On Serre's affineness criterion and other vanishing theorems

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April 5, 2023

Abstract

We introduce the basic concepts of sheaf theory, up to the definition of sheaf cohomology. We then introduce the basics of scheme theory, to get to Serre's result as quickly as possible, while keeping black boxes to a minimum. I ended up avoiding the Čech machinery as an attempt to make this work as self-contained as possible.

Contents

1 Sheaf theory

In this section, X is a topological space, and whenever a topological space is mentionned implicitly, it will always be X.

1.1 Presheaves

Definition 1. A **presheaf of sets** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Set}$ into the category of sets.
Similarly, an **abelian presheaf** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Ab}$ into the

category of abelian groups.
Lastly, a **presheaf of rings** on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Rng}$ into the category of rings.

Remark 2. If U, V and W are open sets such that $W \subset V \subset U$, then we have:

$$\mathrm{res}_{W,U} = \mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}.$$

Thus we often omit the open set we restrict from, and if $s \in \mathscr{F}(U)$, we write $s|_V$ instead of $\mathrm{res}_{V,U}(s)$.

In what follows, we talk about "presheaves" in general because what we say applies to the three kinds that we defined.
So if $\mathscr{F}$ is a presheaf on X, then:

- To each inclusion $V \subset U$ of open sets, $\mathscr{F}$ associates a **restriction map**

$$\mathrm{res}_{V,U} : \mathscr{F}(U) \to \mathscr{F}(V)$$

- For every open set U, we have that $\mathrm{res}_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$.
- For every open sets U, V and W such that $W \subset V \subset U$, we have that: $\mathrm{res}_{W,U} = \mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}$.
- we can define the **restriction** $\mathscr{F}|_U$ of $\mathscr{F}$ over an open set U by $\mathscr{F}|_U(V) := \mathscr{F}(V)$ for all open set $V \subset U$.

Definition 3. Let $\mathscr{F}$ be a presheaf on X and U an open set.
An element of $\mathscr{F}(U)$ is called a **section** of $\mathscr{F}$ over U, so that $\mathscr{F}(U)$ is called the **set of sections** of $\mathscr{F}$ over U.
An element of $\mathscr{F}(X)$ is called a **global section** of $\mathscr{F}$, so that $\mathscr{F}(X)$ is called the **set of global sections** of $\mathscr{F}$.

Remark 4. The notation $\Gamma(U, \mathscr{F})$ can be used to designate $\mathscr{F}(U)$, mostly when $\mathscr{F}$ is regarded as an argument.

Definition 5. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X.
A natural transformation from $\mathscr{F}$ to $\mathscr{G}$ is called a **morphism of presheaves.** We can thus verify that with these morphisms, presheaves on X form categories, denoted by $\mathrm{PSh}(X)$ for presheaves of sets, $\mathrm{AbPSh}(X)$ for abelian presheaves, and $\mathrm{RngPSh}(X)$ for sheaves of rings.

Proposition 6. *Let $\Phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves.*
The following are equivalent:

1. *Φ is a monomorphism (resp. epimorphism).*
2. *For all $U \subset X$ the induced map $\Phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a monomorphism (resp. epimorphism).*

The following definition is of tremendous importance, because it captures the local behavior of a presheaf.

Definition 7. Let $\mathscr{F}$ be a presheaf on X, and $x \in X$.

$$\mathscr{F}_x := \operatorname*{colim}_{\substack{U \text{ open} \\ U \ni x}} \mathscr{F}(U)$$

is called the **stalk** of $\mathscr{F}$ at x.

Before getting into sheaves, let us define a last kind of presheaf.

Definition 8. Let $\mathscr{O}$ be a presheaf of rings.
A **presheaf of $\mathscr{O}$-modules** is an abelian presheaf $\mathscr{F}$ together with a map of presheaves of sets

$$\mathscr{O} \times \mathscr{F} \to \mathscr{F}$$

such that for all open set U, the map $\mathscr{O}(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ defines a structure of $\mathscr{O}(U)$-module on the abelian group $\mathscr{F}(U)$.

Definition 9. A morphism of abelian presheaves $\Phi : \mathscr{F} \to \mathscr{G}$ is called a **morphism of presheaves of $\mathscr{O}$-modules** if it is such that the diagram

$$\begin{tikzcd}
\mathscr{O} \times \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id} \times \Phi"'] & \mathscr{F} \arrow[d, "\Phi"] \\
\mathscr{O} \times \mathscr{G} \arrow[r] & \mathscr{G}
\end{tikzcd}$$

commutes.
The set of these morphisms is denoted by $\mathrm{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$.

Proposition 10. *Together with the morphisms we defined, presheaves of $\mathscr{O}$-modules form a category denoted by $P\mathscr{O}-\mathrm{mod}$.*

Definition 11. The stalks of a presheaf of $\mathscr{O}$-modules are defined to be the stalks of the underlying abelian presheaf.

Remark 12. The definition of a presheaf of $\mathscr{O}$-modules endows $\mathscr{F}_x$ with a structure of $\mathscr{O}_x$-module.

1.2 Sheaves

Presheaves were given a name, but they are nothing more than contravariant functors. In order to be able to transition from local to global and vice versa, let us impose some more properties on our functors. For the moment, we shall restrict to sheaves of sets, abelian sheaves and sheaves of rings, thus leaving sheaves of $\mathscr{O}$-modules for later.

Definition 13. Let $\mathscr{F}$ be a presheaf.
We say that $\mathscr{F}$ is a **sheaf** if it satisfies the following axioms:

- (locality) Suppose U is an open set, $(U_i)_{i \in I}$ is an open covering of U, and $s, t \in \mathscr{F}(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (gluing) Suppose U is an open set, $(U_i)_{i \in I}$ is an open covering of U, and $(s_i)_{i \in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathscr{F}(U_i)$. If for all $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists $s \in \mathscr{F}(U)$ such that for all $i \in I$, $s|_{U_i} = s_i$.

This definition basically requires that glueing locally (but cogently) defined sections is allowed, and that it is unique.

Definition 14. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X.
A morphism of presheaves (or simply a natural transformation) from $\mathscr{F}$ to $\mathscr{G}$ is called a **morphism of sheaves** from $\mathscr{F}$ to $\mathscr{G}$.
We can thus verify that with these morphisms, sheaves on X form categories, denoted by $\mathrm{Sh}(X)$ for sheaves of sets, $\mathrm{AbSh}(X)$ for abelian presheaves, and $\mathrm{RngSh}(X)$ for sheaves of rings.

Remark 15. If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then for each point $x \in X$, Φ induces a morphism on stalks:

$$\Phi_x : \mathscr{F}_x \to \mathscr{G}_x$$

Given a presheaf, it is natural to ask if it can be "turned into a sheaf". We will see that this can be done without adding previously non existent local behavior (*i.e.* new stalks).

Definition 16. Let $\mathscr{F}$ be a presheaf.
For every open set U we define:

$$\mathscr{F}^{\#}(U) = \left\{ (s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x, \right.$$
$$\left. \forall x \in U, \exists V \in \mathrm{Op}(U), x \in V, \exists \sigma \in \mathscr{F}(V), \forall v \in V, \sigma_v = s_v \right\}$$

We also define for every pair of open sets $V \subset U$:

$$\mathrm{res}_{U,V} : \begin{cases} \mathscr{F}^{\#}(U) \to \mathscr{F}^{\#}(V) \\ (s_x)_{x\in U} \mapsto (s_x)_{x\in V} \end{cases}$$

Proposition 17. *Together with these restriction maps, $\mathscr{F}^{\#}$ is a sheaf. It is called the **sheaf associated with $\mathscr{F}$**.*

Proof. Locality: Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $s, t \in \mathscr{F}^{\#}(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s_x = t_x$ for all $x \in \cup_{i\in I} U_i = U$, hence $s = t$.

Gluing: Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $(s_i)_{i\in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathscr{F}^{\#}(U_i)$. If for all $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then $s = (s_x)_{x\in U}$ is such that $s \in \mathscr{F}(U)$ and for all $i \in I$, $s|_{U_i} = s_i$. $\square$

Remark 18. There is a canonical morphism of presheaves $\iota : \mathscr{F} \to \mathscr{F}^{\#}$.

The following propositions are easy, but fundamental. Their proofs will be sketched very briefly.

Proposition 19. *Let $\mathscr{F}$ be a sheaf.*
The canonical morphism of sheaves $\mathscr{F} \to \mathscr{F}^{\#}$ is an isomorphism.

Proof. This is immediate. □

Proposition 20. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf.*
If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, it uniquely factors through $\mathscr{F}^{\#}$.
In other words, there exists a unique morphism of sheaves $\Phi^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\Phi"] \arrow[d, "\iota"'] & \mathscr{G} \\ \mathscr{F}^{\#} \arrow[ur, "\Phi^{\#}"] & \end{tikzcd}$$

Proof. The existence of a factorization is a consequence of the fact that $\mathscr{G}$ is a sheaf, because then the image of a given stalk is unique by the locality axiom.

Its uniqueness follows from the fact that the stalks of a section on an open set determine its image by the glueing axiom.

Another, more concise way to look at it is the following:
We clearly have a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "\Phi"'] & \mathscr{F}^{\#} \arrow[d, "\Phi^{\#}"] \\ \mathscr{G} \arrow[r] & \mathscr{G}^{\#} \end{tikzcd}$$

where the horizontal arrows are canonical.
We saw that the bottom one was an isomorphism, thus the result. □

Proposition 21. *The functor $\# : *\mathrm{PSh}(X) \to *\mathrm{Sh}(X)$ is left-adjoint to the forgetful functor $*\mathrm{Sh}(X) \to *\mathrm{PSh}(X)$.*

Proof. The last proposition essentially amounts to a bijection

$$\mathrm{Hom}_{*\mathrm{PSh}(X)}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{*\mathrm{Sh}(X)}(\mathscr{F}^{\#}, \mathscr{G})$$

The algebraic structure of this bijection, as well as its functoriality with regards to both $\mathscr{F}$ and $\mathscr{G}$ are easy. □

Proposition 22. *Let $\Phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*
The following are equivalent:

1. *Φ is a monomorphism (resp. epimorphism) in $*\mathrm{Sh}(X)$.*
2. *For all $U \subset X$ the induced map $\Phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a monomorphism (resp. epimorphism).*
3. *For all $x \in X$ the induced map on stalks $\Phi_x : \mathscr{F}_x \to \mathscr{G}_x$ is a monomorphism (resp. epimorphism).*

Remark 23. All the categories of sheaves we will be working with allow the term "monomorphism" (resp. "epimorphism") to be replaced by "injective morphism" (resp. "surjective morphism").

Let us now define kernels and cokernels.

Definition 24. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X, and $\Phi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves.
The **kernel** of Φ is a presheaf defined for every open set U by:

$$\mathrm{Ker}(\Phi)(U) := \mathrm{Ker}(\Phi_U) \subset \mathscr{F}(U)$$

Proposition 25. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\mathrm{Ker}(\Phi)$ is a sheaf.*

Proof. Locality: Suppose U is an open set of X, $(U_i)_{i\in I}$ is an open covering of U, and $s, t \in \mathrm{Ker}(\Phi)(U)$ are sections. If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then by the locality axiom for $\mathscr{F}$, $s = t$.

Gluing: Suppose U is an open set, $(U_i)_{i\in I}$ is an open covering of U, and $(s_i)_{i\in I}$ is a family of sections such that for all $i \in I$, $s_i \in \mathrm{Ker}(\Phi)(U_i)$. If for all $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then by the gluing axiom for $\mathscr{F}$ there exists $s \in \mathscr{F}(U)$ such that for all $i \in I$, $s|_{U_i} = s_i$. But for all $i \in I$,

$$\Phi(s)(U_i) = \Phi(s|_{U_i})(U_i) = \Phi(s_i)(U_i) = 0,$$

so that $\Phi(s)$ vanishes over U by the locality axiom for $\mathscr{G}$, thus $s \in \mathrm{Ker}(\Phi)(U)$. □

Proposition 26. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\mathrm{Ker}(\Phi) = 0$ if and only if Φ is injective.*

Proof. If $\mathrm{Ker}(\Phi) = 0$, let $x \in X$ be a point, and let $s \in \mathscr{F}(U_s)$ and $t \in \mathscr{F}(U_t)$ be sections such that $\Phi_x(s_x) = \Phi_x(t_x)$. $\Phi(s)$ and $\Phi(t)$ must then agree on some open set V containing x, thus $\mathrm{Ker}(\Phi)(V) = 0$ implies $s|_V = t|_V$, hence $s_x = t_x$.

If Φ is injective, let U be an open set and $s \in \mathrm{Ker}(\Phi)(U)$. $\Phi(s)$ vanishes on U, so in particular its stalks are those of the zero section. But then injectivity gives that s is zero on a neighborhood of each point of U, which ensures by the gluing axiom that $s = 0$. □

Proposition 27. *Kernels as we just defined them are kernels in the categorical sense.*

Now let us state some analogous results for sujective morphisms.

Definition 28. Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X, and $\Phi : \mathscr{F} \to \mathscr{G}$ a morphism of presheaves.
The **image presheaf** is defined for every open set U by:

$$\mathrm{Im}(\Phi_U) \subset \mathscr{G}(U)$$

The sheaf $\mathrm{Im}(\Phi)$ associated to the image presheaf is called the **image** of Φ.

Remark 29. In general, the image presheaf is not a sheaf and thus is not isomorphic to its associated sheaf.

Proposition 30. *If $\Phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then it induces a morphism of sheaves $\mathrm{Im}(\Phi) \to \mathscr{G}$, which is an isomorphism if and only if Φ is surjective.*

Proof. By the universal property of the sheaf associated to a presheaf, there exists a morphism of sheaves $\phi : \mathrm{Im}(\Phi) \to \mathscr{G}$.

If ϕ is an isomorphism, let $x \in X$ be a point and $g_x \in \mathscr{G}_x$ a stalk at x. By surjectivity of ϕ, there exists a section t of $\mathrm{Im}(\Phi)$ such that $\Phi_x(t) = g_x$. Since sections of $\mathrm{Im}(\Phi)$ are locally induced by sections of the image presheaf, there exists a section s of $\mathscr{F}$ such that

$$\phi_x(s_x) = g_x.$$

If Φ is surjective, then clearly ϕ is surjective. Also the image presheaf is injective, so that ϕ is also injective. Indeed, sections of the associated sheaf locally coincide with sections of the presheaf. $\square$

Let us now treat the special case of sheaves of modules.

Definition 31. Let $\mathscr{O}$ be a sheaf of rings.
A **sheaf of $\mathscr{O}$-modules** is a presheaf of $\mathscr{O}$-modules such that the underlying presheaf of abelian groups is a sheaf.
A morphism of sheaves of $\mathscr{O}$-modules is a morphism of presheaves of $\mathscr{O}$-modules.

Remark 32. This definition could be extended to the case where $\mathscr{O}$ is not a sheaf, but this is useless in practice.

Proposition 33. *Together with the morphisms we defined, sheaves of $\mathscr{O}$-modules form a category denoted by $\mathscr{O}-\mathrm{mod}$.*

Theorem 34. *The categories $\mathrm{AbSh}(X)$ and $\mathscr{O}-\mathrm{mod}$ are abelian.*

Before getting into sheaf cohomology, let us see how sheaves behaves with respect to continuous maps from a topological space to another.

Definition 35. Let Y be a topological space and $f : X \to Y$ be a continuous map.
Then for any sheaf $\mathscr{F}$ on X, we can naturally define a sheaf on Y by setting

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

The sheaf $f_*\mathscr{F}$ is called the **the image sheaf of $\mathscr{F}$.**

Now it is tempting to define similarly $f^*\mathscr{G}(U) = \mathscr{G}(f(U))$, but this does not define a sheaf in general ($f(U)$ is not even open in general). Still, taking the appropriate colimit solves (most of) our problems.

Definition 36. Let Y be a topological space and $f : X \to Y$ be a continuous map.
Then for any sheaf $\mathscr{G}$ on Y, we can (more or less) naturally define a sheaf on X by setting

$$f^{-1}\mathscr{G}(U) := \operatorname*{colim}_{\substack{W \text{ open in X} \\ W \supset U}} \mathscr{G}(W)$$

The sheaf $f^{-1}\mathscr{G}$ is called the **the inverse image sheaf of $\mathscr{G}$.**

1.3 Sheaf cohomology

We saw that a sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

was exact if for all $x \in X$ the induced sequence of stalks at x was exact.

Proposition 37. *Let $\mathscr{F}$ be a sheaf on X, and U an open set.*
$\Gamma(U,-)$ is a functor from AbSh(X) *to* Ab. *This functor is* ***left-exact.***
When $U = X$, it is called the ***global section functor.***

Proof. Let

$$0 \longrightarrow \mathscr{F} \xrightarrow{\Phi} \mathscr{G} \xrightarrow{\Psi} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of sheaves of $\mathscr{O}$-modules.
By applying $\Gamma(U,-)$ we get

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\Phi_U} \mathscr{G}(U) \xrightarrow{\Psi_U} \mathscr{H}(U) \longrightarrow 0$$

Given that $\mathrm{Ker}(\Phi) = 0$, it is clear that $\mathrm{Ker}(\Phi_U) = 0$.
Also, $\Psi \circ \Phi = 0$, hence $\Psi_U \circ \Phi_U = 0$ and $\Im(\Phi_X) \subset \mathrm{Ker}(\Psi_X)$.
Lastly, any section $s \in \mathscr{G}(X)$ which maps to 0 in $\mathscr{H}(X)$ is by definition locally a section of $\mathscr{F}$. But $\mathscr{F}$ is a sheaf, hence by the gluing axiom s is a section in $\mathscr{F}(X)$. □

Proposition 38. *The categories* AbSh(X) *and* $\mathscr{O}-$mod *have* ***enough injectives.***

Proof. The proof for AbSh(X) is essentially a simpler version of the one for $\mathscr{O}-$mod, so let us show only the latter.
Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
For all $x \in X$, let us fix an injective morphism of $\mathscr{O}_x$-modules $\mathscr{F}_x \to I_x$ in an injective $\mathscr{O}_x$-module I_x.
We can then define a sheaf $\mathscr{I}$ by setting

$$\mathscr{I}(U) := \prod_{x \in U} I_x$$

There is an obvious injective morphism of sheaves of $\mathscr{O}$-modules $\mathscr{F} \to \mathscr{I}$.
It now remains to show that $\mathscr{I}$ is indeed injective.
A morphism of sheaves of $\mathscr{O}$-modules $f : \mathscr{A} \to \mathscr{I}$ is equivalent to a collection of maps $f_x : \mathscr{A}_x \to \mathscr{I}_x$ such that for all $x \in X$, the f_x is an injective morphism of $\mathscr{O}_x$-modules.
Given an injective morphism of sheaves of $\mathscr{O}$-modules $i : \mathscr{A} \to \mathscr{B}$, we have injective morphisms $i_x : \mathscr{A}_x \to \mathscr{B}_x$ such that for all $x \in X$, $g_x : \mathscr{B}_x \to \mathscr{I}_x$ is a morphism extending f_x, because $\mathscr{I}_x$ is injective.
But then these morphisms define a morphism of sheaves of $\mathscr{O}$-modules $g : \mathscr{B} \to \mathscr{I}$ extending f, hence the result. □

This last result allows us to define derived functors, and in particular, sheaf cohomology in AbSh(X) and $\mathscr{O}-$mod.

Definition 39. We define **sheaf cohomology** $H^\bullet(X,-)$ as the right-derived functor of the global section functor $\Gamma(X,-)$.

Definition 40. A sheaf of $\mathscr{O}$-modules $\mathscr{F}$ is said to be

- **flasque** if for every open set U, any section $s \in \mathscr{F}(U)$ can be extended to a global section;
- **soft** if for every closed sets C, any section

$$s \in \mathscr{F}(C) := \operatorname*{colim}_{\substack{U \text{ open} \\ U \supset C}} \mathscr{F}(U)$$

  can be extended to a global section;
- **fine** if for every open cover $(U_i)_{i \in I}$ of X there is a family of morphisms $(\phi_\alpha : \mathscr{F} \to \mathscr{F})_{\alpha \in A}$ subordinate to that cover such that locally, at any point $x \in X$, all but finitely many vanish (*i.e.* $(\phi_\alpha)_x = 0$ for all but finitely many $\alpha \in A$), and $\sum_{\alpha \in A} \phi_\alpha = \mathrm{id}$.

Definition 41. We say that a sheaf $\mathscr{A}$ is **acyclic** for the functor Γ (or Γ-acyclic) if $R^p\Gamma(\mathscr{A}) = 0$ for all $p > 0$.

Remark 42. From now on, "acyclic" will always mean "Γ-acyclic", where Γ is the global section functor.

The whole point of acyclic objects is that they behave somewhat like injective objects, while being easier to manipulate (and more abundant in practice).

Proposition 43. *Let $\mathscr{F} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots$ be an acyclic resolution of $\mathscr{F}$. Then for all $p \in \mathbb{N}$, $R^p\Gamma(X, \mathscr{F})$ is equal to $H^p(\Gamma(X, \mathscr{A}^\bullet))$.*

Proof. Let's procede by induction.
We have a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{A}^0 \to \mathscr{B} \to 0$$

where $\mathscr{B} := \mathrm{Coker} d_0$. But $(\mathscr{A}^{\bullet+1}, d_0 : \mathscr{B} \to \mathscr{A}^1)$ is a resolution of $\mathscr{B}$, thus the long exact sequence induced by our short exact sequence amounts to:

$$\begin{cases} R^p\Gamma(\mathscr{F}) = R^{p-1}\Gamma(\mathscr{B}) & \text{for } p > 1 \\ R^1\Gamma(\mathscr{F}) = H^1(\Gamma(\mathscr{A}^\bullet)) \end{cases}.$$

We can now apply the same reasoning to $\mathscr{B}$, thus the result. □

The following proposition sums up the relationships between the different types of sheaves we just defined. The complete proof is quite lengthy and a bit technical, therefore I chose not to include it.

Proposition 44. *Suppose X is paracompact.*

(i) Injective sheaves are flasque.

(ii) Flasque sheaves are soft.

(iii) Soft sheaves are fine.

(iv) Fine sheaves are acyclic.

Proof. See [Taylor]. □

Remark 45. It is required that X be paracompact, which will always be assumed in what follows.

Another very useful property is the following:

Proposition 46. *If the first two terms of an exact sequence of sheaves of $\mathscr{O}$-modules are soft, then so is the third.*

Proof. Again, see [Taylor]. □

1.4 Properties of some classical operations on sheaves

Definition 47. Let $i : Y \to X$ be an embedding, and $\mathscr{G}$ be a sheaf on Y.

(i) If Y is closed in X, then $i_*\mathscr{G}$ defines a sheaf called the **extension of $\mathscr{G}$ to X by zero.** It has the property that

$$(i_*\mathscr{G})_x = \begin{cases} \mathscr{G}_x & \text{if } x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

(ii) If Y is open, $i_*\mathscr{G}$ does not have this last property in general. But we still have that

$$V \mapsto \begin{cases} \mathscr{G}(V) & \text{if } V \subset Y, \\ 0 & \text{otherwise} \end{cases}$$

defines a presheaf on X, whose associated sheaf $\mathscr{G}^X$ (or $i_!\mathscr{G}$) is also called the **extension of $\mathscr{G}$ to X by zero.** It has the property that

$$\mathscr{G}^X_x = \begin{cases} \mathscr{G}_x & \text{if } x \in Y, \\ 0 & \text{otherwise} \end{cases}$$

Proposition 48. *Let $\mathscr{R}$ be a sheaf of rings on X.*
We have the following:

(i) If $\mathscr{I}$ is an injective sheaf of $\mathscr{R}$-modules on X, and U is an open set, then $\mathscr{I}|_U$ is an injective sheaf of $\mathscr{R}|_U$-modules.

(ii) If $f : Y \to X$ is a continuous map and $\mathscr{I}$ an injective sheaf of $f^{-1}\mathscr{R}$-modules, then $f_*\mathscr{I}$ is an injective sheaf of $\mathscr{R}$-modules.

(iii) The direct product of any set of injective sheaves of $\mathscr{R}$-modules is an injective sheaf of $\mathscr{R}$-modules.

Proof. Given that extension by zero is exact, and that $\mathscr{I}$ is injective, $\mathrm{Hom}((-)^X, \mathscr{I})$ is an exact functor. Hence $\mathrm{Hom}_{\mathscr{R}|_U}(-, \mathscr{I}|_U) \cong \mathrm{Hom}_{\mathscr{R}}((-)^X, \mathscr{I})$ implies that $\mathrm{Hom}_{\mathscr{R}|_U}(-, \mathscr{I}|_U)$ is an exact functor, thus proving that $\mathscr{I}|_U$ is injective. This proves (i).

We already saw that, for a sheaf $\mathscr{G}$ of $\mathscr{R}$-modules on X and a sheaf $\mathscr{I}$ of $f^{-1}\mathscr{R}$-modules on Y, we have $\mathrm{Hom}_{f^{-1}\mathscr{R}}(f^{-1}\mathscr{G}, \mathscr{I}) \cong \mathrm{Hom}_{\mathscr{R}}(\mathscr{G}, f_*\mathscr{I})$. The functor f^{-1} is exact, so if $\mathscr{I}$ is injective, then $\mathrm{Hom}_{\mathscr{R}}(-, f_*\mathscr{I})$ is exact, and $f_*\mathscr{I}$ is injective. This proves (ii).

Let $(\mathscr{I}_\alpha)_{\alpha\in A}$ be a set of injective sheaves on X, and let $i : \mathscr{F} \to \mathscr{G}$ be a injective morphism of sheaves. A morphism $f : \mathscr{F} \to \prod_{\alpha\in A} \mathscr{I}_\alpha$ is just a product of morphisms $f_\alpha : \mathscr{F} \to \mathscr{I}_\alpha$. Each of these extends to $g_\alpha : \mathscr{G} \to \mathscr{I}_\alpha$, and $\prod_{\alpha\in A} g_\alpha : \mathscr{G} \to \prod_{\alpha\in A} \mathscr{I}_\alpha$ extends f, hence $\prod_{\alpha\in A} \mathscr{I}_\alpha$ is also injective, which completes the proof. □

Proposition 49. *Let $i : Y \to X$ be an embedding of Y as a closed subspace of X, and $\mathscr{G}$ be a sheaf on Y.*
Then for all $p \in \mathbb{N}$, the canonical morphism

$$H^p(Y, \mathscr{G}) \to H^p(X, i_*\mathscr{G})$$

is an isomorphism.

Proof. The sheaf $i_*\mathscr{G}$ is just $\mathscr{G}$ extended to X by zero.
It is easily seen that i_* is an exact functor. Indeed, given an exact sequence of sheaves on Y, applying i_* and taking stalks at $x \in X$ gives either a canonically isomorphic exact sequence of stalks if $x \in Y$, or a trivial sequence if $x \notin Y$ (because $X \setminus Y$ is open).
Now if $\mathscr{G} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \cdots$ is an injective resolution of $\mathscr{G}$, then by exactness $i_*\mathscr{G} \to i_*\mathscr{I}_0 \to i_*\mathscr{I}_1 \to \cdots$ is a resolution of $i_*\mathscr{G}$. It is moreover injective by the last proposition.
As $\Gamma(X, i_*\mathscr{G}) \cong \Gamma(Y, \mathscr{G})$ and $\Gamma(X, i_*\mathscr{I}_p) \cong \Gamma(Y, \mathscr{I}_p)$ for all $p \in \mathbb{N}$, we have the desired isomorphism. □

1.5 Coherent Sheaves

In the rest of the document, we will use abundantly the fact that regular functions on algebraic varieties form a sheaf $\mathscr{O}$, in much the same way that holomorphic functions (which we will often call analytic) on a complex manifold form a sheaf $\mathscr{H}$.

We begin with the definitions of quasi-coherent and coherent sheaves, which will be omnipresent in the rest of this document.

Definition 50. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
The sheaf $\mathscr{F}$ is called **quasi-coherent** when it has a local presentation at all points, *i.e.* for all $x \in X$ there exists an open set $U \ni x$ and sets I, J such that we have an exact sequence

$$\mathscr{O}^{\oplus I}|_U \longrightarrow \mathscr{O}^{\oplus J}|_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Definition 51. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules.
The sheaf $\mathscr{F}$ is called **coherent** when it satisfies the following:

(i) $\mathscr{F}$ is of finite type, *i.e.* for all $x \in X$ there exists an open set $U \ni x$ and a surjective map $\mathscr{O}^{\oplus m}|_U \to \mathscr{F}|_U$ for some integer $m \in \mathbb{N}$.

(ii) For all open set U, any morphism $\mathscr{O}^{\oplus m}|_U \to \mathscr{F}|_U$ for some integer $m \in \mathbb{N}$ has a finitely generated kernel.

2 Scheme theory

2.1 Basic definitions

Definition 52. A **ringed space** is a topological space X with a sheaf $\mathscr{O}_X$, called the **structure sheaf of** X.
A **locally ringed space** is a ringed space such that for all $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. The only maximal ideal of $\mathscr{O}_{X,x}$ is denoted my $\mathfrak{m}_x$.

Definition 53. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces.
We say that a pair (f, ϕ) is a **morphism of ringed spaces** if $f : X \longrightarrow Y$ is a continuous map and $\phi : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ is a morphism of sheaves.
Moreover we say that (f, ϕ) is a **morphism of locally ringed spaces** if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are locally ringed spaces and if the maps induced by ϕ on stalks are local morphisms.

Definition 54. An **affine scheme** is a locally ringed space that is isomorphic to $\mathrm{Spec}\,(R)$ with the Zariski topology for some ring R.
A **scheme** is a locally ringed space that can be covered by open sets that are affine schemes.
If moreover any cover can be reduced to a finite cover, the scheme is called **quasi-compact.**

Remark 55. This definition of quasi-compact is not standard, but it fits our purpose.

From now on, all schemes will be assumed to be quasi-compact.

Definition 56. Let X and Y be schemes.
Then $f : X \longrightarrow Y$ is called a **morphism of schemes** if it is a morphism of locally ringed spaces.

2.2 Coherent algebraic sheaves

We defined coherent sheaves in a general way, but these definitions are not the most practical when it comes to algebraic geometry.

Definition 57. Let X be an affine scheme and M be an $\Gamma(X, \mathscr{O}_X)$-module.
The rule

$$\tilde{M}(X_f) = M_f$$

defines a sheaf called the sheaf associated to M on X.

In fact, many authors in algebraic geometry use the following as their definition of quasi-coherent and coherent algebraic sheaves.

Proposition 58. *Let X be a scheme and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$-modules on X. If for each open affine subscheme U of X, $\mathscr{F}_U$ is isomorphic to $\widetilde{\mathscr{F}(U)}$ as a sheaf of $\mathscr{O}_X$-modules, then $\mathscr{F}$ is a quasi-coherent sheaf.*

2.3 Serre's theorem

In this part we aim to show an important characterization result of Serre, and a few other related results. The first part of his result is the following:

Theorem 59. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*
Then we have $H^p(X, \mathscr{F}) = 0$ for all $p \geqslant 1$.

Proof. We wish to show that for any finite covering $\mathfrak{U}$ of X by principal open sets, we have $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for all $p \geqslant 1$. □

Actually, a proof that does not use the Čech machinery is also possible. Let us begin with a useful proposition, and then a technical lemma will allow us to conclude. We denote by ${}_U\mathscr{F}$ the extension of $\mathscr{F}_{|U}$ by zero on X.

Proposition 60. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme and*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

be an exact sequence of sheaves of $\mathscr{O}_X$-module, where $\mathscr{F}$ is quasi-coherent.
Then

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H}) \to 0$$

is exact.

Proof. Left exactness is already known, it now remains to show that $\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H})$ is surjective.
Let $s \in \Gamma(X, \mathscr{H})$ be a global section of $\mathscr{H}$.
Since $\mathscr{G} \to \mathscr{H}$ is surjective, for each $x \in X$ there is an open neighborhood X_f of x such that $s_{|X_f}$ lifts to a section $t \in \Gamma(X_f, \mathscr{G})$.
Now let us show that for some $k > 0$, $f^k t$ extends to a global section of $\mathscr{G}$ over X.

We can take a finite open covering $(X_{f_i})_{i \in I}$ of X such that for all $i \in I$, $s_{|X_{f_i}}$ lifts to a section $t_i \in \Gamma(X_{f_i}, \mathscr{G})$.
On $X_f \cap X_{f_i}$, t and t_i both lift s, thus since $\mathscr{F}$ is quasi-coherent, $f^{l_i}(t - t_i)$ extends to a section $u_i \in \Gamma(X_{f_i}, \mathscr{F})$ for some $l_i > 0$.

Indeed if $(X_{g_j})_{j \in I}$ is a suitable covering of X, for all $j \in I$ we have $\mathscr{F}_{|X_{g_j}} = \tilde{M}_j$ and $\Gamma(X_{fg_j}, \mathscr{F}) = (M_j)_f$.
Then by definition for any $j \in I$ we have some $v_j \in M_j$ which restricts to $f^{n_j}(t - t_i)$ on X_{fg_j}, thus we can set $n = \max_{j \in I} n_j$.
Similarly on the intersection $X_{g_j} \cap X_{g_k} = X_{g_j g_k}$, v_j and v_k coincide, and we get that for some $m_{j,k}$ we have $f^m(v_j - v_k) = 0$ on $X_{g_i g_j}$, and wet set $m = \max_{i,j \in I} m_{i,j}$.
Now the $f^m v_j$ glue together to give a global section whose restriction to X_f is $f^{n+m}(t - t_i)$.

Again we pick $l = \max_{i \in I} l_i$ and consider the $t'_i = f^l t_i + u_i$.
They are all lifts of $f^l s$, and by a similar argument we get that for some $p > 0$, $f^p(t'_i - t'_j) = 0$, such that the $f^p t'_i$ glue to give a global lift of f^{p+l}.

Finally, by using the same argument for each f_i, we get an integer k and global sections $t_i \in \Gamma(X, \mathscr{G})$ such that t_i is a lifting of $f_i^n s$.
But the X_{f_i} cover X, hence the f_i^n generate the unit ideal, and in particular we have $1 = \sum_{i \in I} r_i f_i^n$ with $r_i \in R$.
We can now see that $t = \sum r_i t_i$ is a global section of $\mathscr{G}$ whose image in $\Gamma(X, \mathscr{H})$ is $\sum_{i \in I} r_i f_i^n s = s$. □

Lemma 61. *Let $\mathscr{F}$ be a sheaf on X and $\mathfrak{U}$ be a topological basis of X.*
Assume that for some $n \in \mathbb{N}^*$ we have that $H^i(U, \mathscr{F}_U)$ for all $U \in \mathfrak{U}$ and $0 < i < n$.
Then each element of $H^n(X, \mathscr{F})$ admits a covering $\mathfrak{V}$ such that $H^n(X, {}_V\mathscr{F}) = 0$ for all $V \in \mathfrak{V}$.

Proof. Let
$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$
be an exact sequence of sheaves, where $\mathscr{G}$ is flabby (and thus acyclic). For all open set U we have a long exact sequence that gives an exact sequence
$$0 \to \Gamma(U, \mathscr{F}_{|U}) \to \Gamma(U, \mathscr{G}_{|U}) \to \Gamma(U, \mathscr{H}_{|U}) \to H^1(U, \mathscr{F}_{|U}) \to 0$$
and isomorphisms
$$H^p(U, \mathscr{H}_{|U}) \simeq H^{p+1}(U, \mathscr{F}_{|U})$$
for all $p > 0$.

Now let us consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_V\mathscr{F} \arrow[r] & {}_V\mathscr{G} \arrow[r] & {}_V\mathscr{H} &
\end{tikzcd}$$

Given that $\mathscr{G} \to {}_V\mathscr{G}$ and $\mathscr{G} \to \mathscr{H}$ are surjective, the images of ${}_V\mathscr{G} \to {}_V\mathscr{H}$ and $\mathscr{H} \to {}_V\mathscr{H}$ are the same, we denote it by $\mathscr{K}$.

It is clear that ${}_V\mathscr{G}$ is flabby as well, hence for the same reason we have an exact sequence
$$0 \to \Gamma(X, {}_V\mathscr{F}) \to \Gamma(X, {}_V\mathscr{G}) \to \Gamma(X, \mathscr{K}) \to H^1(X, {}_V\mathscr{F}) \to 0$$
and isomorphisms
$$H^p(X, \mathscr{K}) \simeq H^{p+1}(X, {}_V\mathscr{F})$$
for all $p > 0$.

Wo now procede by induction on n.
If $n = 1$, let $\alpha \in H^1(X, \mathscr{F})$.
By the first exact sequence for $U = X$, the connecting map δ is surjective, and $\alpha = \delta(h)$ for some $h \in \Gamma(X, \mathscr{H})$.
But by the second exact sequence, α vanishes in $H^1(X, {}_V\mathscr{F})$ if and only if the image of h in $\Gamma(X, \mathscr{K})$ lifts to an element in $\Gamma(X, {}_V\mathscr{G}) = \Gamma(V, \mathscr{G})$.
The exactness of $\mathscr{G} \to \mathscr{H} \to 0$ allows to conclude that for some V there is indeed such a lift, and we can even ask that $V \in \mathfrak{U}$ because $\mathfrak{U}$ is a basis.

Now if $n > 1$, and $U \subset V$ are open subsets, our first reasoning shows that $\Gamma(U, {}_V\mathscr{G}) \to \Gamma(U, {}_V\mathscr{H})$ is surjective.
Thus we have $\mathscr{K} = {}_V\mathscr{H}$, and the last isomorphisms enable us to conclude by induction. $\square$

Now we can prove our theorem.

Proof. We take $\mathfrak{U} = \{X_f\}_{f\in\Gamma(X,\mathscr{O}_X)}$, which is a basis of the Zariski topology, and notice that ${}_{X_f}\mathscr{F}$ is a quasi-coherent sheaf.
Let us procede by induction on p.
If $p = 1$, the result follows from the last proposition.
If $p > 1$, let $\alpha \in H^n(X, \mathscr{F})$ and $\mathfrak{V} \subset \mathfrak{U}$ be a corresponding subcovering given by our lemma. We can take it to be finite.
Then the image of α in $H^n(X, \bigoplus_{V\in\mathfrak{V}} {}_V\mathscr{F})$ is zero.
Thus by the induction hypothesis, the long exact sequence associated to the short exact sequence of quasi-coherent sheaves

$$0 \to \mathscr{F} \to \bigoplus_{V\in\mathfrak{V}} {}_V\mathscr{F} \to \mathscr{G} \to 0$$

grants that $\alpha \in \delta(H^{n-1}(X, \mathscr{G})) = \{0\}$, hence the result. □

The analog of this theorem for Stein manifolds (see next section for a definition) is called Cartan's theorem B:

Theorem 62 (Cartan's theorem B)**.** *Let X be a Stein manifold and $\mathscr{F}$ be a coherent analytic sheaf on X.*
Then we have $H^p(X, \mathscr{F}) = 0$ for all $p \geqslant 1$.

The proof is even more technical, and can be found in [Taylor]. This theorem is of fundamental importance in complex algebraic geometry: among other things, it served as a basis for Serre's famous GAGA theorems.

Let us now turn to what makes our first theorem even more interesting.

Theorem 63 (Serre's affineness criterion)**.** *Let X be a quasi-compact scheme. If for every quasi-coherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ we have $H^1(X, \mathscr{I}) = 0$ then X is affine.*

We now give a good idea of the proof, omitting some of the technical lemmas underlying it.

Proof. Let $x \in X$ be a closed point and $U = \mathrm{Spec}\,(A)$ and affine neighborhood of x.
We can see that $Z = X \backslash U$ and $Z' = Z \cup \{x\}$ are closed subsets, and moreover it is not hard to show that the rule

$$\mathscr{I}(V) = \{f \in \Gamma(U, \mathscr{O}_X) | f_{Z\cap V} = 0\}$$

defines a quasi-coherent sheaf of ideals. We can define $\mathscr{I}'$ similarly by replacing Z with Z'.
Given that $\mathscr{I}'$ is clearly a subsheaf of $\mathscr{I}$, we have a short exact sequence

$$0 \to \mathscr{I}' \to \mathscr{I} \to \mathscr{I}/\mathscr{I}' \to 0.$$

The restriction of $\mathscr{I}/\mathscr{I}'$ to U corresponds to $A/\mathfrak{m}_x$, but by assumption $H^1(X, \mathscr{I}') = 0$, so there is a global section f of $\mathscr{I}$ that maps to a section of $\mathscr{I}/\mathscr{I}'$ that restricts to $1 \in A/\mathfrak{m}_x$ on U.

Clearly $x \in X_f \subset U$, hence $X_f = U_{f_A}$ where f_A is the image of f in $A = \Gamma(U, \mathscr{O}_X)$.

Now if we take $W = \bigcup X_f$ to be the union of all such affine subsets, we see that W is open in X and contains its closed points.
But $X \backslash W$ is a closed subset of a quasi-compact space, and thus is quasi-compact. As such, if a collection of closed subsets has nonempty finite intersections, their intersection is nonempty.
So we can consider the set of closures of points in $X \backslash W$ and apply the preceding reasoning to get that if it is nonempty, this set admits a minimal element (for inclusion), which has to be a closed point.
So $X \backslash W$ is empty and $X = W$.

We can take $f_1, \cdots, f_n \in \Gamma(X, \mathscr{O}_X)$ such that $X = X_{f_1} \cup \cdots \cup X_{f_n}$ is an affine cover.
If the f_i generate the unit ideal in $\Gamma(X, \mathscr{O}_X)$ the proof will be complete (see Lemma 28.27.3 on [Stacks]).

The sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_X^{\oplus n} \xrightarrow{f_1, \cdots, f_n} \mathscr{O}_X \longrightarrow 0$$

is exact if we set $\mathscr{F}$ equal to the kernel of the second-to-last arrow, because the X_{f_i} cover X.
The obvious filtration on $\mathscr{O}_X^{\oplus n}$ induces one on $\mathscr{F}$:

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n = \mathscr{F}.$$

The quotients $\mathscr{F}_i / \mathscr{F}_{i-1}$ are isomorphic to quasi-coherent sheaves of ideals, thus the hypothesis apply.
We have that $H^1(X, \mathscr{F}_0) = 0$, and for $i > 1$, if $H^1(X, \mathscr{F}_{i-1}) = 0$ then the long exact sequence associated to

$$0 \to \mathscr{F}_{i-1} \to \mathscr{F}_i / \mathscr{F}_{i-1} \to \mathscr{F}_i \to 0$$

gives $H^1(X, \mathscr{F}_i) = 0$.
Hence $H^1(X, \mathscr{F}) = 0$, and the image of our short exact sequence is exact as well, *i.e.*

$$\bigoplus_{i=1}^{n} \Gamma(X, \mathscr{O}_X) \xrightarrow{f_1, \cdots, f_n} \Gamma(X, \mathscr{O}_X)$$

is surjective, which is what we wanted. $\square$

We of course can not help but notice that the hypothesis of this theorem is weaker than the conclusions we obtained earlier by supposing affineness. As a result, we get the following equivalence:

Theorem 64 (Serre's affineness criterion)**.** *Let X be a quasi-compact scheme. Then the following conditions are equivalent:*

- *X is affine;*
- *for every quasi-coherent sheaf of ideals $\mathscr{I}$, $H^1(X, \mathscr{I}) = 0$;*

- *every quasi-coherent sheaf on X is acyclic.*

Several close formulations of Serre's result exist. We also mention the following, where a noetherian scheme is a scheme admitting an affine cover by spectra of noetherian rings:

Theorem 65. *Let X be a noetherian scheme.*
Then X is an affine scheme if and only if $H^1(X, \mathscr{F}) = 0$ for all coherent algebraic sheaf $\mathscr{F}$ on X.

3 Other vanishing results

3.1 Stein manifolds

Definition 66. A complex manifold with structure sheaf $\mathscr{H}_X$ is a Stein manifold if:

- X is holomorphically convex, *i.e.* for evey compact subset $K \subset X$, the holomorphically convex hull
$$\overline{K} = \left\{ z \in X, \middle| \forall f \in \Gamma(X, \mathscr{H}_X), |f(z)| \leqslant \sup_{w \in K} \right\}$$
is also a compact subset of X;
- X is holomorphically separable, *i.e.* if $x \neq y$ there exists $f \in \Gamma(X, \mathscr{H}_X)$ such that $f(x) \neq f(y)$.

A famous example of a characterization result that can be stated by changing three terms in Serre's criterion is the following:

Theorem 67. *Let X be a holomorphic variety.*
Then X is a Stein space if and only if $H^1(X, \mathscr{F}) = 0$ for all coherent analytic sheaf $\mathscr{F}$ on X.

It is interesting to see that the proofs are very much alike, see [Taylor] for details.

There are a plethora of more or less similar vanishing results, but let me just quote two that I like.

3.2 Noetherian spaces

Our first theorem is due to Grothendieck and does look like the vanishing of higher singular cohomology of finite-dimensional manifolds. A noetherian topological space is one in which ascending chains of closed subsets are all stationary, and its dimension is simply the maximum length of a strictly ascending chain of closed subsets minus one.

Theorem 68. *Let X be a noetherian topological space.*
Then for all $p > \dim(X)$ and abelian sheaf $\mathscr{F}$, we have that $H^p(X, \mathscr{F}) = 0$.

3.3 Global definition of meromorphic functions

This last one looks less like the ones we usually see. It gives a sufficient condition to be able to define meromorphic function globally, an issue that often remains mysterious and not particularly intuitive.

Theorem 69. *Let X be a Stein manifold.*
If $H^2(X, \mathbb{Z}) = 0$, then for every meromorphic function m on X there are holomorphic functions f and g on X such that:

$$m = \frac{f}{g}$$

It turns out that all non-compact Riemann surfaces satisfy this condition, see exercise 11.14 of [Fo].

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