Standard Time Series Models

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Perspective

- ETS (Error, Trend, Season) model was developed in the 1950s as algorithms to produce point forecasts
- ETS combines a "level" (l_{t-1}), a "trend" ((b_{t-1})) and a "seasonal" (s_{t-m}) components to describe a time series
- The combination $f(l_{t-1}, b_{t-1}, s_{t-m})$ can be additive, multiplicative, etc.
- The rate of change of the components are controlled by "smoothing" parameters: α for the level, β for the trend, γ for the seasonal
- The researcher has to:
    1. To choose the best values for the smoothing parameters
    2. The initial state of the parameters
- Equivalent ETS state-space models have been developpped in the 1990s and the 2000s

Combining Level, Trend and Seasonal Components

- **Additively:** $y_t = l_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$
- **Multiplicatively:** $y_t = l_{t-1} \times b_{t-1} \times s_{t-m} \times (1 + \epsilon_t)$
- **Mixed:** $y_t = (l_{t-1} + b_{t-1}) \times s_{t-m} + \epsilon_t$

Notations:

- **Error** can be additive ("A") or multiplicative ("M")
- **Trend** can be None ("N"), additive ("A"), multiplicative ("M") or damped ("Ad" or "Md")
- Seasonality can be None ("N"), additive ("A") or multiplicative ("M")

Point forecasts and forecast distribution

- Models generates point forecasts estimates, based on a given conditional value $y_{t+1|y0} = f_y(y0)$
- However, need to generate the forecast distribution to assess the quality of models and select the best. **Model selection**
- A stochastic data generating process (DGP) can generate an entire forecast distribution
- Core idea: the residual (ϵ) is the only stochastic (=random) element in $y_t = l_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$
    - Hence, the distribution of the residuals will determine the distribution of the estimator

Level-Only Model ETS(A, N, N) (Simple Smoothing)

Component Form

- Forecast equation: $\hat{y}_{t+h|t} = l_t$
- Smoothing equation: $l_t = \alpha y_t + (1-\alpha) l_{t-1}$

- l_t is the level (="smoothed value") of the series at time t
- $\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}$

Iterate to get exponentially weighted moving average form:
$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T l_0$

Error Correction Form

Component Form

- Forecast equation: $\hat{y}_{t+h|t} = l_t$
- Smoothing equation: $l_t = \alpha y_t + (1-\alpha) l_{t-1}$

Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - l_{t-1}$

Error Correction Form

- $y_t = l_{t-1} + e_t$
- $l_t = l_{t-1} + \alpha * \underbrace{(y_t - l_{t-1})}_{e_t}$
- Intuition: α updates the next-period estimate based on the forecasting error
- Specify probability distribution for e_t, often assumed that $e_t = \epsilon_t \sim \mathcal{N}(0, \sigma^2)$

- Need to choose the best values for α and l_0
- Similarly to regression, choose optimal parameters by minimizing SSE:

$$\text{SSE} = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2$$

- Unlike regression, there is no closed form solution: need to use numerical optimization

State-Space Representation for Additive Models

State-Space Representation

- **Measurement equation**: $y_t = l_{t-1} + \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$
  - The measurement equation is the relationship between observations (y_t) and state (=structure) l_t
- **State equation** $l_t = l_{t-1} + \alpha * \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$
  - The state equation is evolution of the state variable (l_t) through time

- Both equations have the same error process ϵ

State-Space Representation for Multiplicative Models

- Instead of differential errors, specify relative errors: $\eta_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$
- Some easy algebra, substituting $\hat{y}_{t|t-1} = l_{t-1}$ gives

State-Space Representation

- **Measurement equation**: $y_t = l_{t-1}(1 + \epsilon_t)$
- **State equation**: $l_t = l_{t-1}(1 + \alpha * \epsilon_t)$

Implication: Models with additivative and multiplicative errors with the same parameters generate **the same point forecasts but with different prediction intervals**

Holt's Linear Trend

Component Form

- Level: $l_t = \alpha_t + (1 - \alpha)(l_{t-1} + b_{t-1})$
- Trend: $b_t = \beta * (l_t - l_{t-1}) + (1 - \beta)b_{t-1}$
- Forecast: $\hat{y}_{t+h|t} = l_t + hb_t$

- Two smoothing parameters: α and β $(0 \leqslant \alpha,\ \beta \leqslant 1)$
- l_t level: weighted average between y_t and one-step ahead forecast for time t: $l_{t-1} + b_{t-1} = \hat{y}_{t|t-1}$
- b_t slope: weighted average of $(l_t - l_{t-1})$ and b_{t-1}, current and previous estimate of slope
- Choose $\alpha,\ \beta,\ l_0,\ b_0$ to mininize the SSE (sum squared errors)

Damped Trend Method

Component Form

- $l_t = \alpha y_t + (1-\alpha)(l_{t-1} + \phi b_{t-1})$
- $b_t = \beta * (l_t - l_{t-1}) + (1-\beta)\phi b_{t-1}$
- $\hat{y}_{t+h|t} = l_t + (\phi + \phi^2 + \cdots + \phi^h) b_t$

- ϕ is the damping parameter, $0 \leqslant \phi \leqslant 1$
- If $\phi = 1$, the method boils down to Holt's linear trend
- As $h \to \infty$, then $\hat{y}_{T+h|T} \ \to \ l_T + \frac{\phi b_T}{1-\phi}$
- Application: short-run forecasts are trended, long-run forecasts constant

Holt-Winters Seasonal Model

Holt and Winters extended Holt's method to capture seasonality

Component Form

- Level: $l_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(l_{t-1} + b_{t-1})$
- Trend: $b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$
- Season: $\gamma(y_t - l_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$
- Forecast: $\hat{y}_{t+h|t} = l_t + hb_t + s_{t+h-m(k+1)}$

- m is the period of seasonality (e.g. $m = 4$ for quarterly data)

Seasonal Component

- The seasonal component is usually expressed as:

$$s_t = \gamma * (y_t - l_t) + (1 - \gamma) * s_{t-m} \qquad 0 \ \leqslant \ \gamma \ \leqslant 1$$

- By subtitution, we can derive the dynamic of the seasonal term as:

$$s_t = \gamma * (1 - \alpha)(y_t - l_{t-1} - b_{t-1}) + [1 - \gamma(1 - \alpha)]s_{t-m}$$

Forecasting with ETS models

Traditional point forecasts: iterate the equations for $t = T+1, T+2, \ldots, T+h$. By construction, $\epsilon_t = 0 \;\forall\; t > T$

- Equals to $E[y_{t+h}|x_t]$ in the case of additive seasonality only
- Point forecasts for additive ETS are the same as for multiplicative ETS if the parameters are the same

Prediction Intervals

- They can only be generated using the models
- The prediction intervals will differ between models with additive and multiplicative errors
- Some simple ETS models offer exact formula
- For more complex ETS models, the only solution for generating the confidence intervals is by bootstrapping
  - Simulate future sample paths, conditional on the last estimates of the states
  - Obtain the prediction intervals from the percentiles of these simulated future paths

Main Idea: Control the Rate of Change

- α controls the flexibility of the **level**
  - If $\alpha = 0$, the level never updates (stays at the mean)
  - If $\alpha = 1$, the level updates completely (naive, start from yesterday)
- β controls the flexibility of the **trend**
- If $\beta = 0$, the trend is linear
- If $\beta = 1$, the trend changes suddenly at each observation
- γ controls the flexibility of the **seasonality**
  - If $\gamma = 0$ the seasonality is fixed (seasonal mean)
  - If $\gamma = 1$ the seasonality updates completely (seasonal naive)

Stability and Forecastability

- Stability and forecastability have deep mathematical implications that I will only explain intuitively
  - **Stability:** the weights of the observations decay over time, guaranteeing that the newer ones will have higher weights than old one.
    - *This is the core principle behind exponential weights: the model captures information update as time goes through.*
  - **Forecastability:** The forecastability does not guarantee that the weights decay, but it guarantees that the initial value of the state vector will have a constant impact on forecasts, i.e. will not increase in weight with the increase of the forecast horizon.
    - *Forecastability is a variation around the concept of ergodicity: it implies that, as the model embedds new observations, the impact of "old information" from the old observations does not "pollute" the new information brought by the new observations. In other words, the model is "learning" relevant information*

- These concepts are mathematically rigorously defined. For more information, please refer to Hyndman open-source manual `https://otexts.com/fpp3/`

ARIMA Model

- AR: autoregressive (lagged observations as inputs)
- I: integrated (differencing to make series stationary)
- MA: moving average (lagged errors as inputs)

Intuition

Contrary to an ETS, an ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns

Refresher: Intuitive Definition of Stationarity

Intuitive Characterization

If y_t is a stationary time series, then for any period s in the future, the distribution $\{y_t, \ldots, y_{t+s}\}$ doesn't depend on t

A **stationary series** is:

- Roughly horizontal
- Constant variance
- No predictable patterns in the long term

Stabilization:

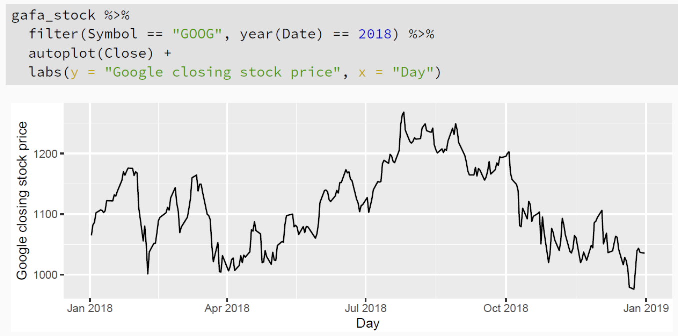
- Transformations help to **stabilize the variance**
- For ARIMA modelling, we also need to **stabilize the mean**

Identifying Non-Stationary Series

Tips:

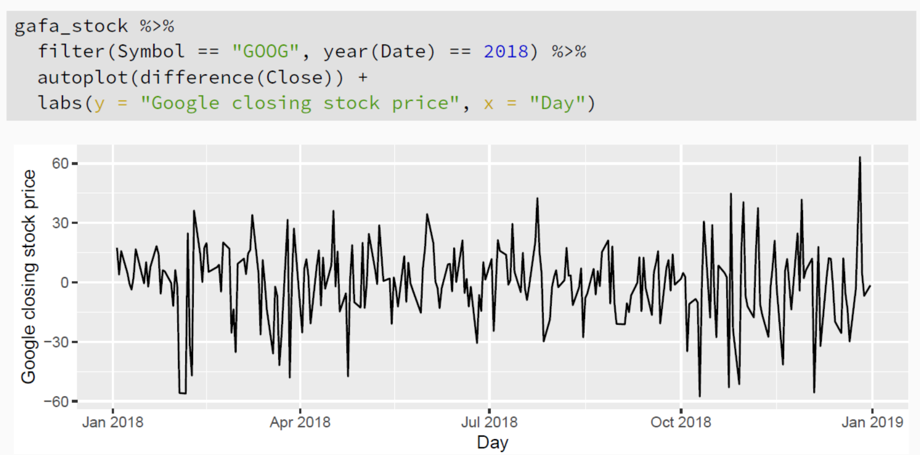
- Time plot
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly
- For non-stationary data, the value of the first coefficient is often large and positive

Stationary?



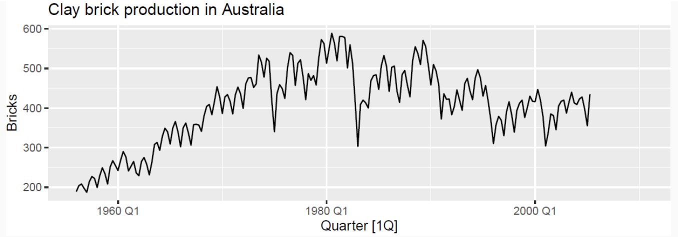
```
gafa_stock %>%
  filter(Symbol == "GOOG", year(Date) == 2018) %>%
  autoplot(Close) +
  labs(y = "Google closing stock price", x = "Day")
```

Stationary?



```
gafa_stock %>%
  filter(Symbol == "GOOG", year(Date) == 2018) %>%
  autoplot(difference(Close)) +
  labs(y = "Google closing stock price", x = "Day")
```

Stationary?



Clay brick production in Australia
600
500
400
300
200
Bricks
1960 Q1
1980 Q1
2000 Q1
Quarter [1Q]

Differencing

- Differencing helps to **stabilize the mean**
- The differenced series is the *change* (or first difference) between each observation in the original series: $y'_t = y_t - y_{t-1}$
- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference y'_1 for the first observation

Random Walk Model

If the differenced series is white noise with zero mean:

Specification

$$y_t - y_{t-1} = \epsilon_t \qquad \text{where } \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

- Very widely used for non-stationary data
- This is the model behind the **naive method**
- Random walks typically have:
  - Long periods of apparent trends up or down
  - Sudden/unpredictable chanes in direction
- In a random walk, the forecast are equal to the last observation
  - Future movements up or down are equally likely

Random Walk with Drift Model

If the differenced series has a drift c:

$$y_t - y_{t-1} = c + \epsilon_t \qquad \text{where } \epsilon_t \ \sim \ \mathcal{N}(0, \sigma^2)$$

- c is the **average change** between consecutive observations
- If $c > 0$, y_t will tend to drift upwards and vice-versa
- This is the model behind the **drift method**

Second-Order Differencing

Occasionally, the differenced data will not appear stationary and it may be necessary to difference the data a second time:

$$y_t'' = y_t' - y_{t-1}' = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

- y_t'' will have $T - 2$ values
- In practice, it is almost never necessary to go beyond second-order differences

Seasonal Differencing

Definition: Seasonal Difference

A seasonal difference is the difference between an observation and the corresponding observation from the previous year

$$y'_t = y_t - y_{t-m}$$

where m = number of seasons

- For monthly data, $m = 12$
- For quarterly data, $m = 4$

Differencing in Practice

When both seasonal and first differences are applied:

- It makes no difference which one is done first - the result will be the same
- If seasonality is strong, we recommend that seasonal differencing be done first because sometimes the resulting series will be stationary and there will be no need for further first difference
- It is important that, if differencing is used, the differences are **interpretable**: for instance, taking lag 3 differences for yearly data is difficult to interpret

Unit Root Tests

Statistical tests can be used to determine the required order of differencing

1. **Augmented Dickey Fuller test**: null hypothesis is that the data is non-stationary and non-seasonal

2. KPSS (Kwiatkowski-Phillips-Schmidt Shin) test: the null hypothesis is that the data is stationary is non-seasonal

3. Other tests are available for seasonal data

Backshift Notation

Notation

The backshift notational device, B is used as follows:

$$By_t = y_{t-1}$$

- B operating on y_t has the effect of **shifting the data back one period**
- Two applications of B to y_t shifts the data back **two periods**

$$B(By_t) = B^2 y_t = y_{t-2}$$

B depends on the period/frequency considered. Shifting monthly data by a year supposes using B^{12}

Relationship with Differencing

Importantly, the backshift operator is convenient for describing differencing

Backshift Operator and Differencing

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1-B)y_t$$

- Likewise, second-order differences are obtained with: $y''_t = (1-B)^2 y_t$
- Pay attention !! Second-order difference is not second difference
  - Second order difference: $(1-B)^2 y_t = y''_t = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$
  - Second difference: $1 - B^2 y_t = y_t - y_{t-2}$

Combined Effects

Assume that you want to combine a first difference with a seasonal difference:

- First difference: $(1 - B)$
- Seasonal difference: $(1 - B^m)$

Then, the backshift operator can be directly combined, as a polynomial, to represent the transformed time series:

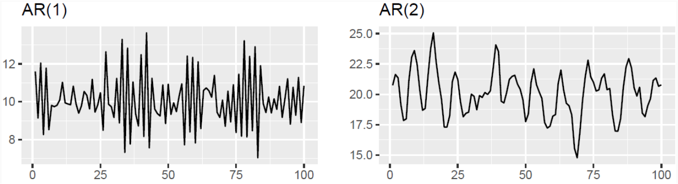
$$(1-B)(1-B^m)y_t = (1-B-B^m+B^{m+1})y_t = y_t - y_{t-1} - y_{t-m} + y_{t-m-1}$$

Autoregressive (AR) Models

Definition

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_P y_{t-p} + \epsilon_t$$

- where ϵ_t is a white noise
- This is a multiple regression with **lagged variables**



AR(1) Model

Specification

$$y_t = c + \phi_1 y_{t-1} + \epsilon_t$$

- When $\phi_1 = 0$, y_t is equivalent to a **white noise**
- When $\phi_1 = 1$ and $c = 0$, y_t is equivalent to a **random walk**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is equivalent to a **random walk with drift**
- When $\phi_1 < 0$ y_t tends to oscillate between positive and negative values

Stationarity Conditions

To restrict AR models to stationary data, some contraints on the coefficients are needed

General Condition for Stationarity

Complex roots of the polynomial $\mathcal{P}(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots \phi_p z^p$ lie outside the unit circle of the complex plane

- Intuition: For an AR(1) model, the backshift polynomial is $y_t = \phi_1 y_{t-1} + \epsilon_t \ \leftrightarrow \ y_t(1 - \phi_1 B) = \epsilon_t$
- $(1 - \phi_1 B) = 0 \ \leftrightarrow \ B = \underbrace{\frac{1}{\phi_1}}_{\text{Not explosive}}$
- To get the AR expression not explosive, we need $|\frac{1}{\phi_1}| < 1$ and therefore $|\phi_1| > 1$

For low lags orders, the stationarity conditions are simply:

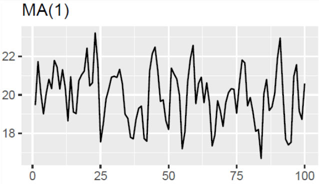
- For $p = 1$: $-1 < \phi_1 < 1$
- For $p = 2$:
    - $-1 < \phi_2 < 1$
    - $\phi_1 + \phi_2 < 1$ and $\phi_2 - \phi_1 < 1$

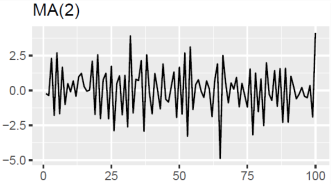
Moving Average Model

Definition: Moving Average Model

$$y_t = c + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

- ϵ_t is a white noise
- This is a multiple regression with **past errors** as predictors
- Do NOT confuse this with *moving average smoothing*!





Wold Decomposition: From AR(p) to MA(∞) Model

Wold Decomposition

It is possible to write any **stationary** AR(p) model as an MA(∞)

- Intuitive: just go backward!
- $y_t = \phi_1 \underbrace{y_{t-1}} + \epsilon_t$
- $y_t = \phi_1(\phi_1 y_{t-1} + \epsilon_{t-1}) + \epsilon_t = \phi_1^2 y_{t-2} + \phi_1 \epsilon_{t-1} + \epsilon_t$
- $\dots$
- Providing that $1 < \phi_1 < 1$:

$$y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \dots$$

Invertibility: From MA(q) to AR(∞)

- Under certain conditions, an MA(q) process can be written as an AR(∞) process

- In this case, the MA model is said to be **invertible**

- Invertible models have some mathematical properties that make them easier to use in practice

- Invertability of an MA model is equivalent to the **forecastability** of an ETS model
    - This is intuitive: AR processes are embedding new information on the most recent lags

Invertibility

General Condition for MA(q) Invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle of the complex plane

- For q =1: $-1 \ < \theta_1 \ < 1$
- For q=2:
  - $-1 \ < \theta_2 \ < 1$
  - $\theta_1 + \theta_2 > -1$ and $\theta_1 - \theta_2 \ < 1$
- More complicated solutions hold for $q \geqslant 3$
- Estimation software takes care of this

ARMA(p, q) Model

Specification

$$y_t = c + \underbrace{\phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}}_{\text{AR}} + \underbrace{\theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}}_{\text{MA}} + \epsilon_t$$

- The predictors include both **lagged values of** y_t and **lagged errors**
- Important specification: the future value of the series depends both on the past values it took (dynamic), as well as recent random noise / error term
- This simple model is "learning" both from the dynamic of the past values and from its inherent randomness
- Conditions on AR coefficients ensure **stationarity**
- Conditions on MA coefficients ensure **invertibility**

Autoregressive Integrated Moving Average (ARIMA)

ARIMA: AR**I**MA stands for: Autoregressive **Integrated** Moving Average model

- Basically, it is a non-stationary model that can be made stationary by differencing
- $(1-B)^d y_t$ follows an ARMA model: d is the degree of differencing
- Once differenced d times, it is stationary and behaves as an ARMA model

Generalization

- $ARIMA(p, d, q)$ where p is the autoregressive order, d the degree of differencing and q the order of the moving average part

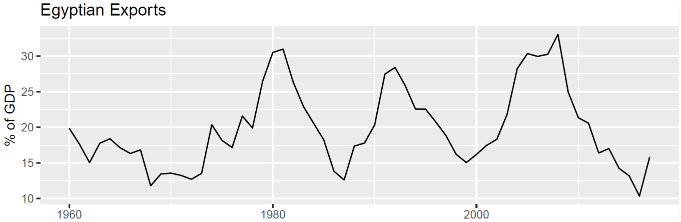
- All linear models we discussed are special cases of the ARIMA model:
  - White noise model: $ARIMA(0, 0, 0)$
  - Random walk: $ARIMA(0, 1, 0)$ with no constant
  - Random walk with drift: $ARIMA(0, 1, 0)$ with constant
  - $AR(p) = ARIMA(p, 0, 0)$, $MA(q) = ARIMA(0, 0, q)$

Backshift Notation for ARIMA

ARIMA(1, 1, 1) model:

$$\underbrace{(1-\phi_1 B)}_{AR(1)} \; \underbrace{(1-B)y_t}_{\text{First difference}} \quad = \quad c + \underbrace{(1+\theta_1 B)\epsilon_t}_{MA(1)}$$

ARIMA Analysis with Egyptian Exports

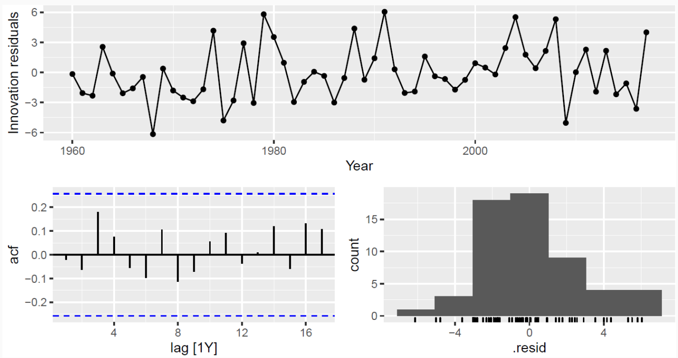


Egyptian Exports Fit

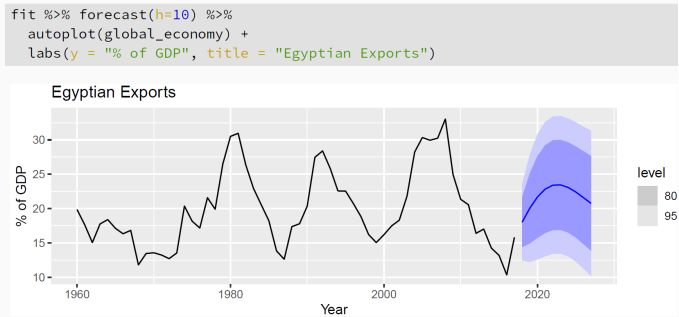
```
fit <- global_economy %>% filter(Code == "EGY") %>%
  model(ARIMA(Exports))
report(fit)
```

```
## Series: Exports
## Model: ARIMA(2,0,1) w/ mean
##
## Coefficients:
##          ar1      ar2     ma1  constant
##        1.676  -0.8034  -0.690     2.562
## s.e.  0.111   0.0928   0.149     0.116
##
## sigma^2 estimated as 8.046:  log likelihood=-142
## AIC=293   AICc=294   BIC=303
```

Egyptian Exports Fit Residuals



Egyptian Exports Forecasts



```
fit %>% forecast(h=10) %>%
  autoplot(global_economy) +
  labs(y = "% of GDP", title = "Egyptian Exports")
```