

Unfitted HHO method with polynomial extension for elliptic interface problems



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Domain decomposition

- **Domain Ω :** $\overline{\Omega} := \overline{\Omega_1} \cup \overline{\Omega_2}$
- **Interface Γ :** $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$
- **Jump across Γ :** $\llbracket u \rrbracket_\Gamma := u|_{\Omega_1} - u|_{\Omega_2}$

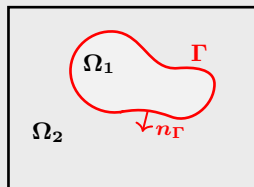


Fig. 1: Model problem

Elliptic interface problem

- **Strong form:** Find $u \in H^1(\Omega_1 \cup \Omega_2)$ such that

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = f & \text{in } \Omega_1 \cup \Omega_2 \\ \llbracket u \rrbracket_\Gamma = g_D & \text{on } \Gamma \\ \llbracket \kappa \nabla u \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma = g_N & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Unfitted methods

- Minimize complexity of mesh generation
- Handle cut cells by doubling unknowns
- Need to integrate polynomials in cut cells (e.g. by submeshing)
- **Price to pay** : Need to stabilize ill-cut cells

Unfitted FEM methods

- Introduced by [Hansbo and Hansbo, 2002]
- Standart technique for stabilization: **Ghost penalty** [Burman, 2010]

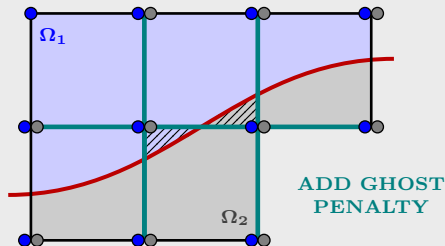


Fig. 2: Doubling of Q_1 -FEM unknowns, ill-cut cells (dashes) and set of ghost-penalty faces

Fitted HHO methods

- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]
- **Main features:**
 - ▶ Design based on cell and face unknowns
 - ▶ General meshes: polyhedral meshes, hanging nodes
 - ▶ Attractive computational cost: Static condensation
 - ▶ Local conservativity at the cell level

Unfitted HHO methods

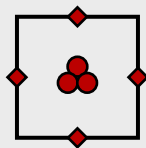
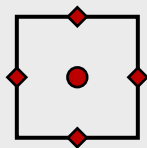
- **Seminal papers:** [Burman and Ern, 2018] [Burman, Cicuttin, Delay, and Ern, 2021]
- **Main features:**
 - ▶ Doubling of cell and face unknowns in cut cells
 - ▶ Cut stabilization by cell agglomeration
- **New approach for cut stabilization: Polynomial extension**
 - ▶ Use of similar technique for unfitted FEM [Badia, Verdugo, and Martín, 2018]

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Degrees of freedom

■ Polynomial unknowns attached to mesh cells and faces



HHO unknowns:

$$\hat{u}_h := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{\mathcal{U}}_h$$

● Cell unknowns, degree $k' \in \{k, k+1\}$ ◆ Face unknowns, degree $k \geq 0$

Fig. 3: Local HHO unknowns. **Left:** $k' = k = 0$. **Right:** $k' = k + 1 = 1$.

► Equal-order: $k' = k$

► Mixed-order: $k' = k + 1$

Global degrees of freedom

■ Mesh \mathcal{T}_h with faces \mathcal{F}_h

■ **Global HHO spaces:**
$$\hat{\mathcal{U}}_h := \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^{k'}(T; \mathbb{R}) \quad \times \quad \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F; \mathbb{R})$$

Design of the local gradient reconstruction operator

■ **Gradient reconstruction operator:**

$$\blacktriangleright (\nabla \mathbf{u})|_T \rightarrow \mathbf{G}_T(\hat{\mathbf{u}}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$$

Design of $\mathbf{G}_T(\hat{\mathbf{u}}_T)$ mimics an integration by parts

$$(\mathbf{G}_T(\hat{\mathbf{u}}_T), \mathbf{q})_T = (\nabla u_T, \mathbf{q})_T - (u_T - u_{\partial T}, \mathbf{q} \cdot \mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbb{P}^k(T; \mathbb{R}^d)$$

Design of the local stabilization operator

$$\blacksquare \text{ **Stabilization operator:}** \quad \delta_{\partial T}(\hat{\mathbf{u}}_T) := \mathbf{u}_{\partial T} - \mathbf{u}_{T|\partial T} \approx \mathbf{0}$$

Matching of cell dofs trace with face dofs (weakly)

► **Equal-order discretization:** Specific stabilization to HHO
(not used in unfitted HHO)

► **Mixed-order discretization:** Same as HDG (Lehrenfeld-Schöberl)

$$s_T(\hat{\mathbf{u}}_T, \hat{w}_T) := \kappa h_T^{-1} (\Pi_{\partial T}^k(u_T - u_{\partial T}), w_T - w_{\partial T})_{\partial T}$$

Main advantages of HHO methods

■ **Improved error estimates for smooth solutions:**

▶ H^1 -error: $\mathcal{O}(h^{k+1})$

▶ L^2 -error: $\mathcal{O}(h^{k+2})$

■ **Attractive computational costs:**

Elimination of cell unknowns by Schur complement (static condensation) :

▶ Global problem couples only face dofs

▶ Cell dofs recovered by local post-processing

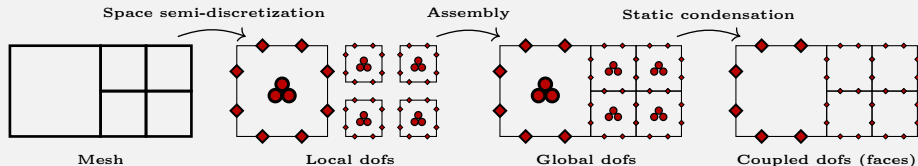


Fig. 4: Assembly and Schur complement procedure in the framework of HHO schemes

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Unfitted meshes and local unknowns

■ **Mesh partitioning:** $\mathcal{T}_h := \mathcal{T}_h^\circ \cup \mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^{\text{KO}}$

► $\mathcal{T}_h^{\text{KO},1} \cup \mathcal{T}_h^{\text{KO},2} = \emptyset$ if mesh fine enough [Burman and Ern, 2018]

■ **Doubling local unknowns in cut cells:**

$$\hat{u}_T := (\hat{u}_{T^1}, \hat{u}_{T^2}) := (u_{T^1}, u_{(\partial T)^1}, u_{T^2}, u_{(\partial T)^2}) \in \hat{\mathcal{U}}_T := \hat{\mathcal{U}}_{T^1} \times \hat{\mathcal{U}}_{T^2}$$

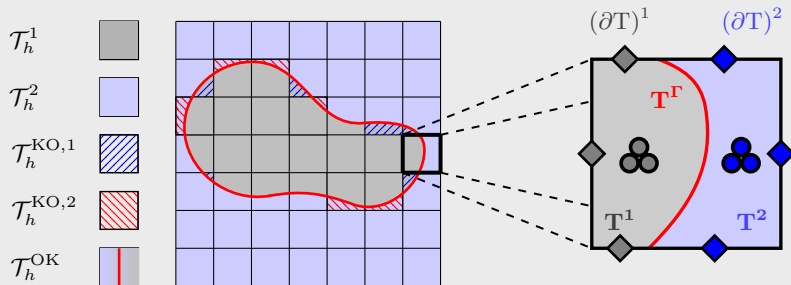


Fig. 5: Left. Types of cells involved in unfitted meshes.

Right. Local dofs in cut cell.

Pairing operator

$$\mathcal{N}_i : \mathcal{T}_h^{\text{KO},i} \ni S \mapsto T \in (\mathcal{T}_h^i \cup \mathcal{T}_h^{\text{OK}} \cup \mathcal{T}_h^{\text{KO},\bar{i}}) \cap \Delta_1(S), \quad \forall i \in \{1,2\}$$

- $\Delta_1(S)$: first layer of neighboring cells of S

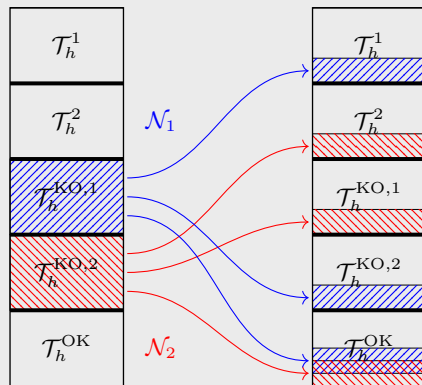


Fig. 6: Pairing of ill-cut cells

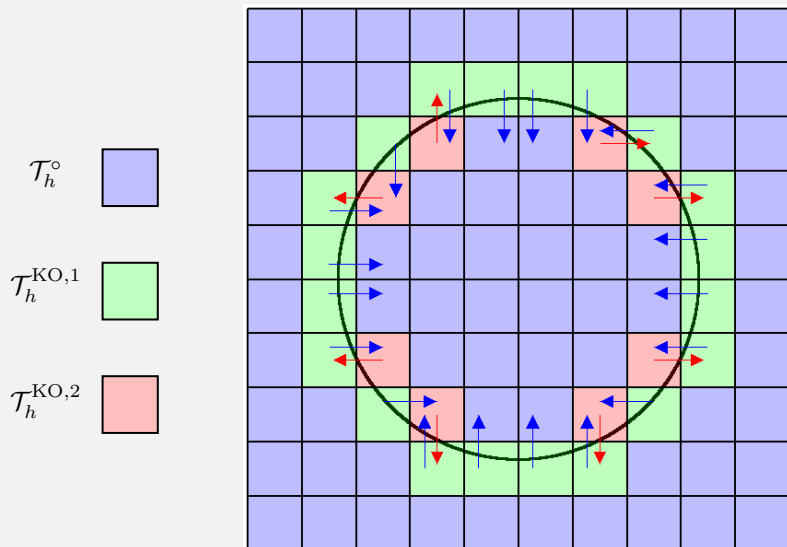


Fig. 7: Exemple of pairing procedure for coarse Cartesian mesh cut by circular interface

Cell agglomeration vs. Polynomial extension

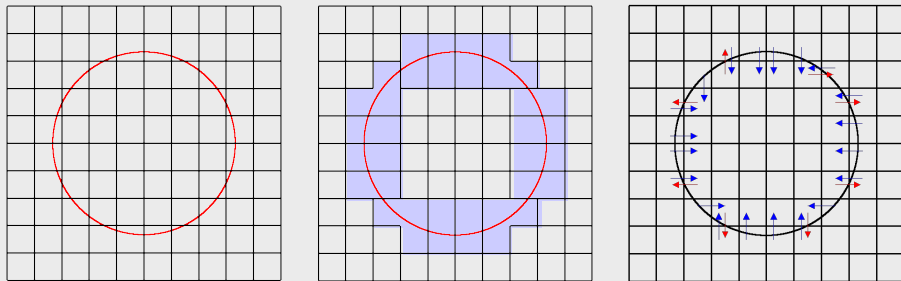


Fig. 8: **Left.** Initial mesh with circular interface. **Middle.** Cell agglomeration. **Right.** Stencil modification for polynomial extension

■ **Cell agglomeration:**

- ✓ Leverages on polyhedral capacity of HHO methods
- ✗ Intrusive on mesh data structure

■ **Polynomial extension:**

- ✓ Works on initial mesh (non-intrusive)
- ✗ Requires modification of the stencil (intrusive at the assembly level)

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■ **UNCUT CELLS:** $T \in \mathcal{T}_h^i$

- Stencil includes
dofs of ill-cut cell(s)

$$\hat{u}_T^+ := (\hat{u}_T, (\hat{u}_S)_{S \in \mathcal{N}^{-1}(T)})$$

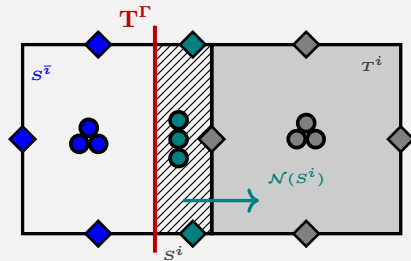


Fig. 9: Pairing configuration

Design of the local gradient reconstruction in the uncut cells

■ **Classical gradient reconstruction:**

$$(\mathbf{G}_T(\hat{u}_T), \mathbf{q})_T = (\nabla u_T, \mathbf{q})_T - (u_T - u_{\partial T}, \mathbf{q} \cdot \mathbf{n}_T)_{\partial T}$$

■ **Gradient reconstruction with polynomial extension:**

$$(\mathbf{G}_T^k(\hat{u}_T^+), \mathbf{q})_T := (\nabla u_T, \mathbf{q})_T - (u_T - u_{\partial T}, \mathbf{q} \cdot \mathbf{n}_T)_{\partial T}$$

$$+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \left\{ (\nabla u_T, \mathbf{q})_{S^i} - (u_T - u_{(\partial S)^i}, \mathbf{q} \cdot \mathbf{n}_S)_{(\partial S)^i} - \delta_{i1} (u_T - u_{S^{\bar{i}}}, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{S^\Gamma} \right\}$$

■ **WELL-CUT CELLS:** $T \in \mathcal{T}_h^{\text{OK}}$

- Stencil includes
dofs of ill-cut cell(s)

$$\hat{u}_{T^i}^+ := (\hat{u}_{T^i}, (\hat{u}_{S^i})_{S \in \mathcal{N}_i^{-1}(T)}) , \quad \forall i \in \{1, 2\}$$

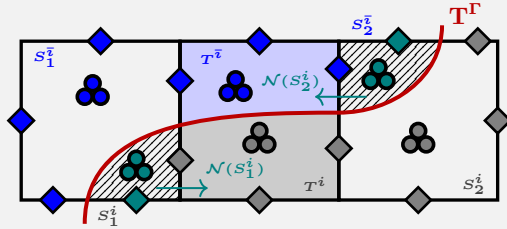


Fig. 10: Pairing configuration

Design of the local gradient reconstruction in the well-cut cells

■ **Classical gradient reconstruction:** $\forall i \in \{1, 2\}$,

$$(\mathbf{G}_{T^i}^k(\hat{u}_T^+), \mathbf{q})_{T^i} := (\nabla u_{T^i}, \mathbf{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \mathbf{q} \cdot \mathbf{n}_T)_{(\partial T)^i} - \delta_{i1} \kappa_1 (u_{T^i} - u_{T^{\bar{i}}}, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{T^\Gamma}$$

- choice $\delta_{i1} \kappa_1$ robust with respect to strong contrast: $\kappa_1 \ll \kappa_2$

■ **Gradient reconstruction with polynomial extension:** $\forall i \in \{1, 2\}$,

$$\begin{aligned} (\mathbf{G}_{T^i}^k(\hat{u}_T^+), \mathbf{q})_{T^i} &:= (\nabla u_{T^i}, \mathbf{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \mathbf{q} \cdot \mathbf{n}_T)_{(\partial T)^i} - \delta_{i1} \kappa_1 (u_{T^i} - u_{T^{\bar{i}}}, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{T^\Gamma} \\ &+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \left\{ (\nabla u_{T^i}, \mathbf{q})_{S^i} - (u_{T^i} - u_{(\partial S)^i}, \mathbf{q} \cdot \mathbf{n}_S)_{(\partial S)^i} - \delta_{i1} \kappa_1 (u_{T^i} - u_{S^{\bar{i}}}, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{S^\Gamma} \right\} \end{aligned}$$

■ **ILL-CUT CELLS:** $T \in \mathcal{T}_h^{\text{KO},i}$

- Stencil of paired cell includes
dofs of ill-cut cell(s)

$$\hat{u}_{T^i}^+ := (\hat{u}_{T^i}, \hat{u}_{\mathcal{N}(T)^i}, (\hat{u}_{S^i})_{S \in \mathcal{N}_i^{-1}(T)})$$

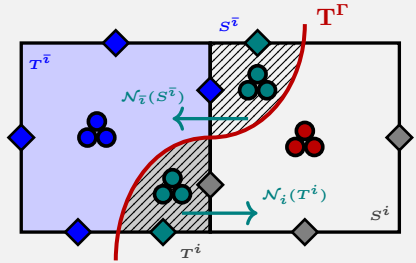


Fig. 11: Pairing configuration

Design of the local gradient reconstruction in the ill-cut cells

■ **Classical gradient reconstruction:** $\forall i \in \{1, 2\},$

$$(G_{T^i}^k(\hat{u}_T^+), \mathbf{q})_{T^i} := (\nabla u_{T^i}, \mathbf{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \mathbf{q} \cdot \mathbf{n}_T)_{(\partial T)^i} - \delta_{i1} \kappa_1 (u_{T^i} - u_{T^i\text{-bar}}, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{T^i\text{-bar}}$$

■ **Gradient reconstruction with polynomial extension:**

$$(G_{T^i}^k(\hat{u}_T^+), \mathbf{q})_{T^i} := 0$$

$$\begin{aligned} (G_{T^i\text{-bar}}^k(\hat{u}_T^+), \bar{\mathbf{q}})_{T^i\text{-bar}} &:= (\nabla u_{T^i\text{-bar}}, \bar{\mathbf{q}})_{T^i\text{-bar}} - (u_{T^i\text{-bar}} - u_{(\partial T)^i\text{-bar}}, \bar{\mathbf{q}} \cdot \mathbf{n}_T)_{(\partial T)^i\text{-bar}} - \delta_{i1} \kappa_1 (u_{T^i\text{-bar}} - u_{\mathcal{N}(T)^i}, \bar{\mathbf{q}} \cdot \mathbf{n}_\Gamma)_{T^i\text{-bar}} \\ &+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \left\{ (\nabla u_{T^i\text{-bar}}, \bar{\mathbf{q}})_{S^i\text{-bar}} - (u_{T^i\text{-bar}} - u_{(\partial S)^i\text{-bar}}, \bar{\mathbf{q}} \cdot \mathbf{n}_S)_{(\partial S)^i\text{-bar}} - \delta_{i1} \kappa_1 (u_{T^i\text{-bar}} - u_{S^i}, \bar{\mathbf{q}} \cdot \mathbf{n}_\Gamma)_{S^i\text{-bar}} \right\} \end{aligned}$$

HHO stabilization

- **Classical HHO stabilization:** $\forall i \in \{1, 2\},$

$$s_{Ti}(\hat{u}_{Ti}, \hat{w}_{Ti}) := \kappa_i h_T^{-1} (\Pi_{(\partial T)^i}^k (u_{Ti} - u_{(\partial T)^i}), w_{Ti} - w_{(\partial T)^i})_{(\partial T)^i}$$

- **Stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\text{OK}}$):** $\forall i \in \{1, 2\},$

$$\begin{aligned} s_{Ti}(\hat{u}_T^+, \hat{w}_T^+) &:= \kappa_i h_T^{-1} (\Pi_{(\partial T)^i}^k (u_{Ti} - u_{(\partial T)^i}), w_{Ti} - w_{(\partial T)^i})_{(\partial T)^i} \\ &+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \kappa_i h_T^{-1} (\Pi_{(\partial S)^i}^k (u_{Ti} - u_{(\partial S)^i}), w_{Ti} - w_{(\partial S)^i})_{(\partial S)^i} \end{aligned}$$

Design of the cut stabilization operator (Nitsche's term)

- **Classical cut stabilization operator:** $\forall i \in \{1, 2\},$

$$s_{Ti}^\Gamma(\hat{u}_T^+, \hat{w}_T^+) := \delta_{i1} \kappa_1 h_T^{-1} (\llbracket u_T \rrbracket_\Gamma, \llbracket w_T \rrbracket_\Gamma)_{T^\Gamma}$$

- **Cut stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\text{OK}}$):** $\forall i \in \{1, 2\},$

$$s_{Ti}^\Gamma(\hat{u}_T^+, \hat{w}_T^+) := \delta_{i1} \kappa_1 h_T^{-1} (\llbracket u_T \rrbracket_\Gamma, \llbracket w_T \rrbracket_\Gamma)_{T^\Gamma} + \sum_{S \in \mathcal{N}_i^{-1}(T)} \delta_{i1} \kappa_1 h_T^{-1} (\llbracket u_S \rrbracket_\Gamma, \llbracket w_S \rrbracket_\Gamma)_{S^\Gamma}$$

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Global discrete problem

$$a_h(\hat{u}_h, \hat{w}_h) = \ell_h(\hat{w}_h) \quad \forall \hat{w}_h \in \hat{\mathcal{U}}_{h0},$$

$$\blacksquare \quad a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} a_{Ti}(\hat{u}_T^+, \hat{w}_T^+)$$

$$\blacktriangleright \quad a_{Ti}(\hat{u}_T^+, \hat{w}_T^+) := \kappa_i(\mathbf{G}_{Ti}^k(\hat{u}_T^+), \mathbf{G}_{Ti}^k(\hat{w}_T^+))_{Ti} + s_{Ti}(\hat{u}_T^+, \hat{w}_T^+) + s_{Ti}^\Gamma(\hat{u}_T^+, \hat{w}_T^+)$$

$$\blacksquare \quad \ell_h(\hat{w}_h) := \sum_{T \in \mathcal{T}_h^\circ} (f, w_{Ti})_{Ti} \\ + \sum_{T \in \mathcal{T}_h^{\text{KO}}} \left\{ (f, w_{\mathcal{N}(T)^i})_{Ti} + (f, w_{T^{\bar{i}}})_{T^{\bar{i}}} \right\} + \sum_{T \in \mathcal{T}_h^{\text{OK}}} \sum_{i \in \{1,2\}} (f, w_{Ti})_{Ti}$$

$$\blacktriangleright \quad \text{For simplicity, we consider } g_D = g_N = 0$$

Algebraic realization for gradient reconstruction

- Algebraic realization of $(\mathbf{G}_{T^i}^k(\hat{u}_T^+), \mathbf{G}_{T^i}^k(\hat{w}_T^+))_{T^i}$ (e.g. $\forall T \in \mathcal{T}_h^{\text{OK}}$):

$$\forall i \in \{1, 2\}, \quad \mathbf{G}_{T^i}^\dagger \mathbf{M}_{T^i}^{-1} \mathbb{G}_{T^i} := \mathbf{G}_{T^i}^\dagger \mathbf{M}_{T^i}^{-1} \mathbf{G}_{T^i} + \sum_{S \in \mathcal{N}_i^{-1}(T)} \{ \mathbf{G}_{S^i}^\dagger \mathbf{M}_{T^i}^{-1} \mathbf{G}_{S^i} \}$$

- $\mathbf{M}_T := (\phi_{T,i}, \phi_{T,j})_T$, $0 \leq i, j < N^k := \dim(\mathbb{P}^k(T; \mathbb{R}))$, (componentwise mass matrix)

- $N_d^k := d \times N^k$
- $N_{\partial T} := \text{number of faces of } T$
- $N_S := \#\mathcal{N}_i^{-1}(T)$
- $N_{\partial S} := \text{number of faces of } S$

► $\mathbb{G}_T := N_d^k$

- Extension of local bilinear form \longrightarrow Modification of assembly

Error analysis

- Based on [Burman, Cicuttin, Delay, and Ern, 2021]
 - ▶ Stability (coercivity)
 - ▶ Consistency
 - ▶ Quasi-optimal error estimates
 - ▶ For smooth solution, H^1 -error: $\mathcal{O}(h^{k+1})$

- Implementation & Analysis in progress

Thank you for your attention !