Linear Programming under Martingale Transport Constraint

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Abstract

This project is concerned with a computational approach to *linear programming* (LP) problems of martingale transport structure.

1 Introduction

Motivated by the model-independent pricing of exotic options, the *martingale optimal transport* (MOT) problem has gained significant momentum both for practitioners and academics. More precisely, given two probability distributions μ, ν on $\mathbb{R}$ and a measurable function $c : \mathbb{R}^2 \to \mathbb{R}$, we seek to maximize/minimize the integral

$$\langle c, \mathbb{P} \rangle \quad := \quad \int_{\mathbb{R}^2} c(\mathsf{x}, \mathsf{y}) \mathbb{P}(d\mathsf{x}, d\mathsf{y})$$

over the set $\mathcal{M}(\mu, \nu)$ of probability measures $\mathbb{P}$ on $\mathbb{R}^2$ such that

$$\begin{aligned} \mathbb{P}[E \times \mathbb{R}] = \mu[E] \;\text{ and }\; \mathbb{P}[\mathbb{R} \times E] = \nu[E], & \qquad \text{for all measurable sets } E \subset \mathbb{R}, \\ \int_{\mathbb{R}} \mathsf{y} \mathbb{P}_{\mathsf{x}}(d\mathsf{y}) = \mathsf{x}, & \qquad \text{for } \mu \text{ - a.e. } \mathsf{x} \in \mathbb{R}, \end{aligned}$$

where $(\mathbb{P}_{\mathsf{x}})_{\mathsf{x} \in \mathbb{R}}$ denotes the disintegration of $\mathbb{P}$ with respect to μ. In financial language, every $\mathbb{P} \in \mathcal{M}(\mu, \nu)$ represents a calibrated market model, and μ, ν describe the initial and final distributions of stock prices. We define the MOT problem as below:

$$\mathsf{S}(\mu, \nu) \left(\text{reps. } \mathsf{I}(\mu, \nu) \right) \quad := \quad \sup_{\mathbb{P} \in \mathcal{M}(\mu, \nu)} \left(\text{reps. } \inf_{\mathbb{P} \in \mathcal{M}(\mu, \nu)} \right) \langle c, \mathbb{P} \rangle. \tag{1}$$

Given the importance of MOT problems for applications in mathematical finance, we focus on solving *linear programming* (LP) problems corresponding to the MOT problem with finitely supported marginal distributions, i.e. μ and ν are two discrete probability distributions given by

$$\mu(d\mathsf{x}) \;=\; \sum_{1 \le i \le m} \alpha_i \delta_{\mathsf{x}^i}(d\mathsf{x}) \quad \text{and} \quad \nu(d\mathsf{y}) \;=\; \sum_{1 \le j \le n} \beta_j \delta_{\mathsf{y}^j}(d\mathsf{y}).$$

Therefore, all the elements $\mathbb{P} \in \mathcal{M}(\mu, \nu)$ are identified by the matrices $(p_{i,j})_{1\le i\le m, 1\le j\le n} \subset \mathbb{R}_+^{mn}$ satisfying

$$\begin{aligned}\sum_{1\le k\le n} p_{i,k} &= \alpha_i, \quad \text{for } i = 1, \cdots, m,\\ \sum_{1\le k\le m} p_{k,j} &= \beta_j, \quad \text{for } j = 1, \cdots, n,\\ \sum_{1\le k\le n} p_{i,k}\mathsf{y}^k &= \alpha_i \mathsf{x}^i, \quad \text{for } i = 1, \cdots, m.\end{aligned}$$

Without loss of generality, we only deal with the maximization problem.

Assumption 1.

$\mu(d\mathsf{x}) = \sum_{1\le i\le m} \alpha_i \delta_{\mathsf{x}^i}(d\mathsf{x})$ and $\nu(d\mathsf{y}) = \sum_{1\le j\le n} \beta_j \delta_{\mathsf{y}^j}(d\mathsf{y})$ are assumed to satisfy

- $\mu \preceq \nu$ and $\mu \neq \nu$;
- $\{\alpha_i\}_{1\le i\le m}$, $\{\beta_j\}_{1\le j\le n} \subset (0, \infty)$.

2 Computational scheme: iterative Bregman projection

The MOT problem $\mathsf{S}(\mu, \nu)$ can be rephrased as the following LP problem:

$$\begin{aligned}&\max_{(p_{i,j})_{1\le i\le m,1\le j\le n}\in\mathbb{R}_+^{mn}} \sum_{1\le i\le m, 1\le j\le n} p_{i,j}c_{i,j}\\ \text{s.t.} &\sum_{1\le k\le n} p_{i,k} \left(\text{resp. } \sum_{1\le k\le m} p_{k,j}\right) = \alpha_i \big(\text{resp. } \beta_j\big), \quad \text{for } i = 1, \cdots, m \Big(\text{resp. } j = 1, \cdots, n\Big),\\ &\sum_{1\le k\le n} p_{i,k}\mathsf{y}^k = \alpha_i \mathsf{x}^i, \quad \text{for } i = 1, \cdots, m,\end{aligned} \tag{2}$$

where $c_{i,j} := c(\mathsf{x}^i, \mathsf{y}^j)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. (2) is a specific LP problem with mn control variables and $2m + n$ linear constraints.

We are concerned with solving (2) by adding a regularization term. Fix an arbitrary $\varepsilon > 0$. For any $p \in \mathbb{R}_+^{mn}$, set $E(p) := \sum_{1\le i\le m, 1\le j\le n} p_{i,j}c_{i,j}$ and

$$E_\varepsilon(p) \ := \ E(p) \ - \ \varepsilon \sum_{1\le i\le m, 1\le j\le n} p_{i,j}\big(\log(p_{i,j}) - 1\big),$$

where we take the convention $0 \times \log(0) := 0$. Then it follows by definition that $E(p) \le E_\varepsilon(p)$ for all $p \in \mathbb{R}_+^{mn}$. Recall that $M(\mu, \nu)$ is the collection of the matrices satisfying the constrains arising in (2).

Proposition 1.

It holds

$$0 \quad \leq \quad \max_{p\in M(\mu,\nu)} E_\varepsilon(p) \;-\; \mathsf{S}(\mu,\nu) \quad \leq \quad \big(\log(mn)+1\big)\varepsilon.$$

Define $q \in \mathbb{R}_+^{mn}$ by $q_{i,j} = e^{c_{i,j}/\varepsilon}$, then one has $E_\varepsilon(p) := \varepsilon \mathrm{KL}(p|q)$, where

$$\mathrm{KL}(p|q) \quad := \quad \sum_{1\leq i\leq m, 1\leq j\leq n} p_{i,j}\left[1 - \log\left(\frac{p_{i,j}}{q_{i,j}}\right)\right].$$

We follow Bregman's idea and separate the linear constraints of $M(\mu,\nu)$. Define by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_{d+2} \subset \mathbb{R}_+^{mn}$ the subsets of matrices $p = (p_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$:

$$\begin{aligned}
\mathcal{C}_1 \quad &:= \quad \left\{ p \in \mathbb{R}_+^{mn} : \quad \sum_{1\leq j\leq n} p_{k,j} \;=\; \alpha_k, \quad \text{for } k = 1, \cdots, m \right\}, \\
\mathcal{C}_2 \quad &:= \quad \left\{ p \in \mathbb{R}_+^{mn} : \quad \sum_{1\leq i\leq m} p_{i,k} \;=\; \beta_k, \quad \text{for } k = 1, \cdots, n \right\}, \\
\mathcal{C}_{2+l} \quad &:= \quad \left\{ p \in \mathbb{R}_+^{mn} : \quad \sum_{1\leq j\leq n} p_{k,j} y_j^l \;=\; \alpha_k x_k^l, \quad \text{for } k = 1, \cdots, m \right\},
\end{aligned}$$

where $l = 1, \ldots, d$. Clearly one has $M(\mu,\nu) = \cap_{1\leq k\leq d+2} \mathcal{C}_k$ and

$$\max_{p\in M(\mu,\nu)} E_\varepsilon(p) \quad = \quad \varepsilon \max_{p\in \cap_{1\leq k\leq d+2}\mathcal{C}_k} \mathrm{KL}(p|q).$$

Notice that $p \mapsto \mathrm{KL}(p|q)$ is strictly concave, then we may apply the iterative Bregman projection as follows. Define the periodic sequence $\{\mathcal{C}_l\}_{l\geq 1}$ by $\mathcal{C}_{l+(d+2)} := \mathcal{C}_l$ for all $l \geq 1$. Let $p^{(0)} = q$, and set for all $l \geq 1$

$$p^{(l)} \quad := \quad \mathrm{argmax}_{p\in\mathcal{C}_l} \, \mathrm{KL}(p|p^{(l-1)}). \tag{3}$$

Then Bregman's theorem yields the desired convergence.

Theorem 1 (Bregman, 67').

With the above notations, the sequence $\big\{p^{(l)}\big\}_{l\geq 1}$ constructed in (3) is convergent and

$$\lim_{l\to\infty} p^{(l)} \quad = \quad \mathrm{argmax}_{p\in\cap_{1\leq k\leq d+2}\mathcal{C}_k} \, \mathrm{KL}(p|q).$$

During this project, we aim to compute numerically $p^{(l)}$ given $p^{(l-1)}$.