{q_k}} &= \gamma_k[b_{k+1}, 0]_\infty, \quad 1 \le k \le n-1 \\
\left[\frac{p_1}{q_1}, \infty\right]_{\frac{p_0}{q_0}} &= \gamma_0\,[b_1, 0]_\infty\,, \\
\left\{\frac{p_k}{q_k}, \frac{p_{k-1}}{q_{k-1}}\right\} &= \gamma_k\{\infty, 0\}, \quad 1 \le k \le n \\
\left\{\frac{p_0}{q_0}, \infty\right\} &= \gamma_0\{\infty, 0\}, \\
\left[\gamma z, \frac{p_{n-1}}{q_{n-1}}\right]_{\gamma\infty} &= \left[\gamma z, \frac{p_{n-1}}{q_{n-1}}\right]_{\gamma\infty}.
\end{aligned}$$

Moreover, we have $\gamma\infty = \frac{a}{c} = \frac{p_n}{q_n} = \gamma_n\infty$, so that

$$\gamma_n = \gamma\begin{pmatrix}1 & m \\ 0 & 1\end{pmatrix},$$

where $m = p_{n-1}d - bq_{n-1}$. Thus, we have $\frac{p_{n-1}}{q_{n-1}} = \gamma_n 0 = \gamma m$ and

$$\left[\gamma z, \frac{p_{n-1}}{q_{n-1}}\right]_{\gamma\infty} = \gamma[z, m]_\infty = [z, m]_\infty,$$

where the last step follows from the fact that $\gamma \in \Gamma$. Similarly, we have $\{\infty, z\} + \{\gamma z, \gamma\infty\} = 0$ in the homology of Y. It follows that

$$\partial\mathcal{P} = [b_0, z]_\infty + [z, m]_{\gamma\infty} - \sum_{k=1}^{n} b_k\gamma_{k-1}[0, 1]_\infty + \sum_{k=1}^{n}\gamma_k\{\infty, 0\} + \{b_0, \infty\} + \{z_0, \gamma z_0\},$$

from which the following theorem follows.

Theorem 2.3. *Under the isomorphism $H_1(Y;\mathbb{Z}) \simeq \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z})$ we can write the cycle as*

$$\mathcal{Z}_\gamma = \begin{cases} b(\gamma)\mathcal{C}_r & \text{if } \gamma \text{ is parabolic,} \\ (b_0 + bq_{n-1} - p_{n-1}d)\mathcal{C}_\infty + \sum_{k=0}^{n-1} b_{k+1}\mathcal{C}_{\gamma_k\infty} \oplus \sum_{k=0}^{n}\gamma_k\{0, \infty\} & \text{if } \gamma \text{ is hyperbolic.} \end{cases}$$

Note that $\sum_{k=0}^{n}\gamma_k\{0, \infty\}$ lies in $\mathcal{MS}_0(\mathbb{Z})$ since

$$\partial\sum_{k=0}^{n}\gamma_k\{0, \infty\} = \sum_{k=1}^{n}\left(\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right) + \left(\frac{p_0}{q_0} - \infty\right) = \frac{p_n}{q_n} - \infty = 0,$$

where the last step follows from the fact that $\frac{p_n}{q_n} = \frac{a}{c} = \gamma\infty \equiv \infty$ modulo Γ.

2.4. Pairing and and cohomology. Let us now take complex coefficients. The integration of 1-cycle on a 1-form gives a pairing

$$H^1(Y;\mathbb{C}) \times H_1(Y;\mathbb{C}) \longrightarrow \mathbb{C}.$$

Since the (co)homology of Y is isomorphic to that of $\overline{Y}^{\mathrm{BS}}$, we also have a pairing.

$$H^1(\overline{Y}^{\mathrm{BS}};\mathbb{C}) \times H_1(\overline{Y}^{\mathrm{BS}};\mathbb{C}) \longrightarrow \mathbb{C}.$$

A differential 1-form on Y can be extended to $\overline{Y}^{\mathrm{BS}}$ by viewing $\omega_r := \lim_{r\to\infty}\omega$ as differential form on the boundary component $\mathcal{C}_r$. The pairing with a cycle

$$\mathcal{Z} = \sum_{r\in C_N} n_r\mathcal{C}_r \oplus \mathcal{M} \in \mathcal{C}(\mathbb{C}) \oplus \mathcal{MS}_0(\mathbb{C})$$

is then

$$\int_{\mathcal{Z}} \omega = \sum_{r\in C_N} n_r \int_{\mathcal{C}_r} \omega_r + \int_{\mathcal{M}} \omega.$$

On the other hand, Poincaré-Lefschetz duality gives

$$H_1(Y;\mathbb{C}) \simeq H^1(Y;\mathbb{C})^\vee \simeq H^1_c(Y;\mathbb{C}), \tag{2.3}$$

where the first isomorphism is induced by the above pairing, and the second isomorphism is induced from the pairing against a compactly supported form.

From a long exact sequence in cohomology (as in (2.1)), one gets the exact map

$$0 \longrightarrow H^0(\overline{Y}^{\mathrm{BS}};\mathbb{C}) \longrightarrow H^0(\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}) \longrightarrow H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}) \longrightarrow H^1(\overline{Y}^{\mathrm{BS}};\mathbb{C}) \longrightarrow H^1(\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}).$$

The last map is the restriction to the boundary, and the *interior cohomology* $H^1_!(Y;\mathbb{C})$ is the image of $H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C})$ in $H^1(Y;\mathbb{C}) = H^1(\overline{Y}^{\mathrm{BS}};\mathbb{C})$. We also have an isomorphism with the compactly supported cohomology $H^1_c(Y;\mathbb{C}) \simeq H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C})$. Similarly to the splitting in modular caps and modular symbols, we have a splitting

$$H^1(Y;\mathbb{C}) \simeq H^1_{\mathrm{Eis}}(Y;\mathbb{C}) \oplus H^1_!(Y;\mathbb{C}). \tag{2.4}$$

By the Eichler-Shimura isomorphism, we have an isomorphism

$$M_2(\Gamma) \oplus \overline{S_2(\Gamma)} \longrightarrow H^1(Y,\mathbb{C}), \quad (f,\overline{g}) \longmapsto \omega_f + \overline{\omega_g}.$$

The subspace $H^1_!(Y;\mathbb{C})$ is the image of $S_2(\Gamma)\oplus\overline{S_2(\Gamma)}$, and $H^1_{\mathrm{Eis}}(Y;\mathbb{C})$ is the image of $E_2(\Gamma)$. We can also embed

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \hookrightarrow H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}), \quad (f,\overline{g}) \longmapsto \omega_f + \overline{\omega_g}.$$

Under Poincaré-Lefschetz duality $H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}) \simeq H^1(Y;\mathbb{C})^\vee$, this subspace is isomorphic to $H^1_!(Y;\mathbb{C})^\vee$. This gives a decomposition

$$H^1(\overline{Y}^{\mathrm{BS}},\partial\overline{Y}^{\mathrm{BS}};\mathbb{C}) = H^1_{\mathrm{Eis}}(Y;\mathbb{C})^\vee \oplus S_2(\Gamma) \oplus \overline{S_2(\Gamma)}. \tag{2.5}$$

Finally, we had the splitting

$$H_1(Y;\mathbb{C}) \simeq \mathcal{C}(\mathbb{C}) \oplus \mathcal{MS}_0(\mathbb{C}). \tag{2.6}$$

The various maps are compatible with the pairing, and the conclusion is that the three splitting (2.4),(2.5) and (2.6) are compatible with the isomorphisms (2.3).

2.5. Hecke operators. We follow [Lan95, p. 111] and define

$$\Delta_1^{(n)}(N) = \left\{ M = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \;\middle|\; \det(M) = n > 0, \;\; a \equiv 1 \mod N \right\}$$

for $n > 0$. The congruence subgroup $\Gamma = \Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$ acts by multiplication on the left and the right of $\Gamma\Delta_1^{(n)}(N)\Gamma$. The left quotient $\Gamma\backslash\Delta_1^{(n)}(N)$ is finite. An explicit choice of representatives is given by matrices

$$\gamma(a,b,d) = \sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \qquad ad = n, \;\; d > 0, \;\; (a,N) = 1,$$

where for each a dividing n and coprime to N we choose a matrix σ_a in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod N.$$

The action of the coset representatives induces the Hecke operator

$$T_n\{\alpha,\beta\} := \sum_{\gamma \in \Gamma\backslash\Delta_1^{(n)}(N)} \{\gamma\alpha, \gamma\beta\}.$$

The Hecke operators also act on differentials forms as follows. If $\omega \in \Omega^1(\mathbb{H})^\Gamma$ is a Γ-invariant form on $\mathbb{H}$, then

$$T_n\omega := \sum_{\gamma \in \Gamma\backslash\Delta_1^{(n)}(N)} X^*\omega$$

is a Γ-invariant form on $\mathbb{H}$. It induces an action of Hecke operators on the cohomology group $H^1(Y)$. If $\omega_{\{\alpha,\beta\}} \in \Omega^1(Y)$ is a Poincaré dual to $\{\alpha,\beta\}$, then $T_n\omega_{\{\alpha,\beta\}} = \omega_{T_n\{\alpha,\beta\}}$. Finally, the Hecke operators act on a modular form $f \in M_2(\Gamma)$ by

$$T_n f := \sum_{\gamma \in \Gamma\backslash\Delta_1^{(n)}(N)} f\big|_{\gamma,2}.$$

If $\omega_f = f(z)dz$, then $T_n(\omega_f) = \omega_{T_n f}$.

2.6. Modular forms and Kronecker-Eisenstein series. Let $e(\alpha) = e^{2i\pi\alpha}$. We start by introducing the Kronecker-Eisenstein series

$$\mathcal{K}_k(s,\tau,\lambda,\mu) := \left(\frac{y}{\pi}\right)^{s-k} \frac{\Gamma(s)}{(-2i\pi)^k} \sum_{\omega \in \mathbb{Z}+\tau\mathbb{Z}}{}' \frac{\overline{\omega+\lambda}^k}{|\omega+\lambda|^{2s}} e\left(\frac{\mathrm{Im}(\omega\bar{\mu})}{y}\right), \qquad \tau = x + iy,$$

defined for a non-negative integer k, and complex numbers λ, μ, s. It converges for $\mathrm{Re}(s) > 1 + \frac{k}{2}$ and the $'$ means that we remove $w = -x$ from the summation if x is in $\mathbb{Z}\tau + \mathbb{Z}$. This is the series considered by Weil in [Wei76, section VIII]. The function admits a meromorphic continuation to the whole plane with only pole at $s = 1$ (if $k = 0$ and λ is in $\mathbb{Z}\tau + \mathbb{Z}$); see [Wei76, section VIII, p. 80]. Moreover, it satisfies the functional equation

$$\mathcal{K}_k(s,\tau,\lambda,\mu) = e\left(\frac{\mathrm{Im}(\bar{\lambda}\mu)}{y}\right) \mathcal{K}_k(1+k-s,\tau,\mu,\lambda).$$

We will consider the case where λ, μ are N-torsion points on $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. For two integers $p, q \in \mathbb{Z}$, we define

$$\begin{aligned} E^{(k,l)}_{p,q}(\tau) &:= \mathcal{K}_{k+l}\left(k, \tau, 0, \frac{p\tau + q}{N}\right) \\ &= \frac{(k-1)!}{(-2i\pi)^{k+l}} \left(\frac{y}{\pi}\right)^{-l} \lim_{s\to 0} \sum_{m,n\in\mathbb{Z}}{}' \frac{\overline{m\tau+n}^{l}}{(m\tau+n)^k|m\tau+n|^{2s}} e\left(\frac{mq-np}{N}\right) \end{aligned}$$

and

$$\begin{aligned} \widehat{E}^{(k,l)}_{p,q}(\tau) &:= \mathcal{K}_{k+l}\left(k, \tau, \frac{p\tau + q}{N}, 0\right) \\ &= N^{k-l} \frac{(k-1)!}{(-2i\pi)^{k+l}} \left(\frac{y}{\pi}\right)^{-l} \lim_{s\to 0} \sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv p \mod N \\ n\equiv q \mod N}}{}' \frac{\overline{m\tau+n}^{l}}{(m\tau+n)^k|m\tau+n|^{2s}}. \end{aligned}$$

For these forms, the functional equation gives

$$\widehat{E}^{(k,l)}_{p,q}(\tau) = E^{(1+l,k-1)}_{p,q}(\tau). \tag{2.7}$$

In particular, when $l = 0$ we get the Eisenstein series

$$\widehat{E}^{(k)}_{p,q}(\tau) := \widehat{E}^{(k,0)}_{p,q}(\tau), \qquad E^{(k)}_{p,q}(\tau) := E^{(k,0)}_{p,q}(\tau).$$

They define holomorphic modular forms in $M_k(\Gamma(N))$ in weight $k \neq 2$. When $k = 2$, the Eisenstein series $E^{(2)}_{p,q}(\tau)$ with $(p,q) \not\equiv (0,0) \mod N$ are holomorphic, whereas $\widehat{E}^{(2)}_{p,q}$ transforms like a modular form of weight 2 but is non-holomorphic.

We have an orthogonal decomposition $M_k(\Gamma(N)) := E_k(\Gamma(N)) \oplus S_k(\Gamma(N))$, where the Eisenstein spaces in weight 1 and 2 (the only weights that we will consider) are

$$\begin{aligned} E_1(\Gamma(N)) &:= \operatorname{span}\left\{ \widehat{E}^{(1)}_{p,q}(\tau) \,\middle|\, (p,q) \in (\mathbb{Z}/N\mathbb{Z})^2 \right\}, \\ E_2(\Gamma(N)) &:= \operatorname{span}\left\{ \widehat{E}^{(2)}_{p,q}(\tau) \,\middle|\, (p,q) \in (\mathbb{Z}/N\mathbb{Z})^2 \right\} \cap M_2(\Gamma(N)) \end{aligned}$$

see [Miy89, Theorem. 7.2.18]. Since only the constant term of $\widehat{E}^{(2)}_{p,q}(\tau)$ is non-holomorphic, we can also characterize the Eisenstein space in weight $k = 2$ as

$$E_2(\Gamma(N)) := \left\{ \sum_{(p,q)\in(\mathbb{Z}/N\mathbb{Z})^2} a_{pq} \widehat{E}^{(2)}_{p,q}(\tau) \,\middle|\, \sum_{(p,q)\in(\mathbb{Z}/N\mathbb{Z})^2} a_{pq} = 0 \right\};$$

see [DS05, Section. 5.11]. In particular, the difference $\widehat{E}^{(2)}_{p,q} - \widehat{E}^{(2)}_{0,0}$ is holomorphic. For a congruence subgroup $\Gamma(N) \subseteq \Gamma$ we set $E_k(\Gamma) := E_k(\Gamma(N)) \cap M_k(\Gamma)$, then we have a decomposition

$$M_k(\Gamma) = E_k(\Gamma) \oplus S_k(\Gamma),$$

which is orthogonal with respect to the Petersson inner product.

Let us also consider the special values

$$G^{(k)}_r(\tau) := E^{(k)}_{r,0}(N\tau), \qquad \widehat{G}^{(k)}_r(\tau) := \widehat{E}^{(k)}_{r,0}(N\tau).$$

Explicitly, they are given by the sums

$$G_r^{(k)}(\tau) = N\frac{(k-1)!}{(-2i\pi)^k}\lim_{s\to 0}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(mN\tau+n)^k|mN\tau+n|^{2s}}e\left(-\frac{rn}{N}\right),$$

$$\widehat{G}_r^{(k)}(\tau) = \frac{(k-1)!}{(-2i\pi)^k}\lim_{s\to 0}\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv r \mod N}}{}' \frac{1}{(m\tau+n)^k|m\tau+n|^{2s}}.$$

They are again holomorphic, except in some cases in weight $k=2$ as above. Note that for $k=1$, we have $G_r^{(1)}(\tau) = \widehat{G}_r^{(1)}(\tau)$ by the functional equation. Finally, let us also define

$$H_{p,q}^{(2)}(\tau) := G_q^{(2)}(\tau) - \delta_{q0}\widehat{G}_p^{(2)}(\tau)$$

where we will use the notation

$$\delta_{q0} = \begin{cases} 1 & \text{if } q\equiv 0 \mod N \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.4. *For $k=1$, the modular forms $G_r^{(1)}(\tau)$ lie in the Eisenstein space $E_1(\Gamma_1(N))$. For $k=2$, the modular form $H_{p,q}^{(2)}(\tau)$ lies in the Eisenstein space $E_2(\Gamma_1(N))$.*

Proof. Notice that

$$G_r^{(k)}(\tau) = \sum_{l\in\mathbb{Z}/N\mathbb{Z}} \zeta_N^{-rl}\widehat{E}_{0,l}^{(k)}(\tau), \qquad \widehat{G}_r^{(k)}(\tau) = \sum_{l\in\mathbb{Z}/N\mathbb{Z}} \widehat{E}_{r,l}^{(k)}(\tau),$$

and that both lie in $M_2(\Gamma_1(N))$. For $k=1$, they lie in $E_1(\Gamma_1(N))$, by definition. For $k=2$, the modular form $H_{p,q}^{(2)}$ is in $E_2(\Gamma_1(N))$ since

$$\sum_{l\in\mathbb{Z}/N\mathbb{Z}} \zeta_N^{-ql} - \delta_{q0}N = 0.$$

□

2.7. Fourier expansions.

Proposition 2.5. *[Bru17, Lemma. 3.3] Let $k\geq 1$ and $(p,q)\not\equiv(0,0) \mod N$. We have the Fourier expansion*

$$E_{p,q}^{(k)}(\tau) = a_0(E_{p,q}^{(k)}) + N^{1-k}\left(\sum_{\substack{m,n\geq 1\\ n\equiv p(N)}} \zeta_N^{mq}n^{k-1}q_\tau^{mn/N} + (-1)^k\sum_{\substack{m,n\geq 1\\ n\equiv -p(N)}} \zeta_N^{-mq}n^{k-1}q_\tau^{mn/N}\right)$$

where we write $q_\tau = e(\tau)$. The constant term is

$$a_0(E_{p,q}^{(1)}) = \begin{cases} \frac{1}{2}\frac{1+\zeta_N^q}{1-\zeta_N^q} & \text{if } p=0,\ q\neq 0 \\ -B_1\left(\left\{\frac{p}{N}\right\}\right) & \text{if } p\neq 0\end{cases}$$

in weight $k=1$, and

$$a_0(E_{p,q}^{(2)}) = -\frac{1}{2}B_2\left(\left\{\frac{p}{N}\right\}\right)$$

in weight $k=2$, where $B_k(t)$ is the Bernoulli polynomial and $\{t\}$ denotes the fractional part of t.

In Theorem 1.5 we stated that

$$G_a^{(1)}G_b^{(1)} + G_b^{(1)}G_c^{(1)} + G_c^{(1)}G_a^{(1)} = G_a^{(2)} + G_b^{(2)} + G_c^{(2)} \tag{2.8}$$

for three integers a, b, c that satisfy $a + b + c \equiv 0 \mod N$ and such that $a, b, c \neq 0 \mod N$. We can verify this relation by comparing the constant terms on both sides. Since $G_r^{(k)}(\tau) = E_{r,0}^{(k)}(N\tau)$, the two Eisenstein series have the same constant term

$$\begin{aligned} a_0(G_r^{(2)}) = a_0(E_{r,0}^{(2)}) = -\frac{1}{2}B_2\left(\left\{\frac{r}{N}\right\}\right), \\ a_0(G_a^{(1)}G_b^{(1)}) = B_1\left(\left\{\frac{a}{N}\right\}\right)B_1\left(\left\{\frac{b}{N}\right\}\right). \end{aligned}$$

Finally, the condition $a + b + c \equiv 0 \mod N$ implies that the numbers $x = \{\frac{a}{N}\}$, $y = \{\frac{b}{N}\}$ and $z = \{\frac{c}{N}\}$ satisfy $x + y + z = 1$ or 2. Finally, taking the constant term of (2.8) is consistent with the following relation between Bernoulli polynomials.

Lemma 2.6. *Let x, y, z be three real numbers such that $x + y + z = 1$ or 2 . Then*

$$B_1(x)B_1(y) + B_1(y)B_1(z) + B_1(z)B_1(x) = -\frac{1}{2}\left(B_2(x) + B_2(y) + B_2(z)\right).$$

Proof. We have $B_1(t) = t - \frac{1}{2}$ and $B_2(t) = t^2 - t + \frac{1}{6} = B_1(t)^2 - \frac{1}{12}$. For three real numbers x, y, z, we have

$$\begin{aligned} (B_1(x) + B_1(y) + B_1(z))^2 = &B_2(x) + B_2(y) + B_2(z) + \frac{1}{4} \\ &+ 2\left(B_1(x)B_1(y) + B_1(y)B_1(z) + B_1(z)B_1(x)\right). \end{aligned}$$

Thus, if $x + y + z = 1$ or 2, then $B_1(x) + B_1(y) + B_1(z) = \pm\frac{1}{2}$ and the equality follows. ☐

3. Theta lift

We recall the construction of [Bra25] in the setting of the modular curve.

3.1. Symmetric space of $\mathrm{GL}_2(\mathbb{R})$.

Let X be the set of positive-definite quadratic forms on $\mathbb{R}^2$, which can also be identified with the space of positive-definite symmetric 2×2 matrices. The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on a symmetric matrix M in X by $M \longmapsto gMg^T$, and the stabilizer of the symmetric matrix $M = \mathbf{1}_2$ is $\mathrm{SO}(2)$. Let $A_\mathbb{R} \simeq \mathbb{R}_{>0}$ be the center in $\mathrm{GL}_2(\mathbb{R})^+$. We have a natural diffeomorphism

$$\mathbb{H} \times A_\mathbb{R} \simeq \mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2) \simeq X, \qquad (z, t) \longmapsto zt = \frac{t^2}{v}\begin{pmatrix} |z|^2 & u \\ u & 1 \end{pmatrix}.$$

Let $z = u + iv$ be the coordinates on $\mathbb{H}$. The first map sends $(z, t) \in \mathbb{H} \times A_\mathbb{R}$ to $tg_z\mathrm{SO}(2)$ where

$$g_z = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}.$$

The second map sends it to the positive definite quadratic form

$$t^2 g_z g_z^T = \frac{t^2}{v}\begin{pmatrix} |z|^2 & u \\ u & 1 \end{pmatrix} \in X.$$

3.2. Tautological bundle. Let V be the vector space $\mathrm{Mat}_2(\mathbb{Q})$. We identify $V \simeq \mathbb{Q}^2 \oplus \mathbb{Q}^2$ by writing a matrix $\mathbf{v} = [m, n]$, where $m = \left[\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right]$ and $n = \left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$ are the columns of $\mathbf{v}$. Let Q be the bilinear form on V defined by

$$Q(\mathbf{v}, \mathbf{v}') := \langle m, n' \rangle + \langle m', n \rangle, \qquad \mathbf{v} = [m, n], \quad \mathbf{v}' = (m', n').$$

The associated quadratic form

$$Q(\mathbf{v}) := \frac{1}{2} Q(\mathbf{v}, \mathbf{v}) = \langle m, n \rangle$$

is of signature $(2, 2)$. The group $\mathrm{GL}_2(\mathbb{R})$ acts on V by

$$\rho_g(\mathbf{v}) = (gm, g^{-t} n), \qquad \mathbf{v} = [m, n].$$

This defines a representation $\rho \colon \mathrm{GL}_2(\mathbb{Q}) \longrightarrow \mathrm{SO}(V) \simeq \mathrm{SO}(2,2)(\mathbb{Q})$. Over $X = \mathbb{H} \times A_{\mathbb{R}}$ we have a rank 2-bundle

$$B = \mathrm{GL}_2(\mathbb{R})^+ \times_{\mathrm{SO}(2)} \mathbb{R}^2,$$

where the fiber is endowed with the Euclidean metric. The bundle is isomorphic to the trivial bundle $X \times \mathbb{R}^2$, where the metric over $z \in X$ is given by the quadratic form z^{-1}.

For $\mathbf{v} = [m, n]$ we define a section

$$s_{\mathbf{v}}(z, t) = \left[t g_z, \frac{t^{-1} g_z^{-1} m - t g_z^T n}{2} \right].$$

We denote by $X_{\mathbf{v}}$ the zero locus of this section, and by $S_{\mathbf{v}}$ its projection onto $\mathbb{H}$.

Definition 3.1. *A vector $\mathbf{v} = [m, n]$ in $V_{\mathbb{R}}^2$ is* regular *if the columns m, n are both nonzero vectors, and* singular *otherwise. We say that $\mathbf{v}$ is positive if $Q(\mathbf{v}) > 0$ (and negative if $Q(\mathbf{v}) < 0$).*

Proposition 3.2. *We have the following.*

1. *The locus $X_{\mathbf{v}}$ satisfies the equivariance $gX_{\mathbf{v}} = X_{\rho(g)\mathbf{v}}$ and $gS_{\mathbf{v}} = S_{\rho(g)\mathbf{v}}$ for all $g \in \mathrm{SL}_2(\mathbb{R})$.*
2. *If $\mathbf{v}$ is a positive regular vector, then the locus $X_{\mathbf{v}}$ is a submanifold of codimension 1 in X. The restriction of the projection $\mathbb{H} \times A_{\mathbb{R}} \longrightarrow \mathbb{H}$ is a diffeomorphism onto $X_{\mathbf{v}} \longrightarrow S_{\mathbf{v}}$. Moreover, if $\mathbf{v} = \left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]$, then $S_{\mathbf{v}}$ is the modular symbol*
$$S_{\mathbf{v}} = \left\{ \frac{n_2}{n_1}, -\frac{m_1}{m_2} \right\}.$$
3. *If $\mathbf{v} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ then $X_{\mathbf{v}} = X$.*
4. *In all other cases ($\mathbf{v}$ nonzero singular or nonpositive regular), we have $X_{\mathbf{v}} = \emptyset$.*

3.3. Special cycles. We start by defining some lattice cosets. Let $V_{\mathbb{Z}} := \mathrm{Mat}_2(\mathbb{Z})$. For a matrix $\mathbf{x}_0 \in V_{\mathbb{Z}}$, let

$$L_{\mathbf{x}_0} := \mathbf{x}_0 + N V_{\mathbb{Z}}.$$

More generally, we denote by $L \subseteq V_{\mathbb{Z}}$ any linear combination of the form

$$L = \sum_{\mathbf{x}_0 \in \mathrm{Mat}_2(\mathbb{Z}/N\mathbb{Z})} n_{\mathbf{x}_0} L_{\mathbf{x}_0}. \tag{3.1}$$

We will mainly use the lattice coset

$$L_{p,q} := \left\{ \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \;\middle|\; \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \equiv \begin{bmatrix} p \\ q \end{bmatrix} \mod N \right\},$$

defined for two integers $(p,q) \in \mathbb{Z}^2$. It is the linear combination

$$L_{p,q} = \sum_{k,l \in \mathbb{Z}/N\mathbb{Z}} L_{\left[\begin{smallmatrix} p & k \\ q & l \end{smallmatrix}\right]}.$$

For a lattice coset L, let Γ be the stabilizer under ρ *i.e.* $\rho_\gamma L = L$ for all $\gamma \in \Gamma$. The stabilizer of $L_{p,q}$ is $\Gamma(N)$ in general, and the stabilizer of $L_{1,0}$ is $\Gamma_1(N)$. By the equivariance, the image $\mathcal{Z}_{[\mathbf{v}]}$ of $S_\mathbf{v}$ in $Y = \Gamma\backslash\mathbb{H}$ only depends on the class $[\mathbf{v}] \in \Gamma\backslash V$. It represents a homology class

$$\mathcal{Z}_{[\mathbf{v}]} \in H_1^{\mathrm{BM}}(Y;\mathbb{Z}) := H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}}; \mathbb{Z}).$$

For a positive integer n, we define

$$\mathcal{Z}_n(L) := \sum_{\substack{[\mathbf{v}] \in \Gamma\backslash L \\ Q(\mathbf{v}) = n}} \mathcal{Z}_{[\mathbf{v}]} \in H_1^{\mathrm{BM}}(Y;\mathbb{Z}). \tag{3.2}$$

Proposition 3.3. *[Bra25, Proposition. 5.1] We have*

$$\mathcal{Z}_n(L_{1,0}) = T_n\{0,\infty\} \in H_1(\overline{Y_1(N)}^{BS}, \partial\overline{Y_1(N)}^{BS}; \mathbb{Z}).$$

Proof. Let $L_{1,0}^{(n)}$ be the set of vectors in $L_{1,0}$ of determinant n. This set coincides with $\Delta_1^{(n)}(N)$ that was used to define the Hecke operators in Section 2.5, but with a different action of $\Gamma_1(N)$. On $\Delta_1^{(n)}(N)$ it acts by left matrix multiplication, whereas on $L_{1,0}^{(n)}$ it acts by

$$\rho_\gamma \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} = \left[\gamma \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \gamma^{-T} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right].$$

We have a bijection between the quotients

$$\Gamma_1(N)\backslash\Delta_1^{(n)}(N) \longrightarrow \Gamma_1(N)\backslash L_{1,0}^{(n)}, \quad \Gamma_1(N)\begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \longmapsto \Gamma_1(N)\begin{bmatrix} m_1 & n_2 \\ -m_2 & n_1 \end{bmatrix}.$$

The map is well-defined since the action of $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ by ρ on the left becomes multiplication by $\left[\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right]$ on the right. Each coset representative $\Delta_1^{(n)}(N)\mathbf{v} = \Delta_1^{(n)}(N)\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]$ sends $\{0,\infty\}$ to $T_\mathbf{v}\{0,\infty\} = \left\{\frac{n_1}{n_2}, \frac{m_1}{m_2}\right\}$. From Proposition 3.2, the submanifold $S_\mathbf{v}$ is then

$$S_{\left[\begin{smallmatrix} m_1 & n_2 \\ -m_2 & n_1 \end{smallmatrix}\right]} = \left\{\frac{n_1}{n_2}, \frac{m_1}{m_2}\right\} = T_\mathbf{v}\{0,\infty\}.$$

$\square$

3.4. Eisenstein class. Let $U \in \Omega^2(B)$ be the Mathai-Quillen Thom form on the vector bundle B, which is closed and $\mathrm{GL}_2(\mathbb{R})^+$-equivariant form of integral 1 along the fiber. Let

$$\varphi^0(z,t,\mathbf{v}) := s_\mathbf{v}^* U \in \Omega^2(\mathbb{H} \times A_\mathbb{R}) \otimes C^\infty(V_\mathbb{R})$$

be the pullback along the section $s_{\mathbf{v}}$. It is closed and satisfies the invariance property

$$g^*\varphi^0(z,t,\mathbf{v}) = \varphi^0(z,t,\rho_{g^{-1}}\mathbf{v}) \tag{3.3}$$

for all $g \in \mathrm{GL}_2(\mathbb{R})^+$. The form $\varphi^0(z,t,\mathbf{v})$ is smooth in $\mathbf{v}$, but not rapidly decreasing. We obtain a rapidly decreasing form by setting

$$\varphi(z,t,\mathbf{v}) \coloneqq e^{-2\pi Q(\mathbf{v})}\varphi^0(z,t,\mathbf{v}) \in \Omega^2(\mathbb{H}\times A_{\mathbb{R}}) \otimes \mathcal{S}(V_{\mathbb{R}}).$$

This form is also closed and satisfies

$$g^*\varphi(z,t,\mathbf{v}) = \varphi(z,t,\rho_{g^{-1}}\mathbf{v}). \tag{3.4}$$

Let $\omega\colon \mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{R}) \longrightarrow U(\mathcal{S}(V_{\mathbb{R}}))$ be the Weil representation defined by

$$\begin{aligned}
&\omega(g,t,1)\phi(\mathbf{v}) = \phi(\rho_{gt}^{-1}\mathbf{v}) && (g,t) \in \mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}},\\
&\omega(1,1,h)\,\phi(\mathbf{v}) = ye(xQ(\mathbf{v}))\phi(\sqrt{y}\mathbf{v}) && h = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y}\end{pmatrix},\\
&\omega(1,1,h)\,\phi(\mathbf{v}) = \int_{V_{\mathbb{R}}} \phi(\mathbf{v}')e\left(-Q(\mathbf{v},\mathbf{v}')\right) d\mathbf{v}' && h = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix},
\end{aligned}$$

for $\phi \in \mathcal{S}(V_{\mathbb{R}}^2)$. We define the theta series

$$\begin{aligned}
\Theta_{\mathbf{x}_0}(z,t,\tau) &\coloneqq j(h_\tau,i)^{-2}\sum_{\mathbf{v}\in L_{\mathbf{x}_0}} \omega(1,1,h_\tau)\varphi(z,t,\mathbf{v}) \\
&= \sum_{\mathbf{v}\in L_{\mathbf{x}_0}} \varphi^0(z,t,\sqrt{y}\mathbf{v})e(\tau Q(\mathbf{v})),
\end{aligned}$$

and $\Theta_L(z,t,\tau)$ for a linear combination L of lattice cosets as in (3.1). It is a Γ-invariant form in z, and transforms like a modular form of weight 2 in τ. Thus it defines an element

$$\Theta_L(z,t,\tau) \coloneqq j(h_\tau,i)^{-1}\sum_{\mathbf{v}\in L}\omega(1,1,h_\tau)\varphi(z,t,\mathbf{v}) \in \left[\Omega^2(\mathbb{H}\times A_{\mathbb{R}})\otimes C^\infty(\mathbb{H})\right]^{\Gamma\times\Gamma}.$$

Let $\pi\colon \mathbb{H}\times A_{\mathbb{R}} \longrightarrow \mathbb{H}$ be the projection onto $\mathbb{H}$. Given a differential 2-form

$$\omega = f_{uv}(z,t)dudv + f_{ut}(z,t)dudt + f_{vt}(z,t)dvdt \in \Omega^2(\mathbb{H}\times A_{\mathbb{R}}),$$

its pushforward is defined by (see [BT82, p.61])

$$\pi_*\omega = du\left(\int_{A_{\mathbb{R}}} f_{ut}(z,t)dt\right) + dv\left(\int_{A_{\mathbb{R}}} f_{vt}(z,t)dt\right) \in \Omega^1(\mathbb{H}).$$

We obtain the Eisenstein series by taking the pushforward (see [BT82, p.61])

$$\mathscr{E}_{\mathbf{x}_0}(z,\tau,s) \coloneqq \pi_*\left(\Theta_{\mathbf{x}_0}(z,t,\tau)t^{2s}\right) \in \left[\Omega^1(\mathbb{H})\otimes C^\infty(\mathbb{H})\right]^{\Gamma\times\Gamma}.$$

The integral converges for any $s \in \mathbb{C}$, and for $\mathrm{Re}(s) \gg 0$ the sum and integral over $A_{\mathbb{R}}$ can be interchanged. We write $\mathscr{E}_L(z,t,\tau)$ for the pushforward of $\Theta_L(z,t,\tau)t^{2s}$.

3.5. Formulas. Let $H_1(t) = 2t$ and $H_2(t) = 4t^2 - 2$ be the first two Hermite polynomials.

Proposition 3.4. *Let $z = u + iv$ be the coordinates on $\mathbb{H}$. We have $\varphi(z, t, \mathbf{v}) = \omega(g_z, t, 1)\varphi(\mathbf{v})$ where*

$$\varphi(\mathbf{v}) = -\left(\varphi_{(2,0)}(\mathbf{v}) + \varphi_{(0,2)}(\mathbf{v})\right) \frac{dudv}{4v^2} + \left(\left(\varphi_{(0,2)}(\mathbf{v}) - \varphi_{(2,0)}(\mathbf{v})\right) \frac{du}{2v} + \varphi_{(1,1)}(\mathbf{v}) \frac{dv}{v} \right) \frac{dt}{t},$$

and the components are

$$\begin{aligned}
\varphi_{(2,0)}(\mathbf{v}) &:= \frac{1}{4\pi} \exp\left(-\pi\|m\|^2 - \pi\|n\|^2\right) H_2(\sqrt{\pi}\langle m+n, e_1\rangle) \\
\varphi_{(0,2)}(\mathbf{v}) &:= \frac{1}{4\pi} \exp\left(-\pi\|m\|^2 - \pi\|n\|^2\right) H_2(\sqrt{\pi}\langle m+n, e_2\rangle) \\
\varphi_{(1,1)}(\mathbf{v}) &:= \frac{1}{4\pi} \exp\left(-\pi\|m\|^2 - \pi\|n\|^2\right) H_1(\sqrt{\pi}\langle m+n, e_1\rangle) H_1(\sqrt{\pi}\langle m+n, e_2\rangle).
\end{aligned}$$

Proof. We briefly recall the computations from [Bra25] that lead to the explicit expression given in the proposition.

Let $\vartheta = (gt)^{-1}d(gt) = t^{-1}dt + g^{-1}dg$ be the Maurer-Cartan form on the principal $\mathrm{SO}(2)$-bundle $\mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}} \longrightarrow \mathbb{H} \times A_{\mathbb{R}}$. Let $\lambda := \frac{1}{2}(\vartheta + \vartheta^t) \in \Omega^1(\mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}}) \otimes \mathrm{Mat}(\mathbb{R}^2)$ and let $\lambda_{ij} \in \Omega^1(\mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}})$ be its (i,j)-entry. In the coordinates $z = u + iv$ on $\mathbb{H}$, then

$$\lambda = \frac{1}{2}\left(\vartheta + \vartheta^t\right) = \frac{1}{2}\left(g^{-1}dg + (g^{-1}dg)^t\right) + \frac{dt}{t} = \begin{pmatrix} \frac{dt}{t} + \frac{dv}{2v} & \frac{du}{2v} \\ \frac{du}{2v} & \frac{dt}{t} - \frac{dv}{2v}. \end{pmatrix}$$

For a function $\sigma \colon \{1,2\} \longrightarrow \{1,2\}$ we define the 2-form

$$\lambda(\sigma) := \lambda_{1\sigma(1)} \wedge \lambda_{2\sigma(2)} \in \Omega^1(\mathrm{SL}_2(\mathbb{R}) \times A_{\mathbb{R}})$$

and the generalized Hermite polynomial $H_\sigma \in \mathbb{C}[\mathbb{R}^2]$ by

$$H_\sigma(a_1, a_2) := H_{d_1}(a_1) H_{d_2}(a_2)$$

where $d_k = |\sigma^{-1}(k)|$ and $H_d(t)$ is the single variable Hermite polynomial. Note that $H_0(t) = 1$. With these notations, it was computed in [Bra25, Proposition. 3.4]

$$\varphi(z, t, \mathbf{v}) = \frac{1}{4\pi} \sum_{\sigma} H_\sigma\left(\sqrt{\pi}(t^{-1}g^{-1}m + tg^T n)\right) \exp\left(-\pi\|t^{-1}g^{-1}m\|^2 - \|tg^T n\|^2\right) \lambda(\sigma),$$

where the sum is over all functions $\sigma \colon \{1,2\} \longrightarrow \{1,2\}$. We have

$$\begin{aligned}
\lambda_{11}\lambda_{21} &= -\frac{dudv}{4v^2} - \frac{dudt}{2vt}, \\
\lambda_{12}\lambda_{22} &= -\frac{dudv}{4v^2} + \frac{dudt}{2vt}, \\
\lambda_{12}\lambda_{21} &= 0, \\
\lambda_{11}\lambda_{22} &= \frac{dvdt}{vt}.
\end{aligned}$$

□

For each Schwartz function $\varphi_{(d_1,d_2)}$, let $\Theta_L(z, t, \tau)_{(d_1,d_2)}$ be the corresponding theta series

$$\Theta_L(z, t, \tau)_{(d_1,d_2)} := j(h_\tau, i)^{-2} \sum_{\mathbf{v} \in L} \omega(g_z, t, h_\tau)\varphi_{(d_1,d_2)}(\mathbf{v}). \tag{3.5}$$

We have

$$\mathscr{E}_L(z,\tau,s) = \left(\mathscr{E}_L(z,\tau,s)_{(0,2)} - \mathscr{E}_L(z,\tau,s)_{(2,0)}\right)\frac{du}{2v} + \mathscr{E}_L(z,\tau,s)_{(1,1)}\frac{dv}{v}$$

where the (d_1,d_2)-component of $\mathscr{E}_L$ is

$$\mathscr{E}_L(z,\tau,s)_{(d_1,d_2)} := \int_0^\infty \Theta_L(z,t,\tau)_{(d_1,d_2)} t^{2s}\frac{dt}{t}.$$

Let L' be the image of L under the involution

$$\begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \longmapsto \begin{bmatrix} -n_2 & -m_1 \\ n_1 & m_2 \end{bmatrix}. \tag{3.6}$$

Proposition 3.5. *The form $\mathscr{E}_L(z,\tau,s)$ satisfies the functional equation*

$$\mathscr{E}_L(z,\tau,-s) = -\mathscr{E}_{L'}(z,\tau,s).$$

In particular, we have $\mathscr{E}_L(z,\tau)=0$ if $L=L'$ (for example if $L=V_{\mathbb{Z}}$).

Proof. From $g^{-T} = S^{-1}gS$ with $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, we get

$$\langle t^{-1}g^{-1}m + tg^T n, e_k\rangle = \langle t^{-1}g^T Sm + tg^{-1}Sn, Se_k\rangle.$$

Since $Se_1 = -e_2$ and $Se_2 = e_1$, we see from the formula in Proposition 3.4 that

$$\omega(g,t,h_\tau)\varphi_{(2,0)}(m,n) = \omega(g,t^{-1},h_\tau)\varphi_{(0,2)}(Sn,Sm),$$
$$\omega(g,t,h_\tau)\varphi_{(1,1)}(m,n) = -\omega(g,t^{-1},h_\tau)\varphi_{(1,1)}(Sn,Sm).$$

The map $(m,n)\mapsto(Sn,Sm)$ is exactly the involution (3.6). Finally, since changing t to t^{-1} in the integral has the same effect as replacing s by $-s$, we find that

$$\mathscr{E}_L(z,\tau,s)_{(2,0)} = \mathscr{E}_{L'}(z,\tau,-s)_{(0,2)},$$
$$\mathscr{E}_L(z,\tau,s)_{(1,1)} = -\mathscr{E}_{L'}(z,\tau,-s)_{(1,1)},$$

and the result follows. □

3.6. Fourier expansion of the theta lift. Under Poincaré-Lefschetz duality

$$H_1(\overline{Y}^{\mathrm{BS}}, \partial\overline{Y}^{\mathrm{BS}};\mathbb{Z}) \simeq H^1(Y;\mathbb{Z}),$$

the cycle $\mathcal{Z}_n(\mathbf{x}_0) := \mathcal{Z}_n(L_{\mathbf{x}_0})$ has a Poincaré dual

$$\mathrm{PD}(\mathcal{Z}_n(\mathbf{x}_0)) \in H^1(Y;\mathbb{Z}),$$

which is characterized by the property that for any closed form $\eta\in\Omega_c^1(Y)$, we have

$$\int_Y \eta\wedge\mathrm{PD}(\mathcal{Z}_n(\mathbf{x}_0)) = \int_{\mathcal{Z}_n(\mathbf{x}_0)}\eta.$$

If $\eta_{\mathcal{Z}}$ represents the Poincaré dual in $H_c^1(Y;\mathbb{Z})$ of a cycle $\mathcal{Z}\in H_1(Y;\mathbb{Z})$, then

$$\int_Y \eta_{\mathcal{Z}}\wedge\mathrm{PD}(\mathcal{Z}_n(\mathbf{x}_0)) = \langle\mathcal{Z},\mathcal{Z}_n(\mathbf{x}_0)\rangle$$

is the intersection number between the two cycles.

Theorem 3.6. *Let $\mathbf{x}_0 = \begin{pmatrix} p & k \\ q & l \end{pmatrix} \in \mathrm{Mat}(\mathbb{Z}/N\mathbb{Z})$ be a nonzero matrix. The cohomology class of $\mathscr{E}_{\mathbf{x}_0}$ has the Fourier expansion*

$$[\mathscr{E}_{\mathbf{x}_0}] = a(\mathbf{x}_0) - \sum_{n=1}^{\infty} \mathrm{PD}(\mathcal{Z}_n(\mathbf{x}_0))e(n\tau) \in H^1(Y(N);\mathbb{Q}) \otimes M_2(\Gamma(N)),$$

where

$$a(\mathbf{x}_0) = \delta_{k0}\delta_{l0}E_{-q,p}(z)dz - \delta_{p0}\delta_{q0}E_{k,l}(z)dz \in H^1(Y_1(N);\mathbb{Q}).$$

Proof. This is a special case of the results in [Bra25], except the computation of the constant term. We briefly recall the proof. Since the theta series is absolutely convergent, we can group the vectors of length n and write

$$\Theta_{\mathbf{x}_0}(z,t,\tau) = \Theta_{\mathbf{x}_0}^{(0)}(z,t,y) + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \Theta_{\mathbf{x}_0}^{(n)}(z,t,y)e(n\tau),$$

where the n-th Fourier coefficient is

$$\Theta_{\mathbf{x}_0}^{(n)}(z,t,y) \coloneqq \sum_{\substack{[\mathbf{v}]\in\Gamma\backslash L_{\mathbf{x}_0} \\ Q(\mathbf{v})=n}} \sum_{\mathbf{w}\in\Gamma\mathbf{v}} \varphi^0(z,t,\sqrt{y}\mathbf{w}).$$

For all regular vectors $\mathbf{v}$, the term

$$\sum_{\mathbf{w}\in\Gamma\mathbf{v}} \varphi^0(z,t,\sqrt{y}\mathbf{w}) \in \Omega^2(Y \times A_{\mathbb{R}}) \otimes C^{\infty}(\mathbb{R}_{>0})$$

is a Poincaré dual to the cycle $X_{[\mathbf{v}]} \in H_2^{\mathrm{BM}}(Y \times A_{\mathbb{R}};\mathbb{Z})$, represented by the image of $X_{\mathbf{v}}$ modulo Γ. Taking the pushforward gives the Fourier expansion

$$\mathscr{E}_{\mathbf{x}_0}(z,\tau) = \mathscr{E}_{\mathbf{x}_0}^{(0)}(z) + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \mathscr{E}_{\mathbf{x}_0}^{(n)}(z,y)e(n\tau)$$

where for $n \neq 0$

$$\mathscr{E}_{\mathbf{x}_0}^{(n)}(z,y) \coloneqq \pi_*\left(\Theta_{\mathbf{x}_0}^{(n)}(z,t,y)\right) \in \Omega^1(Y(N)) \otimes C^{\infty}(\mathbb{R}_{>0}).$$

Interchanging the sum and the integral over $A_{\mathbb{R}}$ is only valid at $s = 0$ for the sum over the regular vectors. For the sum over the singular vectors, the exchange can only be done for $\mathrm{Re}(s)$ large enough.

For regular vectors $\mathbf{v}$, the pushforward

$$\pi_*\left(\sum_{\mathbf{w}\in\Gamma\mathbf{v}} \varphi^0(z,t\sqrt{y}\mathbf{w})\right) \in \Omega^1(Y(N)) \otimes C^{\infty}(\mathbb{R}_{>0})$$

is a Poincaré dual to $\mathcal{Z}_{[\mathbf{v}]}$. In particular, this form is exact if $\mathbf{v}$ is a nonpositive regular vector (since $X_{\mathbf{v}}$ is empty by Proposition 3.2). For positive n, we sum over positive regular vectors and

$$\mathscr{E}_{\mathbf{x}_0}^{(n)}(z,y) = \sum_{\substack{[\mathbf{v}]\in\Gamma\backslash L_{\mathbf{x}_0} \\ Q(\mathbf{v})=n}} \pi_*\left(\sum_{\mathbf{w}\in\Gamma\mathbf{v}} \varphi^0(z,t\sqrt{y}\mathbf{w})\right)$$

is a Poincaré dual to $\mathcal{Z}_n(\mathbf{x}_0)$, by the definition given in (3.2).

Finally, the remaining term is the sum over the singular vectors

$$\mathscr{E}^{(0)}_{\mathbf{x}_0}(z,\tau)=\pi_*\left(\sum_{\substack{\mathbf{v}\in L_{\mathbf{x}_0}\\ \mathbf{v}\text{ singular}}}\varphi^0(z,t,\sqrt{y}\mathbf{v})\right).$$

It splits into the sum over the vectors $\mathbf{v}=\left[\begin{smallmatrix} m_1 & 0\\ m_2 & 0\end{smallmatrix}\right]$ and $\mathbf{v}=\left[\begin{smallmatrix} 0 & n_1\\ 0 & n_2\end{smallmatrix}\right]$ in $L_{\mathbf{x}_0}$ (if $L_{\mathbf{x}_0}$ contains such vectors). If $\left[\begin{smallmatrix} k\\ l\end{smallmatrix}\right]\equiv\left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]$, then there is a contribution from the vectors $\mathbf{v}=\left[\begin{smallmatrix} m_1 & 0\\ m_2 & 0\end{smallmatrix}\right]$, with $m\in\left[\begin{smallmatrix} p+N\mathbb{Z}\\ q+N\mathbb{Z}\end{smallmatrix}\right]$. Taking the decomposition from Proposition 3.4, and the dt/t component of $\varphi^0(z,t,\tau,\mathbf{v})$, we find that the contribution is

$$\begin{aligned}&\sum_{\substack{m_1\in p+N\mathbb{Z}\\ m_2\in q+N\mathbb{Z}}} ye^{-\pi y\frac{|zm_2-m_1|^2}{t^2v}}\left(\frac{v^2m_2^2-(m_1-um_2)^2}{vt^2}\frac{du}{2v}+\frac{(m_1-um_2)vm_2}{vt^2}\frac{dv}{v}\right)\\ &=-\sum_{\substack{m_1\in p+N\mathbb{Z}\\ m_2\in q+N\mathbb{Z}}}\frac{y}{2v^2t^2}e^{-\pi y\frac{|zm_2-m_1|^2}{vt^2}}\mathrm{Re}\left(\overline{zm_2-m_1}^2dz\right).\end{aligned}$$

After adding the term t^{2s} with $\mathrm{Re}(s)\gg 0$ large enough, we can bring the integral inside of the sum, and find that the latter is

$$=-\sum_{\substack{m_1\in p+N\mathbb{Z}\\ m_2\in q+N\mathbb{Z}}}\int_{A_{\mathbb{R}}}\frac{y}{2v^2}e^{-\pi y\frac{|zm_2-m_1|^2}{vt^2}}\mathrm{Re}\left(\overline{zm_2-m_1}^2dz\right)t^{2s-2}\frac{dt}{t}.$$

At $s=0$, this is equal to $\mathrm{Re}\left(\widehat{E}^{(1,1)}_{q,-p}(z)dz\right)$, where we recall that

$$\widehat{E}^{(1,1)}_{a,b}(z):=\mathcal{K}_2\left(1,z,\frac{az+b}{N},0\right)=-\frac{1}{4\pi v}\lim_{s\to 0}\sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv a\mod N\\ n\equiv b\mod N}}\frac{\overline{mz+n}}{(mz+n)|mz+n|^{2s}}.$$

Using the functional equation (2.7), we find that the contribution from the vectors $\mathbf{v}=\left[\begin{smallmatrix} m_1 & 0\\ m_2 & 0\end{smallmatrix}\right]$ is

$$\mathrm{Re}\left(\widehat{E}^{(1,1)}_{q,-p}(z)dz\right)=\mathrm{Re}\left(E^{(2)}_{q,-p}dz\right),$$

which is cohomologous to $E^{(2)}_{q,-p}(z)$ by Lemma 3.7 below. Finally, note that $E^{(2)}_{q,-p}(z)=E^{(2)}_{-q,p}(z)$.

Suppose that $\mathbf{x}_0=\left[\begin{smallmatrix} 0 & k\\ 0 & l\end{smallmatrix}\right]$, with $\left[\begin{smallmatrix} p\\ q\end{smallmatrix}\right]\equiv\left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]$. From Proposition 3.5, we have the functional equation $\mathscr{E}_{\mathbf{x}_0}(z,\tau,s)=-\mathscr{E}_{\mathbf{x}'_0}(z,\tau,-s)$ where $\mathbf{x}'_0=\left[\begin{smallmatrix} -l & 0\\ k & 0\end{smallmatrix}\right]$ and we find that the second contribution is $-E^{(2)}_{k,l}(z)$, from the vectors of the form $\mathbf{v}=\left[\begin{smallmatrix} 0 & n_1\\ 0 & n_2\end{smallmatrix}\right]$. □

For $(a,b)\not\equiv(0,0)\mod N$, the Siegel unit is defined by

$$g_{a,b}(\tau):=q_\tau^{B_2(a/N)/2}\zeta_N^{\frac{b(a-1)}{2}}\prod_{n=1}^{\infty}(1-q_\tau^{n+\frac{a}{N}}\zeta_N^b)(1-q_\tau^{n-\frac{a}{N}}\zeta_N^{-b}),$$

where $q_\tau=e^{2i\pi\tau}$. The function $g_{a,b}(\tau)^{12}$ is a modular function for $\Gamma(N)$, and defines an element in $\mathscr{O}(Y(N))^\times$ (or $\mathscr{O}(Y_1(N))^\times$ if $(a,b)=(0,1)$).

Lemma 3.7. *The forms* $\mathrm{Re}(E^{(2)}_{p,q}(z)dz)$ *and* $E^{(2)}_{p,q}(z)dz$ *are cohomologous.*

Proof. We show that $\mathrm{Im}(E^{(2)}_{p,q}(z)dz)$ is exact, using the same idea as in [BCG23a, Lemma. 9.6]. The weight 0 Kronecker-Eisenstein series (with $(p,q)\neq(0,0)$)

$$\mathcal{K}_0\left(s,z,0,\frac{pz+q}{N}\right)=\frac{\Gamma(s)}{\pi^s}\sum_{m,n\in\mathbb{Z}}{}'\frac{v^s}{|mz+n|^{2s}}e\left(\frac{mq-np}{N}\right)$$

satisfies Kronecker's second limit formula [Lan87, p. 276]

$$\lim_{s\to 1}\mathcal{K}_0\left(s,z,0,\frac{pz+q}{N}\right)=-2\pi\log|g_{p,q}(z)|$$

where $g_{p,q}(z)$ is the Siegel unit. Moreover, from

$$\frac{\partial}{\partial z}\frac{v^s}{|mz+n|^{2s}}=-is\frac{v^{s-1}}{(mz+n)^2|mz+n|^{2s-2}}$$

we deduce that

$$\frac{\partial}{\partial z}\mathcal{K}_0\left(s,z,0,\frac{pz+q}{N}\right)=4is\frac{\pi^{s+1}}{\Gamma(1+s)}\mathcal{K}_2\left(1+s,z,0,\frac{pz+q}{N}\right)$$

for $\mathrm{Re}(s)\gg 0$. Since $(p,q)\neq(0,0)$, the functions admit an analytic continuation to the entire plane and at $s=1$ we have

$$-\frac{1}{2i\pi}\frac{\partial}{\partial z}\log|g_{p,q}(z)|=E^{(2)}_{p,q}(z).$$

Similarly, we have

$$-\frac{1}{2i\pi}\frac{\partial}{\partial\bar z}\log|g_{p,q}(z)|=-\overline{E^{(2)}_{p,q}(z)},$$

from which we conclude that

$$-\frac{1}{2i\pi}d\log|g_{p,q}(z)|=E^{(2)}_{p,q}(z)dz-\overline{E^{(2)}_{p,q}(z)dz}=2i\mathrm{Im}\left(E^{(2)}_{p,q}(z)dz\right).$$

□

3.7. Nonvanishing of L-functions. Let us now consider the lattice cosets $L=L_{p,q}$ with $(p,q)\not\equiv(0,0)\mod N$, where we recall that

$$L_{p,q}=\sum_{k,l\in\mathbb{Z}/N\mathbb{Z}}L_{\left[\begin{smallmatrix}p&k\\q&l\end{smallmatrix}\right]}.$$

It is preserved by $\Gamma=\Gamma(N)$ (or $\Gamma_1(N)$ if $(p,q)=(1,0)$). To simplify notation, let us set

$$\Theta_{p,q}(z,t,\tau):=\Theta_{L_{p,q}}(z,t,\tau),\quad \mathscr{E}_{p,q}(z,\tau):=\mathscr{E}_{L_{p,q}}(z,\tau).$$

From the invariance of the form φ^0 in (3.3), it follows that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ we have

$$\gamma^*\mathscr{E}_{p,q}(z,\tau)=\mathscr{E}_{(p,q)\gamma^{-t}}(z,\tau)=\mathscr{E}_{dp-bq,aq-pc}(z,\tau).\tag{3.7}$$

For $(p,q)=(1,0)$, it follows from Theorem 3.6 and Proposition 3.3 that the Fourier expansion is

$$[\mathscr{E}_{1,0}]=E^{(2)}_{0,1}(z)dz-\sum_{n=1}^{\infty}\mathrm{PD}(T_n\{0,\infty\})e(n\tau)\in H^1(Y_1(N);\mathbb{Q})\otimes M_2(\Gamma_1(N)).$$

By the following lemma, the constant term is equal to $-\frac{1}{2i\pi}d\log(g_{0,1})$, as in the introduction.

Lemma 3.8. *For $(p,q) \neq (0,0)$ we have $2i\pi E^{(2)}_{p,q}(z)dz = -d\log(g_{p,q}(z))$.*

Proof. See [BZ23, Lemma. 37]. The form $E^{(2)}_{p,q}(z)$ is the form $-E^{(2)}_{\mathbf{x}}(z)$ in *loc. cit.* with $\mathbf{x} = (p/N, q/N)$. $\square$

The integral against $\mathscr{E}_{1,0}$ defines a lift

$$\mathscr{E}\colon H^1(\overline{Y_1(N)}^{\mathrm{BS}}, \partial\overline{Y_1(N)}^{\mathrm{BS}}; \mathbb{C}) \longrightarrow M_2(\Gamma_1(N))$$

which sends a cuspidal class $[\omega]$ represented by a closed form ω, to the modular form

$$\int_Y \omega \wedge \mathscr{E}_{1,0}(z,\tau) = \int_{Y_1(N)} \omega \wedge E^{(2)}_{0,1}(z)dz - \sum_{n=1}^{\infty}\left(\int_{\{0,\infty\}} T_n\omega\right) e(n\tau).$$

The restriction of $\mathscr{E}$ to $S_2(\Gamma_1(N))$ is

$$\mathscr{E}(\omega_f) = i\sum_{n=1}^{\infty}\left(\int_0^{\infty}(T_nf)(iy)dy\right)e(n\tau),$$

which is the map $\rho(f)$ considered by Borisov-Gunnells [BG01]. The set

$$\mathscr{B} := \{f(n\tau) \mid f \in S_2^{\mathrm{new}}(\Gamma_1(N/d)), \text{ for all } n \mid d \mid N\}$$

is a basis of $S_2(\Gamma_1(N))$; see [DS05, Theorem. 5.8.3]. Let $S^{\mathrm{new}}_{2,\mathrm{rk}=0}(\Gamma_1(N)) \subset S_2^{\mathrm{new}}(\Gamma_1(N))$ be the space of newforms with nonvanishing L-function, and

$$\mathcal{U} := \operatorname{span}\left\{f(n\tau) \in \mathscr{B} \mid f \in S^{\mathrm{new}}_{2,\mathrm{rk}=0}(\Gamma_1(N/d)), \text{ for all } n \mid d \mid N\right\},$$

spanned by lifts of forms with nonvanishing L-function. We also define the subspace

$$\mathcal{H}^{(2)} := \operatorname{span}\{H^{(2)}_{p,q} \mid p,q \text{ are coprime}\} \subseteq E_2(\Gamma_1(N)),$$

and finally

$$\mathcal{V} := \mathcal{H} \oplus \operatorname{span}\{\mathcal{U}, \mathcal{U}^{\sigma}\} \subseteq M_2(\Gamma_1(N)),$$

where $\mathcal{U}^{\sigma}$ is the space spanned by the complex conjugates of the forms in $\mathcal{U}$ (the complex conjugation on the Fourier coefficients). Note that if f is a newform (so an eigenform for all T_n), we have

$$\mathscr{E}(\omega_f) = -\frac{1}{2i\pi}L(f,1)f, \qquad \mathscr{E}(\overline{\omega_f}) = \frac{1}{2i\pi}L(f^{\sigma},1)f^{\sigma}. \tag{3.8}$$

Using Poincaré duality $H^1(\overline{Y_1(N)}^{\mathrm{BS}}, \partial\overline{Y_1(N)}^{\mathrm{BS}}; \mathbb{C}) \simeq H_1(Y_1(N);\mathbb{C})$, we can also view it as map

$$\mathscr{E}\colon H_1(Y_1(N);\mathbb{C}) \longrightarrow M_2(\Gamma_1(N))$$

sending the class represented by a cycle $\mathcal{Z}$ to

$$\int_{\mathcal{Z}} \mathscr{E}_{1,0}(z,\tau) = \int_{\mathcal{Z}} E^{(2)}_{0,1}(z)dz - \sum_{n=1}^{\infty}\langle \mathcal{Z}, T_n\{0,\infty\}\rangle e(n\tau).$$

Theorem 3.9. *We have*

$$\mathcal{H}^{(2)} \oplus S^{new}_{2,\mathrm{rk}=0}(\Gamma_1(N)) \subseteq \mathrm{Im}(\mathscr{E}) \subseteq \mathcal{V}.$$

In particular, if N is prime, then

$$\mathcal{H}^{(2)} \oplus S^{new}_{2,\mathrm{rk}=0}(\Gamma_1(N)) = \mathrm{Im}(\mathscr{E}).$$

Proof. As in Section 2.4, we can split the cuspidal cohomology

$$H^1(\overline{Y_1(N)}^{\mathrm{BS}}, \partial\overline{Y_1(N)}^{\mathrm{BS}}; \mathbb{C}) = H^1_{\mathrm{Eis}}(Y; \mathbb{C})^\vee \oplus S_2(\Gamma_1(N)) \oplus \overline{S_2(\Gamma_1(N))}.$$

It is proved in [BG01, Proposition. 4.5] that the image of $S_2(\Gamma_1(N))$ is contained in $\mathcal{U}$, and it follows that the image of $\overline{S_2(\Gamma_1(N))}$ is contained in $\mathcal{U}^\sigma$. On the other hand, in homology, the space $H^1_{\mathrm{Eis}}(Y_1(N); \mathbb{C})^\vee$ corresponds to the space of modular caps $\mathcal{C}(\mathbb{C})$. By Proposition 4.2 and Corollary 4.4.1, it follows that the images of modular caps span $\mathcal{H}^{(2)} \subseteq E_2(\Gamma_1(N))$. This shows that the image is contained in $\mathcal{V}$, and the inclusion of $S^{\mathrm{new}}_{2,\mathrm{rk}=0}(\Gamma_1(N))$ follows from (3.8). □

3.8. Poisson summation. For the computations in the next section, we will apply Poisson summation. For $\varphi \in \mathcal{S}(V_\mathbb{R}^2)$, let $\widehat{\varphi}$ be the partial Fourier transform

$$\widehat{\varphi}\left(\begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix}\right) := \int_{\mathbb{R}^2} \varphi\left(\begin{bmatrix} a & n_1 \\ m_2 & b \end{bmatrix}\right) e(an_2 - bm_1) da db. \tag{3.9}$$

Recall that the Weil representation was given by

$$\begin{aligned} \omega(g,t,1)\varphi(\mathbf{v}) &= \varphi(\rho_{gt}^{-1}\mathbf{v}) & (g,t) &\in \mathrm{SL}_2(\mathbb{R}) \times A_\mathbb{R}, \\ \omega(1,1,h)\varphi(\mathbf{v}) &= ye(xQ(\mathbf{v}))\varphi(\sqrt{y}\mathbf{v}) & h &= \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \\ \omega(1,1,h)\varphi(\mathbf{v}) &= \int_{V_\mathbb{R}} \varphi(\mathbf{v}')e\left(-Q(\mathbf{v},\mathbf{v}')\right) d\mathbf{v}' & h &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

We define ω' by

$$\omega'(g,t,h) := \omega(h,t,g).$$

Let $\mathcal{F} \colon \mathcal{S}(V_\mathbb{R}^2) \longrightarrow \mathcal{S}(V_\mathbb{R}^2)$ denote the operator $\mathcal{F}(\varphi) := \widehat{\varphi}$. A direct computation shows that

$$\mathcal{F} \circ \omega = \omega' \circ \mathcal{F}.$$

Applying this to the theta series (3.5) shows that

$$\begin{aligned} \Theta_{\mathbf{x}_0}(z,t,\tau)_{(d_1,d_2)} &= \frac{1}{y} \sum_{\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L_{\mathbf{x}_0}} \omega(g_z, t, h_\tau)\varphi_{(d_1,d_2)}\left(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]\right) \\ &= \frac{1}{N^2 y} \sum_{\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{\mathbf{x}_0}} \omega(h_\tau, t, g_z)\widehat{\varphi}_{(d_1,d_2)}\left(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]\right) e\left(lm_1 - pn_2\right) \end{aligned}$$

where $\mathbf{x}_0 = \left[\begin{smallmatrix} p & k \\ q & l \end{smallmatrix}\right]$ and

$$L^*_{\mathbf{x}_0} := \left\{ \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \in V \;\middle|\; n_1 \in k + N\mathbb{Z},\ m_2 \in q + N\mathbb{Z},\ m_1, n_2 \in \frac{1}{N}\mathbb{Z} \right\}.$$

The Fourier transform of $H_d(\sqrt{\pi}(m+n))e^{-\pi m^2-\pi n^2}$ in n is

$$\int_{\mathbb{R}} H_d(\sqrt{\pi}(m+a))e^{-\pi m^2-\pi a^2}e(an)da = 2^d i^d \sqrt{\pi}^d \overline{(im+n)}e^{-\pi m^2-\pi n^2}. \tag{3.10}$$

So the partial Fourier transform of

$$\varphi_{(d_1,d_2)}\left(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]\right) = \frac{1}{4\pi}H_{d_1}(\sqrt{\pi}(m_1+n_1))H_{d_2}(\sqrt{\pi}(m_2+n_2))e^{-\pi\|m\|^2-\pi\|n\|^2}$$

is

$$\widehat{\varphi}_{(d_1,d_2)}\left(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]\right) = -\overline{(n_2+in_1)}^{d_1}\overline{(im_2-m_1)}^{d_2}e^{-\pi\|m\|^2-\pi\|n\|^2}$$

and

$$\begin{aligned}\omega(h_\tau,t,g_z)\widehat{\varphi}_{(d_1,d_2)}&\left(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]\right) \\ &= -v^2t^{d_1-d_2}\frac{\overline{(n_1\tau+n_2)}^{d_1}\overline{(m_2\tau-m_1)}^{d_2}}{y}e^{-\pi vt^2\frac{|n_1\tau+n_2|^2}{y}-\pi\frac{v}{t^2}\frac{|m_2\tau-m_1|^2}{y}}e^{2i\pi u\langle m,n\rangle}.\end{aligned}$$

It follows that $\Theta_{\mathbf{x}_0}(z,t,\tau)_{(d_1,d_2)}$ is equal to

$$-\frac{v^2t^{d_1-d_2}}{N^2y^2}\sum_{\mathbf{v}}\overline{(n_1\tau+n_2)}^{d_1}\overline{(m_2\tau-m_1)}^{d_2}e\left(u\langle m,n\rangle+lm_1-pn_2\right)e^{-\pi vt^2\frac{|n_1\tau+n_2|^2}{y}-\pi\frac{v}{t^2}\frac{|m_2\tau-m_1|^2}{y}}, \tag{3.11}$$

where the sum is over $\mathbf{v} = \left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{\mathbf{x}_0}$.

4. Integral over modular caps and modular symbols

The next goal will be to compute the image of modular caps and modular symbols under this theta lift. Using the isomorphism $H_1(Y;\mathbb{Z}) \simeq \mathcal{C}(\mathbb{Z}) \oplus \mathcal{MS}_0(\mathbb{Z})$, we can then express the lift of a cycle $\mathcal{Z}$ as a linear combination of lifts of caps and modulars symbols.

4.1. Constant term and integral over modular caps.

We start by computing the constant term of $\mathscr{E}_{p,q}(z,\tau)$, in the boundary component $\mathcal{C}_\infty$.

Proposition 4.1. *We have*

$$\lim_{v\to\infty}\mathscr{E}_{p,q}(z,\tau) = H^{(2)}_{p,q}(\tau)du, \qquad (z=u+iv).$$

Proof. We compute the limit as $v\to\infty$ of each of the components

$$\mathscr{E}_{p,q}(z,\tau,s)_{(d_1,d_2)} := \int_0^\infty \Theta_{p,q}(z,t,\tau)_{(d_1,d_2)}t^{2s}\frac{dt}{t}.$$

We start with the $(2,0)$-component. From (3.11) we write

$$\Theta_{p,q}(z,t,\tau)_{(2,0)} = -\frac{v^2t^2}{Ny^2}\sum_{\mathbf{v}}\overline{(n_1\tau+n_2)}^2e\left(u\langle m,n\rangle-pn_2\right)e^{-\pi vt^2\frac{|n_1\tau+n_2|^2}{y}-\pi\frac{v}{t^2}\frac{|m_2\tau-m_1|^2}{y}},$$

where the sum is over $\mathbf{v} = \left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{p,q}$. We split the sum over regular vectors $\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{p,q}$, for which $m, n \neq 0$, and singular vectors, for which $m = 0$ or $n = 0$. For two real numbers $a, b > 0$ and a complex number $\nu \in \mathbb{C}$, we define the Bessel function by

$$K_\nu(a,b) = \int_0^\infty e^{-(a^2t+b^2/t)} t^\nu \frac{dt}{t}.$$

After integrating the theta series from 0 to ∞, we find that

$$\mathscr{E}_{p,q}(z,\tau,s)_{(2,0)} = \delta_{q0} A(\tau,v,s) + \sum_{d\in\mathbb{Z}} B^{(d)}(\tau,v,s) e(ud),$$

where for $d \in \mathbb{Z}$ the regular term is

$$B^{(d)}(\tau,v,s) = -\frac{v^2}{2Ny^2} \sum_{\substack{\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{p,q} \\ \langle m,n\rangle = d \\ m,n\neq 0}} \overline{(n_1\tau+n_2)}^2 e(-pn_2) K_{1+s}\left(\pi v \frac{|n_1\tau+n_2|}{y}, \pi v \frac{|m_2\tau - m_1|}{y}\right).$$

The regular term is rapidly decreasing for any $s \in \mathbb{C}$, so that

$$\lim_{v\to\infty} \mathscr{E}_{p,q}(z,\tau,s)_{(2,0)} = \delta_{q0} A(\tau,v,s).$$

Since $n_1\tau + n_2 = 0$ if $(n_1,n_2) = (0,0)$, the only singular vectors that contribute to $A(\tau,v,s)$ are the vectors $\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{p,q}$ with $(m_1,m_2) = (0,0)$. This is only possible if $q = 0$ (otherwise $A = 0$), in which case we have is

$$\begin{aligned} A(\tau,v,s) &:= -\frac{v^{1-s}\Gamma(1+s)}{2N\pi^{1+s}y^{1-s}} \sum_{\substack{n_1\in\mathbb{Z} \\ n_2\in\frac{1}{N}\mathbb{Z}}} \frac{\overline{n_1\tau+n_2}}{(n_1\tau+n_2)\left|n_1\tau+n_2\right|^{2s}} e(-pn_2) \\ &= -\frac{v^{1-s}\Gamma(1+s)}{2N^{1-2s}\pi^{1+s}y^{1-s}} \sum_{n_1,n_2\in\mathbb{Z}} \frac{\overline{n_1N\tau+n_2}}{(n_1N\tau+n_2)\left|n_1N\tau+n_2\right|^{2s}} e\left(-\frac{pn_2}{N}\right). \end{aligned}$$

At $s = 0$, it follows from the functional equation that

$$\lim_{s\to 0} A(\tau,v,s) = 2vE^{(1,1)}_{p,0}(N\tau) = 2v\widehat{E}^{(2)}_{p,0}(N\tau) = 2v\widehat{G}^{(2)}_p(\tau)$$

and we conclude that

$$\lim_{v\to\infty} \mathscr{E}_{p,q}(z,\tau,s)_{(2,0)} = \delta_{q0}\widehat{G}^{(2)}_p du.$$

The computation for the $(0,2)$-component is analogous and we find that

$$\mathscr{E}_{p,q}(z,\tau,s)_{(0,2)} = \widetilde{A}(\tau,v,s) + \sum_{d\in\mathbb{Z}} \widetilde{B}^{(d)}(\tau,v,s) e(ud),$$

where the sum over the regular terms is rapidly decreasing, and

$$\widetilde{A}(\tau,v,s) := -\frac{v^{1+s}}{2N\pi^{1-s}y^{1+s}} \Gamma(1-s) \sum_{\substack{m_2\in q+N\mathbb{Z} \\ m_1\in\mathbb{Z}}} \frac{\overline{m_2\tau-m_1}}{(m_2\tau-m_1)\left|m_2\tau-m_1\right|^{-2s}}.$$

At $s=0$ it is equal to $\lim_{s\to 0}\widetilde{A}(\tau,v,s)=2vG_q^{(2)}(\tau)$ and it follows that

$$\lim_{v\to\infty}\mathscr{E}_{p,q}(z,\tau,s)_{(0,2)}=G_q^{(2)}(\tau).$$

Finally, for the component $(1,1)$ there are no singular contributions, since the term $(n_1\tau+n_2)(m_2\tau-m_1)$ forces the vanishing of the series whenever $m=0$ or $n=0$. Thus, the sum is rapidly decreasing as $v\to\infty$ and

$$\lim_{v\to\infty}\mathscr{E}_{p,q}(z,\tau,s)_{(1,1)}=0.$$

□

Proposition 4.2 (Integral over modular caps)**.** *Let $\mathcal{C}_r$ be the closed modular cap at the cusp $r=[m:n]\in\mathbb{P}^1(\mathbb{Q})$. Write $r=\gamma_r\infty$ for some matrix $\gamma_r=\left(\begin{smallmatrix} m & i\\ n & j\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. Then*

$$\int_{\mathcal{C}_r}\mathscr{E}_{1,0}(z,\tau)=H_{j,-n}^{(2)}(\tau).$$

Proof. Since $\int_{\mathcal{C}_\infty}du=1$ and the integral of $\mathscr{E}_{p,q}(z,\tau)$ along the modular cap $\mathcal{C}_r$ at the cusp r is given by the constant term (in z) of $\mathscr{E}_{p,q}(z,\tau)$ at the cusp r, we deduce that

$$\int_{\mathcal{C}_\infty}\mathscr{E}_{p,q}(z,\tau)=H_{p,q}^{(2)}(\tau).$$

From the equivariance of $\mathscr{E}^{(p,q)}$ in (3.7), it follows that

$$\int_{\gamma_r\mathcal{C}_\infty}\mathscr{E}_{1,0}(z,\tau)=\int_{\mathcal{C}_\infty}\gamma_r^*\mathscr{E}_{1,0}(z,\tau)=\int_{\mathcal{C}_\infty}\mathscr{E}_{j,-n}(z,\tau)=H_{j,-n}^{(2)}(\tau),$$

since $\gamma_r^{-1}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)=\left(\begin{smallmatrix}j\\-n\end{smallmatrix}\right)$. □

4.2. Integral over unimodular symbols. It will be enough to compute the integral of $\mathscr{E}^{(p,q)}$ along the modular symbol $\{0,\infty\}$, the computation over a unimodular symbol $\gamma\{0,\infty\}$ follows from translation. Moreover, only the $(1,1)$-component (which is the dv-component) contributes to the sum, so that we have

$$\int_{\{0,\infty\}}\mathscr{E}_{p,q}(z,\tau)=\int_0^\infty\mathscr{E}_{p,q}(iv,\tau)_{(1,1)}\frac{dv}{v}.$$

From the Poisson summation (3.11) it follows that the restriction of the theta series along the modular symbol $\{0,\infty\}$ is

$$\begin{aligned}&\Theta_{p,q}(iv,t,\tau)_{(1,1)}\\ &=-\frac{v^2}{Ny^2}\sum_{\left[\begin{smallmatrix} m_1 & n_1\\ m_2 & n_2\end{smallmatrix}\right]\in L_{p,q}^*}\overline{(n_1\tau+n_2)(m_2\tau-m_1)}e\left(-pn_2\right)e^{-\pi vt^2\frac{|n_1\tau+n_2|^2}{y}-\pi\frac{v}{t^2}\frac{|m_2\tau-m_1|^2}{y}}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\int_{\{0,\infty\}}\mathscr{E}_{p,q}(iv,\tau,s)_{(1,1)}\frac{dv}{v}&=\int_0^\infty\int_0^\infty\Theta_{p,q}(iv,t,\tau)_{(1,1)}t^{2s}\frac{dv}{v}\frac{dt}{t}\\ &=-\frac{1}{Ny^2}\sum_{\left[\begin{smallmatrix} m_1 & n_1\\ m_2 & n_2\end{smallmatrix}\right]\in L_{p,q}^*}\overline{(n_1\tau+n_2)(m_2\tau-m_1)}e\left(-pn_2\right)\\ &\quad\times\int_0^\infty\int_0^\infty e^{-\pi vt^2\frac{|n_1\tau+n_2|^2}{y}-\pi\frac{v}{t^2}\frac{|m_2\tau-m_1|^2}{y}}v^2t^{2s}\frac{dv}{v}\frac{dt}{t}.\end{aligned}$$

Setting $a = vt^2$ and $b = \frac{v}{t^2}$ gives

$$-\frac{1}{4Ny^2} \sum_{\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in L^*_{p,q}} \overline{(n_1\tau + n_2)(m_2\tau - m_1)} e(-pn_2) \\ \times \int_0^\infty \int_0^\infty e^{-\pi a \frac{|n_1\tau+n_2|^2}{y} - \pi b \frac{|m_2\tau - m_1|^2}{y}} a^{1+\frac{s}{2}} b^{1-\frac{s}{2}} \frac{da}{a}\frac{db}{b}$$

$$= -\frac{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{4N\pi^2} \sum_{\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ q+N\mathbb{Z} & \frac{1}{N}\mathbb{Z} \end{smallmatrix}\right]} \frac{1}{(m_2\tau - m_1)|m_2\tau - m_1|^{-\frac{s}{2}}} \frac{e(-pn_2)}{(n_1\tau + n_2)|n_1\tau + n_2|^{\frac{s}{2}}}.$$

By the functional equation, we have $\widehat{E}^{(1)}_{r,0}(N\tau) = E^{(1)}_{r,0}(N\tau)$ where

$$\widehat{E}^{(1)}_{r,0}(N\tau) = -\frac{1}{2i\pi} \lim_{s\to 0} \sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv p \mod N}} \frac{1}{(m\tau+n)|m\tau+n|^{2s}},$$
$$E^{(1)}_{r,0}(N\tau) = -\frac{1}{2i\pi} \lim_{s\to 0} \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(mN\tau+n)|mN\tau+n|^{2s}} e\left(-\frac{np}{N}\right).$$

Thus, we have

$$\int_{\{0,\infty\}} \mathscr{E}_{p,q}(z,\tau) = \widehat{E}^{(1)}_{q,0}(N\tau)E^{(1)}_{p,0}(N\tau) = \widehat{E}^{(1)}_{q,0}(N\tau)\widehat{E}^{(1)}_{p,0}(N\tau) = G^{(1)}_q(\tau)G^{(1)}_p(\tau).$$

Proposition 4.3 (Integral over unimodular symbols). *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The image of the unimodular symbol $\mathcal{M} = \gamma\{0,\infty\}$ is*

$$\int_{\mathcal{M}} \mathscr{E}_{1,0}(z,\tau) = -G^{(1)}_d(\tau)G^{(1)}_c(\tau).$$

Proof. From the equivariance of φ it follows that

$$\int_{\gamma\{0,\infty\}} \mathscr{E}_{1,0}(z,\tau) = \int_{\{0,\infty\}} \gamma^*\mathscr{E}_{1,0}(z,\tau) = \int_{\{0,\infty\}} \mathscr{E}_{d,-c}(z,\tau),$$

where

$$\gamma^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ -c \end{pmatrix}.$$

The result follows from the previous computation. □

4.3. Image of the theta lift. We can now determine the image of the theta lift

$$\mathscr{E} \colon H_1(Y_1(N);\mathbb{C}) \longrightarrow M_2(\Gamma_1(N)).$$

If γ is hyperbolic, let $[b_0,\ldots,b_n]$ be the continued fraction expansion of $\frac{a}{c} = \gamma\infty$ as in (2.2). Let $\frac{p_0}{q_0},\ldots,\frac{p_n}{q_n}$ be the convergents, and $m = p_{n-1}d - bq_{n-1}$. We set $(p_{-1},q_{-1}) \coloneqq (1,0)$.

Theorem 4.4. *If γ is a parabolic matrix in $\Gamma_1(N)$ stabilizing the cusp $r=[m:n]$, then*

$$\mathscr{E}(\mathcal{Z}_\gamma)=b(\gamma)H^{(2)}_{j,-n},$$

where i,j are integers such that $mj-in=1$. If $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is a hyperbolic matrix in $\Gamma_1(N)$, then

$$\mathscr{E}(\mathcal{Z}_\gamma)=(b_0+bq_{n-1}-p_{n-1}d)H^{(2)}_{1,0}+\sum_{k=0}^{n-1}b_{k+1}H^{(2)}_{q_{k-1},q_k}-\sum_{k=0}^{n}G^{(1)}_{q_k}G^{(1)}_{q_{k-1}}.$$

Proof. Recall from Theorem 2.3 that we can write the cycle as

$$\mathcal{Z}_\gamma=\begin{cases} b(\gamma)\mathcal{C}_r & \text{if } \gamma \text{ is parabolic,}\\ (b_0+bq_{n-1}-p_{n-1}d)\mathcal{C}_\infty+\sum_{k=0}^{n-1}b_{k+1}\mathcal{C}_{\gamma_k\infty}\oplus\sum_{k=0}^{n}\gamma_k\{0,\infty\} & \text{if } \gamma \text{ is hyperbolic}\end{cases}$$

under the isomorphism $H_1(Y;\mathbb{Z})\simeq\mathcal{C}(\mathbb{Z})\oplus\mathcal{MS}_0(\mathbb{Z})$. The integral along the unimodular symbol $\gamma_k\{0,\infty\}$ is

$$\int_{\gamma_k\{0,\infty\}}\mathscr{E}_{1,0}(z,\tau)=-G^{(1)}_{q_k}(\tau)G^{(1)}_{q_{k-1}}(\tau)$$

by Proposition 4.3, where $\gamma_k=\left(\begin{smallmatrix} -p_k & p_{k-1}\\ -q_k & q_{k-1}\end{smallmatrix}\right)$. If $\gamma_r\in\mathrm{SL}_2(\mathbb{Z})$ is a matrix such that $[m:n]=r=\gamma_r\infty$, then $\gamma_r=\begin{pmatrix} m & i\\ n & j\end{pmatrix}$ for some integers i,j such that $mj-in=1$. By Proposition 4.2, we have

$$\int_{\mathcal{C}_r}\mathscr{E}_{1,0}(z,\tau)=H^{(2)}_{j,-n}(\tau).$$

In particular, the integral along the modular cap $\mathcal{C}_{\gamma_k\infty}$ is

$$\int_{\mathcal{C}_{\gamma_k\infty}}\mathscr{E}_{1,0}(z,\tau)=H^{(2)}_{q_{k-1},q_k}(\tau).$$

For $\gamma_r=\mathbf{1}$, the integral along $\mathcal{C}_\infty$ is

$$\int_{\mathcal{C}_\infty}\mathscr{E}_{1,0}(z,\tau)=H^{(2)}_{1,0}(\tau).$$

□

Let $\mathcal{H}^{(2)}\subset E_2(\Gamma_1(N))$ be the subspace spanned by the forms $H^{(2)}_{p,q}(\tau)=G^{(2)}_q(\tau)$ with $q\not\equiv 0 \mod N$, and the forms $H^{(2)}_{p,0}$ with $p\in(\mathbb{Z}/N\mathbb{Z})^\times$. Let $\mathcal{H}^{(1,1)}\subset M_2(\Gamma_1(N))$ be the subspace spanned by all the products $G^{(1)}_pG^{(1)}_q$.

Corollary 4.4.1. *We have*

$$\mathcal{H}^{(2)}\oplus S^{new}_{2,\mathrm{rk}=0}(\Gamma_1(N))\subseteq\mathrm{Im}(\mathscr{E})\subseteq\mathrm{span}\{\mathcal{H}^{(2)},\mathcal{H}^{(1,1)}\}.$$

Proof. First, note that the forms $H^{(2)}_{1,0}$ and $H^{(2)}_{q_{k-1},q_k}$ that appear in the previous theorem are all in $\mathcal{H}^{(2)}$,since q_{k-1},q_k are coprime integers This proves $\mathrm{Im}(\mathscr{E})\subseteq\mathrm{span}\{\mathcal{H}^{(2)},\mathcal{H}^{(1,1)}\}$.

Now suppose that $H^{(2)}_{p,q}\in\mathcal{H}^{(2)}$. If $q\equiv 0 \mod N$, then (p,q) are coprime. By Proposition 4.2 we can then find a modular cap such that its image is $H^{(2)}_{p,q}$. If $q\not\equiv 0 \mod N$, then $H^{(2)}_{p,q}$ does not depend on p and $H^{(2)}_{p,q}=H^{(2)}_{1,q}$, which reduces it to the first case. Finally, the inclusion of the rank 0 forms was proved in Theorem 3.9. □

Remark 4.1. In view of [BG01, Theorem. 4.11], it seems reasonable to expect that these inclusions are also equalities, when N is prime. The first equality is Theorem 3.9, but it is not completely clear whether $\mathrm{Im}(\mathscr{E}) = \mathrm{span}\{\mathcal{H}^{(2)}, \mathcal{H}^{(1,1)}\}$ holds. While a single product $G_a^{(1)} G_b^{(1)} \in \mathcal{H}^{(1,1)}$ is the image of a unimodular symbol, one would need to write it as the image of a cycle, so a linear combination of modular caps and modular symbols of degree 0.

4.4. Diagonal restrictions of Eisenstein series. Recall that a matrix is hyperbolic if $|\mathrm{tr}(\gamma)| > 2$.

Proposition 4.5. *We have*

$$\mathrm{Im}(\mathscr{E}) = \mathrm{span}\{\mathscr{E}(\mathcal{Z}_\gamma) \mid \gamma \in \Gamma_1(N) \textit{ hyperbolic}\}.$$

Proof. It follows from the fact that any parabolic matrix can be written as a product of hyperbolic matrices. More precisely, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & t-a \end{smallmatrix}\right)$ is a hyperbolic matrix with $|t| = |\mathrm{tr}(\gamma)| > 2$, then we write

$$\gamma = \gamma\gamma_1\gamma_1^{-1}, \qquad \gamma_1 = \begin{pmatrix} 1+m^2 & m \\ m & 1 \end{pmatrix} \in \Gamma_1(N)$$

where m is any integer divisible by N. We have

$$\mathrm{tr}(\gamma\gamma_1) = t + (b+c)m + am^2.$$

For m large enough, both γ_1 and $\gamma\gamma_1$ are hyperbolic, and

$$\mathcal{Z}_\gamma = \mathcal{Z}_{\gamma\gamma_1} - \mathcal{Z}_{\gamma_1}$$

in homology. Thus, $\mathscr{E}(\mathcal{Z}_\gamma) = \mathscr{E}(\mathcal{Z}_{\gamma\gamma_1}) - \mathscr{E}(\mathcal{Z}_{\gamma_1})$ is a linear combination of images of hyperbolic matrices. $\square$

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a primitive hyperbolic matrix in $\Gamma_1(N)$, and $F = \mathbb{Q}(\gamma)$ the field generated by γ. We have

$$\gamma = \delta_0^{-1} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{pmatrix} \delta_0,$$

where

$$\delta_0 := \begin{pmatrix} 1 & \nu \\ 1 & \nu' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+, \quad \text{with } \nu = \frac{\epsilon - d}{c}, \quad \text{and } \epsilon = \frac{a+d+\sqrt{D}}{2}.$$

The matrix δ_0 induces an isomorphism

$$\mathbb{Q}(\gamma) \longrightarrow \mathbb{Q}(\epsilon) = \mathbb{Q}(\sqrt{D}), \quad \delta_0^{-1} \begin{pmatrix} \mu & 0 \\ 0 & \mu' \end{pmatrix} \delta_0 \longmapsto \mu,$$

with $D = \mathrm{tr}(\gamma)^2 - 4$. Note that D is never a square. The linear map $\delta_0 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is F-equivariant, where it acts on the left as a matrix in $F = \mathbb{Q}(\gamma) \subset \mathrm{GL}_2(\mathbb{Q})$, and on the right $F \simeq \mathbb{Q}(\epsilon)$ acts by $\nu \cdot (x, y) = (\nu x, \nu' y)$. The lattice $\mathbb{Z}^2$ is sent to $\delta_0\mathbb{Z}^2 = \sigma(\mathfrak{f}) = \{(\mu, \mu') \in \mathbb{R}^2 \mid \mu \in \mathfrak{f}\}$, where $\mathfrak{f} := \mathbb{Z} + \nu\mathbb{Z}$. Let $\mathscr{O} \subseteq F$ be the multiplier ring that preserves $\mathfrak{f}$, which is an order in $\mathscr{O}_F$. It corresponds to the stabilizer in $\mathbb{Q}(\gamma)$ of the lattice $\mathbb{Z}^2$, so that $\mathrm{SL}_2(\mathbb{Z}) \cap \mathbb{Q}(\gamma) \simeq \mathscr{O}^{\times,+}$. On the other hand, $\mathbb{Q}(\gamma) \cap \Gamma_1(N)$ is the stabilizer of the coset $\left(\begin{smallmatrix} 1+N\mathbb{Z} \\ N\mathbb{Z} \end{smallmatrix}\right)$. It corresponds to the stabilizer $U := \mathscr{O}^{\times,+} \cap (1 + N\mathfrak{f}) \subset \mathscr{O}^{\times,+}$ of the coset $1 + N\mathfrak{f}$. By Dirichlet's unit theorem, the units

$\mathscr{O}^{\times,+} = \epsilon_0^{\mathbb{Z}}$ are generated by a fundamental unit ϵ_0, which is an eigenvalue of the primitive hyperbolic matrix

$$\gamma_0 = \delta_0^{-1} \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \epsilon_0' \end{pmatrix} \delta_0 \in \mathrm{SL}_2(\mathbb{Z}).$$

Since U is a subgroup of $\mathscr{O}^{\times,+} \simeq \gamma_0^{\mathbb{Z}}$, it is generated by some power $\gamma_1 := \gamma_0^m \in \Gamma_1(N)$, for some $m \geq 1$, and with eigenvalue ϵ_1. Since $\gamma \in \mathbb{Q}(\gamma) \simeq \Gamma_1(N) \simeq U = \gamma_1^{\mathbb{Z}}$, we have that $\gamma = \gamma_1^k$ for some $k \geq 1$. If γ is primitive in $\Gamma_1(N)$ (in the sense that it cannot be written as a power of another hyperbolic matrix in $\Gamma_1(N)$), then $k = 1$ and $\gamma = \gamma_1$.

The computation of the integral is similar to the computation in [Bra25]. Since the integral along $\{z_0, \gamma z_0\}$ does not depend on the basepoint, we pick $z_0 = g_0^{-1} i$. Moreover, the segment $\{i, \epsilon^2 i\}$ is contained in $\{0, \infty\}$, only the dv-component contributes. We have

$$\int_{\{z_0,\gamma z_0\}} \mathscr{E}_{1,0}(z,\tau) = \int_1^{\epsilon^2} (\delta_0^{-1})^* \mathscr{E}_{1,0}(iv,\tau)_{(1,1)} \frac{dv}{v}.$$

The form φ satisfies the equivariance (3.4)

$$g^* \varphi(z,t,\mathbf{v}) = \varphi(z,t,\rho_{g^{-1}}\mathbf{v}),$$

so that we get

$$\int_{\{z_0,\gamma z_0\}} \mathscr{E}_{1,0}(z,\tau,s) = \int_0^\infty \int_1^{\epsilon^2} \frac{1}{y} \sum_{(m,n)\in \rho_{\delta_0} L_{1,0}} \omega(g_{iv},t,h_\tau)\varphi_{(1,1)}(\mathbf{v}) t^{2s} \frac{dt}{t}\frac{dv}{v}.$$

The action of ρ_g on $\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]$ was given by $[gm, g^{-T}n]$. This action becomes $[gm, gn]$ after performing the partial Fourier transform in n

$$\widetilde{\varphi}\left(\begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix}\right) := \int_{\mathbb{R}^2} \varphi\left(\begin{bmatrix} m_1 & a \\ m_2 & b \end{bmatrix}\right) e(an_1 + bm_2)dadb.$$

Note that in (3.9), the Fourier transform $\widehat{\varphi}$ was in the variables m_1, n_2, whereas here it is n_1, n_2. Using the computations in (3.10), one can compute that

$$\widetilde{\varphi}_{(1,1)}(\left[\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right]) = -\overline{(m_1 i + n_1)}^{d_1}\overline{(m_2 i + n_2)}^{d_2} e^{-\pi\|m\|^2 - \pi\|n\|^2}.$$

Applying Poisson summation, we find that (along the geodesic $\{0, \infty\}$)

$$\begin{aligned} &\frac{1}{y}\sum_{\mathbf{v}\in\rho_{\delta_0}L_{1,0}} \omega(g_{iv},t,h_\tau)\varphi_{(1,1)}(\mathbf{v}) \\ &\qquad = -\sqrt{D}\sum_{\mathbf{v}\in\delta_0 L_{1,0}} \frac{\overline{(m_1\tau+n_1)(m_2\tau+n_2)}}{y^2t^4} e^{-\pi\frac{1}{vt^2}\frac{|m_1\tau+n_1|^2}{y} - \pi\frac{v}{t^2}\frac{|m_2\tau+n_2|^2}{y}}, \end{aligned}$$

where $|\det(\delta_0)| = D^{\frac{1}{2}}$. The sum is now over the coset

$$\delta_0\left(\begin{smallmatrix} 1+N\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right) \simeq (1+N\mathfrak{f}) \times \mathfrak{f}$$

and the integral to compute is

$$-2D^{\frac{1}{2}} \int_0^\infty \int_1^\epsilon \sum_{(m,n)\in(1+N\mathfrak{f})\times\mathfrak{f}} \frac{\overline{(m_1\tau+n_1)(m_2\tau+n_2)}}{y^2} e^{-\pi\frac{1}{w^2t^2}\frac{|m_1\tau+n_1|^2}{y} - \pi\frac{w^2}{t^2}\frac{|m_2\tau+n_2|^2}{y}} t^{2s-4}\frac{dt}{t}\frac{dw}{w},$$

after setting $w^2 = v$. Let γ be primitive, so that $U = \epsilon^{\mathbb{Z}}$. Then after unfolding and changing variables to $a = \frac{1}{w^2t^2}$, $b = \frac{w^2}{t^2}$, we get

$$\frac{D^{\frac{1}{2}}}{4}\int_0^\infty\int_0^\infty \sum_{(m,n)\in(1+N\mathfrak{f})\times\mathfrak{f}/U} \frac{\overline{(m_1\tau+n_1)(m_2\tau+n_2)}}{y^2} e^{-\pi a\frac{|m_1\tau+n_1|^2}{y}-\pi b\frac{|m_2\tau+n_2|^2}{y}}(ab)^{1-\frac{s}{2}}\frac{da}{a}\frac{db}{b}$$
$$= \frac{D^{\frac{1}{2}}\Gamma\left(1-\frac{s}{2}\right)^2}{4\pi^{2-s}} \sum_{(m,n)\in(1+N\mathfrak{f})\times\mathfrak{f}/U} \frac{y^{-s}}{\mathrm{N}(m\tau+n)|\mathrm{N}(m\tau+n)|^{-s}}$$
$$= E_{1,\mathfrak{f}}^{(1)}\left(\tau,\tau',-\frac{s}{2}\right),$$

where we recall that

$$E_{1,\mathfrak{f}}^{(k)}(\tau,\tau',s) = (-1)^k\frac{D^{k-\frac{1}{2}}}{(2i\pi)^{2k}} \sideset{}{'}\sum_{(m,n)\in(1+N\mathfrak{f})\times N\mathfrak{f}/U} \frac{(yy')^s}{N(m\tau+n)^k|N(m\tau+n)|^{2s}}$$

is the Hilbert-Eisenstein series of parallel weight k for $\Gamma_1(N\mathfrak{f}) \subset \mathrm{SL}_2(F)$. Note that since $\mathbb{Z} \subset \mathfrak{f}$, we have $\Gamma_1(N) \subset \Gamma_1(N\mathfrak{f}) \cap \mathrm{SL}_2(\mathbb{Z})$. This proves the following.

Proposition 4.6. *Let $\gamma = \gamma_1^m$ be a hyperbolic matrix, for some primitive hyperbolic matrix $\gamma_1 \in \Gamma_1(N)$. Then*

$$\mathscr{E}(\mathcal{Z}_\gamma) = mE_{1,\mathfrak{f}}^{(1)}(\tau,\tau).$$

4.5. Linear relations between Eisenstein series. In this last section, we show how to recover linear relations between the Eisenstein series.

Theorem 4.7. *Let n_1, n_2, n_3 be three integers coprime to N that satisfy $n_1+n_2+n_3 \equiv 0 \mod N$. Then*

$$G_{n_1}^{(1)}G_{n_2}^{(1)} + G_{n_2}^{(1)}G_{n_3}^{(1)} + G_{n_3}^{(1)}G_{n_1}^{(1)} = G_{n_1}^{(2)} + G_{n_2}^{(2)} + G_{n_3}^{(2)}.$$

Proof. The Eisenstein series only depend on the residue of $n_i \mod N$, so we can replace n_i by $n_i - k_iN$. First, we can pick k_2 such that $n_2 - k_2N$ and n_1 are coprime integers: since n_1 and N are coprime, there exists an integer k_2 such that $k_2 \equiv (n_2-1)N^{-1} \mod n_1$, and thus $n_2 - 1 \equiv k_2N \mod n_1$. It follows that $\gcd(n_1, n_2-k_2N) = 1$. Moreover, by assumption, we have $n_1+n_2+n_3 = k_3N$ for some k_3. By replacing n_3 with $n_3' = n_3 - k_3N$, we can assume that the $n_i's$ are such that

$$n_1+n_2+n_3 = 0$$

and n_1, n_2 coprime.

Since n_1, n_2 are coprime, we can find m_1, m_2 such that

$$\gamma_{12} := \begin{pmatrix} m_2 & m_1 \\ -n_2 & -n_1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Setting $m_3 := -m_1 - m_2$, we find that the matrices

$$\gamma_{23} := \begin{pmatrix} m_3 & m_2 \\ -n_3 & -n_2 \end{pmatrix}, \quad \gamma_{31} := \begin{pmatrix} m_1 & m_3 \\ -n_1 & -n_3 \end{pmatrix},$$

are also in $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{M}_{ij} := \gamma_{ij}\{0,\infty\}$ be the corresponding unimodular symbols then

$$\mathcal{M}_{12} = \{r_1, r_2\}, \quad \mathcal{M}_{23} = \{r_2, r_3\}, \quad \mathcal{M}_{31} = \{r_3, r_1\}.$$

with $r_i = -\frac{m_i}{n_i}$. Let $\mathcal{T}$ be the triangle having the oriented modular symbols as sides, and closed by adding the modular caps

$$\mathcal{C}_1 = [r_3, r_2]_{r_1}, \quad \mathcal{C}_2 = [r_1, r_3]_{r_2}, \quad \mathcal{C}_3 = [r_2, r_1]_{r_3};$$

see Figure 5 below. We get

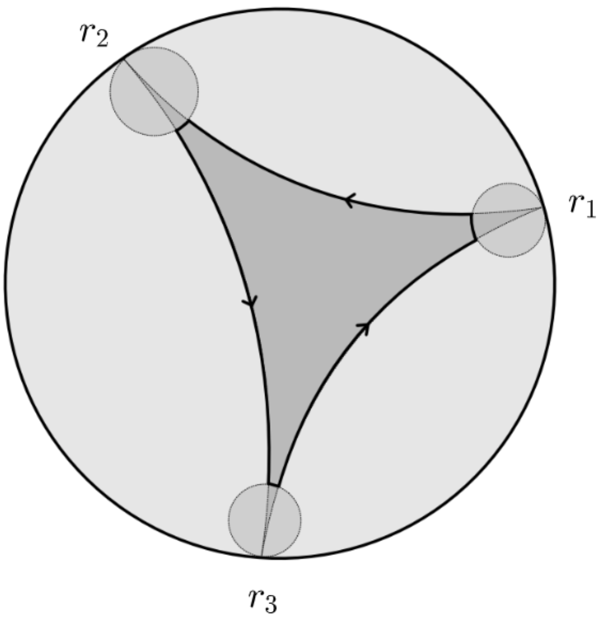

Figure 5: A hyperbolic triangle with unimodular sides and closed by modular caps.

$$\mathcal{M}_{12} + \mathcal{M}_{23} + \mathcal{M}_{31} + \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 = \partial\mathcal{T} = 0 \in H_1(\overline{Y_1(N)}^{\mathrm{BS}}; \mathbb{Z}).$$

It follows that

$$\int_{\mathcal{M}_{12}+\mathcal{M}_{23}+\mathcal{M}_{31}} \mathscr{E}_{1,0}(z,\tau) = -\int_{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3} \mathscr{E}_{1,0}(z,\tau).$$

From Proposition 4.3 we get

$$\int_{\mathcal{M}_{ij}} \mathscr{E}_{1,0}(z,\tau) = -G^{(1)}_{n_i}(\tau)G^{(1)}_{n_j}(\tau).$$

For the modular caps, notice that

$$\gamma_{ij}(0) = -\frac{m_i}{n_i} = r_i, \quad \gamma_{ij}(\infty) = -\frac{m_j}{n_j} = r_j, \quad \gamma_{ij}(1) = -\frac{m_k}{n_k} = r_k.$$

It follows that

$$\mathcal{C}_1 = [r_3, r_2]_{r_1} = \gamma_{31}[0,1]_\infty,$$

and similarly $\mathcal{C}_2 = \gamma_{12}[0,1]_\infty$, and $\mathcal{C}_3 = \gamma_{23}[0,1]_\infty$. From Proposition 4.2, we deduce that

$$\int_{\mathcal{C}_i} \mathscr{E}_{1,0}(z,\tau) = -G^{(2)}_{n_i}(\tau).$$

□

The condition on a, b, c guarantees the existence of a triangle with unimodular sides. The rest of the argument extends naturally to an arbitrary polygon whose vertices are the cusps $\frac{m_1}{n_1}, \ldots, \frac{m_d}{n_d}$ (with $m_i \neq 0 \mod N$), and the sides are unimodular symbols. The integral along each modular symbol is $-G^{(1)}_{n_i} G^{(1)}_{n_{i+1}}$, and over the modular cap at $\frac{m_i}{n_i}$ the integral is $G^{(2)}_{n_i}$.

Corollary 4.7.1. *If there exists a polygon joining cusps $\frac{m_1}{n_1}, \ldots, \frac{m_d}{n_d}$ by unimodular symbols, then*

$$G^{(1)}_{n_1} G^{(1)}_{n_2} + G^{(1)}_{n_2} G^{(1)}_{n_3} + \cdots + G^{(1)}_{n_d} G^{(1)}_{n_1} = G^{(2)}_{n_1} + \cdots + G^{(2)}_{n_d}.$$

References

[AR79] Avner Ash and Lee Rudolph. "The modular symbol and continued fractions in higher dimensions". In: *Inventiones mathematicae* 55.3 (1979), pp. 241–250.

[BCG20] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. "Transgressions of the Euler class and Eisenstein cohomology of $GL_N(\mathbb{Z})$". In: *Japanese Journal of Mathematics* 15.2 (2020), pp. 311–379.

[BCG23a] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. *Cocycles de groupe pour GL_n et arrangements d'hyperplans*. Vol. 39. CRM Monograph Series. American Mathematical Society, Providence, RI, 2023, p. 127.

[BCG23b] Nicolas Bergeron, Pierre Charollois, and Luis E. García. "Eisenstein cohomology classes for GL_N over imaginary quadratic fields". In: *J. Reine Angew. Math.* 797 (2023), pp. 1–40.

[BG01] Lev A. Borisov and Paul E. Gunnells. "Toric modular forms and nonvanishing of L-functions". In: *J. Reine Angew. Math.* 539 (2001), pp. 149–165.

[BT82] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*. 1st ed. Vol. 82. Springer-Verlag New York, 1982.

[Bra25] Romain Branchereau. *Eisenstein classes and generating series of modular symbols in* SL_N. 2025. arXiv: 2411.08690 [math.NT].

[Bru17] François Brunault. "Régulateurs modulaires explicites via la méthode de Rogers-Zudilin". In: *Compos. Math.* 153.6 (2017), pp. 1119–1152.

[Bru25] François Brunault. "On the Borisov–Gunnells relations for products of Eisenstein series". In: *Ramanujan J.* 68.4 (2025), p. 99.

[BZ23] François Brunault and Wadim Zudilin. *Modular regulators and multiple Eisenstein values*. 2023. arXiv: 2303.15554 [math.NT]. URL: https://arxiv.org/abs/2303.15554.

[CS16] Pierre Charollois and Robert Sczech. "Elliptic functions according to Eisenstein and Kronecker: an update". In: *Eur. Math. Soc. Newsl.* 101 (2016), pp. 8–14.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436. ISBN: 0-387-23229-X.

[DN18] Martin Dickson and Michael Neururer. "Products of Eisenstein series and Fourier expansions of modular forms at cusps". In: *Journal of Number Theory* 188 (2018), pp. 137–164.

[KMR17] Kamal Khuri-Makdisi and Wissam Raji. "Periods of modular forms and identities between Eisenstein series". In: *Mathematische Annalen* 367.1-2 (2017), pp. 165–183.

[KM86] Stephen S. Kudla and John J. Millson. "The theta correspondence and harmonic forms. I". In: *Mathematische Annalen* 274.3 (1986), pp. 353–378.

[KM87] Stephen S. Kudla and John J. Millson. "The theta correspondence and harmonic forms. II". In: *Mathematische Annalen* 277.2 (1987), pp. 267–314.

[KM90] Stephen S. Kudla and John J. Millson. "Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables". In: *Publications Mathématiques de l'IHÉS* 71 (1990), pp. 121–172.

[Lan87] Serge Lang. *Elliptic functions*. Second. Vol. 112. Graduate Texts in Mathematics. With an appendix by J. Tate. Springer-Verlag, New York, 1987, pp. xii+326. ISBN: 0-387-96508-4. DOI: 10.1007/978-1-4612-4752-4. URL: https://doi.org/10.1007/978-1-4612-4752-4.

[Lan95] Serge Lang. *Introduction to modular forms*. Vol. 222. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original. Springer-Verlag, Berlin, 1995.

[Li17] Yingkun Li. "Restriction of Coherent Hilbert Eisenstein series". In: *Mathematische Annalen* 368.1 (2017), pp. 317–338.

[MQ86] Varghese Mathai and Daniel Quillen. "Superconnections, thom classes, and equivariant differential forms". In: *Topology* 25.1 (1986), pp. 85–110.

[Miy89] Toshitsune Miyake. "Modular Groups and Modular Forms". In: *Modular Forms*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1989, pp. 96–194.

[RX20] Martin Raum and Jiacheng Xia. "All modular forms of weight 2 can be expressed by Eisenstein series". In: *Res. Number Theory* 6.3 (2020).

[Ste89] Glenn Stevens. "The Eisenstein measure and real quadratic fields". In: *Proceedings of the International Number Theory Conference held at Université Laval, July 5-18, 1987*. De Gruyter, 1989, pp. 887–927.

[Wei76] André Weil. *Elliptic Functions according to Eisenstein and Kronecker*. 1st edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, Heidelberg, 1976. ISBN: 978-3-540-07422-9.

[Xue23] Hui Xue. "Rankin-Cohen brackets of Eisenstein series". In: *International Journal of Number Theory* 19.07 (2023), pp. 1563–1570.

[Zha20] Hao Zhang. "Elliptic Cocycle for $GL_N(\mathbb{Z})$ and Hecke Operators". Sorbonne Université. PhD thesis. 2020. URL: https://www.imj-prg.fr/theses/pdf/hao_zhang.pdf.