

# A regularized theta lift on $\mathrm{SL}_N$

Romain Branchereau

## Abstract

We define a regularized lift from harmonic weak Maass forms of weight  $2 - N$  to differential forms of degree  $N - 1$  on the symmetric space  $\mathrm{SL}_N(\mathbb{R})/\mathrm{SO}(N)$ , that are smooth outside of certain modular symbols. We show that this lift is adjoint to the derivative of a theta lift. We compute the periods of the regularized lift over tori and relate them to Fourier coefficients of Hilbert-Eisenstein series.

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## 1. Introduction

Regularized lifts are theta lifts from harmonic weak Maass forms to functions on orthogonal Grassmannians  $O(p, q)/O(p) \times O(q)$  with singularities along smaller sub-Grassmannians. The regularization is done by taking the limit of an integral over a truncated fundamental. First introduced by Borcherds in his seminal work [Bor95; Bor98], many variations of these regularized lifts (now often called Borcherds lifts) have been considered since then, for example in [Bru12; Gar16; ANS20; FH21; CF23]. Most notably, Bruinier-Funke [BF04] constructed a lift from weak harmonic Maass forms to differential forms with singularities along special cycles, and which is adjoint to the Kudla-Millson lift. Since the work of Bruinier [Bru02; Bru99] and Bruinier-Funke [BF04], it is known that these regularized lifts satisfy certain current equations and play an important role in arithmetic geometry. When the signature is  $(p, 2)$  and the Grassmannian is a hermitian domain, these regularized lifts are Green currents for special cycles. For general signature  $(p, q)$ , the regularized lift of Bruinier-Funke defines a differential character.

When the regularized lift is a function on the Grassmannian, it can be evaluated at special points associated with CM extensions. There is an extensive literature on these special values of regularized lifts, mainly in the case  $O(p, 2)$  – see for example [Sch09; Via15; BKY12; BY09; BEY21; Ehl17; Li22; Li23; BLY25]. This was greatly motivated by the work of Gross-Zagier [GZ86] and Gross-Kohnen-Zagier [GKZ87]. In particular, in signature  $(2, 2)$  these values were conjectured by Gross-Zagier to be logarithms of algebraic numbers, which has been recently proved by Li [Li23] and Bruinier-Li-Yang [BLY25]. An important aspect of these special values is that they are related to Fourier coefficients of derivatives of Eisenstein series; see [BKY12; BY09; BEY21]. This plays an important role in the proof of Li [Li23] of the conjecture of Gross-Zagier. There have been a few results about special values of regularized lifts in signature  $(n, 1)$ . First, by Bruinier-Schwagenscheidt [BS21], and more recently by Herrero-İmamoğlu-Pippich-Schwagenscheidt [Her+24] in signature  $(3, 1)$ .

In this paper, we consider a regularized lift similar to the construction of Bruinier-Funke [BF04], lifting weak harmonic Maass forms to singular differential forms on a symmetric space. Instead of an orthogonal symmetric space, we consider the symmetric space  $S = \mathrm{SL}_N(\mathbb{R})/\mathrm{SO}(N)$ . We show that this lift is adjoint to the derivative of a cohomological lift on  $S$ . In particular, we replace the evaluation at a special point by the integral over a cycle attached to an algebraic torus, and relate it to the derivative of a Hilbert-Eisenstein series.

**1.1. A cohomological theta lift.** Let  $X = \mathrm{GL}_N(\mathbb{R})^+/\mathrm{SO}(N)$  be the set of positive definite symmetric matrices, and  $S = \mathrm{SL}_N(\mathbb{R})/\mathrm{SO}(N)$  be the set of matrices of determinant one. For a torsion-free congruence subgroup  $\Gamma \subset \mathrm{SL}_N(\mathbb{Q})$ , we consider the locally symmetric spaces

$$S_\Gamma = \Gamma \backslash X$$

and  $X_\Gamma = \Gamma \backslash X$ . Let  $V = \mathbb{Q}^N \times \mathbb{Q}^N$  be the quadratic space of signature  $(N, N)$  with quadratic form  $Q(\mathbf{v}) = \langle v, w \rangle$  for a vector  $\mathbf{v} = (v, w)$  in  $V$ . In [Bra25], we used the Mathai-Quillen formalism [MQ86] to construct a closed and  $\mathrm{GL}_N(\mathbb{R})$ -invariant  $N$ -form

$$\varphi(z, \mathbf{v}) \in [\Omega^N(X) \otimes \mathcal{S}(V_{\mathbb{R}})]^{\mathrm{GL}_N(\mathbb{R})},$$

valued in the space of Schwartz functions on  $V_{\mathbb{R}} = \mathbb{R}^N \times \mathbb{R}^N$ . By taking a regularized integral (with a complex parameter  $s \in \mathbb{C}$ ) of a theta series we obtain a form

$$E_\varphi(z_1, \tau, \phi, s) \in \Omega^{N-1}(S)^\Gamma \otimes C^\infty(\mathbb{H}), \quad z_1 \in S, \tau \in \mathbb{H}, \quad (1.1)$$

where  $\phi$  is a  $\Gamma$ -invariant finite Schwartz function. Moreover, in the variable  $\tau$  it transforms like a modular form of weight  $N$  for a congruence subgroup  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ .

Let  $\widetilde{M}_N(\Gamma')$  be the space of smooth functions on  $\mathbb{H}$  that transform like a modular form of weight  $N$  for  $\Gamma'$  and have moderate growth at all cusps. By integrating the above differential form above we obtain a lift

$$\begin{aligned} E_\varphi: C_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow \widetilde{M}_N(\Gamma'), \\ c &\longmapsto E_\varphi(c, \tau, \phi, s) := \int_c E_\varphi(z_1, \tau, \phi, s). \end{aligned}$$

from the group of smooth chains  $C_{N-1}(S_\Gamma; \mathbb{Z})$  to modular forms. The form  $E_\varphi(c, \tau, \phi, s)$  is closed and holomorphic at  $s = 0$ , so that it specializes to a holomorphic lift at the level of cohomology

$$\begin{aligned} E_\varphi: H_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow M_N(\Gamma'), \\ [c] &\longmapsto E_\varphi(c, \tau, \phi) := \int_c E_\varphi(z_1, \tau, \phi). \end{aligned} \quad (1.2)$$

This lift is closely related to the lift in [BCG20; BCG23]. Moreover, if  $\omega_c$  denotes a Poincaré dual to  $c$ , then it was shown in [Bra25] that

$$E_\varphi(c, \tau, \phi) = \int_c a_0 + \sum_{n=1}^{\infty} \left( \int_{S_n(\phi)} \omega_c \right) e(n\tau),$$

where  $a_0$  is a transgression of an Euler form as in [BCG20], and

$$S_n(\phi) = \sum_{\substack{[\mathbf{v}] \in \Gamma \backslash V \\ Q(\mathbf{v})=n}} \phi(\mathbf{v}) S_{[\mathbf{v}]} \in Z_{\frac{N^2-N}{2}}^{\mathrm{BM}}(S_\Gamma, \mathbb{Z}) \quad (1.3)$$

is a locally finite cycle of codimension  $N - 1$  in  $S_\Gamma$ , and  $S_{[\mathbf{v}]}$  is a generalized modular symbol obtained from an embedding of  $\mathrm{GL}_{N-1}(\mathbb{R})$  in  $\mathrm{SL}_N(\mathbb{R})$ . Hence, we can interpret the  $n$ -th Fourier coefficient as the intersection number

$$\int_{S_n(\phi)} \omega_c = \langle S_n(\phi), c \rangle,$$

and this lift can be seen as a Kudla-Millson lift for  $S_\Gamma$ . Compare with [KM86; KM87; KM90] for the lift defined by Kudla and Millson in the orthogonal setting.

*Example.* When  $N = 2$  and  $\phi$  is a suitable finite Schwartz function, the space is  $S_\Gamma = Y_0(p)$  and  $S_n(\phi) = T_n\{0, \infty\}$  is a Hecke translate of the geodesic from 0 to  $\infty$ .

The lift (1.2) vanishes if the class  $[c]$  is a torsion class, but it can also vanish because of the existence of a functional equation – see Section 2.8. When it vanishes, one can instead consider its derivative

$$\begin{aligned} E'_\varphi: C_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow \widetilde{M}_N(\Gamma') \\ c &\longmapsto E'_\varphi(c, \tau, \phi) := \left. \frac{\partial}{\partial s} \right|_{s=0} \int_c E_\varphi(z_1, \tau, \phi, s). \end{aligned} \quad (1.4)$$

We will define a regularized theta lift that is adjoint to  $E'_\varphi$  of this lift, and use it to express the Fourier coefficients of its holomorphic projection.

*Remark 1.1.* In the complex setting of the hermitian space associated with  $O(p, 2)$ , Kudla's programm [Kud97; KRY99; GS19] predicts that derivatives of Eisenstein series are related to height pairings of cycles in the arithmetic Chow group. In the setting of real symmetric spaces, the cohomology of a smooth manifold can be refined by defining its *differential cohomology*, represented by *differential characters*. As explained in [GS89], there are many striking similarities between the arithmetic intersection theory and the theory of differential characters developed by Cheeger and Simons in [CS85]. Hence, one could imagine a version of Kudla's program for real locally symmetric spaces, involving differential cohomology, and linking numbers.

In the complex setting, the Green functions for the arithmetic cycles can be constructed by using regularized theta lifts; see [Bru02; BF04; ES18]. On the other hand, the regularized theta lift constructed by Bruinier-Funke in [BF04] produces a differential character on the real symmetric space associated with  $O(p, q)$ . These observations were another motivation to relate, in this paper, the Fourier coefficients of the derivative of  $E'_\varphi(c, \tau, \phi)$  to a regularized theta lift. Although the lift that we construct does not seem to satisfy the current equation of a different character that one could expect in this setting, we hope to modify the construction in future work to clarify the geometric interpretation of the regularized lift.

**1.2. A regularized lift.** The Mathai-Quillen formalism produces a Thom form, that we used to define the form  $\varphi(z, \mathbf{v})$ . It also defines a transgression form, that we use to define another differential form

$$\psi(z, \mathbf{v}) \in \Omega^N(X) \otimes \mathcal{S}(V_{\mathbb{R}}).$$

By replacing  $\varphi$  by  $\psi$  in the construction of (1.1), we obtain another kernel form

$$E_\psi(z_1, \tau, \phi) \in \Omega^{N-1}(S)^\Gamma \otimes C^\infty(\mathbb{H}),$$

that transforms like a modular form of weight  $N - 2$ .

Let  $H_{2-N}(\Gamma')$  be the space of weak harmonic Maass forms of weight  $2 - N$  for the congruence subgroup  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ . Following the seminal work of Borcherds [Bor95; Bor98] and Bruinier-Funke [BF04], we use  $E_\psi$  to define a regularized lift of  $f \in H_{2-N}(\Gamma')$  by defining

$$\Phi(z_1, f, \phi) := \mathrm{CT}_{\rho=0} \left\{ \lim_{T \rightarrow \infty} \int_{F_T} E_\psi(z_1, \tau, \phi) f(\tau) y^{-\rho} \frac{dxdy}{y^2} \right\}, \quad (1.5)$$

where  $F_T$  is a truncated fundamental domain in  $\mathbb{H}$  for  $\Gamma'$ . For each cusp  $r \in \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$ , there is a matrix  $\gamma_r \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma_r \infty = r$ . Let  $\phi_r \in \mathcal{S}(V_{\mathbb{A}_f})$  be the finite Schwartz function  $\phi_r := \omega_f(\gamma_r^{-1})\phi$ , where  $\omega_f$  is the finite Weil representation. Let  $a_r^+(f, n)$  be the Fourier coefficients of the holomorphic part

$$(f|_{\gamma_r})^+ = \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} a_r^+(f, n) e(n\tau)$$

of  $f|_{\gamma_r}$ . Define the locus

$$S_f(\phi) := \bigcup_{r \in \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})} \bigcup_{\substack{n \in \mathbb{Q}_{>0} \\ a_r^+(f, -n) \neq 0}} S_n(\phi_r) \subset S_\Gamma,$$

where  $S_n(\phi)$  are the special cycles in (1.3).

**Theorem 1.1.** *The right-hand side of (1.5) converges to a differential form*

$$\Phi(z_1, f, \phi) \in \Omega^{N-1}(S_\Gamma \setminus S_f(\phi)),$$

*smooth outside of  $S_f(\phi)$ . Moreover, it is integrable over any smooth chain  $c \in C_{N-1}(S_\Gamma)$  that intersects  $S_f(\phi)$  transversally.*

*Remark 1.2.* When  $N = 2$  and  $S = Y_0(p)$ , then  $S_f(\phi)$  is a collection of infinite geodesics. It would be interesting to understand if there is a relation to the locally harmonic Maass forms that appear in [LS22].

We want to relate this regularized lift to (1.4), the derivative of the previous theta lift. At the cusp  $r \in \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$ , the derivative  $E'_\varphi(c, \tau, \phi)$  has a Fourier expansion

$$E'_\varphi(c, \tau, \phi)|_{\gamma_r} = A(c, \phi_r) + \log(y)B(c, \phi_r) + \sum_{n \in \mathbb{Q}} C_n(c, \phi_r, y) e(n\tau).$$

**Theorem 1.2.** *Let  $f \in H_{2-N}(\Gamma')$  be a harmonic weak Maass form and  $g = \xi_{2-N}(f) \in S_N(\Gamma')$  its image under the  $\xi$ -operator. For any smooth cycle  $c$  transverse to  $S_f(\phi)$  we have*

$$2N(-1)^{N-1} \int_c \Phi(z_1, f, \phi) = \langle E'_\varphi(c, \phi), g \rangle + \sum_{r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})} w_r \kappa_r(c, f, \phi)$$

where

$$\kappa_r(c, f, \phi) := A(c, \phi_r) a_r^+(f, 0) + \lim_{T \rightarrow \infty} \sum_{n \in \mathbb{Q}_{>0}} C_n(c, \phi_r, T) a_r^+(f, -n).$$

A form  $F$  in  $\widetilde{M}_N(\Gamma')$  transforms like a modular form and is of moderate growth at each cusp, so that

$$\langle F, g \rangle = \int_{\Gamma' \setminus \mathbb{H}} F(\tau) \overline{g(\tau)} y^N \frac{dx dy}{y^2}$$

converges for any cusp form  $g \in S_N(\Gamma')$ . In particular, the pairing  $\langle E'_\varphi(c, \phi), g \rangle$  in the previous theorem converges. The pairing defines a functional in  $S_N(\Gamma')^*$ , which is identified with  $S_N(\Gamma')$  via the Petersson inner product between holomorphic modular forms. Hence, there exists a unique cusp form  $F^{\text{hol}} \in S_N(\Gamma')$  such that

$$\langle F, g \rangle = \langle F^{\text{hol}}, g \rangle,$$

where the right-hand side is the Petersson product. We call  $F^{\text{hol}}$  the holomorphic projection of  $F$ .

Let us assume that  $\Gamma' = \Gamma_0(l)$ . For  $n \in \mathbb{N}$ , let  $P_n \in S_N(\Gamma')$  be  $n$ -th Poincaré series of weight  $N$ , that satisfies

$$\langle P_n, g \rangle = \frac{(N-2)!}{(4\pi n)^{N-1}} a_n(g)$$

for any cusp form  $g \in S_N(\Gamma_0(l))$ ; see [Bru02, §1.2]. (Note that when  $N = 2$  then the Poincaré series needs to be regularized). There is a harmonic weak Maass form

$$Q_{-n} = q^{-n} + a^+(Q_{-n}, 0) + O(q) \in H_{2-N}(\Gamma_0(l)),$$

see [BOR08, §6] and [Bru02, §1.3], such that

$$\xi_{2-N}(Q_{-n}) = \frac{(4\pi n)^{N-1}}{(N-2)!} P_n.$$

In particular, we have  $\langle \xi_{2-N}(Q_{-n}), g \rangle = a_n(g)$ , and we deduce the following.

**Corollary 1.2.1.** *The holomorphic projection of  $E'_\varphi(c, \phi)$  has the Fourier expansion*

$$E'_\varphi(c, \phi)^{\text{hol}} = \sum_{n=1}^{\infty} a_n(c, \phi) e(n\tau) \in S_N(\Gamma_0(l)),$$

where

$$a_n(c, \phi) = 2N(-1)^{N-1} \int_c \Phi(z_1, Q_{-n}, \phi) - \sum_{r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})} w_r \kappa_r(c, Q_{-n}, \phi).$$

**1.3. Periods over tori.** In the last section of the paper, we consider the case when  $c$  is a cycle obtained by embedding a totally real field  $F$ . Let  $D_F$  be the discriminant and  $\mathcal{O}$  the ring of integers. Assume that  $F$  is of even degree  $N$  and let<sup>1</sup>  $\chi: \text{Cl}(F)^+ \longrightarrow \{\pm 1\}$  be a totally odd Hecke character on the narrow class

<sup>1</sup>The existence of a totally odd character implies that the field does not contain a unit of negative norm. In particular, it excludes all fields of odd degree  $N$ , since  $-1$  is a unit of norm  $(-1)^N$

group of  $F$ . For an ideal  $\mathfrak{c}$  of  $\mathcal{O}$  dividing a prime  $p$ , consider the non-holomorphic (holomorphic at  $s = 0$ ) weight  $N$  modular form for  $\Gamma' = \Gamma_0(p)$

$$\mathcal{E}_{\chi, \mathfrak{c}}(\tau, s) := C(s) \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{1+2s} \sum_{(m, n) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a}/\mathcal{O}^{\times, +}} \frac{y^{Ns}}{N(v\tau + w)|N(v\tau + w)|^{2s}},$$

where  $C(s)$  is a term involving the discriminant of  $F$ , powers of  $\pi$  and of the Gamma function. With this normalization, it satisfies the functional equation

$$\mathcal{E}_{\chi, \mathfrak{c}}(\tau, -s) = \chi(\mathfrak{d}\mathfrak{c}) N(\mathfrak{d}\mathfrak{c})^{2s} \mathcal{E}_{\chi, \mathfrak{c}}(\tau, s).$$

In particular, if  $\chi(\mathfrak{c}) = -\chi(\mathfrak{d})$ , then the Eisenstein series  $\mathcal{E}_{\chi, \mathfrak{c}}(\tau, s)$  vanishes at  $s = 0$ . Instead, we consider its derivative

$$\mathcal{E}'_{\chi, \mathfrak{c}}(\tau) := \left. \frac{\partial}{\partial s} \right|_{s=0} \mathcal{E}_{\chi, \mathfrak{c}}(\tau, s).$$

It is a non-holomorphic modular form of weight  $N$  for  $\Gamma_0(p)$  and has a Fourier expansion

$$\mathcal{E}'_{\chi, \mathfrak{c}}(\tau) = \log(\alpha(\chi, \mathfrak{c})) + A_\chi + B_\chi \log(y) + \sum_{n=1}^{\infty} \log(J_{\chi, \mathfrak{c}}(n)) e(n\tau) + \sum_{n \in \mathbb{Z}} a_{\chi, \mathfrak{c}}(n, y) e(n\tau)$$

where  $\alpha(\chi, \mathfrak{c})$ ,  $J_{\chi, \mathfrak{c}}(n)$  and  $B_\chi$  are algebraic numbers, and  $A_\chi$  is transcendental. Let  $\epsilon$  be a basis of  $F$  over  $\mathbb{Q}$ , and

$$c_\epsilon \in Z_{N-1}(S_\Gamma)$$

be a cycle obtained from the embedding of the torus  $F^\times$  in  $\mathrm{GL}_N(\mathbb{Q})$  induced by  $\epsilon$ . For a suitable finite Schwartz function  $\phi_{\chi, \mathfrak{c}}$ , the Eisenstein series can be obtained as the image of the cycle  $c_\epsilon$  by the lift (1.2), so that we have

$$\mathcal{E}_{\chi, \mathfrak{c}}(\tau, s) = E_\varphi(c_\epsilon, \tau, \phi_{\chi, \mathfrak{c}}, s).$$

**Theorem 1.3.** *Let  $f \in H_{2-N}(\Gamma_0(p))$  be a harmonic weak Maass form and  $g = \xi_{2-N}(f) \in S_N(\Gamma_0(p))$ . Suppose that the principal parts of  $f$  at all cusps have rational coefficients. Then*

$$\int_{c_\epsilon} \Phi(z_1, f, \phi_{\chi, \mathfrak{c}}) = \log(\alpha(f, \chi, \mathfrak{c})) - \frac{1}{2N} \langle \mathcal{E}'_{\chi, \mathfrak{c}}, g \rangle - \frac{A_\chi}{2N} \left( a_\infty^+(f, 0) + p \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} a_0^+(f, 0) \right),$$

for an algebraic number  $\alpha(f, \chi, \mathfrak{c}) \in \overline{\mathbb{Q}}$ .

The algebraic number  $\alpha(f, \chi, \mathfrak{c})$  is a  $d$ -th root of a rational number, so if we replace  $f$  by  $df$ , then  $\alpha(df, \chi, \mathfrak{c}) = \alpha(f, \chi, \mathfrak{c})^d \in \mathbb{Q}$ . We use the fact that the  $L$ -function  $L(\chi, 0)$  is a rational number, by Shintani's Theorem.

**1.4. Overview.** In Section 2, we recall the construction of the theta lift on  $S$ , using differential forms obtained from the Mathai-Quillen formalism, see 2.2. The special cycles  $S_{[\mathbf{v}]}$  are defined in 2.1. In 2.6, we define the form  $\psi$ , that is used to define the kernel  $E_\psi(z_1, \tau, \phi)$ , and the regularized lift  $\Phi(z_1, f, \phi)$  in 3.1. The integrability of  $\Phi(z_1, f, \phi)$  is proved in Proposition 3.5. The relation to the derivative of  $E_\varphi(z_1, \tau, \phi)$  is proved in Theorem 3.8, and follows from the fact that  $L_N E'_\varphi(z_1, \tau, \phi) + 2N(-1)^{N-1} E_\psi(z_1, \tau, \phi)$  is an exact form by Proposition 3.7, where  $L_N$  is the lowering operator.

## 2. Theta lift on $\mathrm{SL}_N$

Let  $W$  be an  $N$ -dimensional  $\mathbb{Q}$ -vector space. Let  $X$  be the space of positive-definite quadratic forms on  $W_{\mathbb{R}}$ . After fixing a basis  $W \simeq \mathbb{Q}^N$ , we can identify  $X$  with the space of positive-definite symmetric  $N \times N$  matrices. The group  $G := \mathrm{GL}_N(\mathbb{R})^+$  acts on  $X$  by  $z \mapsto g z g^t$ , and the stabilizer of  $\mathbf{1}_{N \times N}$  is  $K := \mathrm{SO}(N)$ . Thus, we have

$$X \simeq G/K.$$

It is a smooth manifold of dimension  $\frac{N^2+N}{2}$ . Let  $S \subset X$  be the submanifold of matrices of determinant one, that can be described as the homogeneous space

$$S \simeq G^1/K,$$

where  $G^1 := \mathrm{SL}_N(\mathbb{R})$ . We have a diffeomorphism

$$S \times \mathbb{R}_{>0} \longrightarrow X, \quad (z_1, u) \longmapsto z = u^2 z_1,$$

where  $u = \det(z)^{\frac{1}{2N}}$ . We view  $X$  as a bundle of rank one over  $S$ , with the projection map

$$\pi: X \longrightarrow S, \quad z_1 = \pi(z) := u^{-2} z.$$

Let  $V := W \times W^\vee$  be the quadratic space with the quadratic form of signature  $(N, N)$

$$Q(\mathbf{v}, \mathbf{v}') := \langle v, w' \rangle + \langle v', w \rangle \quad \mathbf{v} = (v, w), \quad \mathbf{v}' = (v', w'),$$

where  $\langle v, w \rangle := v^t w$ . We set

$$Q(\mathbf{v}) := \frac{1}{2} Q(\mathbf{v}, \mathbf{v}) = \langle v, w \rangle.$$

The group  $G$  acts on a vector  $\mathbf{v} = (v, w)$  in  $V$  by

$$\rho(g)(\mathbf{v}) = (gv, g^\vee w),$$

where  $g^\vee$  is the induced action on the dual space. This gives a representation  $\rho: G \longrightarrow \mathrm{SO}(V_{\mathbb{R}}) = \mathrm{SO}(N, N)$ . From now on we will identify  $W \simeq \mathbb{Q}^N$  and  $W^\vee \simeq \mathbb{Q}^N$  with the standard inner product  $\langle v, w \rangle = v^t w$ , so that  $g^\vee = g^{-t}$ .

**2.1. Special cycles in the locally symmetric space.** For a place  $v$  of  $\mathbb{Q}$  let  $W_{\mathbb{Q}_v} := W \otimes \mathbb{Q}_v$ . Similarly, let  $V_{\mathbb{Q}_v} = V \otimes \mathbb{Q}_v$  be the quadratic space with the quadratic form  $Q$  extended to  $\mathbb{Q}_v$ . We have fixed a basis of  $W$  so that we identify  $V_{\mathbb{Q}_v} \simeq \mathbb{Q}_v^{2N}$ , and for a finite place  $v = p$  let  $V_{\mathbb{Z}_p} = \mathbb{Z}_p^{2N}$ . Let  $V_{\mathbb{A}_f} = \prod'_v V_{\mathbb{Q}_v}$  be the finite adeles, where the product is the restricted product with respect to  $V_{\mathbb{Z}_p}$ .

Let  $\mathcal{S}(V_{\mathbb{A}_f})$  be the space of finite Schwartz function. The group  $\mathrm{SL}_N(\mathbb{Q})$  acts on finite Schwartz function  $\phi \in \mathcal{S}(V_{\mathbb{A}_f})$  by  $\rho$

$$(\rho(g)\phi)(v, w) = \phi(gv, g^{-t}w). \quad (2.1)$$

Let us now fix a finite Schwartz function  $\phi \in \mathcal{S}(V_{\mathbb{A}_f})$  and  $\Gamma \subset \mathrm{SL}_N(\mathbb{Q})$  be a torsion-free subgroup preserving  $\phi$  under  $\rho$ . We consider the locally symmetric spaces

$$S_\Gamma := \Gamma \backslash S \quad \text{and} \quad X_\Gamma := \Gamma \backslash X \simeq S_\Gamma \times \mathbb{R}_{>0}.$$

For each vector  $\mathbf{v} = (v, w) \in V$  we define the submanifold

$$X_\mathbf{v} := \{z \in X \mid zw = v\} \subseteq X.$$

Note that if  $z \in X_\mathbf{v}$  for some vector  $\mathbf{v} \in V$ , then  $v^t zw = v^t v \geq 0$ . A vector  $\mathbf{v} = (v, w)$  is *regular* if  $v, w$  are both nonzero, and *singular* otherwise. Thus, if  $\mathbf{v}$  is a regular vector with  $Q(\mathbf{v}) > 0$ , then  $X_\mathbf{v}$  is a submanifold of codimension  $N$  in  $X$ . If  $\mathbf{v} = 0$ , then  $X_\mathbf{v} = X$ , and otherwise  $X_\mathbf{v}$  is empty. Let

$$S_\mathbf{v} := \pi(X_\mathbf{v}) \subseteq S,$$

be the image under the projection  $\pi: X \longrightarrow S$ . If  $\mathbf{v}$  is a regular vector with  $Q(\mathbf{v}) > 0$ , then the restriction of the projection  $\pi: X \longrightarrow S$  to  $X_\mathbf{v}$  is a diffeomorphism onto  $S_\mathbf{v}$ , which is of codimension  $N - 1$  in  $S$ . The submanifolds  $X_\mathbf{v}$  and  $S_\mathbf{v}$  satisfy the invariance property

$$gX_\mathbf{v} = X_{\rho(g)\mathbf{v}} \quad \text{and} \quad g_1 S_\mathbf{v} = S_{\rho(g_1)\mathbf{v}},$$

where we recall that  $g$  acts on  $z$  by  $gzg^t$ .

*Remark 2.1.* The representation  $\rho: G \rightarrow \mathrm{SO}(V_{\mathbb{R}}) = \mathrm{SO}(N, N)$  induces an embedding of  $X$  as a smooth submanifold of  $\mathbb{D} \simeq \mathrm{SO}(N, N)/K \times K$ , the Grassmannian of negative  $N$ -planes that appears in the work of Kudla-Millson. In particular, we have  $X_{\mathbf{v}} = X \cap \mathbb{D}_{\mathbf{v}}$  where  $\mathbb{D}_{\mathbf{v}}$  are the planes in  $\mathbb{D}$  that are orthogonal to  $\mathbf{v}$  with respect to  $Q$ .

Define the group of locally finite cycles

$$Z_k^{\mathrm{BM}}(S_{\Gamma}, \mathbb{Z}) := Z_k(\overline{S}_{\Gamma}, \{\infty\}, \mathbb{Z}),$$

where  $\overline{S}_{\Gamma} := S_{\Gamma} \cup \{\infty\}$  is the one-point compactification of  $S_{\Gamma}$ . The homology of this complex is the Borel-Moore homology. Let us denote by  $S_{[\mathbf{v}]}$  the image of  $S_{\mathbf{v}}$  under the projection map  $S \rightarrow S_{\Gamma}$ . By the equivariance, it only depends on the class  $[\mathbf{v}]$  of  $\mathbf{v}$  in  $\Gamma \backslash L$ . It defines a locally finite cycle

$$S_{[\mathbf{v}]} \in Z_{\frac{N^2-N}{2}}^{\mathrm{BM}}(S_{\Gamma}; \{\infty\}, \mathbb{Z})$$

of codimension  $N - 1$  in  $S_{\Gamma}$ . For a positive integer  $n$ , we define the special cycle

$$S_n(\phi) := \sum_{\substack{\mathbf{v} \in \Gamma \backslash V \\ Q(\mathbf{v})=n}} \phi(\mathbf{v}) S_{[\mathbf{v}]} \in Z_{\frac{N^2-N}{2}}^{\mathrm{BM}}(S_{\Gamma}, \mathbb{Z}),$$

where the sum is finite.

## 2.2. Differential forms.

Over  $X = G/K$  we have a bundle

$$E = G \times_K V_{\mathbb{R}}$$

that consists equivalence classes of pairs  $[g, v] \in G \times V_{\mathbb{R}}$ , modulo the equivalence relation  $[g, v] \equiv [gk, k^{-1}v]$ . We can view it as a metric bundle by taking the constant metric  $\|[g, v]\|^2 := \|v\|^2$  in each fiber. This bundle is isometric to the tautological metric bundle

$$X \times \mathbb{R}^N, \quad \|(z, v)\|^2 := v^t z^{-1} v,$$

where the metric over a point  $z \in X$  is given by the positive-definite quadratic form corresponding to  $z$ . The isometry is given by the map

$$E \longrightarrow X \times V_{\mathbb{R}}, \quad [g, v] \longmapsto (z, gv),$$

where  $z = gK$ . The group  $G$  acts on  $E$  by  $g \cdot [g', v] = [gg', v]$ , and is  $G$ -equivariant. In particular it is  $\Gamma_{\mathbf{v}}$ -equivariant, where  $\Gamma_{\mathbf{v}}$  is the stabilizer of  $\mathbf{v}$  under the action by  $\rho$ . The quotient  $E_{\Gamma_{\mathbf{v}}} := \Gamma_{\mathbf{v}} \backslash E$  is a bundle over  $X_{\Gamma_{\mathbf{v}}}$ . After fixing an orientation of the fibers, the pushforward along the fiber induces the Thom isomorphism

$$H^{k+N}(E_{\Gamma_{\mathbf{v}}}, E_{\Gamma_{\mathbf{v}}} \setminus E_0; \mathbb{R}) \longrightarrow H^k(X_{\Gamma_{\mathbf{v}}}; \mathbb{R}),$$

where  $E_0$  is the image of the zero section. Taking  $k = 0$ , a canonical class is given by taking the preimage of a generator of  $H^0(X_{\Gamma_{\mathbf{v}}}; \mathbb{R}) \simeq \mathbb{R}$ . It is called the *Thom class* of the vector bundle. Mathai and Quillen constructed in [MQ86] an explicit representative

$$U \in \Omega^N(E)^G$$

of this class, which is closed, rapidly decreasing along the fiber and  $G$ -invariant. In particular, it is  $\Gamma_{\mathbf{v}}$ -invariant and represents the Thom class on  $E_{\Gamma_{\mathbf{v}}}$ . If  $s_{\mathbf{v}}: X_{\Gamma_{\mathbf{v}}} \longrightarrow E_{\Gamma_{\mathbf{v}}}$  is a section that is transverse to the zero section  $E_0$ , then  $s_{\mathbf{v}}^{-1}(E_0)$  is a submanifold of codimension  $N$ , and  $s_{\mathbf{v}}^* U$  is a Poincaré dual to  $s_{\mathbf{v}}^{-1}(E_0)$ .

### 2.2.1. Pullback of the Thom form.

The submanifold  $X_{\mathbf{v}}$  is precisely the zero locus of the section

$$s_{\mathbf{v}}: X \longrightarrow E, \quad z \mapsto \left[ g, \frac{g^{-1}v - g^t w}{2} \right], \tag{2.2}$$

where  $\mathbf{v} = (v, w)$ . The pullback by the section  $s_{\mathbf{v}}$  gives a closed form

$$\varphi^0(z, \mathbf{v}) := s_{\mathbf{v}}^* U \in \Omega^N(X) \otimes C^\infty(V_{\mathbb{R}})$$

which satisfies the invariance property

$$g^* \varphi^0(z, \mathbf{v}) = \varphi^0(z, \rho_{g^{-1}} \mathbf{v}).$$

*Remark 2.2.* Note that section is  $\Gamma_{\mathbf{v}}$ -equivariant, and we can view it as a section  $s_{\mathbf{v}}: X_{\Gamma_{\mathbf{v}}} \longrightarrow E_{\Gamma_{\mathbf{v}}}$ . Then  $\varphi^0(z, \mathbf{v})$  represents the Poincaré dual in  $H^N(X_{\Gamma_{\mathbf{v}}}, \mathbb{R})$ , of the image of  $\Gamma_{\mathbf{v}} \backslash X_{\mathbf{v}} \hookrightarrow \Gamma_{\mathbf{v}} \backslash X$ .

The construction of Mathai-Quillen is explicit and we can compute this form as follows. The Maurer-Cartan form is the 1-form

$$\vartheta := g^{-1}dg \in \Omega^1(G) \otimes \mathfrak{g},$$

where  $\mathfrak{g} \simeq \text{Mat}_N(\mathbb{R})$  is the Lie algebra of  $G$ . Let  $\lambda_{ij} \in \Omega^1(G)$  be the  $(i, j)$ -entry of

$$\lambda := \frac{1}{2}(\vartheta + \vartheta^t) \in \Omega^1(G) \otimes \mathfrak{g}. \quad (2.3)$$

For a subset  $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, N\}$  and a function  $\sigma: I \longrightarrow \{1, \dots, N\}$  we define the  $k$ -form

$$\lambda(\sigma) := \lambda_{i_1\sigma(1)} \wedge \dots \wedge \lambda_{i_k\sigma(N)} \in \Omega^k(G).$$

For a tuple  $\underline{d} = (d_1, \dots, d_N)$  such that  $|\underline{d}| := d_1 + \dots + d_N = |I|$  we set

$$\lambda_I(\underline{d}) := \sum_{\substack{\sigma \\ |\sigma^{-1}(m)|=d_m}} \lambda(\sigma),$$

where the sum is over all functions  $\sigma: I \longrightarrow \{1, \dots, N\}$ . When  $I = \{1, \dots, N\}$  we write

$$\lambda(\underline{d}) := \lambda_{\{1, \dots, N\}}(\underline{d}).$$

We also define a polynomial  $H_{\underline{d}} \in \mathbb{C}[\mathbb{R}^N]$  by

$$H_{\underline{d}}(v) := \prod_{m=1}^N H_{d_m}(\langle v, e_m \rangle),$$

where  $H_d$  denotes the single variable Hermite polynomial

$$H_d(t) := \left(2x - \frac{d}{dt}\right)^d \cdot 1.$$

The first few Hermite polynomials are  $H_1(t) = 2t$ ,  $H_2(t) = 4t^2 - 2$  and  $H_3(t) = 8t^3 - 12t$ . One can compute<sup>2</sup> that

$$\varphi^0(z, \mathbf{v}) = \frac{1}{2^N \pi^{\frac{N}{2}}} e^{-\pi \|g^{-1}v - g^tw\|^2} \sum_{|\underline{d}|=N} H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^tw)) \lambda(\underline{d}). \quad (2.4)$$

Note that here we view the right-hand side as a  $K$ -invariant (basic) form on  $G$  that descends to  $X$ .

The form  $\varphi^0$  is smooth in  $V_{\mathbb{R}}$  but not rapidly decreasing in  $V_{\mathbb{R}}$ . We obtain a rapidly decreasing form by setting

$$\varphi(z, \mathbf{v}) := e^{-2\pi Q(\mathbf{v})} \varphi^0(z, \mathbf{v}) \in \Omega^N(X) \otimes \mathcal{S}(V_{\mathbb{R}}).$$

This form is also closed and satisfies

$$g^* \varphi(z, \mathbf{v}) = \varphi(z, \rho_{g^{-1}} \mathbf{v}).$$

We have

$$\varphi(z, \mathbf{v}) = \frac{1}{2^N \pi^{\frac{N}{2}}} e^{-\pi \|g^{-1}v\|^2 - \pi \|g^tw\|^2} \sum_{|\underline{d}|=N} H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^tw)) \lambda(\underline{d}). \quad (2.5)$$

<sup>2</sup>See [Bra25, Proposition. 3.7]. Note that there is a mistake at several places of *loc. cit.*, where  $H_d(-\sqrt{\pi}(v + w))$  should have been  $H_d(\sqrt{\pi}(v + w))$ .

**2.2.2. The transgression form.** The Thom form of Mathai-Quillen comes with a *transgression form* on  $E$  of degree  $N - 1$ . By pulling back by  $s_{\mathbf{v}}$  it gives rise to a form

$$\alpha^0(z, \mathbf{v}) \in \Omega^{N-1}(X) \otimes C^\infty(V_{\mathbb{R}}),$$

that satisfies the *transgression formula*

$$d_X \alpha^0(z, \sqrt{y}\mathbf{v}) = y \frac{\partial}{\partial y} \varphi^0(z, \sqrt{y}\mathbf{v});$$

see [Bra25, Proposition. 3.12]. Similarly to the form  $\varphi^0$ , the form is given by an explicit formula

$$\alpha^0(z, \mathbf{v}) = -\frac{1}{2^N \pi^{\frac{N-1}{2}}} e^{-\pi \|g^{-1}v - g^t w\|^2} \sum_{l=1}^N (-1)^{l-1} \langle g^{-1}v - g^t w, e_l \rangle \sum_{|\underline{d}|=N-1} H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^t w)) \lambda_{I_l}(\underline{d}) \quad (2.6)$$

where  $I_l := \{1, \dots, N\} \setminus \{l\}$ . The form  $\alpha^0$  is also not rapidly decreasing in  $V_{\mathbb{R}}$  and we define

$$\alpha(z, \mathbf{v}) := \alpha^0(z, \mathbf{v}) \exp(-2\pi Q(\mathbf{v})) \in \Omega^N(X) \otimes \mathcal{S}(V_{\mathbb{R}}).$$

We have

$$\alpha(z, \mathbf{v}) = -\frac{1}{2^N \pi^{\frac{N-1}{2}}} e^{-\pi \|g^{-1}v\|^2 - \pi \|g^t w\|^2} \sum_{l=1}^N (-1)^{l-1} \langle g^{-1}v - g^t w, e_l \rangle \sum_{|\underline{d}|=N-1} H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^t w)) \lambda_{I_l}(\underline{d}).$$

**2.3. Weil representation.** The next step is to construct theta series using the previously constructed  $\mathcal{S}(V_{\mathbb{R}})$ -valued differential forms and the Weil representation. We consider the Weil representation in two different models. Let  $\mathcal{S}(V_{\mathbb{Q}_v})$  be the space of Schwartz-Bruhat functions. In the first model, the representation  $\omega: \mathrm{GL}_N(\mathbb{Q}_v) \times \mathrm{SL}_2(\mathbb{Q}_v) \rightarrow U(\mathcal{S}(V_{\mathbb{Q}_v}))$  is defined by the formulas

$$\begin{aligned} \omega_v(g, h) \varphi_v(\mathbf{v}) &= \omega_v(h) \varphi_v(\rho(g)^{-1} \mathbf{v}), \\ \omega\left(1, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \varphi_v(\mathbf{v}) &= |a|_v^N \varphi_v(a\mathbf{v}), \quad a \in \mathbb{Q}_v^\times, \\ \omega_v\left(1, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) \varphi_v(\mathbf{v}) &= e(nQ(\mathbf{v})) \varphi_v(\mathbf{v}), \quad n \in \mathbb{Q}_v, \\ \omega_v\left(1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \varphi_v(\mathbf{v}) &= \int_{V_{\mathbb{Q}_v}} \varphi_v(\mathbf{v}') e_v(-Q(\mathbf{v}, \mathbf{v}')) d\mathbf{v}', \end{aligned}$$

where  $e_\infty(t) = e^{2i\pi t}$  if  $v = \infty$ , and  $e_p(t) = e^{-2i\pi \{t\}_p}$  if  $v = p$ . A second model  $\omega': \mathrm{GL}_N(V_{\mathbb{Q}_v}) \times \mathrm{SL}_2(\mathbb{Q}_v) \rightarrow U(\mathcal{S}(V_{\mathbb{Q}_v}))$  for the Weil representation is given by

$$\omega'_v(g, h) \varphi_v(\mathbf{v}) := |\det(g)|_v^{-1} \varphi(g^{-1} \mathbf{v} h)$$

where  $h$  acts on the two columns of  $\mathbf{v} = (v, w)$  from the right. The two representations are intertwined by partial Fourier transform in the second variable

$$\mathcal{S}(V_{\mathbb{Q}_v}) \longrightarrow \mathcal{S}(V_{\mathbb{Q}_v}), \quad \varphi \longmapsto \widehat{\varphi}_v(v, w) := \int_{V_{\mathbb{Q}_v}} \varphi_v(v, w') e(\langle w, w' \rangle) dw'.$$

These representations extend to a representations of the space of finite Schwartz function  $\mathcal{S}(V_{\mathbb{A}_f})$ . We will denote by  $\omega_f, \omega'_f$  the Weil representations

$$\omega_f, \omega'_f: \mathrm{GL}_N(V_{\mathbb{A}_f}) \times \mathrm{SL}_2(\mathbb{A}_f) \rightarrow U(\mathcal{S}(V_{\mathbb{A}_f})).$$

Similarly, the partial Fourier transforms extends to a map  $\phi \in \mathcal{S}(V_{\mathbb{A}_f}) \mapsto \widehat{\phi} \in \mathcal{S}(V_{\mathbb{A}_f})$ . By Poisson summation, for  $\varphi \in \mathcal{S}(V_{\mathbb{R}})$  and  $\phi \in \mathcal{S}(V_{\mathbb{A}_f})$ , we have

$$\sum_{\mathbf{v} \in V} (\omega_\infty(g, h)\varphi)(\mathbf{v}) \phi(\mathbf{v}) = \sum_{\mathbf{v} \in V} (\omega'_\infty(g, h)\widehat{\varphi})(\mathbf{v}) \widehat{\phi}(\mathbf{v}).$$

**Proposition 2.1.** [Bra25, Proposition. 4.9] For a vector  $v \in \mathbb{C}^N$  and an  $N$ -tuple  $\underline{d}$  let us write  $v^{\underline{d}} := \langle v, e_1 \rangle^{d_1} \cdots \langle v, e_N \rangle^{d_N}$ , where  $e_l \in \mathbb{R}^N$  is the  $l$ -th basis vector. The partial Fourier transforms of  $\varphi(z, \mathbf{v})$  and  $\alpha(z, \mathbf{v})$  are

$$\widehat{\varphi}(z, \mathbf{v}) = \frac{i^N}{\det(g)} e^{-\pi \|g^{-1}v\|^2 - \pi \|g^{-1}w\|^2} \sum_{|\underline{d}|=N} \overline{g^{-1}(vi+w)}^{\underline{d}} \lambda(\underline{d})$$

and

$$\widehat{\alpha}(z, \mathbf{v}) = -\frac{i^N}{\det(g)} e^{-\pi \|g^{-1}v\|^2 - \pi \|g^{-1}w\|^2} \sum_{l=1}^N (-1)^{l-1} \sum_{|\underline{d}|=N-1} \left[ \frac{2\pi |\langle g^{-1}(vi+w), e_l \rangle|^2 + d_l}{2\pi \langle g^{-1}(vi+w), e_l \rangle} \right] \overline{g^{-1}(vi+w)}^{\underline{d}} \lambda_{I_l}(\underline{d}),$$

where  $z = gg^t$ .

**2.4. Example when  $N = 2$ .** We have  $X \simeq \mathbb{H} \times \mathbb{R}_{>0}$ . Let  $z_1 = x_1 + iy_1$  be the coordinate on  $S \simeq \mathbb{H}$ . Using the formula in Proposition 2.1 we find

$$\begin{aligned} \widehat{\alpha}(z, \mathbf{v}) &= u^{-2} \exp(-u^{-2}\pi \|g_1^{-1}(vi+w)\|^2) \left[ \left( |\langle g^{-1}(vi+w), e_1 \rangle|^2 + \frac{1}{2\pi} \right) \lambda_{I_1}(1, 0) \right. \\ &\quad + \langle g^{-1}(vi+w), e_1 \rangle \overline{\langle g^{-1}(vi+w), e_2 \rangle} \lambda_{I_1}(0, 1) - \overline{\langle g^{-1}(vi+w), e_1 \rangle} \langle g^{-1}(vi+w), e_2 \rangle \lambda_{I_2}(1, 0) \\ &\quad \left. - \left( |\langle g^{-1}(vi+w), e_2 \rangle|^2 + \frac{1}{2\pi} \right) \lambda_{I_2}(0, 1) \right]. \end{aligned}$$

The form  $\lambda$  is

$$\lambda = \frac{1}{2} (g_1^{-1} dg_1 + (g_1^{-1} dg_1)^t) + \frac{du}{u} = \begin{pmatrix} \frac{du}{u} + \frac{dy_1}{2y_1} & \frac{dx_1}{2y_1} \\ \frac{dx_1}{2y_1} & \frac{du}{u} - \frac{dy_1}{2y_1} \end{pmatrix} \in \Omega^2(X) \otimes \text{End}(\mathbb{R}^2)$$

where

$$g_1 = \begin{pmatrix} \sqrt{y_1} & x_1/\sqrt{y_1} \\ 0 & 1/\sqrt{y_1} \end{pmatrix}.$$

For  $I_1 = \{2\}$  and  $I_2 = \{1\}$  we have

$$\begin{aligned} \lambda_{I_1}(1, 0) &= \lambda_{21} = \frac{dx_1}{2y_1}, & \lambda_{I_2}(1, 0) &= \lambda_{11} = \frac{du}{u} + \frac{dy_1}{2y_1}, \\ \lambda_{I_1}(0, 1) &= \lambda_{22} = \frac{du}{u} - \frac{dy_1}{2y_1}, & \lambda_{I_2}(0, 1) &= \lambda_{12} = \frac{dx_1}{2y_1}. \end{aligned}$$

Hence, we can rewrite  $\widehat{\alpha}(z, \mathbf{v})$  as

$$\widehat{\alpha}(z, \mathbf{v}) = \widehat{\alpha}_{x_1}(g, \mathbf{v}) \frac{dx_1}{2y_1} + \widehat{\alpha}_{y_1}(g, \mathbf{v}) \frac{dy_1}{2y_1} + \widehat{\alpha}_u(g, \mathbf{v}) \frac{du}{2u},$$

where

$$\begin{aligned} \widehat{\alpha}_{x_1}(g, \mathbf{v}) &:= u^{-4} e^{-u^{-2}\pi \|g_1^{-1}(vi+w)\|^2} (|\langle g_1^{-1}(vi+w), e_1 \rangle|^2 - |\langle g_1^{-1}(vi+w), e_2 \rangle|^2), \\ \widehat{\alpha}_{y_1}(g, \mathbf{v}) &:= -2u^{-4} e^{-u^{-2}\pi \|g_1^{-1}(vi+w)\|^2} \operatorname{Re} (\langle g_1^{-1}(\overline{vi+w}), e_1 \rangle \langle g_1^{-1}(vi+w), e_2 \rangle), \\ \widehat{\alpha}_u(g, \mathbf{v}) &:= 2iu^{-4} e^{-u^{-2}\pi \|g_1^{-1}(vi+w)\|^2} \operatorname{Im} (\langle g_1^{-1}(\overline{vi+w}), e_1 \rangle \langle g_1^{-1}(vi+w), e_2 \rangle). \end{aligned} \tag{2.7}$$

**2.5. Theta series.** From Proposition 2.1 it follows that

$$\begin{aligned} \omega'_\infty(1, k) \widehat{\varphi}(z, \mathbf{v}) &= j(k, i)^{-N} \widehat{\varphi}(z, \mathbf{v}), \\ \omega'_\infty(1, k) \widehat{\alpha}(z, \mathbf{v}) &= j(k, i)^{-(N-2)} \widehat{\alpha}(z, \mathbf{v}) \end{aligned}$$

for  $k \in \mathrm{SO}(2)$ , where  $j(g, \tau) = c\tau + d$  for  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Since the partial Fourier transform intertwines  $\omega_\infty$  and  $\omega'_\infty$ , we also have

$$\begin{aligned}\omega_\infty(1, k)\varphi(z, \mathbf{v}) &= j(k, i)^{-N}\varphi(z, \mathbf{v}) \\ \omega_\infty(1, k)\alpha(z, \mathbf{v}) &= j(k, i)^{-(N-2)}\alpha(z, \mathbf{v}).\end{aligned}\tag{2.8}$$

For  $\tau \in \mathbb{H}$  and  $\phi \in \mathcal{S}(V_{\mathbb{A}_f})$  we set

$$\Theta_\varphi(z, \tau, \phi) := j(h_\tau, i)^N \sum_{\mathbf{v} \in V} \omega_\infty(1, h_\tau)\varphi(z, \mathbf{v})\phi(\mathbf{v}) \in \Omega^N(X) \otimes C^\infty(\mathbb{H}),$$

where  $h_\tau \in \mathrm{SL}_2(\mathbb{R})$  is any matrix sending  $i$  to  $\tau$ . By (2.8), the right-hand side only depends on  $\tau$  and not on the choice of  $h_\tau$ . Taking the standard matrix  $h_\tau = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$  we get

$$\Theta_\varphi(z, \tau, \phi) = \sum_{\mathbf{v} \in V} \varphi^0(z, \sqrt{y}\mathbf{v})\phi(\mathbf{v})e(Q(\mathbf{v})\tau).$$

The differential form  $\varphi$  satisfies the equivariance  $g^*\varphi(z, \mathbf{v}) = \varphi(z, \rho_{g^{-1}}\mathbf{v}) = \omega_\infty(g, 1)\varphi(z, \mathbf{v})$ . Moreover, the invariance in (2.1) can be written  $\omega_f(\gamma, 1)\phi = \phi$ . Thus, if  $\Gamma \subset \mathrm{SL}_N(\mathbb{Q})$  is a torsion-free subgroup preserving  $\phi$  as in 2.1, then

$$\gamma^*\Theta_\varphi(z, \tau, \phi) = \Theta_\varphi(z, \tau, \omega_f(\gamma^{-1}, 1)\phi) = \Theta_\varphi(z, \tau, \phi) \in \Omega^N(X)^\Gamma \otimes C^\infty(\mathbb{H})$$

defines a  $\Gamma$ -invariant form on  $X$ . On the other hand, for  $\gamma \in \mathrm{SL}_2(\mathbb{Q})$  it satisfies

$$\Theta_\varphi(z, \gamma\tau, \phi) = j(\gamma, \tau)^N \Theta_\varphi(z, \tau, \omega_f(1, \gamma^{-1})\phi).\tag{2.9}$$

Thus, if  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$  is a congruence subgroup that preserves  $\phi$  under  $\omega_f$ , then it transforms like a modular form of weight  $N$  for  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ .

Similarly, we define the theta series

$$\begin{aligned}\Theta_\alpha(z, \tau, \phi) &= j(k, i)^{(N-2)} \sum_{\mathbf{v} \in V} \omega_\infty(1, h_\tau)\alpha(z, \mathbf{v})\phi(\mathbf{v}) \in \Omega^{N-1}(X)^\Gamma \otimes C^\infty(\mathbb{H}) \\ &= y \sum_{\mathbf{v} \in V} \alpha^0(z, \sqrt{y}\mathbf{v})\phi(\mathbf{v})e(Q(\mathbf{v})\tau)\end{aligned}\tag{2.10}$$

which transforms like a modular form of weight  $N - 2$  and level  $\Gamma_0(p)$ . Moreover, we have

$$d_X \alpha^0(z, \sqrt{y}\mathbf{v}) = y^2 \frac{\partial}{\partial y} \varphi^0(z, \sqrt{y}\mathbf{v}) = L_N \varphi^0(z, \sqrt{y}\mathbf{v})$$

where  $L_N = -2iy^2 \frac{\partial}{\partial \bar{\tau}}$ , is the lowering operator and  $d_X$  is the exterior derivative on  $X$ . Thus, we have [Bra25, Theorem. 4.19]

$$d_X \Theta_\alpha(z, \tau, \phi) = L_N \Theta_\varphi(z, \tau, \phi).\tag{2.11}$$

**2.6. Push-forward.** A matrix  $g \in G$  can be written  $g = g_1 u$  where  $g_1 \in G^1$  and  $u = \det(g)^{\frac{1}{N}}$ . This induces an isomorphism

$$X \simeq S \times \mathbb{R}_{>0}, \quad z \mapsto (z_1, u)$$

and we can view  $X$  as a bundle  $\pi: X \rightarrow S$  of rank 1 over  $S$ . A form  $\eta \in \Omega^k(X)$  can be written  $\eta = \eta_0 + \eta_1 \frac{du}{u}$  where  $\eta_0$  is a  $k$ -form on  $X$  of degree 0 in  $u$ , and  $\eta_1 = (-1)^{N-1} \iota_u \frac{\partial}{\partial u} \eta$  is a  $(k-1)$ -form of degree 0 in  $u$ . The pushforward along  $\pi$  is defined by

$$\pi_*(\eta) := \int_0^\infty \eta_1 \frac{du}{u} \in \Omega^{k-1}(S),$$

provided that the integral converges. We define the forms

$$E_\varphi(z_1, \tau, \phi, s) := \pi_* (\Theta_\varphi(z, \tau, \phi) u^{-2Ns}) = (-1)^{N-1} \int_0^\infty \iota_{u \frac{\partial}{\partial u}} \Theta_\varphi(z, \tau, \phi) u^{-2Ns} \frac{du}{u}, \quad (2.12)$$

and  $E_\alpha(z_1, \tau, \phi, s)$  similarly. By [Bra25, Theorem. 4.19], we have

$$d_S E_\alpha(z_1, \tau, \phi) = L_N E_\varphi(z_1, \tau, \phi),$$

where  $d_S$  is the exterior derivative on  $S$ . Instead of the  $(N-1)$ -form  $\alpha(z, \mathbf{v})$  we will rather use the  $N$ -form

$$\psi(z, \mathbf{v}) := \alpha(z, \mathbf{v}) \frac{du}{u}.$$

*Remark 2.3.* The construction of Bruinier-Funke [BF04] would suggest to use the  $(N-1)$ -form  $\alpha$ , whose pushforward is an  $(N-2)$ -form. The motivation to use the form  $\psi$  is its relation to the derivative of  $E_\varphi(z_1, \tau, \phi, s)$ . On the other hand, if we had used the form  $\alpha$ , then the exterior derivative of the corresponding regularized lift would be related to  $E_\varphi(z_1, \tau, \phi)$  by a current equation, as in [BF04, Theorem. 1.5]. In particular, it should define a differential character in the sense of Cheeger-Simons. We do not have such an interpretation for the corresponding regularized lift  $\Phi(z_1, f, \phi)$  obtained from  $\psi$  in (3.1). We hope to modify the construction presented in this paper to combine these two point of views. This could yield an interesting relation between differential characters and derivative of Eisenstein series, in the spirit of Kudla's program.

Let  $\Theta_\psi(z, \tau, \phi)$  be the theta series defined as in (2.10)

$$\Theta_\psi(z, \tau, \phi) := \Theta_\alpha(z, \tau, \phi) \frac{du}{u} = y \sum_{\mathbf{v} \in V} \alpha^0(z, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) e(Q(\mathbf{v})\tau) \frac{du}{u} \in \Omega^N(X)^\Gamma \otimes C^\infty(\mathbb{H}).$$

We will be interested in its pushforward

$$E_\psi(z_1, \tau, \phi, s) := \pi_* (\Theta_\psi(z, \tau, \phi) u^{-2Ns}) \in \Omega^{N-1}(S)^\Gamma \otimes C^\infty(\mathbb{H}).$$

**Proposition 2.2.** *The theta series  $\Theta_\varphi(z, \tau, \phi)$ ,  $\Theta_\alpha(z, \tau, \phi)$  and  $\Theta_\psi(z, \tau, \phi)$  are  $O(e^{-Cu})$  as  $u \rightarrow \infty$  and  $O(e^{-C/u})$  as  $u \rightarrow 0$  (uniformly for  $(z, \tau)$  in a compact set). Hence, the integrals  $E_\varphi(z_1, \tau, \phi, s)$  and  $E_\psi(z_1, \tau, \phi, s)$  converge uniformly on compact sets and define smooth forms on  $S$ .*

*Proof.* It follows from Proposition 4.4 and 4.9 in [Bra25]. By Poisson summation, we have

$$\Theta_\varphi(z, \tau, \phi) = \sqrt{y}^{-N} \sum_{\mathbf{v} \in V} \omega_\infty(1, h_\tau) \varphi(z, \mathbf{v}) \phi(\mathbf{v}) = \sqrt{y}^{-N} \sum_{\mathbf{v} \in V} \omega'_\infty(1, h_\tau) \widehat{\varphi}(z, \mathbf{v}) \widehat{\phi}(\mathbf{v}),$$

where from Lemma 2.1 we have

$$\omega'_\infty(1, h_\tau) \widehat{\varphi}(z, \mathbf{v}) = y^{-N} u^{-2N} i^N e^{-\pi \|u^{-1} g_1^{-1}(v\tau + w)\|^2/y} \sum_{|\underline{d}|=N} g_1^{-1}(\overline{v\tau + w}) \frac{d}{y} \lambda(\underline{d}).$$

The exponential function guarantees the rapid decay, except possibly at  $(v, w) = (0, 0)$ . But since the function  $\overline{g_1^{-1}(v\tau + w)}^{\frac{d}{y}}$  vanishes at  $(0, 0)$ , we have

$$\Theta_\varphi(z, \tau, \phi) = O(e^{-Cu}) \text{ as } u \rightarrow 0,$$

for all  $N \geq 1$ . Similarly, using the Poisson summation in the first variable shows that

$$\Theta_\varphi(z, \tau, \phi) = O(e^{-Cu}) \text{ as } u \rightarrow \infty.$$

For the theta series  $\Theta_\alpha(z, \tau, \phi)$  the idea is the same. The polynomial term in front of, say,  $\lambda_{I_l}(\underline{d})$  is

$$y^{-(N-1)} \sum_{\substack{\underline{d}=(d_1, \dots, d_m) \\ d_1 + \dots + d_N = N-1}} \left[ \frac{2\pi |\langle g^{-1}(v\tau + w), e_l \rangle|^2 + d_l y}{2\pi \langle g^{-1}(v\tau + w), e_l \rangle} \right] \prod_{m=1}^N \langle g^{-1}(\overline{v\tau + w}), e_m \rangle^{d_m}.$$

For  $N > 2$ , the degree of the product on the right is  $N-1 \geq 2$ . Hence the term vanishes at  $(v, w) = (0, 0)$ .

Now let  $N = 2$ . When  $l = 1$  and  $\underline{d} = (d_1, d_2) = (1, 0)$ , then the term in front of  $\lambda_{I_1}(1, 0)$  is

$$\frac{|\langle g^{-1}(v\tau + w), e_1 \rangle|^2}{y} + \frac{1}{2\pi},$$

which is equal to  $1/2\pi$  at  $(v, w) = (0, 0)$ . However, the term  $1/2\pi$  is cancelled out by the term  $1/2\pi$  from  $\lambda_{I_2}(0, 1)$ ; as can be seen in the formulas given in (2.7). Thus, when  $N = 2$  we also have  $\widehat{\alpha}(z, 0) = 0$ .  $\square$

**2.7. Regular and singular vectors.** A vector  $\mathbf{v} = (v, w) \in V$  is *regular* if  $v$  and  $w$  are both non-zero, and *singular* otherwise. Let  $V_{\text{reg}}$  and  $V_{\text{sing}}$  be the set of regular (resp. singular) vectors in  $V$ , so that

$$V = V_{\text{sing}} \sqcup V_{\text{reg}}$$

and

$$\Theta_\varphi(z, \tau, \phi) = \sum_{\mathbf{v} \in V_{\text{sing}}} \varphi^0(z, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) e(Q(\mathbf{v})\tau) + \sum_{\mathbf{v} \in V_{\text{reg}}} \varphi^0(z, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) e(Q(\mathbf{v})\tau).$$

The sum over the regular vectors is absolutely convergent; see [Bra25, Lemma. 4.7]. Hence, we can bring the integral inside the sum and it follows from (2.12) that

$$E_\varphi(z_1, \tau, \phi, s) = y^{Ns} e_{\varphi,1}(z_1, \phi, s) + y^{-Ns} e_{\varphi,2}(z_1, \phi, s) + \sum_{\mathbf{v} \in V_{\text{reg}}} b_\varphi(z_1, \sqrt{y}\mathbf{v}, s) \phi(\mathbf{v}) e(Q(\mathbf{v})\tau) \quad (2.13)$$

where

$$\begin{aligned} b_\varphi(z_1, \mathbf{v}, s) &:= (-1)^{N-1} \int_0^\infty \iota_{u \frac{\partial}{\partial u}} \varphi^0(z, \mathbf{v}) u^s \frac{du}{u}, \\ e_{\varphi,1}(z_1, \phi, s) &:= (-1)^{N-1} \int_0^\infty \sum_{w \in \mathbb{Q}^N} \iota_{u \frac{\partial}{\partial u}} \varphi^0(z, (0, w)) \phi(0, w) u^{-2Ns} \frac{du}{u}, \\ e_{\varphi,2}(z_1, \phi, s) &:= (-1)^{N-1} \int_0^\infty \sum_{v \in \mathbb{Q}^N} \iota_{u \frac{\partial}{\partial u}} \varphi^0(z, (v, 0)) \phi(v, 0) u^{-2Ns} \frac{du}{u}. \end{aligned}$$

Note that the integral does not depend on  $y$  since it absorbed by the integral over  $u$ . For the sum over the singular vectors we can bring the integral inside if  $\mathrm{Re}(s)$  is large enough.

Similarly, we find for  $E_\psi$  (we will only need the value at  $s = 0$ )

$$E_\psi(z_1, \tau, \phi) = y e_\psi(z_1, \phi) + \sum_{\mathbf{v} \in V_{\text{reg}}} y b_\psi(z_1, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) e(Q(\mathbf{v})\tau) \quad (2.14)$$

where

$$b_\psi(z_1, \mathbf{v}) := \int_0^\infty \alpha^0(z, \mathbf{v}) \frac{du}{u}$$

and  $e_\psi(z_1, \phi) = e_{\psi,1}(z_1, \phi) + e_{\psi,2}(z_1, \phi)$  with

$$\begin{aligned} e_{\psi,1}(z_1, \phi) &:= \int_0^\infty \sum_{v \in \mathbb{Q}^N} \alpha^0(z, (v, 0)) \phi(v, 0) \frac{du}{u}, \\ e_{\psi,2}(z_1, \phi) &:= \int_0^\infty \sum_{w \in \mathbb{Q}^N} \alpha^0(z, (0, w)) \phi(0, w) \frac{du}{u}. \end{aligned}$$

We will also need Fourier expansions at other cusps. Let  $\Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$  be the set of cusps of the congruence subgroup  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ . For each  $r \in \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$ , there is a matrix  $\gamma_r \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma \infty = r$ . If  $f$  is a smooth function that transforms like a modular form for  $\Gamma'$ , then  $f|_{\gamma_r}$  transforms like a modular form for  $\gamma_r^{-1} \Gamma' \gamma_r$ . It has a Fourier expansion

$$f|_{\gamma_r} = \sum_{n \in \frac{1}{w_r} \mathbb{Z}} a_r(n, y) e(n\tau), \quad (2.15)$$

where  $w_r$  is the width of the cusp  $r$ . We will call it a *Fourier expansion of  $f$  at the cusp  $r$* .

*Remark 2.4.* If the width is greater than 1, the Fourier expansion is not unique. If we replace  $\gamma_r$  by  $\gamma_r \left( \begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix} \right)$ , then the Fourier coefficients would be  $a_r(n, y) e\left(\frac{nl}{w_r}\right)$ . However, in the computations of the regularized product, the ambiguity will be cancelled.

By (2.9), we deduce that  $E_\psi(z_1, \tau, \phi)|_{\gamma_r} = E_\psi(z_1, \tau, \omega(1, \gamma_r^{-1})\phi)$ . Hence, after setting  $\phi_r := \omega_f(1, \gamma_r^{-1})\phi$ , we find that a Fourier expansion at the cusp  $r$  is

$$E_\psi(z_1, \tau, \phi)|_{\gamma_r} = y e_\psi(z_1, \phi_r) + \sum_{\mathbf{v} \in V_{\text{reg}}} y b_\psi(z_1, \sqrt{y}\mathbf{v}) \phi_r(\mathbf{v}) e(Q(\mathbf{v})\tau) \quad (2.16)$$

**2.8. Functional equation.** Let  $\widetilde{M}_N(\Gamma')$  be the space of smooth functions on  $\mathbb{H}$  that transform like a modular form of weight  $N$  for  $\Gamma'$  and have moderate growth at all cusps. The Eisenstein class  $E_\varphi$  defines a family of lift

$$\begin{aligned} Z_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow \widetilde{M}_N(\Gamma'), \\ c &\longmapsto E_\varphi(c, \tau, \phi, s) := \int_c E_\varphi(z_1, \tau, \phi, s) \end{aligned}$$

from cycles to non-holomorphic modular forms of weight  $N$ . The form  $E_\varphi(c, \tau, \phi, s)$  is closed and holomorphic at  $s = 0$ , so that it specializes to a holomorphic lift at the level of cohomology

$$\begin{aligned} H_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow M_N(\Gamma'), \\ [c] &\longmapsto E_\varphi(c, \tau, \phi) := \int_c E_\varphi(z_1, \tau, \phi). \end{aligned} \tag{2.17}$$

The lift vanishes if the class  $[c]$  is a torsion class. It can also vanish because of a functional equation. We have an involution  $g \longmapsto g^* := g^{-t}$  on  $\mathrm{GL}_N(\mathbb{R})^+$  that induces an involution on  $X$  and  $S$ . Let  $\Gamma^*$  be the image of  $\Gamma$ . It induces an involution

$$Z_{N-1}(S_\Gamma; \mathbb{Z}) \longrightarrow Z_{N-1}(S_{\Gamma^*}; \mathbb{Z}), \quad c \longmapsto c^*.$$

Let  $\phi^* \in \mathcal{S}(V_{\mathbb{A}_f})$  be the Schwartz function defined by  $\phi^*(v, w) := \phi(w, v)$ . Since  $(\rho(\gamma)\phi)^* = \rho(\gamma^*)\phi^*$ , the Schwartz function  $\phi^*$  is preserved by  $\Gamma^*$ .

**Theorem 2.3.** *The theta lift satisfies the functional equation*

$$E_\varphi(c^*, \tau, \phi, s) = (-1)^N E_\varphi(c, \tau, \phi^*, -s).$$

*Proof.* In the definition of  $\varphi(z, \mathbf{v})$  given in (2.5), we see that for the terms  $H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^tw))$  and the exponential term, the effect of replacing  $g$  by  $g^*$  is the same as swapping  $v$  and  $w$ . Moreover, the pullback of  $\vartheta$  is

$$\vartheta^* = g^t dg^* = ((dg^{-1})g)^t = -(g^{-1}dg)^t = -\vartheta^t.$$

Hence, we have  $\lambda^* = -\lambda$  and  $\lambda(\underline{d})^* = (-1)^N \lambda(\underline{d})$ , thus  $\varphi(z, v, w)^* = (-1)^N \varphi(z, w, v)$ . Finally, replacing  $g$  by  $g^*$  changes  $u^s$  to  $u^{-s}$ .  $\square$

In particular, if  $\phi^* = (-1)^N \phi$  then the lift (2.17) vanishes on the subspace of  $H_{N-1}(S_\Gamma; \mathbb{Z})$  that is fixed by the involution.

### 3. The regularized lift

Let  $k \in \frac{1}{2}\mathbb{Z}$  with  $k \neq 1$ . In the cases that we will consider we will have  $k \leq 0$  and  $k \in \mathbb{Z}$ . Let  $\Delta_k$  be the hyperbolic Laplacian

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + kiy \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

A harmonic weak Maass form of weight  $k$  for the congruence subgroup  $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$  is a smooth function  $f: \mathbb{H} \rightarrow \mathbb{C}$  that satisfies:

1.  $f|_{\gamma, k} = f$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$ ,
2.  $\Delta_k f = 0$ ,
3. for each cusp  $r \in \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$  and  $\gamma_r \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma_r \infty = r$ , there is a polynomial  $P_r \in \mathbb{C}[q^{-1}]$  such that

$$f|_{\gamma_r}(\tau) - P_r(\tau) = O(e^{-\epsilon y}) \quad y \rightarrow \infty,$$

for some  $\epsilon > 0$ .

We denote by  $H_k(\Gamma')$  the space of harmonic weak Maass forms of weight  $k$ . From the growth condition we deduce that we can write  $f = f^+ + f^-$  where

$$\begin{aligned} f^+(\tau) &= \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} a^+(f, n)e(n\tau), \\ f^-(\tau) &= \sum_{n \in \mathbb{Q}_{>0}} a^-(f, n)\beta_k(n, y)e(-n\tau), \end{aligned}$$

and

$$\beta_k(n, y) = \int_y^\infty e^{-4\pi nt} t^{-k} dt.$$

By [BF04, Lemma. 3.4], the Fourier coefficients of  $f^-$  satisfy  $a^-(n) = O(|n|^{k/2})$ , so that

$$f^-(\tau) = O(e^{-Cy}) \quad \text{as } y \rightarrow \infty,$$

for some  $C > 0$ . Similarly, a Fourier expansion at the other cusps is

$$f|_{\gamma_r}(\tau) = \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} a_r^+(f, n)e(n\tau) + \sum_{n \in \mathbb{Q}_{>0}} a_r^-(f, n)\beta_k(n, y)e(-n\tau).$$

Note that as in (2.15), the rational numbers that appear in the Fourier expansion have denominator  $w_r$ , the width of the cusp.

The raising and lowering operators  $R_k$  and  $L_k$  are defined by

$$R_k = 2i \frac{\partial}{\partial \tau} + ky^{-1}, \quad L_k = -2iy^2 \frac{\partial}{\partial \bar{\tau}},$$

and the  $\xi_k$  operator by

$$\xi_k(f)(\tau) := y^{k-2} \overline{L_k f(\tau)} = R_{-k} \left( y^k \overline{f(\tau)} \right).$$

The image of a weak Maass form of weight  $k$  by  $\xi_k$  is a cusp form of weight  $2 - k$ . We have the following exact sequence [BF04, Corollary. 3.8]

$$0 \longrightarrow M_k^!(\Gamma') \longrightarrow H_k(\Gamma') \xrightarrow{\xi_k} S_{2-k}(\Gamma') \longrightarrow 0,$$

where  $M_k^!(\Gamma')$  is the space of weakly holomorphic modular forms of weight  $k$  (whose poles are supported at the cusps).

**3.1. Regularized lift.** Let  $D \subset \mathbb{H}$  be a fundamental domain for  $\mathrm{SL}_2(\mathbb{Z})$  in  $\mathbb{H}$ , and

$$F = \bigsqcup_{\gamma \in \Gamma' \setminus \mathrm{SL}_2(\mathbb{Z})} \gamma D$$

a fundamental domain for  $\Gamma'$ . For  $T \geq 1$  let  $U_T \subset D$  be the subset consisting of  $\tau \in D$  with  $\mathrm{Im}(\tau) > T$ . The complement  $D \setminus U_T$  is relatively compact. We can decompose the fundamental domain as

$$F = F_T \cup \bigsqcup_{\gamma \in \Gamma' \setminus \mathrm{SL}_2(\mathbb{Z})} \gamma U_T,$$

where  $F_T$  is relatively compact. For a harmonic weak Maass form of weight  $2 - N$  for  $\Gamma'$  we define the regularized pairing

$$\Phi(z_1, f, \phi) := \mathrm{CT}_{\rho=0} \left\{ \lim_{T \rightarrow \infty} \int_{F_T} E_\psi(z_1, \tau, \phi) f(\tau) y^{-\rho} \frac{dxdy}{y^2} \right\}. \quad (3.1)$$

We view the cycle  $S_n(\phi)$  as a subset of  $S_\Gamma$

$$S_n(\phi) = \bigcup_{\substack{\mathbf{v} \in \Gamma \setminus V \\ Q(\mathbf{v})=n \\ \phi(\mathbf{v}) \neq 0}} S_{[\mathbf{v}]} \subset S_\Gamma,$$

Consider the set

$$S_f(\phi) := \bigcup_{r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})} \bigcup_{\substack{n \in \mathbb{Q}_{>0} \\ a_r^+(f, -n) \neq 0}} S_n(\phi_r) \subset S_\Gamma,$$

where  $a_r^+(f, -n)$  are the coefficients of the principal part of  $f$  at the cusp  $r$ , and  $\phi_r$  is as in (2.16). Note that the unions are both finite.

**Theorem 3.1.** *The regularized integral converges uniformly on compact subsets of  $S \setminus S_f(\phi)$ . Hence, it defines a smooth form*

$$\Phi(z_1, f, \phi) \in \Omega^{N-1}(S \setminus S_f(\phi))^\Gamma.$$

*Proof.* We can split the truncated fundamental domain as

$$F_T = F_1 \cup \bigsqcup_{\gamma \in \Gamma' \setminus \mathrm{SL}_2(\mathbb{Z})} \gamma(U_1 \setminus U_T),$$

where  $\tau \in U_1 \setminus U_T$  if  $1 < \mathrm{Im}(\tau) \leq T$ . We write

$$\Phi(z_1, f, \phi) = \sum_{\gamma \in \Gamma' \setminus \mathrm{SL}_2(\mathbb{Z})} \tilde{\Phi}_\gamma(z_1, f, \phi) + \int_{F_1} E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2}$$

where

$$\tilde{\Phi}_\gamma(z_1, f, \phi) := \mathrm{CT}_{\rho=0} \left\{ \lim_{T \rightarrow \infty} \int_{U_1 \setminus U_T} E_\psi(z_1, \tau, \phi) |_\gamma(f|_\gamma)(\tau) y^{-\rho} \frac{dxdy}{y^2} \right\}.$$

Since the integral over  $F_1$  converges (it is relatively compact), it is enough to show the convergence of  $\tilde{\Phi}_\gamma(z_1, f, \phi)$ . Let us first take  $\gamma = 1$ , and show the convergence of  $\tilde{\Phi}_1(z_1, f, \phi)$ . We write  $f = f^+ + f^-$  and decompose the integral as

$$\tilde{\Phi}_1(z_1, f, \phi) = \tilde{\Phi}_1(z_1, f^+, \phi) + \tilde{\Phi}_1(z_1, f^-, \phi).$$

The integral

$$\tilde{\Phi}_1(z_1, f^-, \phi) = \lim_{T \rightarrow \infty} \int_{U_1 \setminus U_T} E_\psi(z_1, \tau, \phi) f^-(\tau) \frac{dxdy}{y^2}$$

converges by the rapid decrease of  $f^-$  and the moderate growth of  $E_\psi(z_1, \tau, \phi)$  as  $y \rightarrow \infty$ . It is left to show the convergence of  $\tilde{\Phi}_1(z_1, f^+, \phi)$ . The domain  $U_1 \setminus U_T$  is the rectangle with  $-\frac{1}{2} \leq x < \frac{1}{2}$  and  $1 < y \leq T$ , so that we have

$$\lim_{T \rightarrow \infty} \int_{U_1 \setminus U_T} E_\psi(z_1, \tau, \phi) f^+(\tau) y^{-\rho} \frac{dxdy}{y^2} = \lim_{T \rightarrow \infty} \int_1^T \int_{-\frac{1}{2}}^{\frac{1}{2}} E_\psi(z_1, \tau, \phi) f^+(\tau) y^{-\rho} \frac{dxdy}{y^2}.$$

Inserting the Fourier expansions (2.14) gives

$$\begin{aligned} \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} E_\psi(z_1, \tau, \phi) f^+(\tau) y^{-\rho} \frac{dxdy}{y^2} &= a^+(f, 0) e_\psi(z_1, \phi) \int_1^\infty y^{-\rho} \frac{dy}{y} \\ &\quad + \int_1^\infty \sum_{\mathbf{v} \in V_{\mathrm{reg}}} b_\psi(z_1, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) a^+(f, -Q(\mathbf{v})) y^{-\rho} \frac{dy}{y}. \end{aligned} \tag{3.2}$$

The first term does not contribute, since the constant term in the Laurent expansion is

$$\mathrm{CT}_{\rho=0} \left( \lim_{T \rightarrow \infty} \int_1^T \frac{dy}{y^{1+\rho}} \right) = \mathrm{CT}_{\rho=0} \left( \frac{1}{\rho} \right) = 0. \quad (3.3)$$

Recall from (2.6) that

$$\alpha^0(z, \mathbf{v}) = P(g^{-1}v, g^t w) e^{-\pi \|g^{-1}v - g^t w\|^2},$$

where we set

$$P(v, w) := -(-1)^{l-1} \frac{1}{2^N \pi^{\frac{N-1}{2}}} \sum_{l=1}^N (-1)^{l-1} \langle v - w, e_l \rangle \sum_{|\underline{d}|=N-1} H_{\underline{d}}(\sqrt{\pi}(v+w)) \lambda_{I_l}(\underline{d}) \in \mathbb{C}[V] \otimes \Omega^N(X).$$

Note that  $P(v, w) \in \mathbb{C}[V] \otimes \Omega^N(X)$  has degree  $N$ . Writing  $g = g_1 u$ , we can use Lemma 3.3 to bound

$$|b_\psi(z_1, \sqrt{y}\mathbf{v})| \leq C y^{\frac{N}{2}} \|g_1^{-1}v\|^{\frac{N}{2}} \|g_1^t w\|^{\frac{N}{2}} |P(g_1^{-1}v, g_1^t w)| \exp(-2\pi y R(z_1, \mathbf{v})),$$

where

$$R(z_1, \mathbf{v}) := \|g_1^{-1}v\| \|g_1^t w\| - Q(\mathbf{v}).$$

By the Cauchy-Schwartz inequality we have

$$\|g_1^t w\| \|g_1^{-1}v\| \geq |\langle v, w \rangle| = |Q(\mathbf{v})|,$$

where the equality holds if and only if the vectors  $g_1^{-1}v$  and  $g_1^t w$  are parallel. This is the case if and only if  $z_1 \in S_{\mathbf{v}}$ . Hence  $R(z_1, \mathbf{v})$  is always non-negative, since

$$R(z_1, \mathbf{v}) = \|g_1^t w\| \|g_1^{-1}v\| - |Q(\mathbf{v})| + |Q(\mathbf{v})| - Q(\mathbf{v}) \geq 0,$$

and is zero if and only if  $z_1 \in S_{\mathbf{v}}$ . (Recall that  $S_{\mathbf{v}}$  is empty if  $Q(\mathbf{v}) < 0$ .)

For  $\epsilon > 0$  we define the open neighbourhood of  $S_{\mathbf{v}}$

$$S_{\mathbf{v}}^\epsilon := \{z_1 \in S \mid R(z_1, \mathbf{v}) < \epsilon\}.$$

Let  $U \subset S$  be a relatively compact open subset and

$$A_\epsilon(U, f) := \{\mathbf{v} \in V_{\mathrm{reg}} \cap \mathrm{supp}(\phi) \mid a^+(f, -Q(\mathbf{v})) \neq 0 \text{ and } S_{\mathbf{v}}^\epsilon \cap U \neq \emptyset\}.$$

Let  $B_\epsilon(U, f) \subset V_{\mathrm{reg}} \cap \mathrm{supp}(\phi)$  be such that

$$V_{\mathrm{reg}} \cap \mathrm{supp}(\phi) = A_\epsilon(U, f) \sqcup B_\epsilon(U, f).$$

By Lemma 3.4, there is a constant  $C_U > 0$  such that for all vectors  $\mathbf{v} \in V_{\mathrm{reg}}$  with  $a^+(f, -Q(\mathbf{v})) \neq 0$ , we have

$$C_U (\|\mathbf{v}\| - M) \leq C_U \left( \|\mathbf{v}\| - \frac{n_0}{C_U} \right) \leq R(z_1, \mathbf{v}),$$

where  $M := \lceil n_0/C \rceil$ . We split the sum as

$$\sum_{\mathbf{v} \in B_\epsilon(U, f)} = \sum_{\substack{\mathbf{v} \in B_\epsilon(U, f) \\ \|\mathbf{v}\| \leq M}} + \sum_{\substack{\mathbf{v} \in B_\epsilon(U, f) \\ M < \|\mathbf{v}\|}}$$

The first sum is a finite sum, so we can bring the integral inside of the sum. Since  $\mathbf{v} \in B_\epsilon(U, f)$ , we have  $R(z_1, \mathbf{v}) \geq \epsilon$  and

$$b_\psi(z_1, \sqrt{y}\mathbf{v}) = y^{\frac{N}{2}} O(\exp(-2\pi y\epsilon)) \quad \text{as } y \rightarrow \infty,$$

uniformly on  $U$ . Hence

$$\sum_{\substack{\mathbf{v} \in B_\epsilon(U, f) \\ \|\mathbf{v}\| \leq M}} b_\psi(z_1, \sqrt{y}\mathbf{v}) \phi(\mathbf{v}) a^+(f, -Q(\mathbf{v})) y^{-\rho} \frac{dy}{y}$$

converges at  $\rho = 0$  uniformly on  $U$ . On the other hand, since the exponent of

$$b_\psi(z_1, \sqrt{y}\mathbf{v}) = y^{\frac{N}{2}} O(\exp(-2\pi y C_U (\|\mathbf{v}\| - M))) \quad \text{as } y \rightarrow \infty,$$

is positive when  $M < \|\mathbf{v}\|$ , the sum

$$\sum_{\substack{\mathbf{v} \in B_\epsilon(U, f) \\ M < \|\mathbf{v}\|}} b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v})) y^{-\rho} \frac{dy}{y} = O(\exp(-2\pi y \epsilon)) \quad \text{as } y \rightarrow \infty$$

is rapidly decreasing and the integral converges. Let

$$\tilde{S}_f(\phi)_1 := \bigcup_{\substack{n \in \mathbb{Q}_{>0} \\ a^+(f, -n) \neq 0 \\ \phi(\mathbf{v}) \neq 0}} \bigcup_{\substack{\mathbf{v} \in V \\ Q(\mathbf{v}) = n}} S_{\mathbf{v}}.$$

If  $z_1 \in S \setminus \tilde{S}_f(\phi)_1$ , then by Lemma 3.4 we can pick  $\epsilon$  and  $U$  small enough that  $A_\epsilon(U, f)$  is empty, i.e.  $B_\epsilon(U, f) = V_{\text{reg}} \cap \text{supp}(\phi)$ . Thus, we showed that the form  $\tilde{\Phi}_1(z_1, f, \phi)$  is smooth form on  $S$  outside of  $\tilde{S}_f(\phi)_1$ , which descends to a smooth form on  $S_\Gamma$  outside of  $S_f(\phi)_1$ .

For the other terms, let  $r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})$  be the cusp  $r = \gamma\infty$ . By plugging-in the Fourier expansion at  $r$  given in (2.16), the right-hand side of (3.2) is

$$a_r^+(f, 0) e_\psi(z_1, \phi_r) \int_1^\infty y^{-\rho} \frac{dy}{y} + \int_1^\infty \sum_{\mathbf{v} \in V_{\text{reg}}} b_\psi(z_1, \sqrt{y}\mathbf{v}) \phi_r(\mathbf{v}) a_r^+(f, -Q(\mathbf{v})) y^{-\rho} \frac{dy}{y}.$$

In particular, it only depends on the cusp  $r = \gamma\infty$ , and that  $\tilde{\Phi}_r(z_1, f, \phi)$  converges outside of the set

$$\tilde{S}_f(\phi)_r = \bigcup_{\substack{n \in \mathbb{Q}_{>0} \\ a_r^+(f, -n) \neq 0 \\ \phi_r(\mathbf{v}) \neq 0}} \bigcup_{\substack{\mathbf{v} \in V \\ Q(\mathbf{v}) = n}} S_{\mathbf{v}}.$$

The projection of  $\bigcup_r \tilde{S}_f(\phi)_r$  in  $S_\Gamma$  is precisely  $S_f(\phi)$ . □

We also deduce the following from Theorem 3.1.

**Proposition 3.2.** *We have*

$$\Phi(z_1, f, \phi) = \lim_{T \rightarrow \infty} \left( \int_{F_T} E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2} - \log(T) Q_\psi(z_1, f, \phi) \right).$$

where

$$Q_\psi(z_1, f, \phi) := \sum_{r \in \Gamma' \setminus \text{SL}_2(\mathbb{Z})} w_r a_r^+(f, 0) e_\psi(z_1, \phi_r),$$

and  $w_r$  is the width of the cusp  $r$ .

*Proof.* Note that  $\tilde{\Phi}_r(z_1, f, \phi)$  only depends on the cusp  $r = \gamma\infty$ , so that

$$\sum_{\gamma \in \Gamma' \setminus \text{SL}_2(\mathbb{Z})} \tilde{\Phi}_\gamma(z_1, f, \phi) = \sum_{r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})} w_r \tilde{\Phi}_{\gamma_r}(z_1, f, \phi),$$

where  $w_r$  is the width of  $r$  (it is also equal to the number of translates  $\gamma D$  in the fundamental domain  $F$ , that contain the cusp  $r$ ). The constant term  $\mathrm{CT}_{\rho=0}$  in the definition of  $\tilde{\Phi}_{\gamma_r}(z_1, f, \phi)$  was necessary to remove the divergent constant term

$$\lim_{T \rightarrow \infty} a_r^+(f, 0) e_\psi(z_1, \phi_r) \int_1^T y^{-\rho} \frac{dy}{y},$$

see (3.3). It follows from

$$\lim_{T \rightarrow \infty} \int_1^T \frac{dy}{y} = \lim_{T \rightarrow \infty} \log(T)$$

and the rest of the proof of Theorem 3.1 that the regularization can also be done by subtracting

$$\sum_{\gamma \in \Gamma' \backslash \mathrm{SL}_2(\mathbb{Z})} a_r^+(f, 0) e_\psi(z_1, \phi_r) \log(T).$$

□

### 3.2. Two lemmas.

**Lemma 3.3.** *Let  $P(\mathbf{v}) \in \mathbb{C}[V]$  be a polynomial of degree  $N$  and  $\mathbf{v} = (v, w) \in V_{\mathrm{reg}} \cap \mathrm{supp}(\phi)$  a regular vector. There is a constant  $C > 0$  depending on  $P$  such that*

$$\int_0^\infty P(\sqrt{y}u^{-1}v, \sqrt{y}uw) e^{-\pi y \|u^{-1}v - uw\|^2} \frac{du}{u} \leq C \sqrt{y}^N \|v\|^{\frac{N}{2}} \|w\|^{\frac{N}{2}} P(v, w) e^{-2\pi y (\|v\| \|w\| - Q(\mathbf{v}))}$$

for  $y \geq 1$ .

*Proof.* For  $I = \{i_1, \dots, i_N\} \in (\mathbb{Z}_{\geq 0})^N$  we write  $v^I := v_1^{i_1} \cdots v_N^{i_N}$ . We can write  $P$  as a sum of monomials

$$P(v, w) = \sum_{I, J} a_{I, J} v^I w^J,$$

where the sum is over all  $N$ -tuples  $I, J \in (\mathbb{Z}_{\geq 0})^N$  with  $|I| + |J| \leq N$ . We have

$$\begin{aligned} & \int_0^\infty P(\sqrt{y}u^{-1}v, \sqrt{y}uw) e^{-\pi y \|u^{-1}v - uw\|^2} \frac{du}{u} \\ &= \sum_{I, J} a_{I, J} v^I w^J \sqrt{y}^{|I|+|J|} e^{2\pi y Q(\mathbf{v})} \int_0^\infty e^{-\pi y u^{-2} \|v\|^2 - \pi y u^2 \|w\|^2} u^{|J|-|I|} \frac{du}{u} \\ &= \frac{1}{2} \sum_{I, J} a_{I, J} v^I w^J \sqrt{y}^{|I|+|J|} \left( \frac{\|v\|}{\|w\|} \right)^{\frac{|J|-|I|}{2}} e^{2\pi y Q(\mathbf{v})} K_{\frac{|J|-|I|}{2}}(\pi y \|v\| \|w\|). \end{aligned}$$

For a fixed real number  $r$  there is a positive constant  $C_r$  (depending on  $r$ ) such that for all  $a \gg 0$

$$K_r(a) \leq C_r e^{-2a};$$

see [Lan87, K7 p. 271]. Since  $\mathbf{v} \in V_{\mathrm{reg}} \cap \mathrm{supp}(\phi) \subset \frac{1}{D}(\mathbb{Z}^N \times \mathbb{Z}^N)$  for some integer  $D \geq 1$ , we have

$$\left( \frac{\|v\|}{\|w\|} \right)^{\frac{|J|-|I|}{2}} \leq \frac{\|v\|^{\frac{|J|}{2}} \|w\|^{\frac{|I|}{2}}}{\|v\|^{\frac{|I|}{2}} \|w\|^{\frac{|J|}{2}}} \leq D^N \|v\|^{\frac{N}{2}} \|w\|^{\frac{N}{2}}.$$

□

**Lemma 3.4.** *Let  $U \subset S$  be a relatively open compact neighbourhood of a point  $z_1 \in S$ .*

1. *If  $\mathbf{v} \in V_{\mathrm{reg}} \cap \mathrm{supp}(\phi)$  and  $a^+(f, -Q(\mathbf{v})) \neq 0$ , then there is a constant  $C_U > 0$  that depends only on  $U$  and such that*

$$C_U \|\mathbf{v}\| \leq R(z_1, \mathbf{v}) + n_0$$

*for all  $z_1 \in U$  and some  $n_0 \in \mathbb{N}$ .*

2. The set  $A_\epsilon(U, f)$  is finite.

3. If  $z_1 \in S \setminus \tilde{S}_f(\phi)_1$ , then we can take  $\epsilon$  and  $U$  small enough that  $A_\epsilon(U, f)$  is empty.

*Proof.* 1. For  $g_1 \in G^1$  let

$$\|g_1\| := \sup_{v \in \mathbb{R}^N} \frac{\|g_1 v\|}{\|v\|}.$$

Note that  $\|kv\| = \|v\|$  for  $k \in K$ , so that  $z_1 = g_1 K \mapsto \|g_1\|$  is a well-defined continuous function on  $S$ . Let  $D \geq 1$  be an integer such that  $\text{supp}(\phi) \subset \frac{1}{D}(\mathbb{Z}^N \times \mathbb{Z}^N)$  and let  $C_U > 0$  be the constant

$$C_U := \frac{1}{\sqrt{2}D \sup_{z_1 \in U} \|g_1\| \|g_1^{-t}\|}.$$

Since the matrix norm satisfies  $\|Av\| \leq \|A\| \|v\|$ , we have

$$R(z_1, \mathbf{v}) + Q(\mathbf{v}) = \|g_1^{-1}v\| \|g_1^t w\| \geq \sqrt{2} D C_U \|v\| \|w\|$$

for all  $z_1 \in U$  and  $\mathbf{v} \in V_{\text{reg}}$ . The vector  $\mathbf{v}$  is contained in  $\frac{1}{D}(\mathbb{Z}^N \times \mathbb{Z}^N)$  so that  $\|v\|, \|w\| \geq 1/D$  and

$$\frac{1}{\sqrt{2}} \|\mathbf{v}\| = \frac{1}{\sqrt{2}} (\|v\|^2 + \|w\|^2)^{\frac{1}{2}} \leq \max(\|v\|, \|w\|) \leq D \|v\| \|w\|.$$

Since the principal part of  $f^+$  is finite, there is an  $n_0 \in \mathbb{N}$  such that  $a^+(f, -n) = 0$  if  $n > n_0$ . Thus, if  $a^+(f, -Q(\mathbf{v})) \neq 0$ , then  $Q(\mathbf{v}) \leq n_0$ . We deduce that

$$C_U \|\mathbf{v}\| \leq R(z_1, \mathbf{v}) + Q(\mathbf{v}) \leq R(z_1, \mathbf{v}) + n_0. \quad (3.4)$$

2. If  $\mathbf{v} \in A_\epsilon(U, f)$ , then there is a  $z_1 \in U$  such that  $R(z_1, \mathbf{v}) < \epsilon$ . By the first part of the lemma, it implies that  $\|\mathbf{v}\|$  is bounded by

$$\|\mathbf{v}\| \leq \frac{R(z_1, \mathbf{v}) + n_0}{C_U} < \frac{\epsilon + n_0}{C_U}.$$

It follows that  $A_\epsilon(U, f)$  is finite.

3. Now suppose that  $z_1 \in S \setminus \tilde{S}_f(\phi)_1$ . First, note that

$$\tilde{S}_f(\phi)_1 = \bigcup_{\substack{n \in \mathbb{Q}_{>0} \\ a^+(f, -n) \neq 0 \\ \phi(\mathbf{v}) \neq 0}} \bigcup_{\substack{\mathbf{v} \in V \\ Q(\mathbf{v}) = n}} S_{\mathbf{v}} = \bigcup_{\substack{\mathbf{v} \in V \cap \text{supp}(\phi) \\ a^+(f, -Q(\mathbf{v})) \neq 0}} S_{\mathbf{v}},$$

since for vectors with  $Q(\mathbf{v}) < 0$  the set  $S_{\mathbf{v}}$  is empty. Thus, if  $z_1 \in S \setminus \tilde{S}_f(\phi)_1$  then for all  $\mathbf{v} \in V_{\text{reg}}$  such that  $a^+(f, -Q(\mathbf{v})) \neq 0$ , we have  $R(z_1, \mathbf{v}) > 0$ . The function  $R(z_1, \mathbf{v})$  is continuous in  $z_1$ , so for each of the finitely many vectors  $\mathbf{v} \in A_\epsilon(U, f)$  we can find a neighbourhood  $U_{\mathbf{v}} \subset U$  of  $z_1$  such that

$$\inf_{z_1 \in U_{\mathbf{v}}} R(z_1, \mathbf{v}) > \epsilon' > 0$$

for some  $0 < \epsilon' < \epsilon$ . Let  $U' \subseteq U$  be the neighbourhood

$$U' := \bigcap_{\mathbf{v} \in A_\epsilon(U, f)} U_{\mathbf{v}}.$$

Since  $U' \subseteq U$  and  $0 < \epsilon' < \epsilon$ , we have

$$A_{\epsilon'}(U', f) \subseteq A_\epsilon(U, f).$$

We claim that  $A_{\epsilon'}(U', f)$  is empty. Suppose that  $\mathbf{v}_0 \in A_{\epsilon'}(U', f)$ . Then there is  $\tilde{z}_1 \in U'$  such that  $R(\tilde{z}_1, \mathbf{v}_0) < \epsilon'$ . On the other hand, since  $\tilde{z}_1 \in U' \subset U_{\mathbf{v}_0}$ , we also have

$$R(\tilde{z}_1, \mathbf{v}_0) > \inf_{z_1 \in U_{\mathbf{v}_0}} R(z_1, \mathbf{v}_0) > \epsilon',$$

which contradicts  $R(\tilde{z}_1, \mathbf{v}_0) < \epsilon'$ . This shows that  $A_{\epsilon'}(U', f)$  is empty.  $\square$

**3.3. Local integrability.** Let  $\Delta \subset \mathbb{R}^{N-1}$  be an  $(N-1)$ -simplex. A chain  $c: \Delta \rightarrow S_\Gamma$  is smooth if there exists an open neighbourhood  $\Delta \subset U \subset \mathbb{R}^{N-1}$  of the simplex such that the map extends to a smooth immersion  $c: U \rightarrow S_\Gamma$ .

When the cycle  $c$  intersects  $S_\mathbf{v}$  transversally in  $S$ , then the preimage  $\bar{c}$  of  $c$  in  $X$  intersect  $X_\mathbf{v}$  transversally. In a neighbourhood of an intersection point  $X_\mathbf{v} \cap \bar{c}$ , the restriction of the singular form looks like a differential form on a ball  $B_\delta(0) \subseteq \mathbb{R}^N$  with a singularity at most  $1/\|x\|^{N-1}$ . This explains that the form is integrable over transverse cycles, as proved in the following proposition.

**Proposition 3.5.** *Let  $c: \Delta \rightarrow S_\Gamma$  be a smooth  $N-1$  chain that intersects  $S_f(\phi)$  transversally. Then the integral of  $\Phi(z_1, f, \phi)$  over  $c$  converges, and we have*

$$\int_c \Phi(z_1, f, \phi) = \lim_{T \rightarrow \infty} \left( \int_{F_T} \int_c E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2} - \log(T) \int_c Q_\psi(z_1, f, \phi) \right).$$

*Proof.* Let  $\mathcal{F}$  be a fundamental domain for  $\Gamma$  in  $S$ . Let  $\pi_\Gamma: S \rightarrow S_\Gamma$  be the projection. By subdividing  $\Delta$  if necessary, we can write  $c$  as a finite sum of cycles  $c_i: \Delta \rightarrow S_\Gamma$  such that each  $c_i$  is a smooth embedding and the image of  $c_i(\Delta)$  is contained in an open subset  $U_i \subset S_\Gamma$  that is small enough to be diffeomorphic to  $\pi_\Gamma^{-1}(U_i) \subset \mathcal{F}$  in the fundamental domain. The integral over  $c$  is then

$$\int_c \Phi(z_1, f, \phi) = \sum_i \int_{c_i} \Phi(z_1, f, \phi),$$

where we view  $c_i$  as a chain in  $\mathcal{F} \subset S$ , that intersects each submanifold  $S_\mathbf{v} \subset \tilde{S}_f(\phi) = \pi^{-1}(S_f(\phi))$  transversally. Let us show that these integrals converge (we will write  $c$  instead of  $c_i$ ).

We proceed as in Theorem 3.1 and write

$$\Phi(z_1, f, \phi) = \sum_{\gamma \in \Gamma' \setminus \mathrm{SL}_2(\mathbb{Z})} \tilde{\Phi}_\gamma(z_1, f, \phi) + \int_{F_1} E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2}$$

where

$$\tilde{\Phi}_\gamma(z_1, f, \phi) := \mathrm{CT}_{\rho=0} \left\{ \lim_{T \rightarrow \infty} \int_{U_1 \setminus U_T} E_\psi(z_1, \tau, \phi)|_\gamma(f|_\gamma)(\tau) y^{-\rho} \frac{dxdy}{y^2} \right\}.$$

The integral over  $F_1$  defines a smooth form, so integrable over  $c$  and we can interchange the integrals:

$$\int_c \int_{F_1} E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2} = \int_{F_1} \int_c E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2}.$$

Hence, it is enough to show that  $\tilde{\Phi}_\gamma(z_1, f, \phi)$  is integrable. We can take  $\gamma = 1$  without loss of generality. As in Proposition 3.2, we have

$$\tilde{\Phi}_1(z_1, f, \phi) = \lim_{T \rightarrow \infty} \left( \int_{U_1 \setminus U_T} E_\psi(z_1, \tau, \phi) f(\tau) \frac{dxdy}{y^2} - a^+(f, 0) e_\psi(z_1, \phi) \log(T) \right).$$

The integral against  $f^-$  is also a smooth form since  $f^-$  is rapidly decreasing and  $E_\psi$  moderately growing as  $y \rightarrow \infty$ . Thus is enough to show that the integral

$$\int_c \lim_{T \rightarrow \infty} \left( \int_1^T \int_{-\frac{1}{2}}^{\frac{1}{2}} E_\psi(z_1, \tau, \phi) f^+(\tau) \frac{dxdy}{y^2} - a^+(f, 0) e_\psi(z_1, \phi) \log(T) \right)$$

against  $f^+$  converges and we can exchange the two integrals. By plugging in the Fourier expansion and recalling that  $ye_\psi(z_1, \phi)$  is the constant term of  $E_\psi(z_1, \tau, \phi)$  (see (2.14)), we find that the latter integral is equal to

$$\int_c \int_1^\infty \sum_{\mathbf{v} \in V_{\mathrm{reg}}} \phi(\mathbf{v}) b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v})) \frac{dy}{y}. \quad (3.5)$$

By Fubini's theorem, we have

$$\int_c \int_1^\infty \sum_{\mathbf{v} \in V_{\mathrm{reg}}} \phi(\mathbf{v}) b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v})) \frac{dy}{y} = \int_1^\infty \int_c \sum_{\mathbf{v} \in V_{\mathrm{reg}}} \phi(\mathbf{v}) b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v})) \frac{dy}{y}$$

where the integral on the left converges absolutely if and only if the integral on the right converges absolutely. Assuming the convergence, it follows from the Fourier expansion of  $\int_c E_\psi(z_1, \tau, \phi)$  that

$$\begin{aligned} & \int_1^\infty \int_c \sum_{\mathbf{v} \in V_{\text{reg}}} \phi(\mathbf{v}) b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v})) \frac{dy}{y} \\ &= \lim_{T \rightarrow \infty} \left( \int_{U_1 \setminus U_T} \int_c E_\psi(z_1, \tau, \phi) f^+(\tau) \frac{dxdy}{y^2} - a^+(f, 0) \int_c e_\psi(z_1, \phi) \log(T) \right). \end{aligned}$$

This shows that the integral equals the right handside of the proposition.

Finally, we want to show the absolute convergence (3.5) by splitting  $V_{\text{reg}} \cap \text{supp}(\phi) = A_\epsilon(U, f) \sqcup B_\epsilon(U, f)$  as in the proof of Theorem 3.1, where

$$A_\epsilon(U, f) := \{\mathbf{v} \in V_{\text{reg}} \cap \text{supp}(\phi) \mid a^+(f, -Q(\mathbf{v})) \neq 0 \text{ and } R(z_1, \mathbf{v}) < \epsilon \text{ for some } z_1 \in U\}$$

is finite and  $U$  is a relatively compact open set containing the chain  $c$ . By Lemma 3.4, in particular (3.4), if  $Q(\mathbf{v}) < 0$  then  $R(z_1, \mathbf{v})$  is bounded below on  $U$ . By taking  $\epsilon$  small enough, we can assume that  $A_\epsilon(U, f)$  only consists of positive regular vectors. For the sum over  $B_\epsilon(U, f)$ , the absolute convergence follows from the fact  $b(z_1, \mathbf{v})$  is rapidly decreasing as  $\|\mathbf{v}\| \rightarrow \infty$ , uniformly in  $z_1 \in U$ .

For the (finite) sum over  $A_\epsilon(U, f)$  the absolute convergence (3.5) follows from convergence of the right-handside of

$$\begin{aligned} & \int_c \int_1^\infty \sum_{\mathbf{v} \in A_\epsilon(U, f)} |\phi(\mathbf{v}) b_\psi(z_1, \sqrt{y}\mathbf{v}) a^+(f, -Q(\mathbf{v}))| \frac{du}{u} \frac{dy}{y} \\ & \leq M \sum_{\mathbf{v} \in A_\epsilon(U, f)} |a^+(f, -Q(\mathbf{v}))| \int_{\Delta \times \mathbb{R}_{>0}} \int_1^\infty |c^* \alpha^0(z, \sqrt{y}\mathbf{v})| \frac{du}{u} \frac{dy}{y}, \end{aligned}$$

where  $M$  is such that  $|\phi(\mathbf{v})| \leq M$ .

Thus, it is left to show the convergence of the inner integral for a fixed vector  $\mathbf{v} \in A_\epsilon(U, f)$ . We extend the chain  $c: \Delta \rightarrow S$  to a map

$$\bar{c}: \Delta \times \mathbb{R}_{>0} \longrightarrow X \quad \bar{c}(T, u) = u^2 c(T).$$

We can view the form inside of the integral as pullback of  $\alpha^0(z, \sqrt{y}\mathbf{v}) \frac{du}{u} \in \Omega^N(X)$  by  $\bar{c}$ . To fix some coordinates on  $X$ , let  $B \subset G$  be the Borel subgroup of upper triangular matrices. We have an Iwasawa decomposition  $G = NAK$  where  $N \subset B$  are the unipotent matrices and  $A \simeq \mathbb{R}_{>0}^N$  are diagonal matrices. Let  $P := NA \subset B$ . Since  $P \cap K = \{\mathbf{1}\}$  we have a diffeomorphism  $P \simeq X$  mapping  $g$  to  $gg^t$ . Similarly, we have a diffeomorphism  $S \simeq P^1 := P \cap \text{SL}_N(\mathbb{R})$ . We can view  $c$  as a chain in  $P^1$ , and we will write  $g_T \in P$  for the lift of  $c(T)$ , that satisfies  $c(T) = g_T g_T^t \in S$ . We want to compute the integral of the pullback along the map

$$\bar{c}: \Delta \times \mathbb{R}_{>0} \longrightarrow P \quad \bar{c}(T, u) = ug_T,$$

Recall from (2.2) that  $X_\mathbf{v}$  is the zero locus of the section

$$s_\mathbf{v}: X \longrightarrow G \times_K \mathbb{R}^N, \quad s_\mathbf{v}(g) = [g, f_\mathbf{v}^-(g)]$$

where we define

$$f_\mathbf{v}^\pm(g) = \frac{g^{-1}v \pm g^tw}{2}.$$

The composition of the two maps is

$$\xi_\mathbf{v}^\pm := f_\mathbf{v}^\pm \circ \bar{c}: \Delta \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}^N, \quad \xi_\mathbf{v}^\pm(T, u) = \frac{u^{-1}g_T^{-1}v \pm ug_T^tw}{2} \tag{3.6}$$

After identifying  $X$  with  $P$ , we can view  $X_\mathbf{v} \subset P$  as the zero locus of the smooth map  $f_\mathbf{v}^-: P \longrightarrow \mathbb{R}^N$ . If  $\mathbf{v} \in A_\epsilon(U, f)$ , then  $\mathbf{v}$  is regular and positive (for  $\epsilon$  small enough). By Lemma 3.6 below, the map  $f_\mathbf{v}^-$  has rank  $N$ . Moreover, the submanifold  $S_\mathbf{v}$  is the image of  $X_\mathbf{v}$  under the projection  $\pi: X \rightarrow S$ . Since the restriction of the projection to  $X_\mathbf{v}$  is a diffeomorphism onto  $S_\mathbf{v}$  and  $c$  is transversal to  $S_\mathbf{v}$ , we deduce that

the image of  $\bar{c}: \Delta \times \mathbb{R}_{>0} \longrightarrow P$  is transversal to  $X_{\mathbf{v}}$ . This means that at point  $(T, u) \in \Delta \times \mathbb{R}_{>0}$  such that  $p = \bar{c}(T, u) \in X_{\mathbf{v}}$ , we have

$$T_p X \simeq T_p X_{\mathbf{v}} \oplus \mathrm{Im}(d_{(T,u)} \bar{c}) \simeq \ker(d_p f_{\mathbf{v}}^-) \oplus \mathrm{Im}(d_{(T,u)} \bar{c}).$$

It implies that the composition  $\xi_{\mathbf{v}}^- = f_{\mathbf{v}}^- \circ \bar{c}$  has rank  $N$  at any point  $(T, u) \in \bar{c}^{-1}(X_{\mathbf{v}}) \cap (\Delta \times \mathbb{R}_{>0})$  and is a local diffeomorphism.

Recall that the form  $\alpha^0(z, \mathbf{v})$  was given in (2.6) by the formula

$$\begin{aligned} \alpha^0(z, \mathbf{v}) &= -\frac{1}{2^N \pi^{\frac{N-1}{2}}} \exp(-\pi \|g^{-1}v - g^t w\|^2) \\ &\quad \times \sum_{l=1}^N (-1)^{l-1} \langle g^{-1}v - g^t w, e_l \rangle \sum_{|\underline{d}|=N-1} H_{\underline{d}}(\sqrt{\pi}(g^{-1}v + g^t w)) \otimes \lambda_{I_l}(\underline{d}). \end{aligned}$$

The pullback of  $\alpha^0(z, \mathbf{v})du/u$  by  $\bar{c}$  is of the form

$$\bar{c}^* \left( \alpha^0(z, \mathbf{v}) \frac{du}{u} \right) \propto \sum_{l=1}^N \exp(-\pi \|\xi_{\mathbf{v}}^-(T, u)\|^2) \xi_{\mathbf{v}}^-(T, u)_l Q_l(\xi_{\mathbf{v}}^+(T, u)_1, \dots, \xi_{\mathbf{v}}^+(T, u)_N) r(T) \frac{dT du}{Tu},$$

up to some constants, where  $Q_l$  is some polynomial of degree  $N-1$  and  $r(T)$  is some smooth function on  $\Delta$  (in particular it is bounded). By splitting the polynomial  $Q_l$  into polynomials  $Q_{l,d}$  of homogeneous degree  $0 \leq d \leq N-1$ , the convergence of

$$\int_{\Delta \times \mathbb{R}_{>0}} \int_1^\infty \left| \bar{c}^* \left( \alpha^0(z, \sqrt{y}\mathbf{v}) \frac{du}{u} \right) \right| \frac{dy}{y}$$

follows from the convergence of the integral

$$\begin{aligned} &\int_{\Delta \times \mathbb{R}_{>0}} \int_1^\infty \sqrt{y}^{d+1} \exp(-\pi y \|\xi_{\mathbf{v}}^-(T, u)\|^2) |\xi_{\mathbf{v}}^-(T, u)_l| |Q_{l,d}(\xi_{\mathbf{v}}^+(T, u)_1, \dots, \xi_{\mathbf{v}}^+(T, u)_N)| \frac{dT du dy}{uy} \\ &= \int_{\Delta \times \mathbb{R}_{>0}} \beta_{\frac{d+1}{2}}(\pi \|\xi_{\mathbf{v}}^-(T, u)\|^2) |\xi_{\mathbf{v}}^-(T, u)_l| |Q_{l,d}(\xi_{\mathbf{v}}^+(T, u)_1, \dots, \xi_{\mathbf{v}}^+(T, u)_N)| \frac{dT du}{u} \end{aligned}$$

for each  $l$ , where

$$\beta_s(t) = \int_1^\infty e^{-ty} y^{s-1} dy = \frac{1}{t^s} \int_t^\infty e^{-y} y^{s-1} dy.$$

The function

$$\beta_{\frac{d+1}{2}}(\pi \|\xi_{\mathbf{v}}^-(T, u)\|^2)$$

has singularities at the points  $(T, u)$  where

$$\xi_{\mathbf{v}}^-(T, u)_1 = \dots = \xi_{\mathbf{v}}^-(T, u)_N = 0.$$

Equivalently, the are also the points  $(T, u)$  where  $v = u^2 g_T g_T^t w$ . Since  $T$  lies in the compact set  $\Delta$ , the determinant  $u$  must be bounded and the singularities are all contained in a compact  $\Delta \times [a, b]$ .

Let us show that the integral converges outside of this compact. Note that we have [Kud03, Lemma. 3.8]

$$\begin{aligned} \beta_{\frac{d+1}{2}}(r) &= O(r^{-\frac{d+1}{2}}) \text{ as } r \rightarrow 0, \\ \beta_{\frac{d+1}{2}}(r) &= O(e^{-r}) \text{ as } r \rightarrow \infty. \end{aligned}$$

Since  $(v, w)$  is regular and  $g_T$  is invertible, there is always at least one entry of  $g_T^{-1}v$  and  $g_T w$  that is nonzero. Thus the  $\|\xi_{\mathbf{v}}^-(T, u)\|^2$  goes to  $\infty$  as  $u$  goes to 0 and  $\infty$ , and so  $\beta_{\frac{d+1}{2}}(\pi \|\xi_{\mathbf{v}}^-(T, u)\|^2)$  is rapidly decreasing.

Now let us show the convergence in  $\Delta \times [a, b]$ . Since the polynomials  $Q_{d,l}$  are bounded on the compact set  $\Delta \times [a, b]$ , it is left to show the convergence of

$$\int_{\Delta \times [a,b]} \beta_{\frac{d+1}{2}}(\pi \|\xi_{\mathbf{v}}^-(T, u)\|^2) |\xi_{\mathbf{v}}^-(T, u)_l| \frac{dT du}{u} \leq \frac{1}{a} \int_{\Delta \times [a,b]} \frac{|\xi_{\mathbf{v}}^-(T, u)_l|}{\|\xi_{\mathbf{v}}^-(T, u)\|^{d+1}} dT du,$$

where the set  $\Delta \times [a, b]$  contains all the singularities. Around any singularity  $p \in \Delta \times [a, b]$ , the map  $\xi_{\mathbf{v}}^-$  defined in (3.6) is a local diffeomorphism. Thus, after the change of variables  $x_i = \xi_{\mathbf{v}}^-(T, u)_i$  in a small neighbourhood of  $(T, u)$ , the integral is

$$\int_{B_\delta(0)} \frac{|x_l|}{\|x\|^{d+1}} dx_1 \cdots dx_N \leq \int_{B_\delta(0)} \frac{1}{\|x\|^d} dx_1 \cdots dx_N$$

in a small neighbourhood of  $B_\delta(0) \subset \mathbb{R}^N$  of 0. Since  $0 \leq d \leq N - 1$ , the integral on the right converges to  $\frac{\text{vol}(S^{N-1})}{N-d} \delta^{N-d}$ , and thus the integral converges.  $\square$

*Remark 3.1.* One can also show that if  $\omega \in \Omega_c^k(S/\Gamma)$  is a compactly supported form of degree  $k = \frac{N^2+N}{2}$ , then

$$\int_S \Phi(z_1, f, \phi) \wedge \omega$$

converges.

**Lemma 3.6.** *If  $\mathbf{v}$  is a regular vector with  $Q(\mathbf{v}) > 0$ , then  $f_{\mathbf{v}}^-$  has rank  $N$ .*

*Proof.* We identify the tangent space  $T_g \mathrm{GL}_N(\mathbb{R}) \simeq \mathrm{Mat}_{N \times N}(\mathbb{R})$ . The differential at  $g$  is

$$d_g f_{\mathbf{v}}^-(A) = \left. \frac{d}{dr} \right|_{r=0} f_{\mathbf{v}}^-(ge^{rA}) = -\frac{1}{2} (Ag^{-1}v + A^t g^t w).$$

Since  $\mathbf{v} = (v, w)$  is a regular vector with  $Q(\mathbf{v}) \geq 0$  if and only if it holds for  $\rho_{g^{-1}}\mathbf{v} = (g^{-1}v, g^t w)$ , it is enough to show that the map

$$F: \mathrm{Mat}_{N \times N}(\mathbb{R}) \longrightarrow \mathbb{R}^N, \quad A \longmapsto Av + A^t w$$

is surjective. This is equivalent to showing that the dual map  $F^*: (\mathbb{R}^N)^* \longrightarrow \mathrm{Mat}_{N \times N}(\mathbb{R})^*$  is injective. We identify  $(\mathbb{R}^N)^* \simeq \mathbb{R}^N$  via the scalar product  $\langle \cdot, \cdot \rangle$ , so the dual map sends a vector  $x \in \mathbb{R}^N$  to  $F^*(x) \in \mathrm{Mat}_{N \times N}(\mathbb{R})^*$  defined by

$$F^*(x)(A) = \langle x, F(A) \rangle = x^t Av + x^t A^t w = x^t Av + w^t Ax = \mathrm{Tr}(A(vx^t + xw^t)).$$

We identify  $\mathrm{Mat}_{N \times N}(\mathbb{R})^* \simeq \mathrm{Mat}_{N \times N}(\mathbb{R})$  via the trace pairing  $(A, B) \mapsto \mathrm{Tr}(AB)$ , so that we can see the dual as a map

$$F^*: \mathbb{R}^N \longrightarrow \mathrm{Mat}_{N \times N}(\mathbb{R}), \quad x \longmapsto vx^t + xw^t.$$

To show that the map is injective, suppose that  $x$  is such that  $vx^t + xw^t = 0$ . Multiplying by the nonzero vector  $w$  (since  $\mathbf{v}$  is regular) shows that  $x = \alpha v$ , where  $\alpha = -(x^t w) \|w\|^{-2}$ . We have to show that  $\alpha = 0$ . After substituting in the previous equation, we get

$$0 = vx^t + xw^t = \alpha(\|v\|^2 + Q(v, w)).$$

Since  $v$  is nonzero and  $Q(\mathbf{v}) = Q(v, w) > 0$ , it follows that  $\alpha = 0$ .  $\square$

**3.4. Adjointness formula.** When the lift (2.17) vanishes, we want to consider the derivative

$$\begin{aligned} Z_{N-1}(S_\Gamma; \mathbb{Z}) &\longrightarrow \widetilde{M}_N(\Gamma') \\ c &\longmapsto E'_\varphi(c, \tau, \phi) := \left. \frac{\partial}{\partial s} \right|_{s=0} \int_c E_\varphi(z_1, \tau, \phi, s). \end{aligned}$$

From the Fourier expansion (2.13) we find that

$$E'_\varphi(c, \tau, \phi) = A(c, \phi) + \log(y)B(c, \phi) + \sum_{n \in \mathbb{Q}} C_n(c, \phi, y) e(n\tau)$$

where

$$\begin{aligned} A(c, \phi) &= \frac{\partial}{\partial s} \Big|_{s=0} \int_c (e_{\varphi,1}(z_1, \phi, s) + e_{\varphi,2}(z_1, \phi, s)), \\ B(c, \phi) &= N \int_c (e_{\varphi,1}(z_1, \phi) - e_{\varphi,2}(z_1, \phi)), \\ C_n(c, \phi, y) &= (-1)^{N-1} \sum_{\substack{\mathbf{v} \in V_{\mathrm{reg}} \\ Q(\mathbf{v})=n}} \phi(\mathbf{v}) \int_c \int_0^\infty \iota_{u \frac{\partial}{\partial u}} \varphi^0(z, \sqrt{y}\mathbf{v}) \log(u) \frac{du}{u}. \end{aligned}$$

At the other cusps we have

$$E'_\varphi(c, \tau, \phi)|_{\gamma_r} = A(c, \phi_r) + \log(y)B(c, \phi_r) + \sum_{n \in \mathbb{Q}} C_n(c, \phi_r, y) e(n\tau).$$

**Proposition 3.7.** *Let  $c \in Z_{N-1}(S_\Gamma)$  be a smooth cycle. Then*

$$\int_c E_\psi(z_1, \tau, \phi) = \frac{(-1)^{N-1}}{2N} L_N E'_\varphi(c, \tau, \phi).$$

*Proof.* Recall from (2.11) that  $d\Theta_\alpha(z, \tau, \phi) = L_N \Theta_\varphi(z, \tau, \phi)$ . We have

$$d_X \left( \Theta_\alpha(z, \tau, \phi) \log(u) \right) = L_N \Theta_\varphi(z, \tau, \phi) \log(u) + (-1)^{N-1} \Theta_\alpha(z, \tau, \phi) \frac{du}{u}$$

and

$$\frac{\partial}{\partial s} \Big|_{s=0} L_N (\Theta_\varphi(z, \tau, \phi) u^{-2Ns}) = -2NL_N \Theta_\varphi(z, \tau, \phi) \log(u).$$

For a form  $\eta$  on  $X$  that vanishes as  $u$  goes to 0 and  $\infty$  (and is integrable along the fiber of  $X \rightarrow S$ ) we have  $d_S \pi_*(\eta) = \pi_*(d_X \eta)$  (see [Bra25, Lemma. 3.9]). By Proposition 2.2 the form  $\Theta_\alpha(z, \tau, \phi)$  is rapidly decreasing as  $u \rightarrow 0$  and  $\infty$ , and so the same holds for  $\Theta_\alpha(z, \tau, \phi) \log(u)$ . Thus, we find that

$$d_S \pi_* (\Theta_\alpha(z, \tau, \phi) \log(u)) = -\frac{1}{2N} \frac{\partial}{\partial s} \Big|_{s=0} L_N \pi_* (\Theta_\varphi(z, \tau, \phi) u^{-2Ns}) + (-1)^{N-1} E_\psi(z_1, \tau, \phi).$$

After integrating over the cycle  $c$  we find that

$$0 = -\frac{1}{2N} L_N \frac{\partial}{\partial s} \Big|_{s=0} \int_c E_\varphi(z_1, \tau, \phi, s) + (-1)^{N-1} \int_c E_\psi(z_1, \tau, \phi).$$

Note that the integral and the sum appearing in the definition of  $E_\psi(z_1, \tau, \phi, s)$  and  $E_\varphi(z_1, \tau, \phi, s)$  converge uniformly for  $(z_1, \tau, s)$  in a compact set of  $S \times \mathbb{H} \times \mathbb{C}$ , (as well as their derivatives) so that the exchange of the differential operators with the sum and integral is justified.  $\square$

**Theorem 3.8.** *Let  $f \in H_{2-N}(\Gamma')$  be a harmonic weak Maass form and  $g = \xi_{2-N}(f) \in S_N(\Gamma')$  its image under the  $\xi$ -operator. For any smooth cycle  $c$  transverse to  $S_f(\phi)$  we have*

$$2N(-1)^{N-1} \int_c \Phi(z_1, f, \phi) = \langle E'_\varphi(c, \phi), g \rangle + \sum_{r \in \Gamma' \setminus \mathbb{P}^1(\mathbb{Q})} w_r \kappa_r(c, f, \phi)$$

where

$$\kappa_r(c, f, \phi) := A(c, \phi_r) a_r^+(f, 0) + \lim_{T \rightarrow \infty} \sum_{n \in \mathbb{Q}_{>0}} C_n(c, \phi_r, T) a_r^+(f, -n).$$

*Proof.* From Proposition 3.7 we have

$$\int_c E_\psi(z_1, \tau, \phi) = \frac{(-1)^{N-1}}{2N} L_N E'_\varphi(c, \tau, \phi).$$

In particular, by comparing with the constant term of  $E_\psi$  given in (2.14) and using that  $L_N \log(y) = y$  we find that

$$\int_c e_\psi(z_1, \phi) = \frac{(-1)^{N-1}}{2N} B(c, \phi).$$

Combining with Proposition 3.5, we deduce that

$$\begin{aligned} \int_c \Phi(z_1, f, \phi) &= \lim_{T \rightarrow \infty} \left( \int_c E_\psi(z_1, \tau, \phi) \frac{dxdy}{y^2} - \int_c Q_\psi(z_1, f, \phi) \log(T) \right) \\ &= \frac{(-1)^{N-1}}{2N} \lim_{T \rightarrow \infty} \left( \int_{F_T} L_N E'_\varphi(c, \tau, \phi) \frac{dxdy}{y^2} - \log(T) \sum_{r \in \Gamma' \setminus \mathbb{P}_1(\mathbb{Q})} w_r a_r^+(f, 0) B(c, \phi_r) \right). \end{aligned} \quad (3.7)$$

Since  $f$  is of weight  $2 - N$  and  $E'_\varphi(c, \tau, \phi)$  of weight  $N$ , the form  $E'_\varphi(c, \tau, \phi)f(\tau)d\tau \in \Omega^1(\mathbb{H})$  is a 1-form on  $\mathbb{H}$ . Applying the exterior derivative  $d_{\mathbb{H}}$  on both sides gives

$$d_{\mathbb{H}}(E'_\varphi(c, \tau, \phi)f(\tau)d\tau) = -(L_N E'_\varphi(c, \tau, \phi))f(\tau) \frac{dxdy}{y^2} + E'_\varphi(c, \tau, \phi) \overline{\xi_{2-N} f(\tau)} y^N \frac{dxdy}{y^2}.$$

By Stoke's Theorem we have

$$\begin{aligned} \int_{\partial F_T} E'_\varphi(c, \tau, \phi)f(\tau)d\tau &= \int_{F_T} d_{\mathbb{H}}(E'_\varphi(c, \tau, \phi)f(\tau)d\tau) \\ &= - \int_{F_T} (L_N E'_\varphi(c, \tau, \phi))f(\tau) \frac{dxdy}{y^2} + \int_{F_T} E'_\varphi(c, \tau, \phi) \overline{\xi_{2-N} f(\tau)} y^N \frac{dxdy}{y^2}. \end{aligned}$$

Hence, the limit in (3.7) is equal to

$$\begin{aligned} &\langle E'_\varphi(c, \phi), g \rangle + \lim_{T \rightarrow \infty} \left( - \int_{\partial F_T} E'_\varphi(c, \tau, \phi)f(\tau)d\tau - \log(T) \sum_{r \in \Gamma' \setminus \mathbb{P}_1(\mathbb{Q})} w_r a_r^+(f, 0) B(c, \phi_r) \right) \\ &= \langle E'_\varphi(c, \phi), g \rangle + \sum_{r \in \Gamma' \setminus \mathbb{P}_1(\mathbb{Q})} \lim_{T \rightarrow \infty} \left( \int_{-\frac{w_r}{2}}^{\frac{w_r}{2}} E'_\varphi(c, x + iT, \phi_r)(f|_{\gamma_r})^-(x + iT)dx - w_r \log(T) a_r^+(f, 0) B(c, \phi_r) \right). \end{aligned}$$

Note that in the second equality we used that the boundary  $\partial F_T$  of the truncated fundamental domain is oriented counterclockwise, and that the sides cancel out since  $E'_\varphi(c, \tau, \phi)f(\tau)$  is  $\Gamma'$ -invariant. The limit

$$\lim_{T \rightarrow \infty} \int_{-\frac{w_r}{2}}^{\frac{w_r}{2}} E'_\varphi(c, x + iT, \phi_r)(f|_{\gamma_r})^-(x + iT)dx = 0$$

vanishes because  $(f|_{\gamma_r})^-$  is rapidly decreasing as  $T \rightarrow \infty$ . For the integral against  $(f|_{\gamma_r})^+$  we plug-in the Fourier expansions. Using that

$$\int_{-\frac{w_r}{2}}^{\frac{w_r}{2}} e^{2i\pi x \frac{m+n}{w_r}} = w_r \delta_{m+n=0}$$

for two integers  $m, n$ , we find that

$$\begin{aligned} &\int_{-\frac{w_r}{2}}^{\frac{w_r}{2}} E'_\varphi(c, x + iT, \phi_r)(f|_{\gamma_r})^+(x + iT)dx - w_r \log(T) a_r^+(f, 0) B(c, \phi_r) \\ &= w_r A(c, \phi_r) a_r^+(f, 0) + w_r \lim_{T \rightarrow \infty} \sum_{n \in \mathbb{Q}_{>0}} C_n(c, \phi_r, T) a_r^+(f, -n). \end{aligned}$$

□

#### 4. Period over a torus

Let  $F$  be a totally real field with ring of integers  $\mathcal{O}$  and  $\sigma_1, \dots, \sigma_N$  the  $N$  real embeddings. Let  $F^{\times,+}$  be the set of totally positive elements in  $F$ , and  $\mathcal{O}^{\times,+} = \mathcal{O}^\times \cap F^{\times,+}$  the totally positive units. Let  $W := F$ ,

that we view as a  $\mathbb{Q}$ -vector space. Let  $\epsilon = (\epsilon_1, \dots, \epsilon_N)$  be a  $\mathbb{Q}$ -basis of  $F$ , chosen such that

$$g_\epsilon := \begin{pmatrix} \sigma_1(\epsilon_1) & \cdots & \sigma_1(\epsilon_N) \\ \vdots & & \vdots \\ \sigma_N(\epsilon_1) & \cdots & \sigma_N(\epsilon_N) \end{pmatrix} \in \mathrm{GL}_N(\mathbb{R})^+$$

has positive determinant.

**4.0.1. A finite Schwartz function.** Let  $p$  be a prime, and  $\mathfrak{a}, \mathfrak{c}$  be two integral ideals such that  $\mathfrak{c} \mid p$ . With respect to the basis  $\epsilon$ , we identify  $F \simeq \mathbb{Q}^N$ . Then  $L_1 = \mathfrak{a}\mathfrak{c}$  and  $L_2 = \mathfrak{a}$  are two lattices in  $\mathbb{Q}^N$  that satisfy  $pL_2 \subset L_1 \subset L_2 \subset \mathbb{Q}^N$ , and  $L_2^\vee = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$ . Let  $\phi_{\mathfrak{a}, \mathfrak{c}}$  be the finite Schwartz function  $\phi_{\mathfrak{a}, \mathfrak{c}} := \prod_p 1_{L_1 \otimes \mathbb{Z}_p \times L_2^\vee \otimes \mathbb{Z}_p}$  is in  $\mathcal{S}(V_{\mathbb{A}_f})$ . More generally, for a totally odd character  $\chi: \mathrm{Cl}(F)^+ \rightarrow \{\pm 1\}$  we define

$$\phi_{\chi, \mathfrak{c}} := \sum_{\mathfrak{a} \in \mathrm{Cl}(F)^+} \chi(\mathfrak{a}) \phi_{\mathfrak{a}, \mathfrak{c}} \in \mathcal{S}(V_{\mathbb{A}_f}).$$

Let  $\Gamma \subseteq \mathrm{SL}_N(\mathbb{Z})$  be a torsion-free subgroup that preserves  $\phi_{\chi, \mathfrak{c}}$  for all  $\mathfrak{c} \mid p$  under the  $\rho$ -action, i.e.

$$\phi_{\chi, \mathfrak{c}}(\gamma^{-1}v, \gamma^t w) = \phi_{\chi, \mathfrak{c}}(v, w) \quad \forall \gamma \in \Gamma.$$

Let  $R_\epsilon: F^\times \hookrightarrow \mathrm{GL}_N(\mathbb{Q})$  be the regular representation with respect to the basis  $\epsilon$  of  $F$ . Since  $\mathfrak{a}\mathfrak{c}$  is preserved by multiplication with  $\mathcal{O}^\times$ , we have  $R_\epsilon(\mathcal{O}^\times) \subset \mathrm{SL}_N(\mathbb{Q})$ . Let  $\Lambda_\epsilon := R_\epsilon^{-1}(R_\epsilon(\mathcal{O}^{\times,+}) \cap \Gamma)$ .

**4.0.2. The cycle attached to  $F$ .** Let  $A_{\mathbb{R}} := \mathbb{R}_{>0}^N$ , that we view as a subgroup of  $G$  via the map

$$t = (t_1, \dots, t_N) \in \mathbb{R}_{>0}^N \longmapsto a(t) = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_N \end{pmatrix} \in G.$$

Let  $A_{\mathbb{R}}^1 := A_{\mathbb{R}} \cap G^1$  be the subgroup of elements of norm 1. The group  $F^{\times,+}$  acts on  $A_{\mathbb{R}}$  by

$$\nu(t_1, \dots, t_N) = (\sigma_1(\nu)t_1, \dots, \sigma_N(\nu)t_N),$$

and the group  $F^{1,+}$  of elements of norm 1 acts on  $A_{\mathbb{R}}^1$ . Since  $\Lambda_\epsilon := R_\epsilon^{-1}(R_\epsilon(\mathcal{O}^{\times,+}) \cap \Gamma)$  has finite index in  $\mathcal{O}^{\times,+}$ , it follows from Dirichlet's unit theorem that

$$\Lambda_\epsilon \backslash A_{\mathbb{R}}^1$$

is compact. The regular representation is diagonalized by  $g_\epsilon$ , so that for all  $\nu \in F^\times$  we have

$$R_\epsilon(\nu) = g_\epsilon^{-1} \begin{pmatrix} \sigma_1(\nu) & & 0 \\ & \ddots & \\ 0 & & \sigma_N(\nu) \end{pmatrix} g_\epsilon.$$

Thus, the map  $c_\epsilon(t) := g_\epsilon^{-1}a(t)$  is  $\mathcal{O}^{\times,+}$ -equivariant and induces a smooth map

$$c_\epsilon: \Lambda_\epsilon \backslash A_{\mathbb{R}}^1 \longrightarrow \Gamma \backslash S.$$

It defines a smooth cycle  $c_\epsilon \in Z_{N-1}(S_\Gamma, \mathbb{Z})$ .

**Proposition 4.1.** *The cycle  $c_\epsilon$  is transverse to any  $S_{[\mathbf{v}]}$  with  $\mathbf{v} \in V_{\mathrm{reg}}$  and  $Q(\mathbf{v}) > 0$ .*

*Proof.* By translating by  $g_\epsilon$ , we see that the image of  $g_\epsilon^{-1}A_{\mathbb{R}}^1$  is transverse to  $S_{\mathbf{v}}$  in  $S$  for  $\mathbf{v} \in V_{\mathrm{reg}} \subset \mathbb{Q}^{2N}$  if the image of  $A_{\mathbb{R}}^1$  is transverse to  $S_{\mathbf{w}}$  for  $\mathbf{w} = (g_\epsilon v, g_\epsilon^\vee w) \in \sigma(F) \times \sigma(F)^\vee \subset V_{\mathbb{R}}$ . Note that a vector  $\mathbf{w} \in \sigma(F) \times \sigma(F)^\vee \subset V_{\mathbb{R}}$  has nonzero entries if it is regular (because  $\sigma_i(\lambda) = 0$  implies that  $\lambda = 0$ ). As in the proof of Proposition 3.5,  $A_{\mathbb{R}}^1$  is transverse to  $S_{\mathbf{w}}$  if and only  $A_{\mathbb{R}}$  is transverse to  $X_{\mathbf{w}}$ , which follows from the fact that the restriction of  $s_{\mathbf{w}}$  to the image  $A_{\mathbb{R}} \subset X$  has rank  $N$ . A direct computation shows that this is the case for a vector  $(g_\epsilon v, g_\epsilon^\vee w)$ , since such a vector has nonzero entries if  $\mathbf{v}$  is a regular.  $\square$

**4.1. Hecke characters and  $L$ -function.** We briefly recall the main properties of Hecke  $L$ -functions for totally real fields, following [Neu99]. Let  $I_F$  be the group of fractional ideals of  $F$ , and

$$\chi: I_F \longrightarrow \{\pm 1\}$$

be a totally odd Hecke character. *i.e.* such that

$$\chi((\alpha)) = \text{sgn}(N(\alpha))$$

on principal ideals. The existence of such a character implies that  $\mathcal{O}$  does not contain a unit of negative norm, in particular it implies that  $F$  has even degree. These are Dirichlet characters of type  $(q, 0)$  with  $q = (1, \dots, 1)$  in the notation of [Neu99, Definition. 6.8 on p. 478]. Hence, it defines a character  $\chi: \text{Cl}(F)^+ \longrightarrow \{\pm 1\}$  on the narrow class group. Let

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \quad \text{Re}(s) \geq 1,$$

be the associated  $L$ -function. The completed  $L$ -function

$$|D_F|^{\frac{s}{2}} \Gamma\left(\frac{1+s}{2}\right)^N L(\chi, s)$$

admits an analytic continuation to the entire plane and a functional equation - see [Neu99, Theorem. 8.5 on p. 502]. In particular, since the Gamma function is nonzero and has poles at negative integers, we deduce that  $L(\chi, s)$  is analytic on  $\mathbb{C}$  and has a zero of order  $N$  at odd negative integers. More generally, the proof of the following Theorem due to Shintani can be found in [Neu99, Theorem. 9.8 on p. 515].

**Theorem 4.2** (Shintani). *For  $n \geq 1$ , the special value  $L(\chi, 1-n)$  is a rational number.*

Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_h$  be representatives of the narrow class group. If we write  $\mathfrak{a} = (\mu)\mathfrak{a}_i^{-1}$  for some principal ideal  $(\mu)$  then we can split the sum as

$$L(\chi, s) = \sum_{[\mathfrak{a}_i] \in \text{Cl}(F)^+} \chi(\mathfrak{a}_i) N(\mathfrak{a}_i)^s \sum_{\mu \in \mathfrak{a}_i / \mathcal{O}^{\times, +}} \frac{\text{sgn}(N(\mu))}{|N(\mu)|^s}.$$

**4.2. The integral.** For a totally odd character  $\chi: \text{Cl}(F)^+ \longrightarrow \{\pm 1\}$  as above we set

$$\mathcal{E}_{\chi, \mathfrak{c}}(\tau, s) := \sum_{[\mathfrak{a}] \in \text{Cl}(F)^+} \chi(\mathfrak{a}) \mathcal{E}_{\mathfrak{a}, \mathfrak{c}}(\tau, s),$$

where

$$\mathcal{E}_{\mathfrak{a}, \mathfrak{c}}(\tau, s) := \frac{N(\mathfrak{a})^{2s}}{[\mathcal{O}^{\times, +} : \Lambda_\epsilon]} \int_{c_\epsilon} E_\varphi(z_1, \tau, \phi_{\mathfrak{a}, \mathfrak{c}}, s). \quad (4.1)$$

The factor  $N(\mathfrak{a})^{2s}$  ensures that the integral only depends on the class of  $\mathfrak{a}$  in  $\text{Cl}(F)^+$ . It converges for any  $s \in \mathbb{C}$  and let  $\mathcal{E}_{\chi, \mathfrak{c}}(\tau)$  be the value at  $s = 0$ . We have

$$\mathcal{E}_{\mathfrak{a}, \mathfrak{c}}(\tau, s) = 2^N N(\mathfrak{a})^{2s} \sqrt{y}^{-N} \int_{\mathcal{O}^{\times, +} \setminus A_{\mathbb{R}}^1} \int_0^\infty \sum_{\mathbf{v} \in L_1 \times L_2^\vee} \omega(1, h_\tau) c_\epsilon^* \varphi(z, \mathbf{v}) u^{-2Ns} \frac{du}{u}.$$

By the equivariance of  $\varphi(z, \mathbf{v})$  we have

$$c_\epsilon^* \varphi(z, \mathbf{v}) = a^*(g_\epsilon^{-1})^* \varphi(z, \mathbf{v}) = a^* \varphi(z, \rho_{g_\epsilon} \mathbf{v}).$$

The embedding  $\sigma(\mathfrak{a}) := \{(\sigma_1(\lambda), \dots, \sigma_N(\lambda)) \in \mathbb{R}^N \mid \lambda \in \mathfrak{a}\} \subset \mathbb{R}^N$  of  $\mathfrak{a}$  in  $\mathbb{R}^N$  via the  $N$  real embeddings is a lattice in  $\mathbb{R}^N$ . We have

$$\rho(g_\epsilon)(L_1 \times L_2^\vee) = g_\epsilon L_1 \times g_\epsilon^\vee L_2^\vee = \sigma(\mathfrak{a}) \times \sigma(\mathfrak{a}^{-1} \mathfrak{d}^{-1}) \subset \mathbb{R}^N \times \mathbb{R}^N.$$

We will simply write  $\mathfrak{a} \subset \mathbb{R}^N$  for such a lattice, instead of  $\sigma(\mathfrak{a}) \subset \mathbb{R}^N$ . By using the diffeomorphism  $A_{\mathbb{R}} \simeq A_{\mathbb{R}}^1 \times \mathbb{R}_{>0}$  we deduce that

$$\mathcal{E}_{\mathfrak{a},\mathfrak{c}}(\tau, s) = N(\mathfrak{a})^{2s} \sqrt{y}^{-N} \int_{\mathcal{O}^{\times,+} \backslash A_{\mathbb{R}}} \sum_{\mathbf{v} \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}} \omega(1, h_{\tau}) a^* \varphi(z, \mathbf{v}) u^{-2Ns} \frac{du}{u}.$$

The restriction of the 1-form  $\lambda$  (see (2.3)) to  $A_{\mathbb{R}}$  is

$$a^*(\lambda) = \begin{pmatrix} \frac{dt_1}{t_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{dt_N}{t_N} \end{pmatrix}.$$

Hence, the restriction of  $\lambda(\underline{d})$  is only nonzero for  $\underline{d} = (1, 1, \dots, 1)$  and

$$\lambda(\underline{d}) = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_N}{t_N}.$$

Using the formula given in (2.4), the pullback of  $\varphi(z, \mathbf{v})$  to the split torus  $A_{\mathbb{R}}$  is

$$\prod_{k=1}^N (v_k t_k^{-1} + w_k t_k) \exp\left(-\pi (v_k t_k^{-1})^2 - \pi (w_k t_k)^2\right) \frac{dt_k}{t_k},$$

By Proposition 2.1, the partial Fourier transform of this form in the variable  $w$  is

$$i^N \overline{N(vi + w)} \prod_{k=1}^N \exp(-\pi t_k^{-2} |v_k i + w_k|^2) t_k^{-2} \frac{dt_k}{t_k}.$$

Applying Poisson summation, unfolding the integral (for  $\mathrm{Re}(s)$  large enough the sum is absolutely convergent) and translating by  $g_{\epsilon}^{-1}$  gives

$$\begin{aligned} \mathcal{E}_{\mathfrak{a},\mathfrak{c}}(\tau, s) &= |D_F|^{\frac{1}{2}} N(\mathfrak{a})^{1+2s} \sqrt{y}^{-N} \int_{\mathcal{O}^{\times,+} \backslash A_{\mathbb{R}}} \sum_{(v,w) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a}} \omega'(1, h_{\tau}) a^* \widehat{\varphi}(t, v, w) u^{-2Ns} \\ &= |D_F|^{\frac{1}{2}} N(\mathfrak{a})^{1+2s} i^N \frac{\overline{N(v\tau + w)}}{y^N} \sum_{(v,w) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a} / \mathcal{O}^{\times,+}} \int_{A_{\mathbb{R}}} \prod_{k=1}^N \exp\left(-\pi t_k^{-2} \frac{|v_k i + w_k|^2}{y}\right) t_k^{-2-2s} \frac{dt_k}{t_k} \\ &= 2^{-N} i^N \pi^{-N(1+s)} \Gamma(1+s)^N |D_F|^{\frac{1}{2}} \sum_{(v,w) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a} / \mathcal{O}^{\times,+}} \frac{y^{Ns}}{N(v\tau + w) |N(v\tau + w)|^{2s}}. \end{aligned}$$

Hence, we find that

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s) = C(s) E_{\chi,\mathfrak{c}}(\tau, s)$$

where we define

$$E_{\chi,\mathfrak{c}}(\tau, s) := \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{1+2s} \sum_{(v,w) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a} / \mathcal{O}^{\times,+}} \frac{y^{Ns}}{N(v\tau + w) |N(v\tau + w)|^{2s}}$$

and  $C(s) := 2^{-N} i^N \pi^{-N(1+s)} \Gamma(1+s)^N |D_F|^{\frac{1}{2}}$ .

**4.3. Fourier expansion.** We can use the integral representation to find the Fourier expansion. When  $N = 2$ , the computation are similar to [GKZ87]. Using the previous computations (4.1) we deduce that

$$\begin{aligned} \mathcal{E}_{\chi,\mathfrak{c}}(\tau, s) &= \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \sqrt{y}^{-N} \int_{\mathcal{O}^{\times,+} \backslash A_{\mathbb{R}}} \sum_{(v,w) \in \mathfrak{a}\mathfrak{c} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}} \omega(1, h_{\tau}) c_{\epsilon}^* \varphi(z, v, w) u^{-2Ns} \\ &= \sum_{[\mathfrak{a}] \in \mathrm{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \int_{\mathcal{O}^{\times,+} \backslash A_{\mathbb{R}}} \sqrt{y}^N \sum_{(v,w)} \prod_{k=1}^N (v_k t_k^{-1} + w_k t_k) t_k^{-2s} e^{-y\pi(v_k t_k^{-1})^2 - y\pi(w_k t_k)^2} \frac{dt_k}{t_k} e(x\mathrm{tr}(vw)) \end{aligned}$$

where  $Q(\mathbf{v}) = \sum_k v_k w_k = \text{tr}(vw)$ . We find that the constant term of  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  is given by the singular vectors ( $v = 0$  or  $w = 0$ ) and is equal to

$$c_{\chi,\mathfrak{c}}(0, 2s) := 2^{-N} y^{Ns} \chi(\mathfrak{c}) N(\mathfrak{c})^{-2s} \Lambda(\chi, 2s) + 2^{-N} y^{-Ns} \chi(\mathfrak{d}) N(\mathfrak{d})^{-2s} \Lambda(\chi, -2s),$$

where we set

$$\Lambda(\chi, 2s) := \Gamma\left(\frac{1}{2} + s\right)^N \pi^{-N(\frac{1}{2}+s)} L(\chi, 2s).$$

The higher Fourier coefficients are given by the sum over the regular vectors ( $v \neq 0$  and  $w \neq 0$ ). After unfolding we find that

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s) = c_{\chi,\mathfrak{c}}(0, 2s) + \sum_{\substack{\nu \in \mathfrak{cd}^{-1} \\ \nu \neq 0}} c_{\chi,\mathfrak{c}}(\nu, 2s) e(\tau \text{tr}(\nu))$$

where

$$\begin{aligned} c_{\chi,\mathfrak{c}}(\nu, 2s) &:= e^{-2\pi y \text{Tr}(\nu)} \sqrt{y}^N \sum_{[\mathfrak{a}] \in \text{Cl}^+(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \\ &\quad \times \sum_{\substack{(v,w) \in \mathfrak{ac} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathcal{O}^{\times,+} \\ \nu = vw}} \prod_{k=1}^N \int_0^\infty (v_k t_k^{-1} + w_k t_k) t_k^{-2s} e^{-y\pi(v_k t_k^{-1})^2 - y\pi(w_k t_k)^2} \frac{dt_k}{t_k}. \end{aligned}$$

For a nonzero  $a \in \mathbb{R}^\times$  we define

$$k_s(a) := e^{2\pi a} |a|^{\frac{1}{2}+s} \left( K_{\frac{1}{2}+s}(2\pi|a|) + \text{sgn}(a) K_{\frac{1}{2}-s}(2\pi|a|) \right)$$

where  $K_s$  is the  $K$ -Bessel function

$$K_s(a) := \int_0^\infty e^{-a(t+t^{-1})/2} t^s \frac{dt}{t}.$$

We have

$$c_{\chi,\mathfrak{c}}(\nu, 2s) = 2^{-N} y^{-Ns} \prod_{k=1}^N k_s(y\nu_k) \sum_{[\mathfrak{a}] \in \text{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \sum_{\substack{(v,w) \in \mathfrak{ac} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathcal{O}^{\times,+} \\ \nu = vw}} \frac{\text{sgn}(N(v))}{|N(v)|^{2s}}.$$

Moreover, using the exact sequence

$$1 \longrightarrow \mathcal{O}^\times/\mathcal{O}^{\times,+} \longrightarrow F^\times/F^{\times,+} \longrightarrow \text{Cl}(F)^+ \longrightarrow \text{Cl}(F) \longrightarrow 1$$

we can rewrite the previous term as

$$\begin{aligned} &\sum_{[\mathfrak{a}] \in \text{Cl}(F)^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \sum_{\substack{(v,w) \in \mathfrak{ac} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathcal{O}^{\times,+} \\ \nu = vw}} \frac{\text{sgn}(N(v))}{|N(v)|^{2s}} \\ &= [F^\times : F^{\times,+}] \sum_{[\mathfrak{a}] \in \text{Cl}(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{2s} \sum_{\substack{(v,w) \in \mathfrak{ac} \times \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathcal{O}^\times \\ \nu = vw}} \frac{\text{sgn}(N(v))}{|N(v)|^{2s}} \\ &= 2^N \sigma_{\chi,\mathfrak{c}}(\nu, 2s). \end{aligned}$$

where for  $\nu \in F^\times$  we set

$$\sigma_{\chi,\mathfrak{c}}(\nu, 2s) := \sum_{\substack{\mathfrak{n} \mid \nu \mathfrak{d} \\ \mathfrak{c} \mid \mathfrak{n}}} \frac{\chi(\mathfrak{n})}{N(\mathfrak{n})^{2s}}.$$

Note that  $\sigma_{\chi,\mathfrak{c}}(\nu, 2s) = 0$  if  $\nu \notin \mathfrak{cd}^{-1}$ .

**Proposition 4.3.** *The Eisenstein series  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  has the Fourier expansion*

$$\begin{aligned} \mathcal{E}_{\chi,\mathfrak{c}}(\tau, s) &= 2^{-N} y^{Ns} \chi(\mathfrak{c}) N(\mathfrak{c})^{-2s} \Lambda(\chi, 2s) + 2^{-N} y^{-Ns} \chi(\mathfrak{d}) N(\mathfrak{d})^{-2s} \Lambda(\chi, -2s) \\ &\quad + y^{-Ns} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \neq 0}} \sigma_{\chi,\mathfrak{c}}(\nu, 2s) \prod_{k=1}^N k_s(y\nu_k) e(\tau \mathrm{tr}(\nu)). \end{aligned}$$

At  $s = 0$ , we have  $k_0(a) = 1 + \mathrm{sgn}(a)$ . Thus, the Fourier expansion is

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau) = (\chi(\mathfrak{c}) + \chi(\mathfrak{d})) L(\chi, 0) + 2^N \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \sigma_{\chi,\mathfrak{c}}(\nu, 0) e(\tau \mathrm{tr}(\nu)).$$

The modular curve  $Y_0(p)$  has two cusps 0 and  $\infty$ . Let  $\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , which sends  $\infty$  to the cusp 0. A direct computation shows that

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)|_{\gamma_0} = \chi(\mathfrak{c}) N(\mathfrak{c})^{-1-2s} \mathcal{E}_{\chi,\mathfrak{c}^{-1}}(\tau, s),$$

which gives a Fourier expansion of  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  at the cusp 0. If we let  $\bar{\mathfrak{c}} \subset \mathcal{O}$  be the ideal such that  $p = \mathfrak{c}\bar{\mathfrak{c}}$ , then

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)|_{\gamma_0} = p^{Ns} \chi(\mathfrak{c}) N(\mathfrak{c})^{-1-2s} \mathcal{E}_{\chi,\bar{\mathfrak{c}}} \left( \frac{\tau}{p}, s \right),$$

#### 4.4. The functional equation.

**Proposition 4.4.** *The Eisenstein series  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s) = C(s) E_{\chi,\mathfrak{c}}(\tau, s)$  satisfies the functional equation*

$$\mathcal{E}_{\chi,\mathfrak{c}}(\tau, -s) = \chi(\mathfrak{d}\mathfrak{c}) N(\mathfrak{d}\mathfrak{c})^{2s} \mathcal{E}_{\chi,\mathfrak{c}}(\tau, s).$$

In particular, if  $\chi(\mathfrak{c}) = -\chi(\mathfrak{d})$  then  $\mathcal{E}_{\chi}(\tau) = 0$ .

*Proof.* The functional equation can be deduced from Theorem 2.3, but can even easier be seen from the Fourier expansion of  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$ . Indeed, the Fourier coefficient all satisfy

$$c_{\chi,\mathfrak{c}}(\nu, -2s) = \chi(\mathfrak{d}\mathfrak{c}) N(\mathfrak{d}\mathfrak{c})^{2s} c_{\chi,\mathfrak{c}}(\nu, 2s). \quad (4.2)$$

For the constant term this can be seen directly, and for the higher coefficients it follows from the integral representation that replacing  $s$  by  $-s$  has the same effect as swaping  $v$  and  $w$ . The functional equation then follows from reordering the sum over the narrow class group.  $\square$

**4.5. The derivative.** Let  $\mathfrak{c}$  be an ideal dividing  $p$  and such that  $\chi(\mathfrak{c}) = -\chi(\mathfrak{d})$ . The Eisenstein series  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  vanishes at  $s = 0$  and instead we can consider its derivative

$$\mathcal{E}'_{\chi,\mathfrak{c}}(\tau) := \left. \frac{\partial}{\partial s} \right|_{s=0} \mathcal{E}_{\chi,\mathfrak{c}}(\tau, s).$$

The Eisenstein series  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  is a non-holomorphic modular form of weight  $N$  for any  $s \in \mathbb{C}$ , hence  $\mathcal{E}'_{\chi,\mathfrak{c}}(\tau)$  is as well. The function  $\Lambda(\chi, s)$  satisfies  $\Lambda(\chi, 0) = L(\chi, 0)$  and

$$\Lambda'(\chi, 0) = L'(\chi, 0) - \frac{N}{2} \log(4\pi e^\gamma) L(\chi, 0)$$

where  $\gamma := -\Gamma'(1)$  is the Euler-Mascheroni constant. From the Fourier expansion of  $\mathcal{E}_{\chi,\mathfrak{c}}(\tau, s)$  we deduce that

$$\begin{aligned} \mathcal{E}'_{\chi,\mathfrak{c}}(\tau) &= \log(\alpha(\chi, \mathfrak{c})) + A_\chi + B_\chi \log(y) + 2^{N+1} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \sigma'_{\chi,\mathfrak{c}}(\nu, 0) e(\tau \mathrm{tr}(\nu)) \\ &\quad + 2^{N-1} \sum_{l=1}^N \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu_l < 0 \\ \nu_k > 0, k \neq l}} \sigma_{\chi,\mathfrak{c}}(\nu, 0) \left( \left. \frac{\partial}{\partial s} \right|_{s=0} k_s(-y|\nu_l|) \right) e(\tau \mathrm{tr}(\nu)), \end{aligned} \quad (4.3)$$

where the coefficients are

$$\begin{aligned}\log(\alpha(\chi, \mathfrak{c})) &= -2^{2-N} \chi(\mathfrak{d}) \log(N(\mathfrak{cd})) L(\chi, 0) \in \log(\overline{\mathbb{Q}}), \\ A_\chi &= 2^{2-N} \chi(\mathfrak{d}) \Lambda'(\chi, 0), \\ B_\chi &= 2^{1-N} N\chi(\mathfrak{d}) L(\chi, 0) \in \mathbb{Q}, \\ \sigma'_{\chi, \mathfrak{c}}(\nu, 0) &= - \sum_{\substack{\mathfrak{n}|\nu\mathfrak{d} \\ \mathfrak{c}|\mathfrak{n}}} \chi(\mathfrak{n}) \log(N(\mathfrak{n})) \in \log(\overline{\mathbb{Q}})\end{aligned}$$

Note that  $\log(\alpha(\chi, \mathfrak{c}))$  depends on  $\mathfrak{c}$  and is the logarithm of an algebraic number, since  $L(\chi, 0)$  is rational by Shintani's theorem 4.2. On the other hand, the term  $A_\chi$  is transcendental and does not depend on  $\mathfrak{c}$ .

*Remark 4.1.* In (4.3), the higher Fourier coefficients are the value at  $s = 0$  of

$$\sum_{l=1}^N \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \neq 0}} \sigma_{\chi, \mathfrak{c}}(\nu, 2s) k_s(y\nu_1) \cdots \left( \frac{\partial}{\partial s} k_s(y\nu_l) \right) \cdots k_s(y\nu_N).$$

Since  $k_0(a) = 0$  if  $a < 0$ , it follows that the inner sum is only nonzero if  $\nu_k > 0$  for  $k \neq l$ . Finally, if  $\nu_l > 0$  then  $\nu \gg 0$  and  $\sigma_{\chi, \mathfrak{c}}(\nu\mathfrak{d}, 0) = 0$  by the functional equation (4.2).

Recall that

$$\beta_s(a) = \int_1^\infty e^{-at} t^{s-1} dt.$$

**Lemma 4.5.** *We have*

$$\left( \frac{\partial}{\partial s} \Big|_{s=0} k_s(-a) \right) = \beta_0(4\pi a).$$

*Proof.* By the computations in [GZ86, Proposition. 3.3 p. 278], one finds that

$$k_s(a) = \frac{i}{2} e^{-2\pi a} \frac{\Gamma(1+s)}{\pi^{1+s}} V_s(a) \tag{4.4}$$

where

$$V_s(a) = \int_{-\infty}^\infty \frac{e^{-2i\pi ax}}{(x+i)(x^2+1)^s} dx \quad \text{Re}(s) > 0.$$

Moreover, by [GZ86, Proposition. 3.3 e)], we have for  $a > 0$

$$\left. \frac{\partial}{\partial s} \right|_{s=0} V_s(-a) = -2i\pi e^{2\pi a} \int_1^\infty e^{-4\pi at} \frac{dt}{t} = -2i\pi e^{2\pi a} \beta_0(4\pi a).$$

Since  $k_0(a) = 1 + \text{sgn}(a)$ , it follows from (4.4) that  $V_0(-a)$  for  $a > 0$ . Thus, for  $a > 0$  we have

$$\kappa(a) = \left. \frac{\partial}{\partial s} \right|_{s=0} k_s(-a) = \beta_0(4\pi a).$$

□

**Proposition 4.6.** *The derivative  $\mathcal{E}'_{\chi, \mathfrak{c}}(\tau)$  is a non-holomorphic modular form of weight  $N$  for  $\Gamma_0(p)$  with Fourier expansion*

$$\mathcal{E}'_{\chi, \mathfrak{c}}(\tau) = \log(\alpha(\chi, \mathfrak{c})) + A_\chi + B_\chi \log(y) - \sum_{n=1}^{\infty} \log(J_{\chi, \mathfrak{c}}(n)) e(n\tau) + \sum_{n \in \mathbb{Z}} a_{\chi, \mathfrak{c}}(n, y) e(n\tau)$$

where

$$J_{\chi, \mathfrak{c}}(n) := \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0 \\ \text{Tr}(\nu)=n}} \sum_{\substack{\mathfrak{n}|\nu\mathfrak{d} \\ \mathfrak{c}|\mathfrak{n}}} N(\mathfrak{n})^{\chi(\mathfrak{n})2^{N+1}} \in \mathbb{Q}, \quad a_{\chi, \mathfrak{c}}(n, y) := 2^{N-1} \sum_{l=1}^N \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \text{Tr}(\nu)=n \\ \nu_l < 0 \\ \nu_k > 0, k \neq l}} \sigma_{\chi, \mathfrak{c}}(\nu, 0) \beta_0(4\pi y|\nu_l|).$$

Since  $\chi(\mathfrak{c}) = \chi(\bar{\mathfrak{c}})$ , the Eisenstein series  $\mathcal{E}_{\chi, \bar{\mathfrak{c}}}(\tau, s)$  also vanishes at  $s = 0$ . From

$$\mathcal{E}_{\chi, \mathfrak{c}}(\tau, s)|_{\gamma_0} = p^{Ns} \chi(\mathfrak{c}) N(\mathfrak{c})^{-1-2s} \mathcal{E}_{\chi, \bar{\mathfrak{c}}} \left( \frac{\tau}{p}, s \right),$$

it follows that

$$\mathcal{E}'_{\chi, \mathfrak{c}}(\tau)|_{\gamma_0} = \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} \mathcal{E}'_{\chi, \bar{\mathfrak{c}}} \left( \frac{\tau}{p} \right).$$

When  $N = 2$  the computation are as in [GZ85], with a slight difference for the constant term (the two expressions are related by the functional equation). The sum appearing in the definition of  $J_{\chi, \mathfrak{c}}(n)$  is finite. For  $\nu \in \mathfrak{d}^{-1}$ , let  $w_\nu \in \mathbb{R}^N$  be the vector whose entries are the  $N$  embeddings of  $\nu$ . The set  $L$  of all vectors  $w_\nu$  is a lattice in  $\mathbb{R}^N$ . Requiring that  $\nu_i = \langle w_\nu, e_i \rangle$  is positive for all  $1 \leq i \leq N$  means that we only consider the vectors  $w_\nu$  that are in the quadrant spanned by the vector  $e_1, \dots, e_N$ . On the other hand, requiring that

$$\mathrm{tr}(\nu) = \langle w_\nu, e_1 + \dots + e_N \rangle = n$$

means that we only take the lattice vectors on a hyperplane that intersects the quadrant in a compact set.

On the other hand, the sum appearing in  $a_{\chi, \mathfrak{c}}(n, y)$  is infinite, but converges due to the following Lemma and the fact that  $\beta_0(t) = O(e^{-Ct})$ .

**Lemma 4.7.** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f(x) = O(|x|^{-N-\epsilon})$  for some  $\epsilon > 0$ . Then*

$$\sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{Tr}(\nu)=n \\ \nu_l < 0 \\ \nu_k > 0, k \neq l}} f(|\nu_l|)$$

converges absolutely.

*Proof.* Assume without loss of generality that  $l = N$ . Let  $L = \{\sigma(\nu) \in \mathbb{R}^N \mid \nu \in \mathfrak{d}^{-1}\}$  be the lattice as above. The subset

$$L_{<0} := \{w = \sigma(\nu) \in L \mid w_1, \dots, w_{N-1} > 0, w_N < 0, \mathrm{tr}(\nu) = n\}$$

is the intersection of the hyperplane  $\mathrm{tr}(\nu) = n$  with the quadrant spanned by the vector  $e_1, \dots, e_{N-1}$  and  $-e_N$  (instead of  $e_N$ ). Since the first  $N - 1$  embeddings are positive, we have

$$n = \mathrm{tr}(\nu) = |\nu_1| + \dots + |\nu_{N-1}| - |\nu_N| = \|w_\nu\|_1 - 2|\nu_N|,$$

where  $\|w\|_1$  is the  $L^1$ -norm. For  $d \geq 1$  let  $r(d)$  be the number of vectors in  $L_{<0}$  that satisfy  $d \leq \|w\|_1 < (d+1)$ . If we denote by  $m := \min_{w \in L} \|w\|_1$  the length of the shortest vector in  $L$ , then for each vector  $w$  there is a unique integer  $d$  such that  $dm \leq \|w\|_1 < m(d+1)$ . Thus, the sum can be bounded by

$$\sum_{d>n/m} \frac{r(dm)}{(dm-n)^{N+\epsilon}}.$$

Since  $r(d) = O(d^{N-1})$ , this proves the claim.  $\square$

Let  $f \in H_{2-N}(\Gamma_0(p))$  be a harmonic weak Maass form and

$$f|_{\gamma_0}(\tau) = \sum_{\substack{n \in \frac{1}{p}\mathbb{Z} \\ n \gg -\infty}} a_0^+(f, n) e(n\tau) + \sum_{n \in \frac{1}{p}\mathbb{N}} a_0^-(f, n) \beta_k(n, y) e(-n\tau).$$

a Fourier expansion at 0. Note that the width  $w_0$  of the cusp 0 is  $p$ .

**Theorem 4.8.** *Let  $f \in H_{2-N}(\Gamma_0(p))$  be a harmonic weak Maass form and  $g = \xi_{2-N}(f) \in S_N(\Gamma_0(p))$ . Suppose that the principal parts of  $f$  at both cusps have rational coefficients. Then*

$$\int_{c_\epsilon} \Phi(z_1, f, \phi_{\chi, \mathfrak{c}}) = \log(\alpha(f, \chi, \mathfrak{c})) - \frac{1}{2N} \langle \mathcal{E}'_{\chi, \mathfrak{c}}, g \rangle - \frac{A_\chi}{2N} \left( a_0^+(f, 0) + p \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} a_0^+(f, 0) \right),$$

for some algebraic number  $\alpha(f, \chi, \mathfrak{c})$ .

*Proof.* From Proposition 4.6 we have

$$\begin{aligned}\mathcal{E}'_{\chi,\mathfrak{c}}(\tau) &= \log(\alpha(\chi, \mathfrak{c})) + A_\chi + B_\chi \log(y) + 2^{N+1} \sum_{n=1}^{\infty} \log(J_{\chi,\mathfrak{c}}(n)) e(n\tau) + \sum_{n \in \mathbb{Z}} a_{\chi,\mathfrak{c}}(n, y) e(n\tau) \\ \mathcal{E}'_{\chi,\mathfrak{c}}|_{\gamma_0}(\tau) &= \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} \left( \log(\alpha(\chi, \bar{\mathfrak{c}})) + A_\chi + B_\chi \log(y/p) + \sum_{n=1}^{\infty} \log(J_{\chi,\bar{\mathfrak{c}}}(n)) e\left(\frac{n\tau}{p}\right) + \sum_{n \in \mathbb{Z}} a_{\chi,\bar{\mathfrak{c}}}(n, y) e\left(\frac{n\tau}{p}\right) \right)\end{aligned}$$

since  $\mathcal{E}'_{\chi,\mathfrak{c}}|_{\gamma_0}(\tau) = \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} \mathcal{E}'_{\chi,\bar{\mathfrak{c}}} \left( \frac{\tau}{p} \right)$ . From the proof of Theorem 3.8 it follows that

$$-2N \int_{c_\epsilon} \Phi(z_1, f, \phi) = \langle \mathcal{E}'_{\chi,\mathfrak{c}}, g \rangle + \sum_{r \in \Gamma_0(p) \setminus \mathbb{P}^1(\mathbb{Q})} w_r \kappa_r(c, f, \phi_{\chi,\mathfrak{c}})$$

where the width at the cusp 0 is  $w_0 = p$ , and

$$\begin{aligned}\kappa_\infty(f, \chi, \mathfrak{c}) &= (\log(\alpha(\chi, \mathfrak{c})) + A_\chi) a_\infty^+(f, 0) + \lim_{T \rightarrow \infty} \sum_{n \geq 1} (\log(J_{\chi,\mathfrak{c}}(n)) + a_{\chi,\mathfrak{c}}(n, T)) a_\infty^+(f, -n), \\ \kappa_0(f, \chi, \mathfrak{c}) &= \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} (\log(\alpha(\chi, \bar{\mathfrak{c}})) + A_\chi - B_\chi \log(p)) a_0^+(f, 0) \\ &\quad + \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} \lim_{T \rightarrow \infty} \sum_{n \geq 1} (\log(J_{\chi,\bar{\mathfrak{c}}}(pn)) + a_{\chi,\bar{\mathfrak{c}}}(pn, T)) a_0^+(f, -n).\end{aligned}$$

Since  $\beta_0(4\pi T|\nu_l|) = O(e^{-\epsilon T})$  as  $T \rightarrow \infty$ , we have

$$\lim_{T \rightarrow \infty} a_{\chi,\mathfrak{c}}(n, T) = \lim_{T \rightarrow \infty} \sum_{l=1}^N \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \text{Tr}(\nu)=n \\ \nu_l < 0 \\ \nu_k > 0, k \neq l}} \sigma_{\chi,\mathfrak{c}}(\nu, 0) \beta_0(4\pi T|\nu_l|) = 0.$$

Thus, the contribution from  $\infty$  is

$$\kappa_\infty(f, \chi, \mathfrak{c}) = A_\chi a_\infty^+(f, 0) + \log(\alpha_\infty(f, \chi, \mathfrak{c}))$$

where

$$\alpha_\infty(f, \chi, \mathfrak{c}) = \alpha(\chi, \mathfrak{c})^{a_\infty^+(f, 0)} \prod_{n \geq 1} J_{\chi,\mathfrak{c}}(n)^{a_\infty^+(f, -n)}$$

and the contribution from 0 is

$$\kappa_0(f, \chi, \mathfrak{c}) = \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} A_\chi a_0^+(f, 0) + \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} \log(\alpha_0(f, \chi, \mathfrak{c}))$$

where

$$\alpha_0(f, \chi, \mathfrak{c}) = \alpha(\chi, \bar{\mathfrak{c}})^{a_0^+(f, 0)} p^{-B_\chi a_0^+(f, 0)} \prod_{n \geq 1} J_{\chi,\bar{\mathfrak{c}}}(pn)^{a_0^+(f, -n)}.$$

Thus, we get

$$\kappa_\infty(f, \chi, \mathfrak{c}) + p\kappa_0(f, \chi, \mathfrak{c}) = A_\chi \left( a_\infty^+(f, 0) + p \frac{\chi(\mathfrak{c})}{N(\mathfrak{c})} a_0^+(f, 0) \right) + \log(\alpha(f, \chi, \mathfrak{c})),$$

where

$$\alpha(f, \chi, \mathfrak{c}) := \alpha_\infty(f, \chi, \mathfrak{c}) \alpha_0(f, \chi, \mathfrak{c})^{p\chi(\mathfrak{c})N(\mathfrak{c})^{-1}} \in \overline{\mathbb{Q}}.$$

□

## References

- [ANS20] Claudia Alfes-Neumann and Markus Schwagenscheidt. “Traces of reciprocal singular moduli”. In: *Bull. Lond. Math. Soc.* 52.4 (2020), pp. 641–656.
- [BCG20] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. “Transgressions of the Euler class and Eisenstein cohomology of  $GL_N(\mathbb{Z})$ ”. In: *Japanese Journal of Mathematics* 15.2 (2020), pp. 311–379.
- [BCG23] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. *Cocycles de groupe pour  $GL_n$  et arrangements d’hyperplans*. Vol. 39. CRM Monograph Series. American Mathematical Society, Providence, RI, 2023, p. 127.
- [Bor95] Richard E. Borcherds. “Automorphic forms on  $O_{s+2,2}(\mathbf{R})$  and infinite products”. In: *Invent. Math.* 120.1 (1995), pp. 161–213.
- [Bor98] Richard E. Borcherds. “Automorphic forms with singularities on Grassmannians”. In: *Invent. Math.* 132.3 (1998), pp. 491–562.
- [Bra25] Romain Branchereau. *Eisenstein classes and generating series of modular symbols in  $\mathrm{SL}_N$* . 2025. arXiv: [2411.08690 \[math.NT\]](https://arxiv.org/abs/2411.08690).
- [Bru02] Jan H. Bruinier. *Borcherds products on  $O(2, l)$  and Chern classes of Heegner divisors*. Vol. 1780. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. viii+152.
- [BLY25] Jan H. Bruinier, Yingkun Li, and Tonghai Yang. “Deformations of Theta Integrals and A Conjecture of Gross-Zagier”. In: *Forum of Mathematics, Sigma* 13 (2025). DOI: [10.1017/fms.2024.139](https://doi.org/10.1017/fms.2024.139).
- [BOR08] Jan H. Bruinier, Ken Ono, and Robert C. Rhoades. “Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues”. In: *Math. Ann.* 342.3 (2008), pp. 673–693.
- [Bru99] Jan Hendrik Bruinier. “Borcherds products and Chern classes of Hirzebruch-Zagier divisors”. In: *Invent. Math.* 138.1 (1999), pp. 51–83.
- [Bru12] Jan Hendrik Bruinier. “Regularized theta lifts for orthogonal groups over totally real fields”. In: *J. Reine Angew. Math.* 672 (2012).
- [BEY21] Jan Hendrik Bruinier, Stephan Ehlen, and Tonghai Yang. “CM values of higher automorphic Green functions for orthogonal groups”. In: *Invent. Math.* 225.3 (2021), pp. 693–785.
- [BF04] Jan Hendrik Bruinier and Jens Funke. “On two geometric theta lifts”. In: *Duke Mathematical Journal* 125.1 (2004), pp. 45–90.
- [BKY12] Jan Hendrik Bruinier, Stephen S. Kudla, and Tonghai Yang. “Special values of Green functions at big CM points”. In: *Int. Math. Res. Not. IMRN* 9 (2012), pp. 1917–1967.
- [BS21] Jan Hendrik Bruinier and Markus Schwagenscheidt. “Theta lifts for Lorentzian lattices and coefficients of mock theta functions”. In: *Math. Z.* 297.3-4 (2021), pp. 1633–1657. ISSN: 0025-5874. DOI: [10.1007/s00209-020-02572-y](https://doi.org/10.1007/s00209-020-02572-y). URL: <https://doi.org/10.1007/s00209-020-02572-y>.
- [BY09] Jan Hendrik Bruinier and Tonghai Yang. “Faltings heights of CM cycles and derivatives of  $L$ -functions”. In: *Invent. Math.* 177.3 (2009), pp. 631–681.
- [CS85] Jeff Cheeger and James Simons. “Differential characters and geometric invariants”. In: *Geometry and topology (College Park, Md., 1983/84)*. Vol. 1167. Lecture Notes in Math. Springer, Berlin, 1985, pp. 50–80.
- [CF23] Jonathan Crawford and Jens Funke. “The Shimura-Shintani correspondence via singular theta lifts and currents”. In: *Int. J. Number Theory* 19.10 (2023), pp. 2383–2417.
- [Ehl17] Stephan Ehlen. “Errata for Stephan Ehlen, “CM values of regularized theta lifts and harmonic weak Maassforms of weight 1,” [MR3703434]”. In: *Duke Math. J.* 166.14 (2017).
- [ES18] Stephan Ehlen and Siddarth Sankaran. “On two arithmetic theta lifts”. In: *Compos. Math.* 154.10 (2018), pp. 2090–2149.
- [FH21] Jens Funke and Eric Hofmann. “The construction of Green currents and singular theta lifts for unitary groups”. In: *Trans. Amer. Math. Soc.* 374.4 (2021), pp. 2909–2947.
- [Gar16] Luis E. Garcia. “Regularized theta lifts and  $(1,1)$ -currents on GSpin Shimura varieties”. In: *Algebra Number Theory* 10.3 (2016), pp. 597–644.

- [GS19] Luis E. Garcia and Siddarth Sankaran. “Green forms and the arithmetic Siegel-Weil formula”. In: *Invent. Math.* 215.3 (2019), pp. 863–975.
- [GS89] Henri Gillet and Christophe Soulé. “Arithmetic Chow Groups and Differential Characters”. In: *Algebraic K-Theory: Connections with Geometry and Topology*. Dordrecht: Springer Netherlands, 1989, pp. 29–68.
- [GKZ87] B. Gross, W. Kohnen, and D. Zagier. “Heegner points and derivatives of L-series. II”. In: *Math. Ann.* 278.1-4 (1987), pp. 497–562.
- [GZ85] Benedict H. Gross and Don B. Zagier. “On singular moduli.” In: *Journal für die reine und angewandte Mathematik* 355 (1985), pp. 191–220.
- [GZ86] Benedict H. Gross and Don B. Zagier. “Heegner points and derivatives of L-series”. In: *Invent. Math.* 84.2 (1986), pp. 225–320.
- [Her+24] Sebastián Herrero et al. *Special values of Green’s functions on hyperbolic 3-space*. 2024. URL: <https://arxiv.org/abs/2405.01219>.
- [Kud97] Stephen S. Kudla. “Central Derivatives of Eisenstein Series and Height Pairings”. In: *Annals of Mathematics* 146.3 (1997), pp. 545–646.
- [Kud03] Stephen S. Kudla. “Integrals of Borcherds Forms”. In: *Compositio Mathematica* 137.3 (2003), 293–349.
- [KM86] Stephen S. Kudla and John J. Millson. “The theta correspondence and harmonic forms. I”. In: *Mathematische Annalen* 274.3 (1986), pp. 353–378.
- [KM87] Stephen S. Kudla and John J. Millson. “The theta correspondence and harmonic forms. II”. In: *Mathematische Annalen* 277.2 (1987), pp. 267–314.
- [KM90] Stephen S. Kudla and John J. Millson. “Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables”. In: *Publications Mathématiques de l’IHÉS* 71 (1990), pp. 121–172.
- [KRY99] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. “On the derivative of an eisenstein series of weight one”. In: *International Mathematics Research Notices* 1999.7 (Jan. 1999), pp. 347–385.
- [Lan87] Serge Lang. *Elliptic functions*. Second. Vol. 112. Graduate Texts in Mathematics. With an appendix by J. Tate. Springer-Verlag, New York, 1987, pp. xii+326. ISBN: 0-387-96508-4. DOI: [10.1007/978-1-4612-4752-4](https://doi.org/10.1007/978-1-4612-4752-4). URL: <https://doi.org/10.1007/978-1-4612-4752-4>.
- [Li22] Yingkun Li. “Average CM-values of higher Green’s function and factorization”. In: *Amer. J. Math.* 144.5 (2022), pp. 1241–1298.
- [Li23] Yingkun Li. “Algebraicity of higher Green functions at a CM point”. In: *Invent. Math.* 234.1 (2023), pp. 375–418.
- [LS22] Steffen Löbrich and Markus Schwagenscheidt. “Locally harmonic Maass forms and periods of meromorphic modular forms”. In: *Trans. Amer. Math. Soc.* 375.1 (2022), pp. 501–524.
- [MQ86] Varghese Mathai and Daniel Quillen. “Superconnections, thom classes, and equivariant differential forms”. In: *Topology* 25.1 (1986), pp. 85–110.
- [Neu99] Jürgen Neukirch. *Algebraic number theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.
- [Sch09] Jarad Schofer. “Borcherds forms and generalizations of singular moduli”. In: *Journal für die reine und angewandte Mathematik* 2009.629 (2009), pp. 1–36.
- [Via15] Maryna Viazovska. “CM values of higher Green’s functions and regularized Petersson products”. In: *Arithmetic and geometry*. Vol. 420. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2015, pp. 493–503.