De Rham cohomology, analytic and algebraic

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Abstract

We introduce the basic concepts of sheaf theory, up to the definition of sheaf cohomology and acyclic resolution. We then introduce both the analytic and the algebraic de Rham cohomology, in a way that hopefully makes the analogy between the two more visible. We then give some results on coherent sheaves, while skipping some proofs and the more technical commutative algebra involved, so as to prove Serre's first GAGA theorem. We then conclude by stating Grothendieck's equivalence theorem for smooth algebraic varieties over $\mathbb{C}$.

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Introduction

The treatment of sheaves at the beginning is very elementary, but things get significantly more complicated in the last two sections. Sheaves were the main subject of the EA, so I found appropriate to treat them in a very beginner-friendly and detailed manner at the beginning. Due to my choice of subject, I quickly could not afford such a level of details, and decided to let the reader enjoy the pleasure of filling the gaps in the proofs by themselves. Also, a right index tendinitis suggested that I relegate category theory and algebraic geometry to prerequisites, which I did. That being said, nothing beyond the definition of an abelian category or of a smooth (abstract) algebraic variety will be necessary.

1 Sheaf theory

In this section, X is a topological space, and whenever a topological space is mentionned implicitly, it will always be X.

1.1 Presheaves

Definition 1. A **presheaf of sets** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Set}$ into the category of sets.
Similarly, an **abelian presheaf** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups.
Lastly, a **presheaf of rings** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Rng}$ into the category of rings.

Remark 2. If U, V and W are open sets such that $W \subset V \subset U$, then we have:

$$\mathrm{res}_{W,U} = \mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}.$$

Thus we often omit the open set we restrict from, and if $s \in \mathscr{F}(U)$, we write $s|_V$ instead of $\mathrm{res}_{V,U}(s)$.

In what follows, we talk about "presheaves" in general because what we say applies to the three kinds that we defined.
So if $\mathscr{F}$ is a presheaf on X, then:

- To each inclusion $V \subset U$ of open sets, $\mathscr{F}$ associates a **restriction map**

$$\mathrm{res}_{V,U} : \mathscr{F}(U) \to \mathscr{F}(V)$$

- For every open set U, we have that $\mathrm{res}_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$.
- For every open sets U, V and W such that $W \subset V \subset U$, we have that: $\mathrm{res}_{W,U} = \mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}$.
- we can define the **restriction** $\mathscr{F}|_U$ of $\mathscr{F}$ over an open set U by $\mathscr{F}|_U(V) := \mathscr{F}(V)$ for all open set $V \subset U$.

Definition 3. Let $\mathscr{F}$ be a presheaf on X and U an open set.
An element of $\mathscr{F}(U)$ is called a **section** of $\mathscr{F}$ over U, so that $\mathscr{F}(U)$ is called the **set of sections** of $\mathscr{F}$ over U.
An element of $\mathscr{F}(X)$ is called a **global section** of $\mathscr{F}$, so that $\mathscr{F}(X)$ is called the **set of global sections** of $\mathscr{F}$.

Remark 4. The notation $\Gamma(U, \mathscr{F})$ can be used to designate $\mathscr{F}(U)$, mostly when $\mathscr{F}$ is regarded as an argument.

Definition 5. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X.
A natural transformation from $\mathscr{F}$ to $\mathscr{G}$ is called a **morphism of presheaves.**
We can thus verify that with these morphisms, presheaves on X form categories, denoted by $\mathrm{PSh}(X)$ for presheaves of sets, $\mathrm{AbPSh}(X)$ for abelian presheaves, and $\mathrm{RngPSh}(X)$ for sheaves of rings.

Proposition 6. *Let $\Phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves.*
The following are equivalent:

1. *Φ is a monomorphism (resp. epimorphism).*
2. *For all $U \subset X$ the induced map $\Phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a monomorphism (resp. epimorphism).*

The following definition is of tremendous importance, because it captures the local behavior of a presheaf.

Definition 7. Let $\mathscr{F}$ be a presheaf on X, and $x \in X$.

$$\mathscr{F}_x := \operatorname*{colim}_{\substack{U \text{ open} \\ U \ni x}} \mathscr{F}(U)$$

is called the **stalk** of $\mathscr{F}$ at x.

Before getting into sheaves, let us define a last kind of presheaf.

Definition 8. Let $\mathscr{O}$ be a presheaf of rings.
A **presheaf of $\mathscr{O}$-modules** is an abelian presheaf $\mathscr{F}$ together with a map of presheaves of sets

$$\mathscr{O} \times \mathscr{F} \to \mathscr{F}$$

such that for all open set U, the map $\mathscr{O}(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ defines a structure of $\mathscr{O}(U)$-module on the abelian group $\mathscr{F}(U)$.

Definition 9. A morphism of abelian presheaves $\Phi : \mathscr{F} \to \mathscr{G}$ is called a **morphism of presheaves of $\mathscr{O}$-modules** if it is such that the diagram

$$\begin{tikzcd} \mathscr{O} \times \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id} \times \Phi"'] & \mathscr{F} \arrow[d, "\Phi"] \\ \mathscr{O} \times \mathscr{G} \arrow[r] & \mathscr{G} \end{tikzcd}$$

commutes.
The set of these morphisms is denoted by $\mathrm{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$.

Proposition 10. *Together with the morphisms we defined, presheaves of $\mathscr{O}$-modules form a category denoted by $P\mathscr{O}-\mathrm{mod}$.*

Definition 11. The stalks of a presheaf of $\mathscr{O}$-modules are defined to be the stalks of the underlying abelian presheaf.

Remark 12. The definition of a presheaf of $\mathscr{O}$-modules endows $\mathscr{F}_x$ with a structure of $\mathscr{O}_x$-module.

1.2 Sheaves

Presheaves were given a name, but they are nothing more than contravariant functors. In order to be able to transition from local to global and vice versa, let us impose some more properties on our functors. For the moment, we shall restrict to sheaves of sets, abelian sheaves and sheaves of rings, thus leaving sheaves of $\mathscr{O}$-modules for later.

Definition 13. Let $\mathscr{F}$ be a presheaf.
We say that $\mathscr{F}$ is a **sheaf** if it satisfies the following axioms:

- (locality) Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $s, t \in \mathscr{F}(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $(s_i)_{i\in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathscr{F}(U_i)$. If for all $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i\cap U_j}$, then there exists $s \in \mathscr{F}(U)$ such that for all $i \in I$, $s|_{U_i} = s_i$.

This definition basically requires that glueing locally (but cogently) defined sections is allowed, and that it is unique.

Definition 14. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X.
A morphism of presheaves (or simply a natural transformation) from $\mathscr{F}$ to $\mathscr{G}$ is called a **morphism of sheaves** from $\mathscr{F}$ to $\mathscr{G}$.
We can thus verify that with these morphisms, sheaves on X form categories, denoted by $\mathrm{Sh}(X)$ for sheaves of sets, $\mathrm{AbSh}(X)$ for abelian presheaves, and $\mathrm{RngSh}(X)$ for sheaves of rings.

Remark 15. If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then for each point $x \in X$, Φ induces a morphism on stalks:

$$\Phi_x : \mathscr{F}_x \to \mathscr{G}_x$$

Given a presheaf, it is natural to ask if it can be "turned into a sheaf". We will see that this can be done without adding previously non existent local behavior (*i.e.* new stalks).

Definition 16. Let $\mathscr{F}$ be a presheaf.
For every open set U we define:

$$\mathscr{F}^{\#}(U) = \Bigg\{ (s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x, \\ \forall x \in U, \exists V \in \mathrm{Op}(U), x \in V, \exists \sigma \in \mathscr{F}(V), \forall v \in V, \sigma_v = s_v \Bigg\}$$

We also define for every pair of open sets $V \subset U$:

$$\mathrm{res}_{U,V} : \begin{cases} \mathscr{F}^{\#}(U) \to \mathscr{F}^{\#}(V) \\ (s_x)_{x\in U} \mapsto (s_x)_{x\in V} \end{cases}$$

Proposition 17. *Together with these restriction maps, $\mathscr{F}^{\#}$ is a sheaf. It is called the **sheaf associated with $\mathscr{F}$**.*

Proof. Locality: Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $s, t \in \mathscr{F}^{\#}(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s_x = t_x$ for all $x \in \cup_{i\in I} U_i = U$, hence $s = t$.

Gluing: Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $(s_i)_{i\in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathscr{F}^{\#}(U_i)$. If for all $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then $s = (s_x)_{x\in U}$ is such that $s \in \mathscr{F}(U)$ and for all $i \in I$, $s|_{U_i} = s_i$. $\square$

Remark 18. There is a canonical morphism of presheaves $\iota : \mathscr{F} \to \mathscr{F}^{\#}$.

The following propositions are easy, but fundamental. Their proofs will be sketched very briefly.

Proposition 19. *Let $\mathscr{F}$ be a sheaf.*
The canonical morphism of sheaves $\mathscr{F} \to \mathscr{F}^{\#}$ is an isomorphism.

Proof. This is immediate. □

Proposition 20. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf.*
If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, it uniquely factors through $\mathscr{F}^{\#}$.
In other words, there exists a unique morphism of sheaves $\Phi^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\Phi"] \arrow[d, "\iota"'] & \mathscr{G} \\ \mathscr{F}^{\#} \arrow[ur, "\Phi^{\#}"] & \end{tikzcd}$$

Proof. The existence of a factorization is a consequence of the fact that $\mathscr{G}$ is a sheaf, because then the image of a given stalk is unique by the locality axiom.

Its uniqueness follows from the fact that the stalks of a section on an open set determine its image by the glueing axiom.

Another, more concise way to look at it is the following:
We clearly have a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "\Phi"'] & \mathscr{F}^{\#} \arrow[d, "\Phi^{\#}"] \\ \mathscr{G} \arrow[r] & \mathscr{G}^{\#} \end{tikzcd}$$

where the horizontal arrows are canonical.
We saw that the bottom one was an isomorphism, thus the result. □

Proposition 21. *The functor $\#$: $*\mathrm{PSh}(X) \to *\mathrm{Sh}(X)$ is left-adjoint to the forgetful functor $*\mathrm{Sh}(X) \to *\mathrm{PSh}(X)$.*

Proof. The last proposition essentially amounts to a bijection

$$\mathrm{Hom}_{*\mathrm{PSh}(X)}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{*\mathrm{Sh}(X)}(\mathscr{F}^{\#}, \mathscr{G})$$

The algebraic structure of this bijection, as well as its functoriality with regards to both $\mathscr{F}$ and $\mathscr{G}$ are easy. □

Proposition 22. *Let $\Phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*
The following are equivalent:

1. *Φ is a monomorphism (resp. epimorphism) in $*\mathrm{Sh}(X)$.*

2. *For all $U \subset X$ the induced map $\Phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a monomorphism (resp. epimorphism).*

3. *For all $x \in X$ the induced map on stalks $\Phi_x : \mathscr{F}_x \to \mathscr{G}_x$ is a monomorphism (resp. epimorphism).*

Remark 23. All the categories of sheaves we will be working with allow the term "monomorphism" (resp. "epimorphism") to be replaced by "injective morphism" (resp. "surjective morphism").

Let us now define kernels and cokernels.

Definition 24. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X, and $\Phi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves.
The **kernel** of Φ is a presheaf defined for every open set U by:

$$\mathrm{Ker}(\Phi)(U) := \mathrm{Ker}(\Phi_U) \subset \mathscr{F}(U)$$

Proposition 25. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then* $\mathrm{Ker}(\Phi)$ *is a sheaf.*

Proof. Locality: Suppose U is an open set of X, $(U_i)_{i \in I}$ is an open covering of U, and $s, t \in \mathrm{Ker}(\Phi)(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then by the locality axiom for $\mathscr{F}$, $s = t$.

Gluing: Suppose U is an open set, $(U_i)_{i \in I}$ is an open covering of U, and $(s_i)_{i \in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathrm{Ker}(\Phi)(U_i)$. If for all $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then by the gluing axiom for $\mathscr{F}$ there exists $s \in \mathscr{F}(U)$ such that for all $i \in I$, $s|_{U_i} = s_i$. But for all $i \in I$,

$$\Phi(s)(U_i) = \Phi(s|_{U_i})(U_i) = \Phi(s_i)(U_i) = 0,$$

so that $\Phi(s)$ vanishes over U by the locality axiom for $\mathscr{G}$, thus $s \in \mathrm{Ker}(\Phi)(U)$. $\square$

Proposition 26. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then* $\mathrm{Ker}(\Phi) = 0$ *if and only if Φ is injective.*

Proof. If $\mathrm{Ker}(\Phi) = 0$, let $x \in X$ be a point, and let $s \in \mathscr{F}(U_s)$ and $t \in \mathscr{F}(U_t)$ be sections such that $\Phi_x(s_x) = \Phi_x(t_x)$. $\Phi(s)$ and $\Phi(t)$ must then agree on some open set V containing x, thus $\mathrm{Ker}(\Phi)(V) = 0$ implies $s|_V = t|_V$, hence $s_x = t_x$.

If Φ is injective, let U be an open set and $s \in \mathrm{Ker}(\Phi)(U)$. $\Phi(s)$ vanishes on U, so in particular its stalks are those of the zero section. But then injectivity gives that s is zero on a neighborhood of each point of U, which ensures by the gluing axiom that $s = 0$. $\square$

Proposition 27. *Kernels as we just defined them are kernels in the categorical sense.*

Now let us state some analogous results for sujective morphisms.

Definition 28. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X, and $\Phi : \mathscr{F} \to \mathscr{G}$ a morphism of presheaves.
The **image presheaf** is defined for every open set U by:

$$\operatorname{Im}(\Phi_U) \subset \mathscr{G}(U)$$

The sheaf $\operatorname{Im}(\Phi)$ associated to the image presheaf is called the **image** of Φ.

Remark 29. In general, the image presheaf is not a sheaf and thus is not isomorphic to its associated sheaf.

Proposition 30. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then it induces a morphism of sheaves $\operatorname{Im}(\Phi) \to \mathscr{G}$, which is an isomorphism if and only if Φ is surjective.*

Proof. By the universal property of the sheaf associated to a presheaf, there exists a morphism of sheaves $\phi : \operatorname{Im}(\Phi) \to \mathscr{G}$.

If ϕ is an isomorphism, let $x \in X$ be a point and $g_x \in \mathscr{G}_x$ a stalk at x. By surjectivity of ϕ, there exists a section t of $\operatorname{Im}(\Phi)$ such that $\Phi_x(t) = g_x$. Since sections of $\operatorname{Im}(\Phi)$ are locally induced by sections of the image presheaf, there exists a section s of $\mathscr{F}$ such that

$$\phi_x(s_x) = g_x.$$

If Φ is surjective, then clearly ϕ is surjective. Also the image presheaf is injective, so that ϕ is also injective. Indeed, sections of the associated sheaf locally coincide with sections of the presheaf. $\square$

Let us now treat the special case of sheaves of modules.

Definition 31. Let $\mathscr{O}$ be a sheaf of rings.
A **sheaf of $\mathscr{O}$-modules** is a presheaf of $\mathscr{O}$-modules such that the underlying presheaf of abelian groups is a sheaf.
A morphism of sheaves of $\mathscr{O}$-modules is a morphism of presheaves of $\mathscr{O}$-modules.

Remark 32. This definition could be extended to the case where $\mathscr{O}$ is not a sheaf, but this is useless in practice.

Proposition 33. *Together with the morphisms we defined, sheaves of $\mathscr{O}$-modules form a category denoted by $\mathscr{O}-\mathrm{mod}$.*

Theorem 34. *The categories $\mathrm{AbSh}(X)$ and $\mathscr{O}-\mathrm{mod}$ are abelian.*

Before getting into sheaf cohomology, let us see how sheaves behaves with respect to continuous maps from a topological space to another.

Definition 35. Let Y be a topological space and $f : X \to Y$ be a continuous map.
Then for any sheaf $\mathscr{F}$ on X, we can naturally define a sheaf on Y by setting

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

The sheaf $f_*\mathscr{F}$ is called the **the image sheaf of $\mathscr{F}$.**

Now it is tempting to define similarly $f^*\mathscr{G}(U) = \mathscr{G}(f(U))$, but this does not define a sheaf in general ($f(U)$ is not even open in general). Still, taking the appropriate colimit solves (most of) our problems.

Definition 36. Let Y be a topological space and $f : X \to Y$ be a continuous map.
Then for any sheaf $\mathscr{G}$ on Y, we can (more or less) naturally define a sheaf on X by setting

$$f^{-1}\mathscr{G}(U) := \operatorname*{colim}_{\substack{W \text{ open in X} \\ W \supset U}} \mathscr{G}(W)$$

The sheaf $f^{-1}\mathscr{G}$ is called the **the inverse image sheaf of $\mathscr{G}$.**

1.3 Sheaf cohomology

We saw that a sequence of sheaves

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

was exact if for all $x \in X$ the induced sequence of stalks at x was exact.

Proposition 37. *Let $\mathscr{F}$ be a sheaf on X, and U an open set.*
$\Gamma(U, -)$ is a functor from AbSh(X) *to* Ab. *This functor is **left-exact.***
*When $U = X$, it is called the **global section functor.***

Proof. Let

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathbb{H} \longrightarrow 0$$

be a short exact sequence of sheaves of $\mathscr{O}$-modules.
By applying $\Gamma(U, -)$ we get

$$0 \longrightarrow \mathscr{F}(X) \xrightarrow{\Phi_U} \mathscr{G}(U) \xrightarrow{\Psi_U} \mathbb{H}(U) \longrightarrow 0$$

Given that $\mathrm{Ker}(\Phi) = 0$, it is clear that $\mathrm{Ker}(\Phi_U) = 0$.
Also, $\Psi \circ \Phi = 0$, hence $\Psi_U \circ \Phi_U = 0$ and $\Im(\Phi_X) \subset \mathrm{Ker}(\Psi_X)$.
Lastly, any section $s \in \mathscr{G}(X)$ which maps to 0 in $\mathbb{H}(X)$ is by definition locally a section of $\mathscr{F}$. But $\mathscr{F}$ is a sheaf, hence by the glueing axiom s is a section in $\mathscr{F}(X)$. $\square$

Proposition 38. *The categories* AbSh(X) *and* $\mathscr{O}-$mod *have **enough injectives.***

Proof. The proof for AbSh(X) is essentially a simpler version of the one for $\mathscr{O}-$mod, so let us show only the latter.
Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
For all $x \in X$, let us fix an injective morphism of $\mathscr{O}_x$-modules $\mathscr{F}_x \to I_x$ in an injective $\mathscr{O}_x$-module I_x.
We can then define a sheaf $\mathscr{I}$ by setting

$$\mathscr{I}(U) := \prod_{x \in U} I_x$$

There is an obvious injective morphism of sheaves of $\mathscr{O}$-modules $\mathscr{F} \to \mathscr{I}$.
It now remains to show that $\mathscr{I}$ is indeed injective.

A morphism of sheaves of $\mathscr{O}$-modules $f : \mathscr{A} \to \mathscr{I}$ is equivalent to a collection of maps $f_x : \mathscr{A}_x \to \mathscr{I}_x$ such that for all $x \in X$, the f_x is an injective morphism of $\mathscr{O}_x$-modules.
Given an injective morphism of sheaves of $\mathscr{O}$-modules $i : \mathscr{A} \to \mathscr{B}$, we have injective morphisms $i_x : \mathscr{A}_x \to \mathscr{B}_x$ such that for all $x \in X$, $g_x : \mathscr{B}_x \to \mathscr{I}_x$ is a morphism extending f_x, because $\mathscr{I}_x$ is injective.
But then these morphisms define a morphism of sheaves of $\mathscr{O}$-modules $g : \mathscr{B} \to \mathscr{I}$ extending f, hence the result. □

This last result allows us to define derived functors, and in particular, sheaf cohomology in $\mathrm{AbSh}(X)$ and $\mathscr{O}-$mod.

Definition 39. We define **sheaf cohomology** $H^\bullet(X, -)$ as the right-derived functor of the global section functor $\Gamma(X, -)$.

Definition 40. A sheaf of $\mathscr{O}$-modules $\mathscr{F}$ is said to be

- **flasque** if for every open set U, any section $s \in \mathscr{F}(U)$ can be extended to a global section;
- **soft** if for every closed sets C, any section

  $$s \in \mathscr{F}(C) := \operatorname*{colim}_{\substack{U \text{ open} \\ U \supset C}} \mathscr{F}(U)$$

  can be extended to a global section;
- **fine** if for every open cover $(U_i)_{i \in I}$ of X there is a family of morphisms $(\phi_\alpha : \mathscr{F} \to \mathscr{F})_{\alpha \in A}$ subordinate to that cover such that locally, at any point $x \in X$, all but finitely many vanish (*i.e.* $(\phi_\alpha)_x = 0$ for all but finitely many $\alpha \in A$), and $\sum_{\alpha \in A} \phi_\alpha = \mathrm{id}$.

Definition 41. We say that a sheaf $\mathscr{A}$ is **acyclic** for the functor Γ (or Γ-acyclic) if $R^p\Gamma(\mathscr{A}) = 0$ for all $p > 0$.

Remark 42. From now on, "acyclic" will always mean "Γ-acyclic", where Γ is the global section functor.

The whole point of acyclic objects is that they behave somewhat like injective objects, while being easier to manipulate (and more abundant in practice).

Proposition 43. *Let $\mathscr{F} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots$ be an acyclic resolution of $\mathscr{F}$. Then for all $p \in \mathbb{N}$, $R^p\Gamma(X, \mathscr{F})$ is equal to $H^p(\Gamma(X, \mathscr{A}^\bullet))$.*

Proof. Let's procede by induction.
We have a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{A}^0 \to \mathscr{B} \to 0$$

where $\mathscr{B} := \mathrm{Coker} d_0$. But $(\mathscr{A}^{\bullet+1}, d_0 : \mathscr{B} \to \mathscr{A}^1)$ is a resolution of $\mathscr{B}$, thus the long exact sequence induced by our short exact sequence amounts to:

$$\begin{cases} R^p\Gamma(\mathscr{F}) = R^{p-1}\Gamma(\mathscr{B}) & \text{for } p > 1 \\ R^1\Gamma(\mathscr{F}) = H^1(\Gamma(\mathscr{A}^\bullet)) \end{cases}.$$

We can now apply the same reasoning to $\mathscr{B}$, thus the result. □

The following proposition sums up the relationships between the different types of sheaves we just defined. The complete proof is quite lengthy and a bit technical, therefore I chose not to include it.

Proposition 44. *Suppose X is paracompact.*

(i) Injective sheaves are flasque.

(ii) Flasque sheaves are soft.

(iii) Soft sheaves are fine.

(iv) Fine sheaves are acyclic.

Proof. See [Taylor]. □

Remark 45. It is required that X be paracompact, which will always be assumed in what follows.

Another very useful property is the following:

Proposition 46. *If the first two terms of an exact sequence of sheaves of $\mathscr{O}$-modules are soft, then so is the third.*

Proof. Again, see [Taylor]. □

2 Analytic de Rham cohomology

In this section, M will denote a smooth manifold.

Definition 47. A **tangent vector** of M at a point $x \in M$ is defined to be a function

$$\partial : \mathscr{C}^\infty(M) \to \mathbb{R}$$

such that:

$$\begin{cases} \partial c = 0 & \text{for } c \text{ constant} \\ \partial(f+g) = \partial(f) + \partial(g) & \text{for } f, g \text{ smooth} \\ \partial(fg) = f(x)\partial(g) + g(x)\partial(f) & \text{for } f, g \text{ smooth} \end{cases}$$

Clearly, tangent vectors at x form a $\mathbb{R}$-vector space, denoted by TM_x and called the **tangent space to M at x.**

Remark 48. Given a smooth map of smooth manifolds $\phi : M \to N$ and a point $x \in M$, we clearly have a linear map

$$D\phi_x : \begin{cases} TM_x \to TN_{\phi(x)} \\ \partial \mapsto (f \mapsto \partial(f \circ \phi)) \end{cases}$$

Lemma 49. *Let $f \in \mathscr{C}^\infty(M)$ and $\partial \in TM_x$.*
If there exists an open set $U \subset M$ such that $x \in U$ and $f|_U = 0$, then $\partial f = 0$.

Proof. There exists an open set $V \subset U$ and $g \in \mathscr{C}^\infty(M)$ such that $g|_V = 0$ and $g|_{M \setminus U} = 1$. Then $f = fg$ and

$$\partial f = \partial(fg) = f(x)\partial g + g(x)\partial f = 0$$

$\square$

This justifies that $TM_x \cong TU_x$ for any open set U containing x. We can then take charts, so it is sufficient to work on open subset of $\mathbb{R}^n$.

Given a vector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, it is easy to see that

$$\partial_v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n},$$

where the $\frac{\partial}{\partial x_i}$ are the usual partial derivatives, defines a tangent vector at x. As we will see, they can all be defined like this.

Lemma 50. *Let $U \subset R^n$ be an open set and $x \in U$.*
The map

$$\begin{cases} \mathbb{R}^n \to TU_x \\ v \mapsto \partial_v \end{cases}$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. Without loss of generality we suppose $x = 0$.
Linearity and injectivity are clear.
Let $f \in \mathscr{C}^\infty(U)$. We have that for $u = (u_1, \cdots, u_n)^T \in U$:

$$f(u) = f(0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) u_i + \sum_{i,j=1}^{n} u_i u_j \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tu) \mathrm{d}t$$

We can now apply ∂, and see that:

- $\partial f(0) = 0$
- For $i \in [\![1, n]\!]$, $\partial \left(\frac{\partial f}{\partial x_i}(0) \pi_i \right) = \frac{\partial f}{\partial x_i}(0)$.
- For $i, j \in [\![1, n]\!]$,

$$\partial \left((\pi_i(\cdot)) \left(\pi_j(\cdot) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(t \cdot) \mathrm{d}t \right) \right) = 0$$

  because both parenthesized terms are smooth and vanish at 0.

Thus surjectivity is established, which completes the proof. $\square$

Remark 51. We can see that $(\mathrm{d}x_i)_{1 \leqslant i \leqslant n}$ is the dual basis of $\left(\frac{\partial}{\partial x_i} \right)_{1 \leqslant i \leqslant n}$.
Also, if $x, y \in U$, we have a canonical isomorphism $TU_x \cong TM_y$ which allows to define smooth maps $U \to TU$ of the form

$$\begin{cases} U \to TU \\ u \mapsto g_1(u) \frac{\partial}{\partial x_1} + \cdots + g_n(u) \frac{\partial}{\partial x_n} \end{cases}$$

Such maps form a free $\mathscr{C}^\infty(U)$-module on the basis $\left(\frac{\partial}{\partial x_i} \right)_{1 \leqslant i \leqslant n}$, denoted by $\mathrm{Vect}(U)$ and called the **space of vector fields on U.**

Definition 52. The dual of TM_x is denoted by TM_x^* and called the **cotangent space of M at x.**

We can now define differential forms.

Definition 53. We can now set

- $\Omega^0(U) = \mathscr{C}^\infty(U)$
- $\Omega^1(U) = \mathrm{Hom}_{\mathscr{C}^\infty(U)}(\mathrm{Vect}(U), \mathscr{C}^\infty(U))$
- and
$$\Omega^k(U) = \bigwedge\nolimits_{\mathscr{C}^\infty(U)}^k \Omega^1(U)$$

$\Omega^k(U)$ is called the **set of k-forms on U.**

The following proposition will show that they are indeed functorial with respect to smooth maps.

Proposition 54. *Let $V \subset \mathbb{R}^m$ be an open subset, $\phi : U \to V$ be a smooth map, and $\alpha \in \Omega^k(V)$.*
The formula

$$(\phi^{-1}\alpha)_x(\partial_1, \cdots, \partial_n) := \alpha_{\phi(x)}(D\phi_x(\partial_1), \cdots, D\phi_x(\partial_n))$$

*(where $x \in U$ and $\partial_1, \cdots, \partial_n \in TU_x$) defines a k-form on U called the **pullback of α on U.***

Definition 55. The functoriality of k-forms with respect to smooth maps allows one to define corresponding sheaves on open subsets of $\mathbb{R}^n$, and then on M by glueing:

- $\underline{\Omega}_M^0 = \mathscr{C}_M^\infty$
- $\underline{\Omega}_M^1 = \mathrm{Hom}_{\mathscr{C}_M^\infty}(\underline{\mathrm{Vect}}_M, \mathscr{C}_M^\infty)$
- and
$$\underline{\Omega}_M^k = \bigwedge\nolimits_{\mathscr{C}_M^\infty}^k \underline{\Omega}_M^1$$

An easy proof by induction gives the following central result:

Proposition 56. *For all $k \in \mathbb{N}$ there exists a unique morphism of abelian sheaves $\mathrm{d} : \underline{\Omega}_M^k \to \underline{\Omega}_M^{k+1}$ such that:*

(i) Given an open set U, a section $f \in \underline{\Omega}_M^0(U)$, a point $x \in U$ and a tangent vector $\partial \in TM_x$,
$$(\mathrm{d}f)_x(\partial) = \partial(f)$$

(ii) Given an open set U and sections $\omega \in \underline{\Omega}_M^m(U)$, $\eta \in \underline{\Omega}_M^l(U)$,
$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^m \omega \wedge \mathrm{d}\eta$$

(iii) We have
$$\mathrm{d} \circ \mathrm{d} = 0$$

Proof. See [Kriz^2]. □

This application, which we will call the **exterior differential,** has nice compatibility relationships with the wedge product and pullbacks.

Proposition 57. *Let α and β be forms on V.*
We have the following compatibility relationships:

$$\begin{cases} \phi^{-1}(\mathrm{d}\alpha) = \mathrm{d}(\phi^{-1}\alpha) \\ \phi^{-1}(\alpha \wedge \beta) = \phi^{-1}(\alpha) \wedge \phi^{-1}(\beta) \end{cases}$$

Proposition 58 (analytic Poincaré lemma)**.** *The de Rham complex*

$$0 \to \mathbb{K} \to \Omega^0(\mathbb{K}^n) \to \Omega^1(\mathbb{K}^n) \to \cdots \to \Omega^n(\mathbb{K}^n) \to 0$$

is exact.

Proof. We procede by induction on n.
The result is clear for $n = 0$.
If $n > 0$, let $p \in [\![1, n]\!]$ and $\omega \in \Omega^p(\mathbb{K}^n)$ be such that $\mathrm{d}\omega = 0$, and let's show that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{p-1}(\mathbb{K}^n)$.
We write

$$\omega = \omega_1 \mathrm{d}x_1 + \omega_2$$

where ω_1 and ω_2 do not involve $\mathrm{d}x_1$. We can "integrate" ω_1 "with respect to x_1", *i.e.* replace smooth functions g by $\int_0^{x_1} g(t, x_2, \cdots, x_n)\mathrm{d}t$ wherever they occur, and set

$$\eta_1 = \int \omega_1 \mathrm{d}x_1$$

so that

$$\mathrm{d}\eta_1 = \omega_1 \mathrm{d}x_1 + \omega_3$$

where ω_3 does not involve $\mathrm{d}x_1$. But then $\omega - \mathrm{d}\eta_1 = \omega_2 - \omega_3$ does not involve $\mathrm{d}x_1$ either, so we reduce to the case where ω does not involve $\mathrm{d}x_1$, in which case

$$\omega = \sum_{1 < i_1 < \cdots < i_p} f_{i_1, \cdots, i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

But since $\mathrm{d}\omega = 0$, we have

$$\frac{\partial f_{i_1, \cdots, i_p}}{\partial x_1} = 0$$

for all $1 < i_1 < \cdots < i_p$, the smooth functions $f_{i_1, \cdots, i_p}$ do not involve x_1 at all. The case of $\mathbb{K}^{n-1}$, for which the result is true by induction, allows us to conclude. □

Theorem 59. *Let X be a smooth $\mathbb{K}$-variety.*
The de Rham sheaf complex is a resolution of the constant sheaf $\underline{\mathbb{K}}$.
Moreover, it is acyclic, thus for all $p \in \mathbb{N}$:

$$H^p_{\mathrm{dR}}(X, \mathbb{K}) = H^p(X, \underline{\mathbb{K}})$$

Proof. The Poincaré lemma already shows that the de Rham sheaf complex is a resolution of the constant sheaf, by taking stalks (colimits) at each point.
Moreover, we can show it is a fine (thus acyclic because X is paracompact and Hausdorff) resolution, using a partition of unity. □

Proof. See [Taylor]. □

3 Algebraic de Rham cohomology

The obvious starting point of any attempt to compare analytic and algebraic de Rham cohomology is that... it makes sense to compare them. As we will see, an smooth algebraic variety over $\mathbb{C}$ can be endowed with the structure of a smooth complex manifold (to which our definition of analytic de Rham cohomology applies).

Proposition 60. *A smooth algebraic variety X over $\mathbb{C}$ gives rise to a complex manifold X_{an}.*

Proof. Indeed, the closed points of X can be locally identified with the set of solutions of a system of equations

$$\begin{cases} f_1(z_1, \cdots, z_n) = 0 \\ \qquad \vdots \\ f_m(z_1, \cdots, z_n) = 0 \end{cases}$$

where $f_1, \cdots, f_m$ are rational functions with non-zero denominators on some Zariski (hence analytically) open set $U \subset \mathbb{C}^n$.
Also, $n \geqslant m$ and the ideal of $\mathscr{O}|_U$ generated by determinants of $m \times m$ submatrices of

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{pmatrix}$$

contains 1, which means that these determinants can not all vanish simultaneously.
Hence for any point $(z_1, \cdots, z_n) \in V$, there exists $j_1 < \cdots < j_m$ and a neighborhood on which the determinant with columns $j_1 < \cdots < j_m$ is non-zero.
It is then clear from the implicit function theorem that the system defines the graph of an analytic function of the $n-m$ variables z_i such that $i \neq j_k$ for some $k \in [\![1, m]\!]$, which is a complex manifold. It is denoted X_{an}, and called X with the analytic topology. $\square$

Let us begin with the affine case.

Definition 61. Let $A \to B$ be a morphism of commutative rings (in other words, let B be an A-algebra).
A **differentiation** on B over A consists of a B-module M and a morphism of A-modules $\mathrm{d} : B \to M$ such that

$$\begin{cases} \mathrm{d}(1) = 0 \\ \forall x, y \in B, \mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x \end{cases}$$

Definition 62. Let F_B be the free B-module on $(\mathrm{d}(b))_{b \in B}$, and D be the submodule generated by elements of the form $\mathrm{d}(a)$ for $a \in A$ or $\mathrm{d}(xy) - x\mathrm{d}y - y\mathrm{d}x$ for $x, y \in B$.
The quotient B-module

$$\Omega_{B/A} := F_B / D$$

is called the **B-module of Kähler differentials of B over A.**

The following proposition easily follows from the definitions.

Proposition 63. *The map*

$$\mathrm{d} : \begin{cases} B \to \Omega_{B/A} \\ b \mapsto \mathrm{d}(b) \end{cases}$$

is a ***universal*** *differentiation, i.e. for any differentiation* $\mathrm{d}' : B \to M$*, there exists a* ***unique*** *morphism of* B*-modules* $f : \Omega_{B/A} \to M$ *such that* $\mathrm{d}' = f \circ \mathrm{d}$.

Proposition 64 (algebraic Poincaré lemma)**.** *Let* k *be a field with characteristic* 0 *and* $B = k[x_1, \cdots, x_n]$ *(such that* $\operatorname{Spec} B = \mathbb{A}_k^n$*).*
Then the de Rham complex

$$0 \to k \to \Omega^0_{B/k} \to \Omega^1_{B/k} \to \cdots \to \Omega^n_{B/k} \to 0$$

is exact.

Proof. We procede by induction on n.
The result is clear for $n = 0$.
If $n > 0$, let $p \in [\![1, n]\!]$ and $\omega \in \Omega^p_{B/k}$ be such that $\mathrm{d}\omega = 0$, and let's show that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{p-1}_{B/k}$.
We write

$$\omega = \omega_1 \mathrm{d}x_1 + \omega_2$$

where ω_1 and ω_2 do not involve $\mathrm{d}x_1$. We can "integrate" ω_1 "with respect to x_1", *i.e.* integrate polynomials formally with respect to x_1 wherever they occur, and set

$$\eta_1 = \int \omega_1 \mathrm{d}x_1$$

so that

$$\mathrm{d}\eta_1 = \omega_1 \mathrm{d}x_1 + \omega_3$$

where ω_3 does not involve $\mathrm{d}x_1$. But then $\omega - \mathrm{d}\eta_1 = \omega_2 - \omega_3$ does not involve $\mathrm{d}x_1$ either, so we reduce to the case where ω does not involve $\mathrm{d}x_1$, in which case

$$\omega = \sum_{1 < i_1 < \cdots < i_p} f_{i_1, \cdots, i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

But since $\mathrm{d}\omega = 0$, we have

$$\frac{\partial f_{i_1, \cdots, i_p}}{\partial x_1} = 0$$

for all $1 < i_1 < \cdots < i_p$, the polynomials $f_{i_1, \cdots, i_p}$ do not involve x_1 at all.
The case of $k[x_2, \cdots, x_n]$, for which the result is true by induction, allows us to conclude. ☐

Remark 65. This proof is almost the exact same as for the analytic Poincaré lemma.

Proposition 66. *Let* A *be a Noetherian ring and suppose that* B *is a standard smooth* A*-algebra of dimension* d*.*
Then $\Omega_{B/A}$ *is a rank* d *projective* B*-module.*

Proof. See [Kriz²]. □

Now if X is a smooth algebraic variety over k, there is an open cover $(U_i)_{i\in I}$ such that for all $i \in I$, $U_i = \mathrm{Spec}\,(A_i)$ where A_i is a standard smooth k-algebra.

We will not get into the details of the matter, but Kähler differentials are compatible with localization, *i.e.* for all $p \in \mathbb{N}$ and $g \in A_i$, $\Omega^p(g^{-1}A_i/k) \cong g^{-1}\Omega^p(A_i/k)$.

Given that the corresponding localized open subsets U_{i_g} form a basis of the Zariski topology, this compatibility allows to glue the Ω^p into a coherent algebraic sheaf $\underline{\Omega}^p_{X/k}$ on X.

Definition 67. Similarly to the analytic case we set:

- $\underline{\Omega}^0_{X/k} = \mathscr{O}$
- $\underline{\Omega}^1_{X/k} = \underline{\Omega}_{X/k}$
- and
$$\underline{\Omega}^p_{X/k} = \bigwedge\nolimits^p_{\mathscr{O}} \underline{\Omega}^1_{X/k}$$

Analogous results allow us to define an exterior differential:

Proposition 68. *For all $p \in \mathbb{N}$ there exists a unique morphism of abelian sheaves $\mathrm{d} : \underline{\Omega}^k_{X/k} \to \underline{\Omega}^{k+1}_{X/k}$ such that:*

(i) We have a morphism of sheaves of k-modules $\mathrm{d} : \underline{\Omega}^0_{X/k} \in \underline{\Omega}^1_{X/k}$

(ii) Given an open set U and sections $\omega \in \underline{\Omega}^m_{K/k}(U)$, $\eta \in \underline{\Omega}^l_{X/k}(U)$,
$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^m \omega \wedge \mathrm{d}\eta$$

(iii) We have
$$\mathrm{d} \circ \mathrm{d} = 0$$

Proof. We can use the fact that $\underline{\Omega}^1_{X/k}$ is locally free, keep in mind the proof of the analytic case and proceed analogously. □

Last but not least, the desired cohomology:

Definition 69. The algebraic de Rham cohomology of X is the hypercohomology
$$H^p_{\mathrm{dR}}(X) := \mathbb{H}(X, \underline{\Omega}^\bullet_{X/k})$$

4 More sheaf theory

4.1 Properties of some classical operations on sheaves

Definition 70. Let $i : Y \to X$ be an embedding, and $\mathscr{G}$ be a sheaf on Y.

(i) If Y is closed in X, then $i_*\mathscr{G}$ defines a sheaf called the **extension of $\mathscr{G}$ to X by zero.** It has the property that

$$(i_*\mathscr{G})_x = \begin{cases} \mathscr{G}_x & \text{if} x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

(ii) If Y is open, $i_*\mathscr{G}$ does not have this last property in general. But we still have that

$$V \mapsto \begin{cases} \mathscr{G}(V) & \text{if} V \subset Y, \\ 0 & \text{otherwise} \end{cases}$$

defines a presheaf on X, whose associated sheaf $\mathscr{G}^X$ (or $i_!\mathscr{G}$) is also called the **extension of $\mathscr{G}$ to X by zero.** It has the property that

$$\mathscr{G}_x^X = \begin{cases} \mathscr{G}_x & \text{if} x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

Proposition 71. *Let $\mathscr{R}$ be a sheaf of rings on X.*
We have the following:

(i) If $\mathscr{I}$ is an injective sheaf of $\mathscr{R}$-modules on X, and U is an open set, then $\mathscr{I}|_U$ is an injective sheaf of $\mathscr{R}|_U$-modules.

(ii) If $f : Y \to X$ is a continuous map and $\mathscr{I}$ an injective sheaf of $f^{-1}\mathscr{R}$-modules, then $f_*\mathscr{I}$ is an injective sheaf of $\mathscr{R}$-modules.

(iii) The direct product of any set of injective sheaves of $\mathscr{R}$-modules is an injective sheaf of $\mathscr{R}$-modules.

Proof. Given that extension by zero is exact, and that $\mathscr{I}$ is injective, $\mathrm{Hom}((-)^X, \mathscr{I})$ is an exact functor. Hence $\mathrm{Hom}_{\mathscr{R}|_U}(-, \mathscr{I}|_U) \cong \mathrm{Hom}_{\mathscr{R}}((-)^X, \mathscr{I})$ implies that $\mathrm{Hom}_{\mathscr{R}|_U}(-, \mathscr{I}|_U)$ is an exact functor, thus proving that $\mathscr{I}|_U$ is injective. This proves (i).

We already saw that, for a sheaf $\mathscr{G}$ of $\mathscr{R}$-modules on X and a sheaf $\mathscr{I}$ of $f^{-1}\mathscr{R}$-modules on Y, we have $\mathrm{Hom}_{f^{-1}\mathscr{R}}(f^{-1}\mathscr{G}, \mathscr{I}) \cong \mathrm{Hom}_{\mathscr{R}}(\mathscr{G}, f_*\mathscr{I})$. The functor f^{-1} is exact, so if $\mathscr{I}$ is injective, then $\mathrm{Hom}_{\mathscr{R}}(-, f_*\mathscr{I})$ is exact, and $f_*\mathscr{I}$ is injective. This proves (ii).

Let $(\mathscr{I}_\alpha)_{\alpha \in A}$ be a set of injective sheaves on X, and let $i : \mathscr{F} \to \mathscr{G}$ be a injective morphism of sheaves. A morphism $f : \mathscr{F} \to \prod_{\alpha \in A} \mathscr{I}_\alpha$ is just a product of morphisms $f_\alpha : \mathscr{F} \to \mathscr{I}_\alpha$. Each of these extends to $g_\alpha : \mathscr{G} \to \mathscr{I}_\alpha$, and $\prod_{\alpha \in A} g_\alpha : \mathscr{G} \to \prod_{\alpha \in A} \mathscr{I}_\alpha$ extends f, hence $\prod_{\alpha \in A} \mathscr{I}_\alpha$ is also injective, which completes the proof. □

Proposition 72. *Let $i : Y \to X$ be an embedding of Y as a closed subspace of X, and $\mathscr{G}$ be a sheaf on Y.*
Then for all $p \in \mathbb{N}$, the canonical morphism

$$H^p(Y, \mathscr{G}) \to H^p(X, i_*\mathscr{G})$$

is an isomorphism.

Proof. The sheaf $i_*\mathscr{G}$ is just $\mathscr{G}$ extended to X by zero.
It is easily seen that i_* is an exact functor. Indeed, given an exact sequence of sheaves on Y, applying i_* and taking stalks at $x \in X$ gives either a canonically isomorphic exact sequence of stalks if $x \in Y$, or a trivial sequence if $x \notin Y$ (because $X \setminus Y$ is open).
Now if $\mathscr{G} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \cdots$ is an injective resolution of $\mathscr{G}$, then by exactness $i_*\mathscr{G} \to i_*\mathscr{I}_0 \to i_*\mathscr{I}_1 \to \cdots$ is a resolution of $i_*\mathscr{G}$. It is moreover injective by the last proposition.
As $\Gamma(X, i_*\mathscr{G}) \cong \Gamma(Y, \mathscr{G})$ and $\Gamma(X, i_*\mathscr{I}_p) \cong \Gamma(Y, \mathscr{I}_p)$ for all $p \in \mathbb{N}$, we have the desired isomorphism. □

4.2 Analytic-algebraic correspondence

In the rest of the document, we will use abundantly the fact that regular functions on algebraic varieties form a sheaf $\mathscr{O}$, in much the same way that holomorphic functions (which we will often call analytic) on a complex manifold form a sheaf $\mathscr{H}$.

We begin with the definitions of quasi-coherent and coherent sheaves, which will be omnipresent in the rest of this document.

Definition 73. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
The sheaf $\mathscr{F}$ is called **quasi-coherent** when it has a local presentation at all points, *i.e.* for all $x \in X$ there exists an open set $U \ni x$ and sets I, J such that we have an exact sequence

$$\mathscr{O}^{\oplus I}|_U \longrightarrow \mathscr{O}^{\oplus J}|_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Definition 74. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
The sheaf $\mathscr{F}$ is called **coherent** when it satisfies the following:

(i) $\mathscr{F}$ is of finite type, *i.e.* for all $x \in X$ there exists an open set $U \ni x$ and a surjective map $\mathscr{O}^{\oplus m}|_U \to \mathscr{F}|_U$ for some integer $m \in \mathbb{N}$.

(ii) For all open set U, any morphism $\mathscr{O}^{\oplus m}|_U \to \mathscr{F}|_U$ for some integer $m \in \mathbb{N}$ has a finitely generated kernel.

Remark 75. Determining whether or not sheaves like $\mathscr{H}$ are coherent or not is far from trivial, and in that particular case, it is called the Oka coherence theorem.

Let us now work towards defining the analytification functor.

There is a continuous map $\iota : X_{\text{an}} \to X$. Thus, given an algebraic sheaf $\mathscr{F}$, we are allowed to define its inverse image $\mathscr{F}' := \iota^{-1}\mathscr{F}$ on X_{an}. We then have, for U open in X:

$$\mathscr{F}'(U) = \operatorname*{colim}_{\substack{W \text{ open in } X \\ W \supset U}} \mathscr{F}(U)$$

Notice that at each point, the stalks are the same.

Definition 76. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on X.
We can define a corresponding analytic sheaf $\mathscr{F}_{\text{an}}$ on X_{an} by setting

$$\mathscr{F}_{\text{an}} := \mathscr{H} \otimes_{\mathscr{O}'} \mathscr{F}'$$

Remark 77. We have that for $x \in X$:

$$\mathscr{F}_{\text{an}x} = \operatorname*{colim}_{U \ni x} \mathscr{H}(U) \otimes_{\mathscr{O}'(U)} \mathscr{F}'(U) = \mathscr{H}_x \otimes_{\mathscr{O}'_x} \mathscr{F}'_x = \mathscr{H}_x \otimes_{\mathscr{O}_x} \mathscr{F}_x$$

Theorem 78. *Analytification as we defined it is a functor. Moreover:*

(i) It takes $\mathscr{O}^{\oplus m}$ to $\mathscr{H}^{\oplus m}$ for all $m \in \mathbb{N}$.

(ii) It takes coherent algebraic sheaves to coherent analytic sheaves.

(iii) It takes the ideal sheaf in $\mathscr{O}$ of an algebraic subvariety V to the ideal sheaf in $\mathscr{H}$ of the corresponding subvariety V_{an}.

Proof. Part (i) is trivial since for U open in X: $\mathscr{H}(U) \otimes_{\mathscr{O}(U)} \mathscr{O}(U) = \mathscr{H}(U)$.

Let $\mathscr{F}$ be a coherent algebraic sheaf on X.
If U is an affine open subset of X, then there is a finitely generated $\mathscr{O}(U)$-module M such that $\mathscr{F}|_U \cong \mathscr{O} \otimes_{\mathscr{O}(U)} M$. Also, we have a surjective morphism $\mathscr{O}(U)^m \to M$ for some M. Moreover its kernel is finitely generated since $\mathscr{O}(U)$ is Noetherian. Thus for some $k \in \mathbb{N}$ there is an exact sequence of modules

$$\mathscr{O}(U)^k \to \mathscr{O}(U)^m \to M \to 0$$

If we apply the functor $\mathscr{O} \otimes_{\mathscr{O}(U)} (-)$, we get an exact sequence of sheaves

$$\mathscr{O}^{\oplus k}|_U \to \mathscr{O}^{\oplus m}|_U \to \mathscr{F}|_U \to 0$$

to which we can apply the analytification functor and (i) to get the following:

$$\mathscr{H}^{\oplus k}|_U \to \mathscr{H}^{\oplus m}|_U \to \mathscr{F}_{\text{an}}|_U \to 0$$

Hence $\mathscr{F}_{\text{an}}$ is coherent, which proves (ii).

Part (iii) involves technical results not proven here, so its proof with be omitted. ☐

The details can be found in [Taylor], along with the following results.

Theorem 79. *The analytification functor is a faithful exact functor from the category of sheaves of $\mathscr{O}$-modules on X to the category of sheaves of $\mathscr{H}$-modules on X_{an}.*

Proof. It is clearly a functor from the category of sheaves of $\mathscr{O}$-modules on X to the category of sheaves of $\mathscr{H}$-modules on X_{an}.
The rest of the proof relies on a rather technical result, which is that $\mathscr{H}_x$ is faithfully flat over $\mathscr{O}_x$, and shall be omitted. ☐

Proposition 80. *Let $i : Y \to X$ be a closed embedding of algebraic varieties, and $\mathscr{F}$ be a coherent algebraic sheaf on Y.*
We have that

$$(i_*\mathscr{F})_{\text{an}} = i_*(\mathscr{F}_{\text{an}})$$

We can now prove Serre's first GAGA theorem:

Theorem 81 (GAGA). *Let X be a smooth projective variety and $\mathscr{F}$ a coherent algebraic sheaf on X.*
Then for all $p \in \mathbb{N}$, the canonical morphism

$$H^p(X, \mathscr{F}) \to H^p(X_{\mathrm{an}}, \mathscr{F}_{\mathrm{an}})$$

is an isomorphism.

Proof. Given that X is projective, we have an embedding $i : X \to \mathbb{P}^n_{\mathbb{C}}$ for some $n \in \mathbb{N}$.
We saw that $i_*\mathscr{F}$ was a coherent algebraic sheaf on $\mathbb{P}^n_{\mathbb{C}}$ which has the same cohomology groups as $\mathscr{F}$.
Analogously, $i_*\mathscr{F}_{\mathrm{an}}$ is a coherent analytic sheaf on $\mathbb{P}^n_{\mathbb{C}}$ which induces the same cohomology groups as $\mathscr{F}_{\mathrm{an}}$.
Also, $(i_*\mathscr{F})_{\mathrm{an}} = i_*\mathscr{F}_{\mathrm{an}}$, hence the theorem follows from the special case of projective spaces.

Thus we may restrict our attention to the case where $X = \mathbb{P}^n_{\mathbb{C}}$ for some $n \in \mathbb{N}$.
We established that $\mathscr{F}$ was a quotient of $\mathscr{O}(k)^{\oplus m}$ for some $m \in \mathbb{N}$. If $\mathscr{K}$ is the kernel of $\mathscr{O}(k)^{\oplus m} \to \mathscr{F}$, we have an exact sequence of coherent algebraic sheaves:

$$0 \to \mathscr{K} \to \mathscr{O}(k)^{\oplus m} \to \mathscr{F} \to 0$$

We can apply the analytification functor and obtain a corresponding exact sequence of coherent analytic sheaves

$$0 \to \mathscr{K}_{\mathrm{an}} \to \mathscr{H}(k)^{\oplus m} \to \mathscr{F}_{\mathrm{an}} \to 0$$

by remembering that $\mathscr{O}(k)^{\oplus m}_{\mathrm{an}} = \mathscr{H}(k)^{\oplus m}$.
This functor also induces morphisms on the corresponding long exact sequences:

$$\begin{tikzcd}
H^p(\mathbb{K}) \arrow[r] \arrow[d, "\epsilon_1"'] & H^p(\mathscr{O}(k)^{\oplus m}) \arrow[r] \arrow[d, "\epsilon_2"'] & H^p(\mathscr{F}) \arrow[r] \arrow[d, "\epsilon_3"'] & H^{p+1}(\mathbb{K}) \arrow[r] \arrow[d, "\epsilon_4"'] & H^{p+1}(\mathscr{O}(k)^{\oplus m}) \arrow[d, "\epsilon_5"'] \\
H^p(\mathbb{K}_{\mathrm{an}}) \arrow[r] & H^p(\mathscr{H}(k)^{\oplus m}) \arrow[r] & H^p(\mathscr{F}_{\mathrm{an}}) \arrow[r] & H^{p+1}(\mathbb{K}_{\mathrm{an}}) \arrow[r] & H^{p+1}(\mathscr{H}(k)^{\oplus m})
\end{tikzcd}$$

(where "$\mathbb{P}^n_{\mathbb{C}}$" was suppressed to make things readable).
Let us now proceed by downward recurrence on p.
Given that $H^p(\mathbb{P}^n_{\mathbb{C}}, \mathscr{G}) \cong H^p(\mathbb{P}^n_{\mathbb{C}}, \mathscr{G}_{\mathrm{an}}) \cong 0$ for all $p > n$, the result is obvious for $p > n$.
Now let $p \leqslant n$ and suppose the result is true in degree $p+1$, *i.e.*

$$H^{p+1}(\mathbb{P}^n_{\mathbb{C}}, \mathscr{G}) \cong H^{p+1}(\mathbb{P}^n_{\mathbb{C}}, \mathscr{G}_{\mathrm{an}})$$

for any coherent algebraic sheaf $\mathscr{G}$.
In particular, ϵ_4 is an isomorphism.
We saw earlier that $H^p(\mathbb{P}^n_{\mathbb{C}}, \mathscr{O}(k)) \cong H^p(\mathbb{P}^n_{\mathbb{C}}, \mathscr{H}(k))$, thus ϵ_2 and ϵ_5 are isomorphisms, and ϵ_3 is surjective by the first four lemma. The same reasoning allows us to find that ϵ_1 is surjective as well.
But then the second four lemma garantees that ϵ_3 is injective, and thus is an isomorphism, which completes our proof. $\square$

Finally, let us state the famous equivalence theorem of Grothendieck:

Theorem 82. *Let X be a smooth algebraic variety over $\mathbb{C}$.*
Then the analytification functor induces an isomorphism between the algebraic de Rham cohomology of X and the holomorphic de Rham cohomology of X_{an}:

$$H^\bullet_{\mathrm{dR}}(X) \cong H^\bullet_{\mathrm{dR}}(X_{\mathrm{an}}, \mathbb{C})$$

Proof. Serre's theorem can be extended to chain complexes of coherent sheaves non trivial in finitely many degrees, with arguments involving spectral sequences. Another key result used in the proof is that any smooth complex variety is isomorphic to a quotient of a smooth projective variety. We refer the reader to [Kriz^2], or simply Grothendieck's original article [Gro]. □

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