The Kudla-Millson form via the Mathai-Quillen formalism

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Abstract

In [6], Kudla and Millson constructed a q-form φ_{KM} on an orthogonal symmetric space using Howe's differential operators. It is a crucial ingredient in their theory of theta lifting. This form can be seen as a Thom form of a real oriented vector bundle. In [9], Mathai and Quillen constructed a *canonical* Thom form and we show how to recover the Kudla-Millson form via their construction. A similar result was obtained by Garcia in [3] for signature $(2,q)$, in case the symmetric space is hermitian and we extend it to an arbitrary signature.

Contents

1 Introduction

Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature (p,q) and let G be its orthogonal group. Let $\mathbb{D}$ be the space of *oriented* negative q-planes in $V(\mathbb{R})$ and $\mathbb{D}^+$ one of its connected components. It is a Riemannian manifold of dimension pq and an open subset of the Grassmannian. The Lie group $G(\mathbb{R})^+$ is the connected component of the identity and acts transitively on $\mathbb{D}^+$. Hence we can identify $\mathbb{D}^+$ with $G(\mathbb{R})^+/K$, where K is a compact subgroup of $G(\mathbb{R})^+$ and is isomorphic to $\mathrm{SO}(p)\times\mathrm{SO}(q)$. Moreover let L be a lattice in $V(\mathbb{Q})$ and Γ be a torsion free subgroup of $G(\mathbb{R})^+$ preserving L.

For every vector v in $V(\mathbb{R})$ there is a totally geodesic submanifold $\mathbb{D}_v^+$ of codimension q consisting of all the negative q-planes that are orthogonal to v. Let Γ_v denote the stabilizer of v in Γ. We can view $\Gamma_v\backslash\mathbb{D}^+$ as a rank q vector bundle over $\Gamma_v\backslash\mathbb{D}_v^+$, so that the natural embedding $\Gamma_v\backslash\mathbb{D}_v^+$ in $\Gamma_v\backslash\mathbb{D}^+$ is the zero section. In [6], Kudla and Millson constructed a closed $G(\mathbb{R})^+$-invariant differential form

$$\varphi_{KM}\in\left[\Omega^q(\mathbb{D}^+)\otimes\mathscr{S}(V(\mathbb{R}))\right]^{G(\mathbb{R})^+}, \tag{1.1}$$

where $G(\mathbb{R})^+$ acts on the Schwartz space $\mathscr{S}(V(\mathbb{R}))$ from the left by $(gf)(v)\coloneqq f(g^{-1}v)$ and on $\Omega^q(\mathbb{D}^+)\otimes\mathscr{S}(V(\mathbb{R}))$ from the right by $g\cdot(\omega\otimes f)\coloneqq g^*\omega\otimes(g^{-1}f)$. In particular $\varphi_{KM}(v)$ is a Γ_v-invariant form on

$\mathbb{D}^+$. The main property of the Kudla-Millson form is its Thom form property: if ω in $\Omega_c^{pq-q}(\Gamma_v\backslash\mathbb{D}^+)$ is a compactly supported form, then

$$\int_{\Gamma_v\backslash\mathbb{D}^+} \varphi_{KM}(v)\wedge\omega = 2^{-\frac{q}{2}}e^{-\pi Q(v,v)}\int_{\Gamma_v\backslash\mathbb{D}_v^+}\omega. \tag{1.2}$$

Another way to state it is to say that in cohomology we have

$$[\varphi_{KM}(v)] = 2^{-\frac{q}{2}}e^{-\pi Q(v,v)}\,\mathrm{PD}(\Gamma_v\backslash\mathbb{D}_v^+)\in H^q\left(\Gamma_v\backslash\mathbb{D}^+\right), \tag{1.3}$$

where $\mathrm{PD}(\Gamma_v\backslash\mathbb{D}_v^+)$ denotes the Poincaré dual class to $\Gamma_v\backslash\mathbb{D}_v^+$.

Kudla-Millson theta lift. In order to motivate the interest in the Kudla-Millson form, let us briefly recall how it is used to construct a theta correspondence between certain cohomology classes and modular forms. Let ω be the Weil representation of $\mathrm{SL}_2(\mathbb{R})$ in $\mathscr{S}(V(\mathbb{R}))$. We extend it to a representation in $\Omega^q(\mathbb{D}^+)\otimes\mathscr{S}(V(\mathbb{R}))$ by acting in the second factor of the tensor product. Building on the work of [10], Kudla and Millson [7, 8] used their differential form to construct the theta series

$$\Theta_{KM}(\tau) \coloneqq y^{-\frac{p+q}{4}}\sum_{v\in L}\Big(\omega(g_\tau)\varphi_{KM}\Big)(v)\in\Omega^q(\mathbb{D}^+), \tag{1.4}$$

where $\tau = x+iy$ is in $\mathbb{H}$ and g_τ is the matrix $\begin{pmatrix}\sqrt{y} & x\sqrt{y}^{-1}\\ 0 & \sqrt{y}^{-1}\end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$ that sends i to τ by Möbius transformation. This form is Γ-invariant, closed and holomorphic in cohomology. Kudla and Millson showed that if we integrate this closed form on a *compact* q-cycle C in $\mathcal{Z}_q(\Gamma\backslash\mathbb{D}^+)$, then

$$\int_C\Theta_{KM}(\tau) = c_0(C)+\sum_{n=1}^{\infty}\langle C, C_{2n}\rangle e^{2i\pi n\tau} \tag{1.5}$$

is a modular form of weight $\frac{p+q}{2}$, where

$$C_n \coloneqq \sum_{\substack{v\in\Gamma\backslash L\\ Q(v,v)=n}} C_v \tag{1.6}$$

and the *special cycles* C_v are the images of the composition

$$\Gamma_v\backslash\mathbb{D}_v^+ \hookrightarrow \Gamma_v\backslash\mathbb{D}^+ \longrightarrow \Gamma\backslash\mathbb{D}^+. \tag{1.7}$$

Thus, the Kudla-Millson theta series realizes a lift between the (co)-homology of $\Gamma\backslash\mathbb{D}^+$ and the space of weight $\frac{p+q}{2}$ modular forms.

The result. Let E be a $G(\mathbb{R})^+$-equivariant vector bundle of rank q over $\mathbb{D}^+$ and E_0 the image of the zero section. By the equivariance we also have a vector bundle $\Gamma_v\backslash E$ over $\Gamma_v\backslash\mathbb{D}^+$. The *Thom class* of the vector bundle is a characteristic class $\mathrm{Th}(\Gamma_v\backslash E)$ in $H^q(\Gamma_v\backslash E,\Gamma_v\backslash(E-E_0))$ defined by the Thom isomorphism; see Subsection 3.6. A *Thom form* is a form representing the Thom class. It can be shown that the Thom class is also the Poincaré dual class to $\Gamma_v\backslash E_0$. Let $s_v\colon\Gamma_v\backslash\mathbb{D}^+\longrightarrow\Gamma_v\backslash E$ be a section whose zero locus is $\Gamma_v\backslash\mathbb{D}_v^+$, then

$$s_v^*\,\mathrm{Th}(\Gamma_v\backslash E)\in H^q\left(\Gamma_v\backslash\mathbb{D}^+,\Gamma_v\backslash(\mathbb{D}^+-\mathbb{D}_v^+)\right). \tag{1.8}$$

Viewing it as a class in $H^q(\Gamma_v\backslash\mathbb{D}^+)$ it is the Poincaré dual class of $\Gamma_v\backslash\mathbb{D}_v^+$. Since the Poincaré dual class is unique, property (1.3) implies that

$$[\varphi_{KM}(v)] = 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} s_v^* \operatorname{Th}(\Gamma_v\backslash E) \in H^q\left(\Gamma_v\backslash\mathbb{D}^+\right), \tag{1.9}$$

on the level of cohomology.

For arbitrary oriented real metric vector bundles, Mathai and Quillen used the Chern-Weil theory to construct in [9] a canonical Thom forms on E. We denote by U_{MQ} the canonical Thom form in $\Omega^q(E)$ of Mathai and Quillen. Since U_{MQ} is Γ-invariant, it is also a Thom form for the bundle $\Gamma_v\backslash E$ for every vector v. The main result is the following.

Theorem. *(Theorem 4.5) We have* $\varphi_{KM}(v) = 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} s_v^* U_{MQ}$ *in* $\Omega^q(\Gamma_v\backslash\mathbb{D}^+)$

For signature $(2,q)$, the spaces are hermitian and the result was obtained by a similar method in [3] using the work of Bismut-Gillet-Soulé.

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2 The Kudla-Millson form

2.1 The symmetric space $\mathbb{D}$

Let (V,Q) be a rational quadratic space and let (p,q) be the signature of $V(\mathbb{R})$. Let $e_1,\ldots,e_{p+q}$ be an orthogonal basis of $V(\mathbb{R})$ such that

$$\begin{aligned} Q(e_\alpha,e_\alpha) &= 1 \quad \text{for} \quad 1\le\alpha\le p, \\ Q(e_\mu,e_\mu) &= -1 \quad \text{for} \quad p+1\le\mu\le p+q. \end{aligned} \tag{2.1}$$

Note that we will always use letters α and β for indices between 1 and p, and letters μ and ν for indices between $p+1$ and $p+q$. A plane z in $V(\mathbb{R})$ is *a negative plane* if $Q\big|_z$ is negative definite. Let

$$\mathbb{D} \coloneqq \{z\subset V(\mathbb{R}) \mid\ z \text{ is an oriented negative plane of dimension } q\} \tag{2.2}$$

be the set of negative oriented q-planes in $V(\mathbb{R})$. For each negative plane there are two possible orientations, yielding two connected components $\mathbb{D}^+$ and $\mathbb{D}^-$ of $\mathbb{D}$. Let z_0 in $\mathbb{D}^+$ be the negative plane spanned by the vectors $e_{p+1},\ldots,e_{p+q}$ together with a fixed orientation. The group $G(\mathbb{R})^+$ acts transitively on $\mathbb{D}^+$ by sending z_0 to gz_0. Let K be the stabilizer of z_0, which is isomorphic to $\mathrm{SO}(p)\times\mathrm{SO}(q)$. Thus we have an identification

$$\begin{aligned} G(\mathbb{R})^+/K &\longrightarrow \mathbb{D}^+ \\ gK &\longmapsto gz_0. \end{aligned} \tag{2.3}$$

For z in $\mathbb{D}^+$ we denote by g_z any element of $G(\mathbb{R})^+$ sending z_0 to z.

For a positive vector v in $V(\mathbb{R})$ we define

$$\mathbb{D}_v := \left\{ z \in \mathbb{D} \mid z \subset v^\perp \right\}. \tag{2.4}$$

It is a totally geodesic submanifold of $\mathbb{D}$ of codimension q. Let $\mathbb{D}_v^+$ be the intersection of $\mathbb{D}_v$ with $\mathbb{D}^+$.

Let z in $\mathbb{D}^+$ be a negative plane. With respect to the orthogonal splitting of $V(\mathbb{R})$ as $z^\perp \oplus z$ the quadratic form splits as

$$Q(v,v) = Q\big|_{z^\perp}(v,v) + Q\big|_{z}(v,v). \tag{2.5}$$

We define the *Siegel majorant at* z to be the positive definite quadratic form

$$Q_z^+(v,v) := Q\big|_{z^\perp}(v,v) - Q\big|_{z}(v,v). \tag{2.6}$$

2.2 The Lie algebras $\mathfrak{g}$ *and* $\mathfrak{k}$

Let

$$\mathfrak{g} := \left\{ \left(\begin{array}{cc} A & x \\ {}^t x & B \end{array} \right) \middle| A \in \mathfrak{so}(z_0^\perp),\ B \in \mathfrak{so}(z_0),\ x \in \operatorname{Hom}(z_0, z_0^\perp) \right\}, \tag{2.7}$$

$$\mathfrak{k} := \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \middle| A \in \mathfrak{so}(z_0^\perp),\ B \in \mathfrak{so}(z_0) \right\} \tag{2.8}$$

be the Lie algebras of $G(\mathbb{R})^+$ and K where $\mathfrak{so}(z_0)$ is equal to $\mathfrak{so}(q)$. The latter is the space of skew-symmetric q by q matrices. Similarly we have $\mathfrak{so}(z_0^\perp)$ equals $\mathfrak{so}(p)$. Hence we have a decomposition of $\mathfrak{k}$ as $\mathfrak{so}(z_0^\perp) \oplus \mathfrak{so}(z_0)$ that is orthogonal with respect to the Killing form. Let ϵ be the Lie algebra involution of $\mathfrak{g}$ mapping X to $-\,{}^tX$. The $+1$-eigenspace of ϵ is $\mathfrak{k}$ and the -1-eigenspace is

$$\mathfrak{p} := \left\{ \left(\begin{array}{cc} 0 & x \\ {}^t x & 0 \end{array} \right) \middle| x \in \operatorname{Hom}(z_0, z_0^\perp) \right\}. \tag{2.9}$$

We have a decomposition of $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$ and it is orthogonal with respect to the Killing form. We can identify $\mathfrak{p}$ with $\mathfrak{g}/\mathfrak{k}$. Since ϵ is a Lie algebra automorphism we have that

$$[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}, \qquad\qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}. \tag{2.10}$$

We identify the tangent space of $\mathbb{D}^+$ at eK with $\mathfrak{p}$ and the tangent bundle $T\mathbb{D}^+$ with $G(\mathbb{R})^+ \times_K \mathfrak{p}$ where K acts on $\mathfrak{p}$ by the Ad-representation. We have an isomorphism

$$\begin{aligned} T\colon\ \wedge^2 V(\mathbb{R}) &\longrightarrow \mathfrak{g} \\ e_i \wedge e_j &\longmapsto T(e_i \wedge e_j)e_k := Q(e_i,e_k)e_j - Q(e_j,e_k)e_i. \end{aligned} \tag{2.11}$$

A basis of $\mathfrak{g}$ is given by the set of matrices

$$\{\, X_{ij} := T(e_i \wedge e_j) \in \mathfrak{g} |\, 1 < i < j < p+q \} \tag{2.12}$$

and we denote by ω_{ij} its dual basis in the dual space $\mathfrak{g}^*$. Let E_{ij} be the elementary matrix sending e_i to e_j and the other e_k's to 0. Then $\mathfrak{p}$ is spanned by the matrices

$$X_{\alpha\mu} = E_{\alpha\mu} + E_{\mu\alpha} \tag{2.13}$$

and $\mathfrak{k}$ is spanned by the matrices

$$\begin{aligned} X_{\alpha\beta} &= E_{\alpha\beta} - E_{\beta\alpha}, \\ X_{\nu\mu} &= -E_{\nu\mu} + E_{\mu\nu}. \end{aligned} \tag{2.14}$$

2.3 Poincaré duals

Let M be an arbitrary m-dimensional real orientable manifold without boundary. The integration map yields a non-degenerate pairing [2, Theorem. 5.11]

$$\begin{aligned} H^q(M) \otimes_{\mathbb{R}} H_c^{m-q}(M) &\longrightarrow \mathbb{R} \\ [\omega] \otimes [\eta] &\longmapsto \int_M \omega \wedge \eta, \end{aligned} \tag{2.15}$$

where $H_c(M)$ denotes the cohomology of compactly supported forms on M. This yields an isomorphism between $H^q(M)$ and the dual $H_c^{m-q}(M)^* = \mathrm{Hom}(H_c^{m-q}(M), \mathbb{R})$. If C is an immersed submanifold of codimension q in M then C defines a linear functional on $H_c^{m-q}(M)$ by

$$\omega \longmapsto \int_C \omega. \tag{2.16}$$

Since we have an isomorphism between $H_c^{m-q}(M)^*$ and $H^q(M)$ there is a unique cohomology class $\mathrm{PD}(C)$ in $H^q(M)$ representing this functional *i.e.*

$$\int_M \omega \wedge \mathrm{PD}(C) = \int_C \omega \tag{2.17}$$

for every class $[\omega]$ in $H_c^{m-q}(M)$. We call $\mathrm{PD}(C)$ *the Poincaré dual class to* C, and any differential form representing the cohomology class $\mathrm{PD}(C)$ *a Poincaré dual form to* C.

2.4 The Kudla-Millson form

The tangent plane at the identity $T_{eK}\mathbb{D}^+$ can be identified with $\mathfrak{p}$ and the cotangent bundle $(T\mathbb{D}^+)^*$ with $G(\mathbb{R})^+ \times_K \mathfrak{p}^*$, where K acts on $\mathfrak{p}^*$ by the dual of the Ad-representation. The basis $e_1, \dots, e_{p+q}$ identifies $V(\mathbb{R})$ with $\mathbb{R}^{p+q}$. With respect to this basis the Siegel majorant at z_0 is given by

$$Q_{z_0}^+(v,v) \coloneqq \sum_{i=1}^{p+q} x_i^2. \tag{2.18}$$

Recall that $G(\mathbb{R})^+$ acts on $\mathscr{S}(\mathbb{R}^{p+q})$ from the left by $(g \cdot f)(v) = f(g^{-1}v)$ and on $\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(\mathbb{R}^{p+q})$ from the right by $g \cdot (\omega \otimes f) \coloneqq g^*\omega \otimes (g^{-1}f)$. We have an isomorphism

$$\begin{aligned} \left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+} &\longrightarrow \left[\bigwedge^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^K \\ \varphi &\longrightarrow \varphi_e \end{aligned} \tag{2.19}$$

by evaluating φ at the basepoint eK in $G(\mathbb{R})^+/K$, corresponding to the point z_0 in $\mathbb{D}^+$. We define the *Howe operator*

$$D\colon \bigwedge^\bullet \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}) \longrightarrow \bigwedge^{\bullet+q} \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}) \tag{2.20}$$

by

$$D \coloneqq \frac{1}{2^q} \prod_{\mu=p+1}^{p+q} \sum_{\alpha=1}^{p} A_{\alpha\mu} \otimes \left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) \tag{2.21}$$

where $A_{\alpha\mu}$ denotes left multiplication by $\omega_{\alpha\mu}$. The Kudla-Millson form is defined by applying D to the Gaussian:

$$\varphi_{KM}(v)_e \coloneqq D \exp\left(-\pi Q_{z_0}^+(v,v)\right) \in \bigwedge^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}). \tag{2.22}$$

Kudla and Millson showed that this form is K-invariant. Hence by the isomorphism (2.19) we get a form

$$\varphi_{KM} \in \left[\Omega^q(\mathbb{D}^+) \otimes \mathscr{S}(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+}. \tag{2.23}$$

In particular it is Γ_v-invariant and defines a form on $\Gamma_v \backslash \mathbb{D}^+$. It is also closed and satisfies the Thom form property: for every compactly supported form ω in $\Omega_c^{pq-q}(\Gamma_v \backslash \mathbb{D}^+)$ we have

$$\int_{\Gamma_v \backslash \mathbb{D}^+} \omega \wedge \varphi_{KM}(v) = 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} \int_{\Gamma_v \backslash \mathbb{D}_v^+} \omega. \tag{2.24}$$

3 The Mathai-Quillen formalism

We begin by recalling a few facts about principal bundles, connections and associated vector bundles. For more details we refer to [1] and [5]. The Mathai-Quillen form is defined in Subsection 3.7 following [1]; see also [4].

3.1 K-principal bundles and principal connections

Let K be $\mathrm{SO}(p) \times \mathrm{SO}(q)$ as before and P be a smooth principal K-bundle. Let

$$\begin{aligned} R\colon K \times P &\longrightarrow P \\ (k,p) &\longmapsto R_k(p) \end{aligned} \tag{3.1}$$

be the smooth right action of K on P and

$$\pi\colon P \longrightarrow P/K \tag{3.2}$$

the projection map. For a fixed p in P consider the map

$$\begin{aligned} R_p\colon K &\longrightarrow P \\ k &\longmapsto R_k(p). \end{aligned} \tag{3.3}$$

Let V_pP be the image of the derivative at the identity

$$d_eR_p\colon \mathfrak{k} \longrightarrow T_pP, \tag{3.4}$$

which is injective. It coincides with the kernel of the differential $d_p\pi$. A vector in V_pP is called a *vertical vector*. Using this map we can view a vector X in $\mathfrak{k}$ as a vertical vector field on P. The space P can a priori be arbitrary, but in our case we will consider either

1. P is $G(\mathbb{R})^+$ and R_k the natural right action sending g to gk. Then P/K can be identified with $\mathbb{D}^+$,

2. P is $G(\mathbb{R})^+ \times z_0$ and the action R_k maps (g,w) to $(gk, k^{-1}w)$. In this case P/K can be identified with $G(\mathbb{R})^+ \times_K z_0$. It is the vector bundle associated to the principal bundle $G(\mathbb{R})^+$ as defined below.

A *principal K-connection* on P is a 1-form θ_P in $\Omega^1(P, \mathfrak{k})$ such that

- $\iota_X \theta_P = X \quad$ for any X in $\mathfrak{k}$,
- $R_k^* \theta_P = Ad(k^{-1})\theta_P \quad$ for any k in K,

where ι_X is the interior product

$$\begin{aligned} \iota_X \colon \Omega^k(P) &\longrightarrow \Omega^k(P) \\ \omega &\longmapsto (\iota_X\omega)(X_1, \dots, X_{p-1}) := \omega(X, X_1, \dots, X_{p-1}). \end{aligned} \tag{3.5}$$

and we view X as a vector field on P. Geometrically these conditions imply that the kernel of θ_P defines a horizontal subspace of TP that we denote by HP. It is a complement to the vertical subspace *i.e.* we get a splitting of T_pP as $V_pP \oplus H_pP$.

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})^+$ and let p be the orthogonal projection from $\mathfrak{g}$ on $\mathfrak{k}$. After identifying $\mathfrak{g}^*$ with the space $\Omega^1(G(\mathbb{R})^+)^{G(\mathbb{R})^+}$ of $G(\mathbb{R})^+$-invariant forms we define a natural 1-form

$$\sum_{1 \le i < j \le p+q} \omega_{ij} \otimes X_{ij} \in \Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{g} \tag{3.6}$$

called the *Maurer-Cartan form*, where X_{ij} is the basis of $\mathfrak{g}$ defined earlier and ω_{ij} its dual in $\mathfrak{g}^*$. After projection onto $\mathfrak{k}$ we get a form

$$\theta := p\left(\sum_{1 \le i < j \le p+q} \omega_{ij} \otimes X_{ij}\right) \in \Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{k} \tag{3.7}$$

where we identify $\Omega^1(G(\mathbb{R})^+, \mathfrak{k})$ with $\Omega^1(G(\mathbb{R})^+) \otimes \mathfrak{k}$. A direct computation shows that it is a principal K-connection on P when P is $G(\mathbb{R})^+$.

If P is $G(\mathbb{R})^+ \times z_0$ then the projection

$$\pi \colon G(\mathbb{R})^+ \times z_0 \longrightarrow G(\mathbb{R})^+ \tag{3.8}$$

induces a pullback map

$$\pi^* \colon \Omega^1(G(\mathbb{R})^+) \longrightarrow \Omega^1(G(\mathbb{R})^+ \times z_0). \tag{3.9}$$

The form

$$\widetilde{\theta} := \pi^*\theta \in \Omega^1(G(\mathbb{R})^+ \times z_0) \otimes \mathfrak{k} \tag{3.10}$$

is a principal connection on $G(\mathbb{R})^+ \times z_0$.

3.2 The associated vector bundles

Since z_0 is preserved by K we have an orthogonal K-representation

$$\begin{aligned} \rho \colon K &\longrightarrow \mathrm{SO}(z_0) \\ k &\longmapsto \rho(k)w := k\big|_{z_0} w, \end{aligned} \tag{3.11}$$

where we will usually simply write kw instead of $k\big|_{z_0} w$. We can consider the *associated vector bundle* $P \times_K z_0$ which is the quotient of $P \times z_0$ by K, where K acts by sending (p, w) to $(R_k(p), \rho(k)^{-1}w)$. Hence an element $[p, w]$ of $P \times_K z_0$ is an equivalence class where the equivalence relation identifies (p, w) with $(R_k(p), \rho(k)^{-1}w)$. This is a vector bundle over P/K with projection map sending $[p, w]$ to $\pi(p)$. Let $\Omega^i(P/K, P \times_K z_0)$ be the space of i-forms valued in $P \times_K z_0$, when i is zero it is the space of smooth sections of the associated bundle.

In the two cases of interest to us we define

$$\begin{aligned} E &:= G(\mathbb{R})^+ \times_K z_0, \\ \widetilde{E} &:= \big(G(\mathbb{R})^+ \times z_0\big) \times_K z_0. \end{aligned} \tag{3.12}$$

Note that in both cases P admits a left action of $G(\mathbb{R})^+$ and that the associated vector bundles are $G(\mathbb{R})^+$-equivariant. Morever it is a Euclidean bundle, equipped with the inner product

$$\langle v, w \rangle := -Q\big|_{z_0}(v, w) \tag{3.13}$$

on the fiber. Let $\Omega^i(P, z_0)$ be the space of z_0-valued differential i-forms on P. A differential form α in $\Omega^i(P, z_0)$ is said to be *horizontal* if $\iota_X \alpha$ vanishes for all vertical vector fields X. There is a left action of K on a differential form α in $\Omega^i(P, z_0)$ defined by

$$k \cdot \alpha := \rho(k)(R_k^* \alpha), \tag{3.14}$$

and α is K-invariant if it satisfies $k \cdot \alpha = \alpha$ for any k in K *i.e.* we have $R_k^* \alpha = \rho(k^{-1})\alpha$. We write $\Omega^i(P, z_0)^K$ for the space of K-invariant z_0-valued forms on P. Finally a form that is horizontal and K-invariant is called a *basic form* and the space of such forms is denoted by $\Omega^i(P, z_0)_{bas}$.

Let $X_1, \dots, X_N$ be tangent vectors of P/K at $\pi(p)$ and $\widetilde{X}_i$ be tangent vectors of P at p that satisfy $d_p\pi(\widetilde{X}_i) = X_i$. There is a map

$$\begin{aligned} \Omega^i(P, z_0)_{bas} &\longrightarrow \Omega^i(P/K, P \times_K z_0) \\ \alpha &\longmapsto \omega_\alpha \end{aligned} \tag{3.15}$$

defined by

$$\omega_\alpha\big|_{\pi(p)}(X_1 \wedge \cdots \wedge X_N) = \alpha\big|_p(\widetilde{X}_1 \wedge \cdots \wedge \widetilde{X}_N). \tag{3.16}$$

Proposition 3.1. *The map is well-defined and yields an isomorphism between $\Omega^i(P/K, P \times_K z_0)$ and $\Omega^i(P, z_0)_{bas}$. In particular if z_0 is 1-dimensional then $\Omega^i(P/K)$ is isomorphic to $\Omega^i(P)_{bas}$.*

Proof. In the case where i is zero the horizontally condition is vacuous and the isomorphism simply identifies $\Omega^0(P/K, P \times_K z_0)$ with $\Omega^0(P, z_0)^K$. We have a map

$$\begin{aligned} \Omega^0(P, z_0)^K &\longrightarrow \Omega^0(P/K, P \times_K z_0) \\ f &\longmapsto s_f(\pi(p)) := [p, f(p)], \end{aligned} \tag{3.17}$$

which is well defined since

$$f(R_k(p)) = \rho(k)^{-1} f(p). \tag{3.18}$$

Conversely every smooth section s in $\Omega^0(P/K, P \times_K z_0)$ is given by

$$s(\pi(p)) = [p, f_s(p)] \tag{3.19}$$

for some smooth function f_s in $\Omega^0(P, z_0)^K$. The map sending s to f_s is inverse to the previous one. The proof is similar for positive i. □

3.3 Covariant derivatives

A *covariant derivative* on the vector bundle $P \times_K z_0$ is a differential operator

$$\nabla_P \colon \Omega^0(P/K, P \times_K z_0) \longrightarrow \Omega^1(P/K, P \times_K z_0) \tag{3.20}$$

such that for every smooth function f in $C^\infty(P/K)$ we have

$$\nabla_P(fs) = df \otimes s + f\nabla_P(s). \tag{3.21}$$

The inner product on $P \times_K z_0$ defines a pairing

$$\begin{aligned} \Omega^i(P/K, P \times_K z_0) \times \Omega^j(P/K, P \times_K z_0) &\longrightarrow \Omega^{i+j}(P/K) \\ (\omega_1 \otimes s_1, \omega_2 \otimes s_2) &\longmapsto \langle \omega_1 \otimes s_1, \omega_2 \otimes s_2 \rangle = \omega_1 \wedge \omega_2 \langle s_1, s_2 \rangle, \end{aligned} \tag{3.22}$$

and we say that the derivative is compatible with the metric if

$$d\langle s_1, s_2 \rangle = \langle \nabla_P s_1, s_2 \rangle + \langle s_1, \nabla_P s_2 \rangle \tag{3.23}$$

for any two sections s_1 and s_2 in $\Omega^0(P/K, P \times_K z_0)$. There is a covariant derivative that is induced by a principal connection θ_P in $\Omega^1(P) \otimes \mathfrak{k}$ as follows. The derivative of the representation gives a map

$$d\rho \colon \mathfrak{k} \longrightarrow \mathfrak{so}(z_0) \subset \mathrm{End}(z_0), \tag{3.24}$$

which we also denote by ρ by abuse of notation. Note that for the representation (3.11) this is simply the map

$$\begin{aligned} \rho \colon \mathfrak{k} &\longrightarrow \mathfrak{so}(z_0) \\ X &\longmapsto X\big|_{z_0} \end{aligned} \tag{3.25}$$

since $\mathfrak{k}$ splits as $\mathfrak{so}(z_0^\perp) \oplus \mathfrak{so}(z_0)$. Composing the principal connection with ρ defines an element

$$\rho(\theta_P) \in \Omega^1(P, \mathfrak{so}(z_0)). \tag{3.26}$$

In particular, if s is a section of $P \times_K z_0$ then we can identify it with a K-invariant smooth map f_s in $\Omega^0(P, z_0)^K$. Since $\rho(\theta_P)$ is a $\mathfrak{so}(z_0)$-valued form and $\mathfrak{so}(z_0)$ is a subspace of $\mathrm{End}(z_0)$ we can define

$$df_s + \rho(\theta_P) \cdot f_s \in \Omega^1(P, z_0). \tag{3.27}$$

Lemma 3.2. *The form $df_s + \rho(\theta_P) \cdot f_s$ is basic, hence gives a $P \times_K z_0$-valued form on P/K. Thus $d + \rho(\theta_P)$ defines a covariant derivative on $P \times_K z_0$. Moreover, it is compatible with the metric.*

Proof. See [1, p. 24]. For the compatibility with the metric, it follows from the fact that the connection $\rho(\theta_P)$ is valued in $\mathfrak{so}(z_0)$ that

$$\langle \rho(\theta_P) f_{s_1}, f_{s_2}\rangle + \langle f_{s_1}, \rho(\theta_P) f_{s_2}\rangle = 0. \tag{3.28}$$

Hence if we denote by ∇_P is the covariant derivative defined by $d + \rho(\theta_P)$ then

$$\langle \nabla_P s_1, s_2\rangle + \langle s_1, \nabla_P s_2\rangle = \langle df_{s_1}, f_{s_2}\rangle + \langle f_{s_1}, df_{s_2}\rangle = d\langle f_{s_1}, f_{s_2}\rangle = d\langle s_1, s_2\rangle. \tag{3.29}$$

□

Let us denote by ∇_P the covariant derivative $d + \rho(\theta_P)$. It can be extended to a map

$$\nabla_P \colon \Omega^i(P/K, P \times_K z_0) \longrightarrow \Omega^{i+1}(P/K, P \times_K z_0) \tag{3.30}$$

by setting

$$\nabla_P(\omega \otimes s) \coloneqq d\omega \otimes s + (-1)^i \omega \wedge \nabla_P(s), \tag{3.31}$$

where

$$\omega \otimes s \in \Omega^i(P/K) \otimes \Omega^0(P/K, P \times_K z_0) \simeq \Omega^i(P/K, P \times_K z_0). \tag{3.32}$$

We define the *curvature* R_P in $\Omega^2(P, \mathfrak{k})$ by

$$R_P(X, Y) \coloneqq [\theta_P(X), \theta_P(Y)] - \theta_P([X, Y]) \tag{3.33}$$

for two vector fields X and Y on P. It is basic by [1, Proposition. 1.13] and composing with ρ gives an element

$$\rho(R_P) \in \Omega^2(P, \mathfrak{so}(z_0))_{bas}, \tag{3.34}$$

so that we can view it as an element in $\Omega^2(P/K, P \times_K \mathfrak{so}(z_0))$ where K acts on $\mathfrak{so}(z_0)$ by the Ad-representation. For a section s in $\Omega^0(P/K, P \times_K z_0)$ we have [1, Proposition. 1.15]

$$\nabla_P^2 s = \rho(R_p) s \in \Omega^2(P/K, P \times_K z_0). \tag{3.35}$$

From now on we denote by ∇ and $\widetilde{\nabla}$ the covariant derivatives on E and $\widetilde{E}$ associated to θ and $\widetilde{\theta}$ defined in (3.7) and (3.10). Let R and $\widetilde{R}$ be their respectives curvatures.

3.4 Pullback of bundles

The pullback of E by the projection map gives a canonical bundle

$$\pi^* E \coloneqq \{(e, e') \in E \times E \mid \pi(e) = \pi(e')\} \tag{3.36}$$

over E. We have the following diagram

$$\begin{tikzcd} \pi^* E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & \mathbb{D}^+. \end{tikzcd} \tag{3.37}$$

The projection induces a pullback of the sections

$$\pi^*\colon \Omega^i(\mathbb{D}, E) \longrightarrow \Omega^i(E, \widetilde{E}). \tag{3.38}$$

We can also pullback the covariant derivative ∇ to a covariant derivative

$$\pi^*\nabla\colon \Omega^0(E, \pi^*E) \longrightarrow \Omega^1(E, \pi^*E) \tag{3.39}$$

on π^*E. It is characterized by the property

$$(\pi^*\nabla)(\pi^*s) = \pi^*(\nabla s). \tag{3.40}$$

Proposition 3.3. *The bundles $\widetilde{E}$ and π^*E are isomorphic, and this isomorphism identifies $\widetilde{\nabla}$ and $\pi^*\nabla$.*

Proof. By definition $([g_1, w_1], [g_2, w_2])$ are elements of π^*E if and only if $g_1^{-1}g_2$ is in K. We have a $G(\mathbb{R})^+$-equivariant morphism

$$\begin{aligned} \pi^*E &\longrightarrow \widetilde{E} \\ ([g_1, w_1], [g_2, w_2]) &\longrightarrow [(g_1, g_1^{-1}g_2w_2), w_1]. \end{aligned} \tag{3.41}$$

This map is well defined and has as inverse

$$\begin{aligned} \widetilde{E} &\longrightarrow \pi^*E \\ [(g, w_1), w_2] &\longrightarrow ([g, w_2], [g, w_1]). \end{aligned} \tag{3.42}$$

The second statement follows from the fact that $\widetilde{\theta}$ is $\pi^*\theta$. □

3.5 A few operations on the vector bundles

We extend the K-representation z_0 to $\bigwedge^j z_0$ by

$$k(w_1 \wedge \cdots \wedge w_j) = (kw_1) \wedge \cdots \wedge (kw_j). \tag{3.43}$$

We consider the bundles $P\times_K \wedge^j z_0$ and $P\times_K \wedge z_0$ over P/K, where $\bigwedge z_0$ is defined as $\bigoplus_i \bigwedge^i z_0$. Denote the space of differential forms valued in $P \times_K \wedge^j z_0$ by

$$\Omega_P^{i,j} \coloneqq \Omega_P^i(P/K, P \times_K \wedge^j z_0) = \Omega_P^i(P/K) \otimes \Omega^0(P/K, P \times_K \wedge^j z_0). \tag{3.44}$$

The total space of differential forms

$$\Omega(P/K, P \times_K \wedge z_0) = \bigoplus_{i,j} \Omega_P^{i,j} \tag{3.45}$$

is an (associative) bigraded $C^\infty(P/K)$-algebra where the product is defined by

$$\begin{aligned} \wedge\colon \Omega_P^{i,j} \times \Omega_P^{k,l} &\longrightarrow \Omega_P^{i+k,j+l} \\ (\omega \otimes s, \eta \otimes t) &\longmapsto (\omega \otimes s) \wedge (\eta \otimes t) \coloneqq (-1)^{jk}(\omega \wedge \eta) \otimes (s \wedge t). \end{aligned} \tag{3.46}$$

This algebra structure allows us to define an *exponential map* by

$$\begin{aligned}\exp\colon \Omega(P/K, P\times_K \wedge z_0) &\longrightarrow \Omega(P/K, P\times_K \wedge z_0)\\ \omega &\longmapsto \exp(\omega) := \sum_{k\geq 0} \frac{\omega^k}{k!}\end{aligned} \tag{3.47}$$

where ω^k is the k-fold wedge product $\omega\wedge\cdots\wedge\omega$.

Remark 3.1. Suppose that ω and η commute. Then the binomial formula

$$(\omega+\eta)^k = \sum_{l=0}^{k} \binom{k}{l} \omega^l \eta^{k-l} \tag{3.48}$$

holds and one can show that $\exp(\omega+\eta) = \exp(\omega)+\exp(\eta)$ in the same way as for the real exponential map. In particular the diagonal subalgebra $\bigoplus \Omega_P^{i,i}$ is a commutative since for two forms ω and η in Ω_P we have

$$\omega\wedge\eta = (-1)^{\deg(\omega)+\deg(\eta)}\eta\wedge\omega \tag{3.49}$$

and similarly for two sections $s$and t in $\Omega^0(P/K, P\times_K z_0)$.

The inner product $\langle -,-\rangle$ on z_0 can be extended to an inner product on $\bigwedge z_0$ by

$$\langle\, v_1\wedge\cdots\wedge v_k \;,\; w_1\wedge\cdots\wedge w_l \,\rangle := \begin{cases} 0 & \text{if} \quad k\neq l,\\ \det\langle v_i, w_j\rangle_{i,j} & \text{if} \quad k=l.\end{cases} \tag{3.50}$$

If $e_1,\ldots,e_q$ is an orthonormal basis of z_0, then the set

$$\{e_{i_1}\wedge\cdots\wedge e_{i_k} \mid 1\leq k\leq q,\; i_1<i_2<\cdots<i_k\} \tag{3.51}$$

is an orthonormal basis of $\bigwedge z_0$. We define the *Berezin integral* $\int^B$ to be the orthogonal projection onto the top dimensional component, that is the map

$$\begin{aligned}\int^B\colon \bigwedge z_0 &\longrightarrow \mathbb{R}\\ w &\longmapsto \langle w \;,\; e_1\wedge\cdots\wedge e_q\rangle.\end{aligned} \tag{3.52}$$

The Berezin integral can then be extended to

$$\begin{aligned}\int^B\colon \Omega(P/K, P\times_K \wedge z_0) &\longrightarrow \Omega(P/K)\\ \omega\otimes s &\longmapsto \omega\int^B s\end{aligned} \tag{3.53}$$

where $\int^B s$ in $C^\infty(P/K)$ is the composition of the section with the Berezinian in every fiber. Let $s_1,\ldots,s_q$ be a local orthonormal frame of $P\times_K z_0$. Then $s_1\wedge\cdots\wedge s_q$ is in $\Omega^0(P/K, \wedge^q P\times_K z_0)$ and defines a global section. Hence for α in $\Omega(P/K, P\times_K\wedge z_0)$ we have

$$\int^B \alpha = \langle \alpha, s_1\wedge\cdots\wedge s_q\rangle. \tag{3.54}$$

Finally, for every section s in $\Omega^{0,1}$ we can define the *contraction*

$$\begin{aligned} i(s)\colon \Omega_P^{i,j} &\longrightarrow \Omega_P^{i,j-1} \\ \omega\otimes s_1\wedge\cdots\wedge s_j &\longmapsto \sum_{k=1}^{j}(-1)^{i+k-1}\langle s,s_k\rangle\omega\otimes s_1\wedge\cdots\wedge\widehat{s_k}\wedge\cdots\wedge s_j \end{aligned} \tag{3.55}$$

and extended by linearity, where the symbol $\widehat{\cdot}$ means that we remove it from the product. Note that when j is zero then $i(s)$ is defined to be zero. The contraction $i(s)$ defines a derivation on $\oplus\widetilde{\Omega}^{i,j}$ that satisfies

$$i(s)(\alpha\wedge\alpha') = (i(s)\alpha)\wedge\alpha' + (-1)^{i+j}\alpha\wedge(i(s)\alpha') \tag{3.56}$$

for α in $\widetilde{\Omega}^{i,j}$ and α' in $\widetilde{\Omega}^{k,l}$.

3.6 Thom forms

We denote by E the bundle $G(\mathbb{R})^+\times_K z_0$. On the fibers of the bundle we have the inner product given by $\langle w,w'\rangle \coloneqq -Q(w,w')$. Let v be arbitrary vector in L and Γ_v its stabilizer. Since the bundle is $G(\mathbb{R})^+$-equivariant we have a bundle

$$\Gamma_v\backslash E\longrightarrow \Gamma_v\backslash\mathbb{D}^+, \tag{3.57}$$

and let $\mathrm{D}(\Gamma_v\backslash E)$ be the closed disk bundle. If we have a closed $(q+i)$-form on $\Gamma_v\backslash E$ whose support is contained in $\mathrm{D}(\Gamma_v\backslash E)$, then it has compact support in the fiber and represents a class in $H^{q+i}(\Gamma_v\backslash E,\Gamma_v\backslash E-\mathrm{D}(\Gamma_v\backslash E))$. The cohomology group $H^\bullet(\Gamma_v\backslash E,\Gamma_v\backslash E-\mathrm{D}(\Gamma_v\backslash E))$ is equal to the cohomology group $H^\bullet(\Gamma_v\backslash E,\Gamma_v\backslash(E-E_0))$ that we used in the introduction, where E_0 is the zero section. Fiber integration induces an isomorphism on the level of cohomology

$$\begin{aligned} \mathrm{Th}\colon H^{q+i}(\Gamma_v\backslash E,\Gamma_v\backslash E-\mathrm{D}(\Gamma_v\backslash E)) &\longrightarrow H^i(\Gamma_v\backslash\mathbb{D}^+) \\ [\omega] &\longmapsto \int_{\text{fiber}}\omega \end{aligned} \tag{3.58}$$

known as the *Thom isomorphism* [2, Theorem. 6.17]. When i is zero then $H^i(\Gamma_v\backslash\mathbb{D}^+)$ is $\mathbb{R}$ and we call the preimage of 1

$$\mathrm{Th}(\Gamma_v\backslash E)\coloneqq \mathrm{Th}^{-1}(1)\in H^q(\Gamma_v\backslash E,\Gamma_v\backslash E-\mathrm{D}(\Gamma_v\backslash E)) \tag{3.59}$$

the *Thom class*. Any differential form representating this class is called a *Thom form* , in particular every closed q-form on $\Gamma_v\backslash E$ that has compact support in every fiber and whose integral along every fiber is 1 is a Thom form. One can also view the Thom class as the Poincaré dual class of the zero section E_0 in E, in the same sense as for (2.24).

Let ω in $\Omega^j(E)$ be a form on the bundle and let ω_z be its restriction to a fiber $E_z=\pi^{-1}(z)$ for some z in $\mathbb{D}^+$. After identifying z_0 with $\mathbb{R}^q$ we see ω_z as an element of $C^\infty(\mathbb{R}^q)\otimes\wedge^j(\mathbb{R}^q)^*$. We say that ω is *rapidly decreasing in the fiber* if ω_z lies in $\mathscr{S}(\mathbb{R}^q)\otimes\wedge^j(\mathbb{R}^q)^*$ for every z in $\mathbb{D}^+$. We write $\Omega^j_{\mathrm{rd}}(E)$ for the space of such forms.

Let $\Omega^\bullet_{\mathrm{rd}}(\Gamma_v\backslash E)$ be the complex of rapidly decreasing forms in the fiber. It is isomorphic to the complex $\Omega^\bullet_{\mathrm{rd}}(E)^{\Gamma_v}$ of rapidly decreasing Γ_v-invariant forms on E. Let $H_{\mathrm{rd}}(\Gamma_v\backslash E)$ the cohomology of this complex. The map

$$\begin{aligned} h\colon \Gamma_v\backslash E &\longrightarrow \Gamma_v\backslash E \\ w &\longrightarrow \frac{w}{\sqrt{1-\|w\|^2}} \end{aligned} \tag{3.60}$$

is a diffeomorphism from the open disk bundle $\mathrm{D}(\Gamma_v\backslash E)^\circ$ onto $\Gamma_v\backslash E$. It induces an isomorphism by pullback

$$h^*\colon H_{\mathrm{rd}}(\Gamma_v\backslash E) \longrightarrow H(\Gamma_v\backslash E, \Gamma_v\backslash E - \mathrm{D}(\Gamma_v\backslash E)), \tag{3.61}$$

which commutes with the fiber integration. Hence we have the following version of the Thom isomorphism

$$H^{q+i}_{\mathrm{rd}}(\Gamma_v\backslash E) \longrightarrow H^i(\Gamma_v\backslash \mathbb{D}^+). \tag{3.62}$$

The construction of Mathai and Quillen produces a Thom form

$$U_{MQ} \in \Omega^q_{\mathrm{rd}}(E) \tag{3.63}$$

which is $G(\mathbb{R})^+$-invariant (hence Γ_v-invariant) and closed. We will recall their construction in the next section.

3.7 The Mathai-Quillen construction

As earlier let $\widetilde{E}$ be the bundle $(G(\mathbb{R})^+ \times z_0) \times_K z_0$. Let $\wedge^j \tilde{E}$ be the bundle $(G(\mathbb{R})^+ \times z_0) \times_K \wedge^j z_0$ and

$$\begin{aligned} \Omega^{i,j} &:= \Omega^i(\mathbb{D}^+, \wedge^j E) \\ \widetilde{\Omega}^{i,j} &:= \Omega^i(E, \wedge^j \widetilde{E}). \end{aligned} \tag{3.64}$$

First consider the tautological section $\mathbf{s}$ of E defined by

$$\mathbf{s}[g,w] := [(g,w),w] \in \widetilde{E}. \tag{3.65}$$

This gives a canonical element $\mathbf{s}$ of $\widetilde{\Omega}^{0,1}$. Composing with the norm induced from the inner product we get an element $\|\mathbf{s}\|^2$ in $\widetilde{\Omega}^{0,0}$.

The representation ρ on z_0 induces a representation on $\wedge^i z_0$ that we also denote by ρ. The derivative at the identity gives a map

$$\rho\colon \mathfrak{k} \longrightarrow \mathfrak{so}(\wedge^i z_0). \tag{3.66}$$

The connection form $\rho(\widetilde{\theta})$ in $\Omega^1(G(\mathbb{R})^+ \times z_0, \wedge^j z_0)$ defines a covariant derivative

$$\widetilde{\nabla}\colon \widetilde{\Omega}^{0,j} \longrightarrow \widetilde{\Omega}^{1,j} \tag{3.67}$$

on $\wedge^j \widetilde{E}$. We can extend it to a map

$$\widetilde{\nabla}\colon \widetilde{\Omega}^{i,j} \longrightarrow \widetilde{\Omega}^{i+1,j} \tag{3.68}$$

by setting

$$\widetilde{\nabla}(\omega \otimes s) := d\omega \otimes s + (-1)^i \omega \wedge \widetilde{\nabla}(s), \tag{3.69}$$

as in (3.30). The connection on $\widetilde{\Omega}^{i,j}$ is compatible with the metric. Finally, the covariant derivative $\widetilde{\nabla}$ defines a derivation on $\oplus\widetilde{\Omega}^{i,j}$ that satisfies

$$\widetilde{\nabla}(\alpha \wedge \alpha') = (\widetilde{\nabla}\alpha) \wedge \alpha' + (-1)^{i+j} \alpha \wedge (\widetilde{\nabla}\alpha') \tag{3.70}$$

for any α in $\widetilde{\Omega}^{i,j}$ and α' in $\widetilde{\Omega}^{k,l}$.

Taking the derivative of the tautological section gives an element

$$\widetilde{\nabla}\mathbf{s} = d\mathbf{s} + \rho(\widetilde{\theta})\mathbf{s} \in \widetilde{\Omega}^{1,1}. \tag{3.71}$$

Let $\mathfrak{so}(\widetilde{E})$ denote the bundle $(G(\mathbb{R})^+ \times z_0) \times_K \mathfrak{so}(z_0)$ and consider the curvature $\rho(\widetilde{R})$ in $\Omega^2(\widetilde{E}, \mathfrak{so}(\widetilde{E}))$. We have an isomorphism

$$\begin{aligned} T^{-1}\big|_{z_0} \colon \mathfrak{so}(z_0) &\longrightarrow \wedge^2 z_0 \\ A &\longmapsto \sum_{i<j} \langle Ae_i, e_j\rangle e_i \wedge e_j. \end{aligned} \tag{3.72}$$

The inverse sends $v \wedge w$ to the endomorphism $u \mapsto \langle v, u\rangle w - \langle w, u\rangle v$, and is the isomorphism from (2.11) restricted to z_0. Note that we have

$$T(v \wedge w)u = \iota(u)v \wedge w. \tag{3.73}$$

Using this isomorphism we can also identify $\mathfrak{so}(\widetilde{E})$ and $\wedge^2 \widetilde{E}$ so that we can view the curvature as an element

$$\rho(\widetilde{R}) \in \widetilde{\Omega}^{2,2}. \tag{3.74}$$

Lemma 3.4. *The form $\omega := 2\pi\|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} - \rho(\widetilde{R})$ lying in $\widetilde{\Omega}^{0,0} \oplus \widetilde{\Omega}^{1,1} \oplus \widetilde{\Omega}^{2,2}$ is annihilated by $\widetilde{\nabla} + 2\sqrt{\pi}i(\mathbf{s})$. Moreover*

$$d\int^B \alpha = \int^B \widetilde{\nabla}\alpha, \tag{3.75}$$

for every form α in $\widetilde{\Omega}^{i,j}$. Hence $\int^B exp(-\omega)$ is a closed form.

Proof. We have

$$\begin{aligned} &\left(\widetilde{\nabla} + 2\sqrt{\pi}i(\mathbf{s})\right)\left(2\pi\|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} - \rho(\widetilde{R})\right) \\ &= 2\pi\widetilde{\nabla}\|\mathbf{s}\|^2 + 4\pi^{\frac{3}{2}}i(\mathbf{s})\|\mathbf{s}\|^2 + 2\sqrt{\pi}\widetilde{\nabla}^2\mathbf{s} + 4\pi i(x)\widetilde{\nabla}\mathbf{s} - \widetilde{\nabla}\rho(\widetilde{R}) - 2\sqrt{\pi}i(\mathbf{s})\rho(\widetilde{R}). \end{aligned} \tag{3.76}$$

It vanishes because we have the following:

- $i(\mathbf{s})\|\mathbf{s}\|^2 = 0$ since $\|\mathbf{s}\|$ is in $\widetilde{\Omega}^{0,0}$,
- $\widetilde{\nabla}\rho(\widetilde{R}) = 0$ by Bianchi's identity,
- $\widetilde{\nabla}\|\mathbf{s}\|^2 = 2\langle\widetilde{\nabla}\mathbf{s}, \mathbf{s}\rangle = -2i(\mathbf{s})\widetilde{\nabla}\mathbf{s}$,
- $\widetilde{\nabla}^2\mathbf{s} = \rho(\widetilde{R})\mathbf{s} = i(\mathbf{s})\rho(\widetilde{R})$.

For the last point we used (3.73) where we view $\rho(\widetilde{R})$ as an element of $\Omega^2(E, \mathfrak{so}(\widetilde{E}))$, respectively of $\Omega^2(E, \wedge^2\widetilde{E})$.

Let $s_1 \wedge \cdots \wedge s_q$ in $\Omega^0(E, \wedge^q \widetilde{E})$ be a global section where $s_1, \ldots, s_q$ is a local orthonormal frame for $\widetilde{E}$. Then for any α in $\widetilde{\Omega}^{i,j}$ we have

$$\int^B \alpha = \langle \alpha, s_1 \wedge \cdots \wedge s_q \rangle. \tag{3.77}$$

This vanishes if j is different from q, hence we can assume α is in $\widetilde{\Omega}^{i,q}$. If we write α as $\beta s_1 \wedge \cdots \wedge s_q$ for some β in $\Omega^i(E)$ then

$$\int^B \alpha = \beta. \tag{3.78}$$

On the other hand, since the connection on $\widetilde{\Omega}^{i,q}$ is compatible with the metric, we have

$$0 = d\langle s_1 \wedge \cdots \wedge s_q, s_1 \wedge \cdots \wedge s_q \rangle = 2\langle \widetilde{\nabla}(s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q \rangle. \tag{3.79}$$

Then we have

$$\begin{aligned} \int^B \widetilde{\nabla}\alpha &= \langle \widetilde{\nabla}\alpha, s_1 \wedge \cdots \wedge s_q \rangle \\ &= \langle d\beta \otimes s_1 \wedge \cdots \wedge s_q + (-1)^i \beta \wedge \widetilde{\nabla}(s_1 \wedge \cdots \wedge s_q), s_1 \wedge \cdots \wedge s_q \rangle \\ &= d\beta \\ &= d \int^B \alpha. \end{aligned} \tag{3.80}$$

Since $\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})$ is a derivation that annihilates ω we have

$$\left(\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})\right) \omega^k = 0 \tag{3.81}$$

for positive k. Hence it follows that

$$\begin{aligned} d \int^B \exp(-\omega) &= \int^B \widetilde{\nabla} \exp(-\omega) \\ &= \int^B \left(\widetilde{\nabla} + 2\sqrt{\pi} i(\mathbf{s})\right) \exp(-\omega) \\ &= 0. \end{aligned} \tag{3.82}$$

□

In [9] Mathai and Quillen define the following form

$$U_{MQ} := (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} \int^B \exp\left(-2\pi \|\mathbf{s}\|^2 - 2\sqrt{\pi}\widetilde{\nabla}\mathbf{s} + \rho(\widetilde{R})\right) \in \Omega^q_{rd}(E). \tag{3.83}$$

We call it the *Mathai-Quillen form.*

Proposition 3.5. *The Mathai-Quillen form is a Thom form.*

Proof. From the previous lemma it follows that the form is closed. It remains to show that its integral along the fibers is 1. The restriction of the form U_{MQ} along the fiber $\pi^{-1}(eK)$ is given by

$$\begin{aligned} U_{MQ} &= (-1)^{\frac{q(q+1)}{2}}(2\pi)^{-\frac{q}{2}}e^{-2\pi\|\mathbf{s}\|^2}\int^B \exp(-2\sqrt{\pi}d\mathbf{s}) \\ &= (-1)^{\frac{q(q+1)}{2}}2^{\frac{q}{2}}e^{-2\pi\|\mathbf{s}\|^2}(-1)^q\int^B (dx_1\otimes e_1)\wedge\cdots\wedge(dx_q\otimes e_q) \\ &= 2^{\frac{q}{2}}e^{-2\pi\|\mathbf{s}\|^2}dx_1\wedge\cdots\wedge dx_q, \end{aligned} \tag{3.84}$$

and its integral over the fiber $\pi^{-1}(eK)$ is equal to 1. □

3.8 Transgression form

For $t > 0$ consider the map $t\colon E \longrightarrow E$ given by multiplication by t in the fibers. Consider the K-invariant vector field

$$X \coloneqq \sum_{i=1}^{q} x_i \frac{\partial}{\partial x_i} \tag{3.85}$$

on $G(\mathbb{R})^+ \times \mathbb{R}^q$. Since it is K-invariant it also induces a vector field on E. We define the *transgression form* ψ in $\Omega^{q-1}(E)$ to be $\iota_X U_{MQ}$, where ι_X is the interior product.

Proposition 3.6 (Transgression formula)**.** *The transgression satisfies:*

$$\left(\frac{d}{dt}t^*U_{MQ}\right)_{t=t_0} = -\frac{1}{t_0}d(t_0^*\psi). \tag{3.86}$$

Proof. This is due to Mathai and Quillen. Let us view the multiplication map by t as a map

$$\begin{aligned} m\colon E\times\mathbb{R}_{>0} &\longrightarrow E \\ (e,t) &\longmapsto et. \end{aligned} \tag{3.87}$$

The differential $\tilde{d}$ on $E\times\mathbb{R}_{>0}$ splits as $d + d_{\mathbb{R}_{>0}}$. Since U_{MQ} is closed (hence its pullback) we have

$$0 = \tilde{d}(m^*U_{MQ}) = d(m^*U_{MQ}) + \frac{d}{dt}(m^*U_{MQ})dt. \tag{3.88}$$

Moreover the pushforward of the vector field $t\frac{\partial}{\partial t}$ by m is X, hence for the contraction we have

$$\iota_{\frac{\partial}{\partial t}}m^*U_{MQ} = \frac{1}{t}m^*\iota_X U_{MQ}. \tag{3.89}$$

Since the differential d is independent of t it commutes with the contraction $\iota_{\frac{\partial}{\partial t}}$. Combining with (3.88) yields

$$\frac{d}{dt}(m^*U_{MQ}) = -\frac{1}{t}d(m^*\psi). \tag{3.90}$$

Finally, pulling back by the section

$$\begin{aligned} t_0 \colon E &\longrightarrow E \times \mathbb{R}_{>0} \\ e &\longmapsto (e, t_0) \end{aligned} \tag{3.91}$$

gives the desired formula. □

Let Γ_v be the stabilizer of v in Γ, which acts on the left on E. By the $G(\mathbb{R})^+$-invariance (hence Γ_v-invariance) of U_{MQ}, it is also a form in $\Omega^q(\Gamma_v \backslash E)$. Let S_0 denote the image $\Gamma_v \backslash E_0$ of the zero section in $\Gamma_v \backslash E$.

Proposition 3.7. *The form U_{MQ} represents the Poincaré dual of S_0 in $\Gamma_v \backslash E$.*

Sketch of proof. For $0 < t_1 < t_2$ we have

$$\begin{aligned} t_2^* U_{MQ} - t_1^* U_{MQ} &= \int_{t_1}^{t_2} \left(\frac{d}{dt} t^* U_{MQ} \right) dt \\ &= -\int_{t_1}^{t_2} d(t^* \psi) \frac{dt}{t} \\ &= -d \int_{t_1}^{t_2} t^* \psi \frac{dt}{t} \end{aligned} \tag{3.92}$$

so that $t_2^* U_{MQ}$ and $t_1^* U_{MQ}$ represent the same cohomology class in $H^q(\Gamma_v \backslash E)$. Then, one can show that

$$\lim_{t \to \infty} t^* U_{MQ} = \delta_{S_0} \tag{3.93}$$

where δ_{S_0} is the current of integration along S_0. Hence if ω is a form in $\Omega_c^m(E)$, where m is the dimension of $\mathbb{D}^+$, then

$$\begin{aligned} \int_{\Gamma_v \backslash E} U_{MQ} \wedge \omega &= \lim_{t \to \infty} \int_{\Gamma_v \backslash E} t^* U_{MQ} \wedge \omega \\ &= \int_{S_0} \omega. \end{aligned} \tag{3.94}$$

□

4 Computation of the Mathai-Quillen form

4.1 The section s_v

Let pr denote the orthogonal projection of $V(\mathbb{R})$ on the plane z_0. Consider the section

$$\begin{aligned} s_v \colon \mathbb{D}^+ &\longrightarrow E \\ z &\longmapsto [g_z, \mathrm{pr}(g_z^{-1} v)], \end{aligned} \tag{4.1}$$

where g_z is any element of $G(\mathbb{R})^+$ sending z_0 to z. Let us denote by L_g the left action of an element g in $G(\mathbb{R})^+$ on $\mathbb{D}^+$. We also denote by L_g the action on E given by $L_g[g_z, v] = [gg_z, v]$. The bundle is $G(\mathbb{R})^+$-equivariant with respect to these actions.

Proposition 4.1. *The section s_v is well-defined and Γ_v-equivariant. Moreover its zero locus is precisely $\mathbb{D}_v^+$.*

Proof. The section is well-defined, since replacing g_z by $g_z k$ gives

$$s_v(z) = [g_z k, \mathrm{pr}(k^{-1} g_z^{-1} v)] = [g_z k, k^{-1} \mathrm{pr}(g_z^{-1} v)] = [g, \mathrm{pr}(g_z^{-1} v)] = s_v(z). \tag{4.2}$$

Suppose that z is in the zero locus of s_v, that is to say $\mathrm{pr}(g_z^{-1} v)$ vanishes. Then $g_z^{-1} v$ is in $z_0^{\perp}$. It is equivalent to the fact that $z = g_z z_0$ is a subspace of $v^{\perp}$, which means that z is in $\mathbb{D}_v^+$. Hence the zero locus of s_v is exactly $\mathbb{D}_v^+$. For the equivariance, note that we have

$$s_v \circ L_g(z) = [g g_z, \mathrm{pr}(g_z^{-1} g^{-1} v)] = L_g \circ s_{g^{-1} v}(z). \tag{4.3}$$

Hence if γ is an element of Γ_v we have

$$s_v \circ L_\gamma = L_\gamma \circ s_v. \tag{4.4}$$

□

We define the pullback $\varphi^0(v) \coloneqq s_v^* U_{MQ}$ of the Mathai-Quillen form by s_v. It defines a form

$$\varphi^0 \in C^\infty(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+. \tag{4.5}$$

It is only rapidly decrasing on $\mathbb{R}^q$, and in order to make it rapidly decreasing everywhere we set

$$\varphi(v) \coloneqq e^{-\pi Q(v,v)} \varphi^0(v). \tag{4.6}$$

It defines a form $\varphi \in \mathscr{S}(\mathbb{R}^{p+q}) \otimes \Omega^q(\mathbb{D})^+$

Proposition 4.2. *1. For fixed v in $V(\mathbb{R})$ the form $\varphi^0(v)$ in $\Omega^q(\mathbb{D}^+)$ is given by*

$$\varphi^0(v) = (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} \exp\left(2\pi Q\big|_{z_0}(v,v)\right) \int^B \exp\left(-2\sqrt{\pi}\nabla s_v + \rho(R)\right). \tag{4.7}$$

2. It satisfies $L_g^* \varphi^0(v) = \varphi^0(g^{-1} v)$, hence

$$\varphi^0 \in \left[\Omega^q(\mathbb{D}^+) \otimes C^\infty(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+}. \tag{4.8}$$

3. It is a Poincaré dual of $\Gamma_v \backslash \mathbb{D}_v^+$ in $\Gamma_v \backslash \mathbb{D}^+$.

Proof. 1. Recall that $\widetilde{\nabla} = \pi^* \nabla$ and $\widetilde{R} = \pi^* R$. We pullback by s_v

$$\begin{tikzcd} E \simeq s_v^* \widetilde{E} \arrow[r] \arrow[d] & \widetilde{E} \arrow[d, "\pi"] \\ \mathbb{D}^+ \arrow[r, "s_v"] & E. \end{tikzcd}$$

Since $\pi \circ s_v$ is the identity we have

$$s_v^* \widetilde{\nabla} = s_v^* \pi^* \nabla = \nabla. \tag{4.9}$$

Hence, the pullback connection $s_v^* \widetilde{\nabla}$ satisfies

$$s_v^*(\widetilde{\nabla}\mathbf{s}) = (s_v^*\widetilde{\nabla})(s_v^*\mathbf{s}) = \nabla s_v \tag{4.10}$$

since $s_v^*\mathbf{s} = s_v$. We also have $s_v^*\widetilde{R} = R$ and

$$s_v^*\|\mathbf{s}\|^2 = \|s_v\|^2 = \langle s_v, s_v\rangle = -Q\big|_{z_0}(v,v). \tag{4.11}$$

The expression for φ^0 then follows from the fact that exp and s_v^* commute.

2. The bundle E is $G(\mathbb{R})^+$ equivariant. By construction the Mathai-Quillen is $G(\mathbb{R})^+$-invariant, so $L_g^* U_{MQ} = U_{MQ}$. On the other hand we also have

$$s_v \circ L_g(z) = L_g \circ s_{g^{-1}v}(z), \tag{4.12}$$

and thus

$$L_g^*\varphi^0(v) = L_g^* s_v^* U_{MQ} = \varphi^0(g^{-1}v). \tag{4.13}$$

3. Since s_v is Γ_v-equivariant we view it as a section

$$s_v \colon \Gamma_v\backslash\mathbb{D}^+ \longrightarrow \Gamma_v\backslash E, \tag{4.14}$$

whose zero locus is precisely $\Gamma_v\backslash\mathbb{D}_v^+$. Let S_0 (respectively S_v) be the image in $\Gamma_v\backslash E$ of the section s_v (respectively the zero section). By Proposition 3.7 the Thom form U_{MQ} is a Poincaré dual of S_0. For a form ω in $\Omega_c^{m-q}(\Gamma_v\backslash\mathbb{D}^+)$ we have

$$\begin{aligned} \int_{\Gamma_v\backslash\mathbb{D}^+} \varphi^0(v)\wedge\omega &= \int_{\Gamma_v\backslash\mathbb{D}^+} s_v^*\left(U_{MQ}\wedge\pi^*\omega\right) \\ &= \int_{S_v} U_{MQ}\wedge\pi^*\omega \\ &= \int_{S_v\cap S_0} \pi^*\omega \\ &= \int_{\Gamma_v\backslash\mathbb{D}_v^+} \omega. \end{aligned} \tag{4.15}$$

The last step follows from the fact $\pi^{-1}(S_v\cap S_0)$ equals $\Gamma_v\backslash\mathbb{D}_v^+$.

□

As in (2.19) we have an isomorphism

$$\left[\Omega^q(\mathbb{D}^+)\otimes C^\infty(\mathbb{R}^{p+q})\right]^{G(\mathbb{R})^+} \longrightarrow \left[\bigwedge^q \mathfrak{p}^*\otimes C^\infty(\mathbb{R}^{p+q})\right]^K \tag{4.16}$$

by evaluating at the basepoint eK of $G(\mathbb{R})^+/K$ that corresponds to z_0 in $\mathbb{D}^+$. We will now compute $\varphi^0\big|_{eK}$.

4.2 The Mathai-Quillen form at the identity

From now on we identify $\mathbb{R}^{p+q}$ with $V(\mathbb{R})$ by the orthonormal basis of (2.1), and let z_0 be the negative spanned by the vectors $e_{p+1}, \cdots, e_{p+q}$. Hence we identify z_0 with $\mathbb{R}^q$ and the quadratic form is

$$Q\big|_{z_0}(v,v) = -\sum_{\mu=p+1}^{p+q} x_\mu^2 \tag{4.17}$$

where $x_{p+1}, \dots, x_{p+q}$ are the coordinates of the vector v.

Let f_v in $\Omega^0(G(\mathbb{R})^+, z_0)^K$ be the map associated to the section s_v, as in Proposition 3.1. It is defined by

$$f_v(g) = \mathrm{pr}(g^{-1}v). \tag{4.18}$$

Then $df_v + \rho(\theta)f_v$ is the horizontal lift of ∇s_v, as discussed in Section 3.1. Let X be a vector in $\mathfrak{g}$ and let $X_{\mathfrak{p}}$ and $X_{\mathfrak{k}}$ be its components with respect to the splitting of $\mathfrak{g}$ as $\mathfrak{p} \oplus \mathfrak{k}$. We have

$$(df_v + \rho(\theta)f_v)_e(X) = d_e f_v(X_{\mathfrak{p}}). \tag{4.19}$$

In particular we can evaluate on the basis $X_{\alpha\mu}$ and get:

$$\begin{aligned} d_e f_v(X_{\alpha\mu}) &= \frac{d}{dt}\bigg|_{t=0} f_v(\exp tX_{\alpha\mu}) \\ &= -\,\mathrm{pr}(X_{\alpha\mu}v) \\ &= -\,\mathrm{pr}(x_\mu e_\alpha + x_\alpha e_\mu) \\ &= -x_\alpha e_\mu. \end{aligned} \tag{4.20}$$

So as an element of $\mathfrak{p}^* \otimes z_0$ we can write

$$d_e f_v = -\sum_{\mu=p+1}^{p+q}\left(\sum_{\alpha=1}^{p} x_\alpha \omega_{\alpha\mu}\right) \otimes e_\mu = -\sum_{\alpha=1}^{p} x_\alpha \eta_\alpha, \tag{4.21}$$

with

$$\eta_\alpha := \sum_{\mu=p+1}^{p+q} \omega_{\alpha\mu} \otimes e_\mu \in \Omega^{1,1}. \tag{4.22}$$

Proposition 4.3. *Let $\rho(R_e)$ in $\wedge^2\mathfrak{p}^* \otimes \mathfrak{so}(z_0)$ be the curvature at the identity. Then after identifying $\mathfrak{so}(z_0)$ with $\wedge^2 z_0$ we have*

$$\rho(R_e) = -\frac{1}{2}\sum_{\alpha=1}^{p} \eta_\alpha^2 \in \wedge^2\mathfrak{p}^* \otimes \wedge^2 z_0, \tag{4.23}$$

where $\eta_\alpha^2 = \eta_\alpha \wedge \eta_\alpha$.

Proof. Using the relation $E_{ij}E_{kl} = \delta_{il}E_{kj}$ one can show that

$$[X_{\alpha\mu}, X_{\beta\nu}] = \delta_{\mu\nu}X_{\alpha\beta} + \delta_{\alpha\beta}X_{\mu\nu} \tag{4.24}$$

for two vectors $X_{\alpha\nu}$ and $X_{\beta\mu}$ in $\mathfrak{p}$. Hence we have

$$\begin{aligned} R_e(X_{\alpha\nu} \wedge X_{\beta\mu}) &= [\theta(X_{\alpha\nu}), \theta(X_{\beta\mu})] - \theta([X_{\alpha\nu}, X_{\beta\mu}]) \\ &= -\theta([X_{\alpha\nu}, X_{\beta\mu}]) \\ &= -p\left(\delta_{\alpha\beta} X_{\nu\mu} + \delta_{\nu\mu} X_{\alpha\beta}\right) \\ &= -\delta_{\alpha\beta} X_{\nu\mu}. \end{aligned} \tag{4.25}$$

On the other hand, since $\eta_i(X_{jr}) = \delta_{ij} e_r$, we also have

$$\begin{aligned} \sum_{i=1}^{p} \eta_i^2(X_{\alpha\nu} \wedge X_{\beta\mu}) &= \sum_{i=1}^{p} \eta_i(X_{\alpha\nu}) \wedge \eta_i(X_{\beta\mu}) - \eta_i(X_{\beta\mu}) \wedge \eta_i(X_{\alpha\nu}) \\ &= 2\delta_{\alpha\beta} e_\nu \wedge e_\mu. \end{aligned} \tag{4.26}$$

The lemma follows since $\rho(X_{\nu\mu}) = T(e_\nu \wedge e_\mu)$ in $\mathfrak{so}(z_0)$, because

$$Q(\rho(X_{\nu\mu})e_\nu, e_\mu) e_\nu \wedge e_\mu = -Q(e_\mu, e_\mu) e_\nu \wedge e_\mu = e_\nu \wedge e_\mu. \tag{4.27}$$

□

Using the fact that the exponential satisfies $\exp(\omega + \eta) = \exp(\omega)\exp(\eta)$ on the subalgebra $\bigoplus \Omega^{i,i}$ - see Remark 3.1 - we can write

$$\varphi^0\big|_e(v) = (-1)^{\frac{q(q+1)}{2}} (2\pi)^{-\frac{q}{2}} \exp\left(2\pi Q\big|_{z_0}(v,v)\right) \int^B \prod_{\alpha=1}^{p} \exp\left(2\sqrt{\pi} x_\alpha \eta_\alpha - \frac{1}{2}\eta_\alpha^2\right). \tag{4.28}$$

We define the *n-th Hermite polynomial* by

$$H_n(x) \coloneqq \left(2x - \frac{d}{dx}\right) \cdot 1 \in \mathbb{R}[x]. \tag{4.29}$$

The first three Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$.

LEMMA 4.4. *Let η be a form in $\bigoplus \Omega^{i,i}$. Then*

$$\exp(2x\eta - \eta^2) = \sum_{n \geq 0} \frac{1}{n!} H_n(x) \eta^n, \tag{4.30}$$

where H_n is the n-th Hermite polynomial.

Proof. Since η and η^2 are in $\bigoplus \Omega^{i,i}$, they commute and we can use the binomial formula:

$$\begin{aligned} \exp(2x\eta - \eta^2) &= \sum_{k \geq 0} \frac{1}{k!} \left(2x\eta - \eta^2\right)^k \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x\eta)^{k-l} \left(-\eta^2\right)^l \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (2x)^{k-l} (-1)^l \eta^{l+k} \\ &= \sum_{n \geq 0} P_n(x) \eta^n, \end{aligned} \tag{4.31}$$

where

$$P_n(x) \coloneqq \sum_{\substack{0\leq l\leq k\leq n\\ k+l=n}} \frac{(-1)^l}{l!(k-l)!}(2x)^{k-l}. \tag{4.32}$$

The conditions on k and l imply that n is less than or equal to $2k$. First suppose that n is even. Then we have that k is between $\frac{n}{2}$ and n, so that the sum above can be written

$$\sum_{k=\frac{n}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!}(2x)^{2k-n} = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-m}}{(\frac{n}{2}-m)!(2m)!}(2x)^{2m} = \frac{1}{n!}H_n(x), \tag{4.33}$$

where in the second step we let m be $k-\frac{n}{2}$. If n is odd then k is between $\frac{n+1}{2}$ and n, so that the sum can be written

$$\sum_{k=\frac{n+1}{2}}^{n} \frac{(-1)^{n-k}}{(n-k)!(2k-n)!}(2x)^{2k-n} = \sum_{m=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-m}}{(\frac{n-1}{2}-m)!(2m+1)!}(2x)^{2m+1} = \frac{1}{n!}H_n(x). \tag{4.34}$$

□

Applying the lemma to (4.28) we get

$$\begin{aligned}
&\int^B \prod_{\alpha=1}^{p} \exp\left(2\sqrt{\pi}x_\alpha\eta_\alpha - \frac{1}{2}\eta_\alpha^2\right)\\
=&\int^B \prod_{\alpha=1}^{p} \exp\left(2\sqrt{2\pi}x_\alpha\frac{\eta_\alpha}{\sqrt{2}} - \left(\frac{\eta_\alpha}{\sqrt{2}}\right)^2\right)\\
=&\int^B \prod_{\alpha=1}^{p} \sum_{n\geq 0} \frac{2^{-n/2}}{n!} H_n\left(\sqrt{2\pi}x_\alpha\right)\eta_\alpha^n\\
=&\sum_{n_1,\ldots,n_p} \frac{2^{-\frac{n_1+\cdots+n_p}{2}}}{n_1!\cdots n_p!} H_{n_1}\left(\sqrt{2\pi}x_1\right)\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right)\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}.
\end{aligned} \tag{4.35}$$

If $n_1+\cdots+n_p$ is different from q, then the Berezinian of $\eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}$ vanishes and we get

$$\begin{aligned}
&\sum_{n_1,\ldots,n_p} \frac{2^{-\frac{n_1+\cdots+n_p}{2}}}{n_1!\cdots n_p!} H_{n_1}\left(\sqrt{2\pi}x_1\right)\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right)\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}\\
=&2^{-\frac{q}{2}} \sum_{n_1+\cdots+n_p=q} \frac{H_{n_1}\left(\sqrt{2\pi}x_1\right)\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right)}{n_1!\cdots n_p!}\int^B \eta_1^{n_1}\wedge\cdots\wedge\eta_p^{n_p}.
\end{aligned} \tag{4.36}$$

Note that

$$\begin{aligned}
\eta_\alpha^{n_\alpha} &= \left(\sum_{\mu=p+1}^{p+q} \omega_{\alpha\mu}\otimes e_\mu\right)^{n_\alpha}\\
&= \sum_{\mu_1,\ldots,\mu_{n_\alpha}} (\omega_{\alpha\mu_1}\otimes e_{\mu_1})\wedge\cdots\wedge(\omega_{\alpha\mu_{n_\alpha}}\otimes e_{\mu_{n_\alpha}})\\
&= n_\alpha! \sum_{\mu_1<\cdots<\mu_{n_\alpha}} (\omega_{\alpha\mu_1}\otimes e_{\mu_1})\wedge\cdots\wedge(\omega_{\alpha\mu_{n_\alpha}}\otimes e_{\mu_{n_\alpha}}),
\end{aligned} \tag{4.37}$$

where the sums are over all μ_i's between $p+1$ and $p+q$. If $n_1 + \cdots + n_p$ is equal to q we have

$$\begin{aligned}
&\int^B \eta_1^{n_1} \wedge \cdots \wedge \eta_p^{n_p} \\
&= \int^B \prod_{\alpha=1}^{p} \left(\sum_{\mu=p+1}^{p+q} \omega_{\alpha\mu} \otimes e_\mu \right)^{n_\alpha} \\
&= \int^B \prod_{\alpha=1}^{p} n_\alpha! \sum_{\mu_1 < \cdots < \mu_{n_\alpha}} (\omega_{\alpha\mu_1} \otimes e_{\mu_1}) \wedge \cdots \wedge (\omega_{\alpha\mu_{n_\alpha}} \otimes e_{\mu_{n_\alpha}}) \\
&= n_1! \cdots n_p! \sum \int^B (\omega_{\alpha(p+1)} \otimes e_1) \wedge \cdots \wedge (\omega_{\alpha(p+q)} \otimes e_q) \\
&= (-1)^{\frac{q(q+1)}{2}} n_1! \cdots n_p! \sum \omega_{\alpha_1(p+1)} \wedge \cdots \wedge \omega_{\alpha_q(p+q)},
\end{aligned} \tag{4.38}$$

where the sums in the last two lines go over all tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_q)$ with α between 1 and p, and the value α appears exactly n_α-times in $\underline{\alpha}$. Hence

$$\begin{aligned}
\varphi^0\big|_e(v) = 2^{-q}\pi^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \wedge \cdots \wedge \omega_{\alpha_q(p+q)} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \\
\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(2\pi Q\big|_{z_0}(v,v)\right).
\end{aligned} \tag{4.39}$$

After multiplying by $\exp\left(-\pi Q(v,v)\right)$ we get

$$\begin{aligned}
\varphi\big|_e(v) = 2^{-q}\pi^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \wedge \cdots \wedge \omega_{\alpha_q(p+q)} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \\
\cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(-\pi Q_{z_0}^+(v,v)\right).
\end{aligned} \tag{4.40}$$

The form is now rapidly decreasing in v, since the Siegel majorant is positive definite. We have

$$\varphi\big|_e \in \left[\bigwedge^q \mathfrak{p}^* \otimes \mathscr{S}(\mathbb{R}^{p+q}) \right]^K. \tag{4.41}$$

Theorem 4.5. *We have $2^{-\frac{q}{2}}\varphi(v) = \varphi_{KM}(v)$.*

Proof. It is a straightforward computation to show that

$$(2\pi)^{-n_\alpha/2} H_{n_\alpha}\left(\sqrt{2\pi}x_\alpha\right) \exp(-\pi x_\alpha^2) = \left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha} \right)^{n_\alpha} \exp(-\pi x_\alpha^2). \tag{4.42}$$

Hence applying this we find that the Kudla-Millson form, defined by the Howe operators in (2.22), is

$$\begin{aligned}
\varphi_{KM}\big|_e(v) &= 2^{-q}(2\pi)^{-\frac{q}{2}} \sum \omega_{\alpha_1(p+1)} \wedge \cdots \wedge \omega_{\alpha_q(p+q)} \otimes H_{n_1}\left(\sqrt{2\pi}x_1\right) \\
&\qquad \cdots H_{n_p}\left(\sqrt{2\pi}x_p\right) \exp\left(-\pi Q\big|_{z_0}(v,v)\right) \\
&= 2^{-\frac{q}{2}} e^{-\pi Q(v,v)} \varphi^0\big|_e(v).
\end{aligned} \tag{4.43}$$

□

5 Examples

1. Let us compute the Kudla-Millson as above in the simplest setting of signature $(1,1)$. Let $V(\mathbb{R})$ be the quadratic space $\mathbb{R}^2$ with the quadratic form $Q(v,w) = x'y + xy'$ where x and x' (respectively y and y') are the components of v (respectively of w). Let $e_1 = \frac{1}{\sqrt{2}}(1,1)$ and $e_2 = \frac{1}{\sqrt{2}}(1,-1)$. The 1-dimensional negative plane z_0 is $\mathbb{R}e_2$. If r denotes the variable on z_0 then the quadratic form is $Q\big|_{z_0}(r) = -r^2$. The projection map is given by

$$\begin{aligned} \mathrm{pr}\colon V(\mathbb{R}) &\longrightarrow z_0 \\ v = (x,x') &\longmapsto \frac{x-x'}{\sqrt{2}}. \end{aligned} \tag{5.1}$$

The orthogonal group of $V(\mathbb{R})$ is

$$G(\mathbb{R})^+ = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t > 0 \right\}, \tag{5.2}$$

and $\mathbb{D}^+$ can be identified with $\mathbb{R}_{>0}$. The associated bundle E is $\mathbb{R}_{>0} \times \mathbb{R}$ and the connection ∇ is simply d since the bundle is trivial. Hence the Mathai-Quillen form is

$$U_{MQ} = \sqrt{2} e^{-2\pi r^2} dr \in \Omega^1(E), \tag{5.3}$$

as in the proof of Proposition 3.5. The section $s_v \colon \mathbb{R}_{>0} \to E$ is given by

$$s_v(t) = \left(t, \frac{t^{-1}x - tx'}{\sqrt{2}} \right), \tag{5.4}$$

where x and x' are the components of v. We obtain

$$s_v^* U_{MQ} = e^{-\pi\left(\frac{x}{t} - tx'\right)^2} \left(\frac{x}{t} + tx' \right) \frac{dt}{t}. \tag{5.5}$$

Hence after multiplication by $2^{-\frac{1}{2}} e^{-\pi Q(v,v)}$ we get

$$\varphi_{KM}(x,x') = 2^{-\frac{1}{2}} e^{-\pi\left[\left(\frac{x}{t}\right)^2 + (tx')^2\right]} \left(\frac{x}{t} + tx' \right) \frac{dt}{t} \tag{5.6}$$

2. The second example illustrates the functorial properties of the Mathai-Quillen form. Suppose that we have an orthogonal splitting of $V(\mathbb{R})$ as $\bigoplus_i^r V_i(\mathbb{R})$. Let (p_i, q_i) be the signature of $V_i(\mathbb{R})$. We have

$$\mathbb{D}_1 \times \cdots \times \mathbb{D}_r \simeq \left\{ z \in \mathbb{D} \mid z = \bigoplus_{i=1}^r z \cap V_i(\mathbb{R}) \right\}. \tag{5.7}$$

Suppose we fix $z_0 = z_0^1 \oplus \cdots \oplus z_0^r$ in $\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ \subset \mathbb{D}$, where z_0^i is a negative q_i-plane in $V_i(\mathbb{R})$. Let $G_i(\mathbb{R})$ be the subgroup preserving $V_i(\mathbb{R})$, let K_i the stabilizer of z_0^i and $\mathbb{D}_i$ be the symmetric space associated to $V_i(\mathbb{R})$.
Over $\mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+$ the bundle E splits as an orthogonal sum $E_1 \oplus \cdots \oplus E_r$, where E_i is the bundle $G_i(\mathbb{R})^+ \times_{K_i} z_0^i$. Moreover the restriction of the Mathai-Quillen form to this subbundle is

$$U_{MQ}\big|_{E_1 \times \cdots \times E_r} = U_{MQ}^1 \wedge \cdots \wedge U_{MQ}^r, \tag{5.8}$$

where U^i_{MQ} is the Mathai-Quillen form on E_i. The section s_v also splits as a direct sum $\oplus s_{v_i}$ where v_i is the projection of v onto v_i. In summary the following diagram commutes

$$\begin{tikzcd} E_1 \oplus \cdots \oplus E_r \arrow[r, hook] & E \\ \mathbb{D}_1^+ \times \cdots \times \mathbb{D}_r^+ \arrow[r, hook] \arrow[u, "\oplus s_{v_i}"] & \mathbb{D}^+ \arrow[u, "s_v"'] \end{tikzcd}, \tag{5.9}$$

and we can conclude that

$$\varphi_{KM}(v)\big|_{\mathbb{D}_1^+\times\cdots\times\mathbb{D}_r^+} = \varphi^1_{KM}(v_1)\wedge\cdots\wedge\varphi^r_{KM}(v_r) \tag{5.10}$$

where φ^i_{KM} is the Kudla-Millson form on $\mathbb{D}_i^+$.

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