

Hybrid high-order methods for the numerical simulation of elasto-acoustic wave propagation



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Goal

Accurate modeling and simulation of seismo-acoustic waves through **heterogeneous domains with complex geometries**

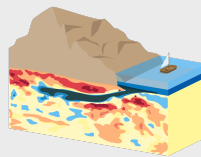
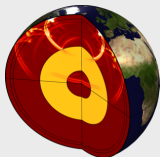


Fig. 1: Global seismic wave propagation **Fig. 2:** Local heterogeneities of the Earth

continuous Galerkin methods using spectral elements

- Commonly used numerical tool:
 - ▶ Hexahedral/quadrangular meshes \rightarrow use of tensorized polynomial basis
- **Main issue:** Complex mesh generation for realistic geological structures

cG vs. dG methods

Main advantages of dG methods

- **Mesh flexibility:** Handling of **unstructured / polyhedral meshes**
- **Local conservativity** at the element level
- **Same order of convergence** as cG for smooth solutions:
 - ▶ H^1 -error: $\mathcal{O}(h^k)$
 - ▶ L^2 -error: $\mathcal{O}(h^{k+1})$

Drawbacks of dG methods

- **Higher computational cost** and **memory requirement**

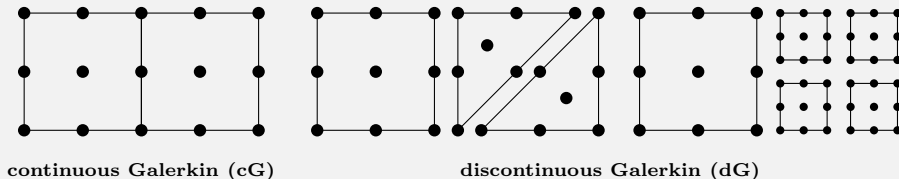


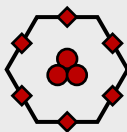
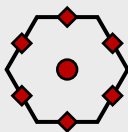
Fig. 3: Discrete unknowns for cG and dG methods

Introduction to HHO methods

- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]

Degrees of freedom

- **Polynomial unknowns attached to mesh cells and faces**



HHO unknowns:

$$\hat{u}_h := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{\mathcal{U}}_h$$

- **Cell unknowns, degree $k' \in \{k, k+1\}$**
- ◆ **Face unknowns, degree $k \geq 0$**

Fig. 4: Local HHO unknowns. **Left:** $k' = k = 0$. **Right:** $k' = k + 1 = 1$.

- Equal-order: $k' = k$
- Mixed-order: $k' = k + 1$

Design

■ Gradient reconstruction operator:

$$(\nabla \mathbf{u})|_T \rightarrow \mathbf{G}_T(\hat{\mathbf{u}}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$$

Design of $\mathbf{G}_T(\hat{\mathbf{u}}_T)$ mimics an integration by parts

■ Stabilization operator: $\delta_{\partial T}(\hat{\mathbf{u}}_T) := \mathbf{u}_{\partial T} - \mathbf{u}_{T|\partial T} \approx \mathbf{0}$

Matching of cell dofs trace with face dofs (weakly)

Advantages of HHO over dG methods

■ Improved error estimates for smooth solutions

▶ H^1 -error: $\mathcal{O}(h^{k+1})$

▶ L^2 -error: $\mathcal{O}(h^{k+2})$

■ Attractive computational costs

Elimination of cell unknowns by **static condensation**

- ▶ Global problem couples only face dofs
- ▶ Cell dofs recovered by local post-processing

Link to other methods

$$\text{HHO} \equiv \text{HDG} \equiv \text{WG} \equiv \text{ncVEM}$$

[Cockburn, Di Pietro, and Ern, 2016] [Lemaire, 2020] [Cicuttin, Ern, and Pignet, 2021]

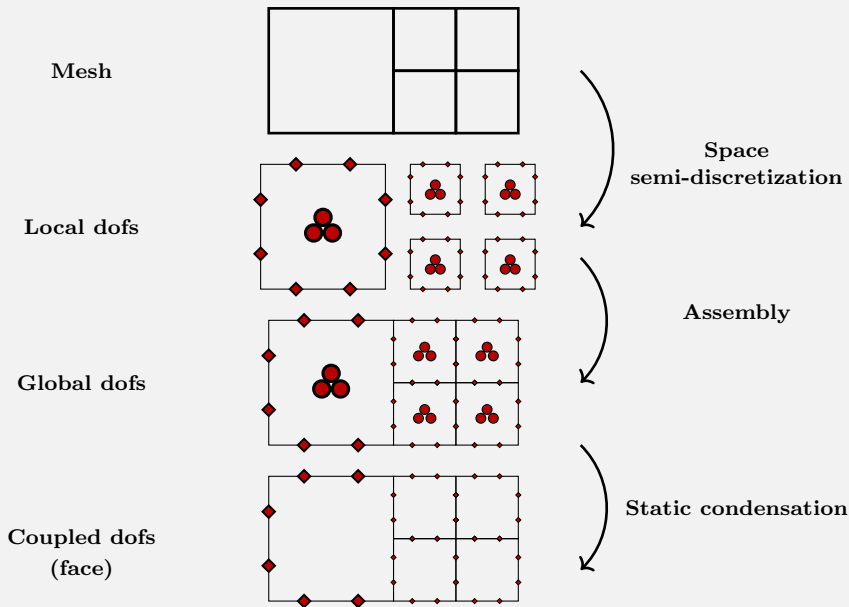
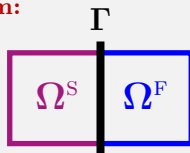


Fig. 5: Assembly and static condensation procedure in HHO framework

■ **Model problem:**



$$\Omega := \Omega^S \cup \Omega^F$$

Elasto-acoustic interface Γ

Fig. 6: Setting for elasto-acoustic coupling

Strong form of acoustic and elastic wave equation in 1st order formulation

$$\begin{cases} \partial_t \varepsilon - \nabla_s \mathbf{v}^S = \mathbf{0} \\ \rho^S \partial_t \mathbf{v}^S - \nabla \cdot (\mathbf{C} : \varepsilon) = \mathbf{f}^S \end{cases}$$

Unknowns

- ▶ \mathbf{v}^S elastic velocity field
- ▶ $\varepsilon := \nabla_s \mathbf{u}$ linearized strain tensor

Parameters

- ▶ $\rho^S, \mathbf{C}(\lambda, \mu)$ (Lamé coefficients)
- ▶ $c_p^S := \sqrt{\frac{\lambda + 2\mu}{\rho^S}}, c_s := \sqrt{\frac{\mu}{\rho^S}}$

$$\begin{cases} \rho^F \partial_t \mathbf{v}^F - \nabla p = \mathbf{0} \\ \frac{1}{\kappa} \partial_t p - \nabla \cdot \mathbf{v}^F = f^F \end{cases}$$

- ▶ p scalar pressure field
- ▶ \mathbf{v}^F acoustic velocity field

- ▶ ρ^F, κ

$$\text{▶ } c_p^F := \sqrt{\frac{\kappa}{\rho^F}}$$

Coupling conditions

$$\begin{cases} \mathbf{v}^S \cdot \mathbf{n}_\Gamma = \mathbf{v}^F \cdot \mathbf{n}_\Gamma & \blacktriangleright \text{Balance of mass} \\ (\mathbf{C}:\boldsymbol{\varepsilon}) \cdot \mathbf{n}_\Gamma = p \mathbf{n}_\Gamma & \blacktriangleright \text{Balance of forces} \end{cases}$$

Initial and boundary conditions

- Initial conditions on (ρ^S, \mathbf{v}^S) and (ρ^F, \mathbf{v}^F)
- Homogeneous Dirichlet boundary conditions on $\partial\Omega$ for simplicity

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HHO space semi-discretization

- **Elastic domain:** $\mathcal{Z}_{\mathcal{T}^s}^{k'} := \underbrace{\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T; \mathbb{R}_{\text{sym}}^{d \times d})}_{\text{space for } \varepsilon}, \quad \widehat{\mathbf{u}}_h^s := \underbrace{\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^{k'}(T; \mathbb{R}^d) \times \bigtimes_{F \in \mathcal{F}_h^s} \mathbb{P}^k(F; \mathbb{R}^d)}_{\text{space for } \mathbf{v}^s}$
- **Acoustic domain:** $\mathcal{V}_{\mathcal{T}^F}^k := \underbrace{\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T; \mathbb{R}^d)}_{\text{space for } \mathbf{v}^F}, \quad \widehat{\mathbf{u}}_h^F := \underbrace{\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^{k'}(T; \mathbb{R}) \times \bigtimes_{F \in \mathcal{F}_h^F} \mathbb{P}^k(F; \mathbb{R})}_{\text{space for } p}$

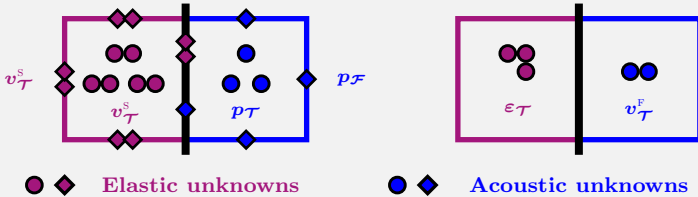


Fig. 7: Elasto-acoustic unknowns with $k' = 1$ and $k = 0$. **Left:** HHO unknowns for \mathbf{v}^s and p . **Right:** dG unknowns for ε and \mathbf{v}^F .

References

- Same discretization as for acoustic [Burman, Duran, and Ern, 2022] and elastic [Burman, Duran, Ern, and Steins, 2021] problems, but **with coupling terms**

Local reconstruction operators

- **Strain reconstruction:** $\mathbf{E}_T(\hat{\mathbf{v}}_T^s) \in \mathbb{P}^k(T; \mathbb{R}_{\text{sym}}^{d \times d})$ s.t. for all $\hat{\mathbf{v}}_T^s \in \hat{\mathcal{U}}_T^s$,

$$(\mathbf{E}_T(\hat{\mathbf{v}}_T^s), \boldsymbol{\zeta})_T = (\nabla_s \mathbf{v}_T^s, \boldsymbol{\zeta})_T - (\mathbf{v}_T^s - \mathbf{v}_{\partial T}^s, \boldsymbol{\zeta} \cdot \mathbf{n}_T)_{\partial T}, \quad \forall \boldsymbol{\zeta} \in \mathbb{P}^k(T; \mathbb{R}_{\text{sym}}^{d \times d})$$

- **Gradient reconstruction:** $\mathbf{G}_T(\hat{p}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$ s.t. for all $\hat{p}_T \in \hat{\mathcal{U}}_T^f$,

$$(\mathbf{G}_T(\hat{p}_T), \mathbf{q})_T = (\nabla p_T, \mathbf{q})_T - (p_T - p_{\partial T}, \mathbf{q} \cdot \mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbb{P}^k(T; \mathbb{R}^d)$$

Local stabilization operators

- **Mixed-order discretization: Stabilization in HDG** (Lehrenfeld-Schöberl)

$$S_{\partial T}(\hat{p}_T) := \Pi_{\partial T}^k(p_T - p_{\partial T}) \quad \mathbf{S}_{\partial T}(\hat{\mathbf{v}}_T^s) := \boldsymbol{\Pi}_{\partial T}^k(\mathbf{v}_T^s - \mathbf{v}_{\partial T}^s)$$

- **Equal-order discretization: Specific stabilization to HHO**

- ▶ Needs additional velocity and pressure reconstructions

HHO space semi-discretization for the elasto-acoustic coupling

■ Elastic wave equation: $\mathbf{E}_{\mathcal{T}}(\hat{\mathbf{v}}_h^s)|_T := \mathbf{E}_T(\hat{\mathbf{v}}_T^s)$

$$(\partial_t \boldsymbol{\varepsilon}_{\mathcal{T}}(t), \mathbf{z}_{\mathcal{T}})_{\Omega_{scs}} - (\mathbf{E}_{\mathcal{T}}(\hat{\mathbf{v}}_h^s(t)), \mathbf{z}_{\mathcal{T}})_{\Omega_{scs}} = 0$$

$$(\rho^s \partial_t \mathbf{v}_{\mathcal{T}^s}^s(t), \mathbf{w}_{\mathcal{T}})_{\Omega_{scs}} + (\mathbf{C}:\boldsymbol{\varepsilon}_{\mathcal{T}}, \mathbf{E}_{\mathcal{T}}(\hat{\mathbf{w}}_h))_{\Omega_{scs}} + s_h^s(\hat{\mathbf{v}}_h^s, \hat{\mathbf{w}}_h) + (p_{\mathcal{F}}(t), \mathbf{w}_{\mathcal{F}} \cdot \mathbf{n}_{\Gamma})_{\Gamma} = (\mathbf{f}^s(t), \mathbf{w}_{\mathcal{T}})_{\Omega_{scs}}$$

■ Acoustic wave equation: $\mathbf{G}_{\mathcal{T}}(\hat{p}_h)|_T := \mathbf{G}_T(\hat{p}_T)$

$$(\rho^F \partial_t \mathbf{v}_{\mathcal{T}}^F(t), \mathbf{r}_{\mathcal{T}})_{\Omega_{scf}} + (\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t)), \mathbf{r}_{\mathcal{T}})_{\Omega_{scf}} = 0$$

$$(\frac{1}{\kappa} \partial_t p_{\mathcal{T}}(t), q_{\mathcal{T}})_{\Omega_{scf}} - (\mathbf{v}_{\mathcal{T}}^F(t), \mathbf{G}_{\mathcal{T}}(\hat{q}_h))_{\Omega_{scf}} + s_h^F(\hat{p}_h(t), \hat{q}_h) - (\mathbf{v}^F s(t) \cdot \mathbf{n}_{\Gamma}, q_{\mathcal{F}})_{\Gamma} = (f^F(t), q_{\mathcal{T}})_{\Omega_{scf}}$$

Global stabilization forms

$$s_h^s(\hat{\mathbf{v}}_h^s, \hat{\boldsymbol{\zeta}}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^s(\mathbf{S}_{\partial T}(\hat{\mathbf{v}}_T^s), \mathbf{S}_{\partial T}(\hat{\boldsymbol{\zeta}}_T))_{\partial T}$$

$$s_h^F(\hat{p}_h, \hat{q}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^F(\mathbf{S}_{\partial T}(\hat{p}_T), \mathbf{S}_{\partial T}(\hat{q}_T))_{\partial T}$$

■ with two strategies: $\tau_T^s = \mathcal{O}(1) = \tau_T^F$ or $\tau_T^s = \mathcal{O}(1/h) = \tau_T^F$

Algebraic realization

■ Static coupling between cell and face unknowns

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & -\mathbf{G}_{\mathcal{F}} & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & \Sigma_{\mathcal{F}\mathcal{F}}^F & 0 & 0 & \mathbf{C}_{\Gamma} \\ \hline 0 & 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^s & \Sigma_{\mathcal{T}\mathcal{F}}^s \\ 0 & 0 & -\mathbf{C}_{\Gamma}^\dagger & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^s & \Sigma_{\mathcal{F}\mathcal{F}}^s \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^F \\ 0 \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^s}^s \\ 0 \end{bmatrix}$$

■ Rearrangement of dofs: cell unknowns first and then face unknowns

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^s & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^s & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^s \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}_{\Gamma} \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^s & -\mathbf{C}_{\Gamma}^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^s \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^F \\ 0 \\ \hline \mathbf{F}_{\mathcal{T}^s}^s \\ 0 \\ 0 \end{bmatrix}$$

SDIRK($s, s + 1$) schemes

- Generic ODE with $f : J \times \mathbb{R}^m \rightarrow \mathbb{R}^m$,

$$\begin{cases} y'(t) = f(t, y(t)), & \forall t \in J := [0, T) \\ y|_{t=0} = y_0 \in \mathbb{R}^m \end{cases}$$

- SDIRK($s, s + 1$) consists

- ▶ in solving sequentially for all $i \in \{1, \dots, s\}$,

$$u_{\dot{i}}^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^i a_{ij} f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

- ▶ and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^s b_j f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

c_1	a_{*}	0	\dots	0
c_2	a_{21}	a_{*}	\ddots	0
\vdots	\vdots	\ddots	\ddots	\vdots
c_s	a_{s1}	\dots	$a_{s,s-1}$	a_{*}
	b_1	\dots	b_{s-1}	b_s

Algebraic realization of SDIRK-HHO

- Face-based sparse linear system to be solved at each stage
- We solve sequentially for all $i \in \{1, \dots, s\}$,

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^F}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,i} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,i} \\ \mathbf{P}_{\mathcal{F}^F}^{n,i} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^F}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n-1} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n-1} \\ \mathbf{P}_{\mathcal{F}^F}^{n-1} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n-1} \end{bmatrix}$$

$$+ \Delta t \sum_{j=1}^i a_{ij} \left(\begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^F}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^F}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,j} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n,j} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,j} \end{bmatrix} \right)$$

- The upper 4×4 submatrix associated with all the cell unknowns is block-diagonal
 - Schur complement procedure

ERK(s) schemes

■ ERK(s) consists

- ▶ in updating sequentially for all $i \in \{1, \dots, s\}$,

$$u_i^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^{i-1} a_{ij} f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

- ▶ and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^s b_j f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

c_1	0	\cdots	\cdots	0
c_2	a_{21}	0	\cdots	0
\vdots	\vdots	\ddots	\ddots	\vdots
c_s	a_{s1}	\cdots	$a_{s,s-1}$	0
	b_1	\cdots	b_{s-1}	b_s

HHO-ERK scheme

■ Coupling of face unknowns at the interface Γ

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^F}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,i} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,i} \\ \mathbf{P}_{\mathcal{F}^F}^{n,i} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^F}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n-1} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n-1} \\ \mathbf{P}_{\mathcal{F}^F}^{n-1} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n-1} \end{bmatrix}$$

$$+ \Delta t \sum_{j=1}^{i-1} a_{ij} \left(\begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^S}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^F}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,j} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n,j} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,j} \end{bmatrix} \right)$$

■ Key observation:

$$\begin{bmatrix} \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \text{ has a block-diagonal structure for } \text{mixed-order HHO}$$

Rearrangement of the face terms for the inversion of coupling block

- Distinguish between internal faces in $\Omega^S \cup \Omega^F$ and faces located on Γ

$$\begin{bmatrix}
 \Sigma_{\mathcal{FF}}^F & 0 & 0 & 0 \\
 0 & \Sigma_{\mathcal{FF}}^S & 0 & 0 \\
 \hline
 0 & 0 & \Sigma_{\mathcal{FF}}^F & C_\Gamma \\
 0 & 0 & -C_\Gamma^\dagger & \Sigma_{\mathcal{FF}}^S
 \end{bmatrix}
 \begin{bmatrix}
 P_{\mathcal{F}_h^{\circ F}} \\
 V_{\mathcal{F}_h^{\circ S}}^S \\
 \hline
 P_{\mathcal{F}_h^{\circ \Gamma}} \\
 V_{\mathcal{F}_h^{\circ \Gamma}}^S
 \end{bmatrix}
 =
 \begin{bmatrix}
 \Sigma_{F^1}^F & C_{F^1} & 0 & 0 & 0 & 0 \\
 -C_{F^1}^\dagger & \Sigma_{F^1}^S & 0 & 0 & 0 & 0 \\
 \hline
 0 & 0 & \ddots & & 0 & 0 \\
 0 & 0 & & \ddots & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & \Sigma_{F^n}^F & C_{F^n} \\
 0 & 0 & 0 & 0 & -C_{F^n}^\dagger & \Sigma_{F^n}^S
 \end{bmatrix}
 \begin{bmatrix}
 P_{F^1} \\
 V_{F^1}^S \\
 \hline
 \vdots \\
 \vdots \\
 \hline
 P_{F^n} \\
 V_{F^n}^S
 \end{bmatrix}$$

Table of Contents

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Computational parameters

- Space refinement: $h = 0.1 \times 2^{-\ell}$ (in each subdomain)
- Time refinement: $\Delta t = 0.1 \times 2^{-n}$

Meshes

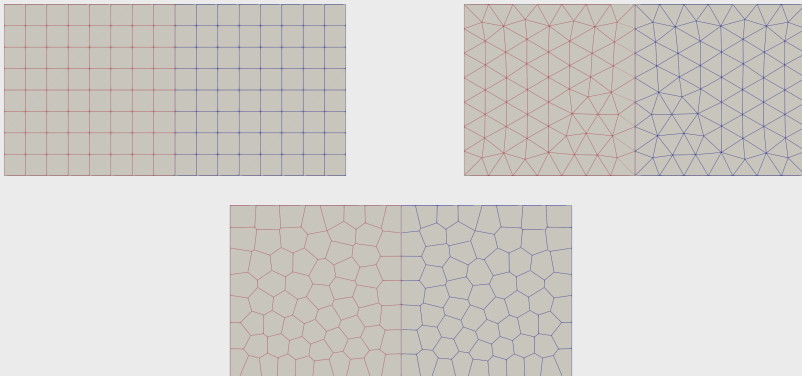


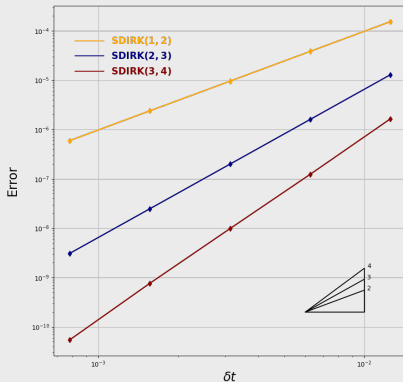
Fig. 8: Cartesian, simplicial and polyhedral meshes for $\ell = 0$

Convergence rates in time

► **Analytical solution:** polynomial in space, sinusoidal in time

► **SDIRK-HHO scheme**

- $k' = k + 1 = 6$
- $\ell = 2$
- $n \in \{3, 4, 5, 6, 7\}$
- $\tau^F = \mathcal{O}(1) = \tau^S$



► **ERK-HHO scheme**

- $k' = k + 1 = 5$
- $\ell = 1$
- $n \in \{6, 7, 8, 9\}$
- $\tau^F = \mathcal{O}(1) = \tau^S$

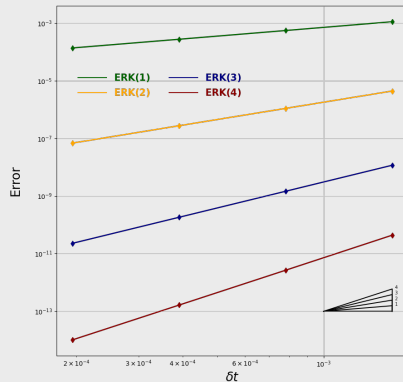


Fig. 9: L^2 -errors for HHO-RK schemes as a function of the time-step

Convergence rates in space

► **Analytical solution:** polynomial in time, sinusoidal in space

► **SDIRK(3,4)-HHO scheme**

► $n = 8$

► $\ell \in \{0, 1, 2, 3, 4\}$

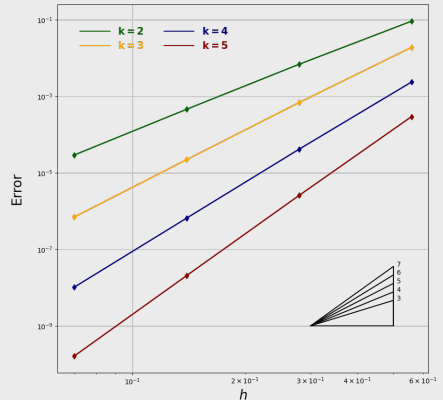
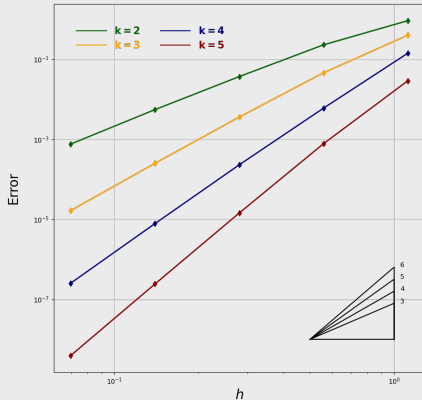


Fig. 10: L^2 -errors for the HHO-SDIRK(3,4) schemes as a function of the mesh-size. **Left:** $\tau_T^F = \mathcal{O}(1) = \tau_T^S$. **Right:** $\tau_T^F = \mathcal{O}(h_T^{-1}) = \tau_T^S$

Ricker wavelet

- **SDIRK(3,4)**, $k = 1$, $\ell = 7$, $n = 9$ ► Homogeneous Dirichlet boundary conditions
- **Initial condition:** velocity Ricker wavelet centered at point $(x_c, y_c) \in \Omega_{scf}$,

$$\mathbf{v}_0(x, y) := \theta e^{-\pi^2 \frac{r^2}{\lambda^2}} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

Academic test case

- **Homogeneous material properties:** $\rho^F = \rho^S = 1$, $c_P^S = \sqrt{3}$, $c_P^F = c_S = 1$

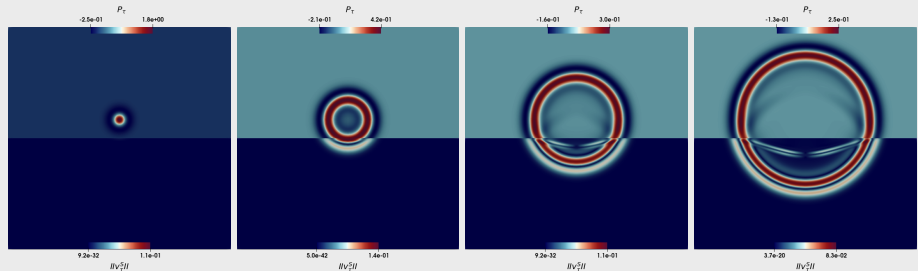


Fig. 11: Acoustic pressure (upper side) and elastic velocity norm (lower side) at times $t \in \{0, 0.025, 0.075, 0.15\}$

Realistic test case with strong property contrast: Granite-Water

■ **Material properties:**

- ▶ **Granite:** $\rho^S = 2800 \text{ kg.m}^{-3}$, $c_p^S = 5000 \text{ m.s}^{-1}$, $c_s = 3000 \text{ m.s}^{-1}$
- ▶ **Water:** $\rho^F = 997 \text{ kg.m}^{-3}$, $\kappa = 2.1 \times 10^9 \text{ Pa}$, $c_p^F = 1450 \text{ m.s}^{-1}$

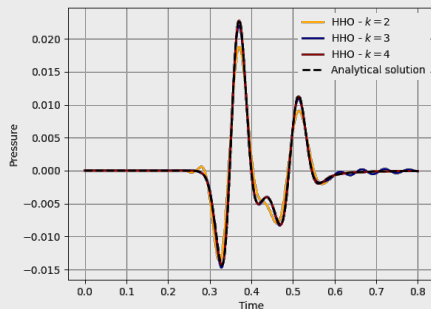
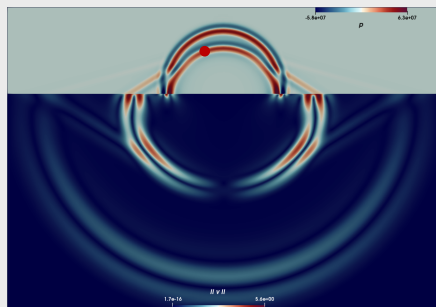
■ **Computational parameters:** SDIRK(3,4), $n = 8$, $l = 7$, $k = 2$ 

Fig. 12: Left panel: Acoustic pressure (upper side) and elastic velocity norm (lower side) at time $t = 0.375\text{s}$. Right panel: Comparison to analytical solution (Gar6more).

Propagation of an elastic pulse in sedimentary basin and atmosphere

■ **Material properties:**

- ▶ **Sedimentary basin:** $\rho^S = 1200 \text{ kg.m}^{-3}$, $c_p^S = 3400 \text{ m.s}^{-1}$, $c_s = 1400 \text{ m.s}^{-1}$
- ▶ **Bedrock:** $\rho^S = 5350 \text{ kg.m}^{-3}$, $c_p^S = 3090 \text{ m.s}^{-1}$, $c_s = 2570 \text{ m.s}^{-1}$
- ▶ **Air:** $\rho^F = 1.292 \text{ kg.m}^{-3}$, $c_p^F = 340 \text{ m.s}^{-1}$

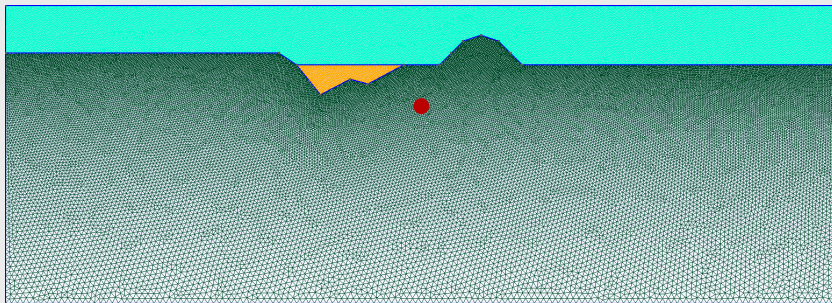
■ **Computational parameters:** SDIRK(3,4), $k = 1$, $\ell = 8$, $n = 9$ ■ **Homogeneous Dirichlet boundary conditions**■ **Initial condition:** velocity Ricker wavelet centered at point $(x_c, y_c) \in \Omega_{scs}$ 

Fig. 13: Mesh of sedimentary basin

Propagation of elastic pulse in sedimentary basin and atmosphere

■ Energy transfer enhancement above sedimentary basin

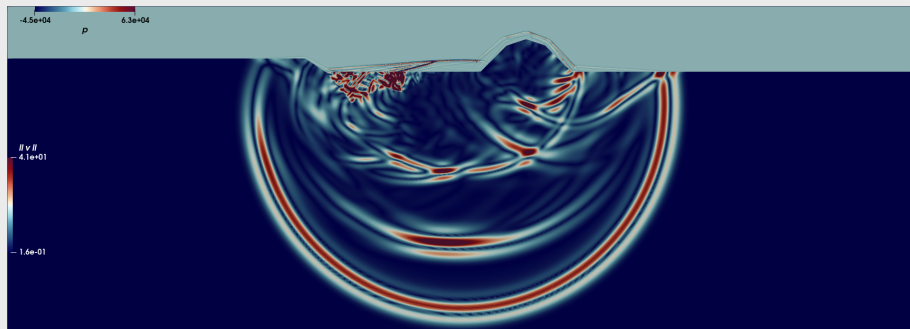


Fig. 14: Propagation of elastic pulse in sedimentary basin and atmosphere

Thank you for your attention !