Introduction to Time Series Econometrics

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Outline

This short crash-course is divided in three parts covering:

1. **Data concepts**: population, sample, data types, data generating process, etc.
2. **Statistical Inference**: stochastic process, estimator, distributions, convergence, etc.
3. **Statistical Properties of Financial Time Series**: stationarity, autocorrelation, heavy tails, etc.

NB: this slide-deck follows the excellent course of `https://sites.google.com/view/christophe-hurlin/teaching-resources` Christophe Hurlin

Overview

Financial econometrics (including time-series econometrics) are based on four main elements:

1. A sample of data
2. An econometric model, based on a theory or not
3. An estimation method to estimate the coefficients of the model
4. Inference/testing approach to validate the estimation

Population vs. Sample

Definition: Population

A **population** is defined as including all entities (e.g. banks or firms) or all the time periods of the processus that has to be explained

- In most cases, it is impossible to observe the entire statistical population, due to constraints (recording period, cost, etc.)
- A researcher would instead observe a **statistical sample** from the population. He will estimate an econometric model to understand the **properties on the population as a whole**.

Data Generating Process

Definition: Data Generating Process

A **Data Generating Process (DGP)** is a process in the real world that "generates" the data (or the sample) of interest

Example: Data Generating Process

Let us assume that there is a linear relationship between interest rates in two countries (R, R^*), their forward (F) and their spot exchange rate (S).

$$\frac{F}{S} = \frac{1+R}{1+R^*}$$

This non-arbitrage relationship (CIP) can be used in the foreign exchange market to determine the forward exchange rate

$$\mathbb{E}[F|S=s, R=r, R^*=r^*] = s * \frac{1+r}{1+r^*}$$

This relationship is the **Data Generating Process** for F

The equivalent of population for time series econometrics is the DGP.
NB: note that I use R to describe the random variable and r to describe its realization

Econometrics Challenge

The challenge of econometrics is to draw conclusions about a DGP (or population), after observing only one realization $\{x_1, \dots X_N\}$ of a random sample (the dataset).

Data Types

In econometrics, sets can be mainly distinguished in three types:

1. Cross-sectional data

2. Time series data

3. Panel data

Cross-Sectional Data

Cross-sectional data are the most common type of data encountered in statistics and econometrics.

- Data at the entities level: banks, countries, individuals, households, etc.
- **No time dimension**: only one "wave" or multiple waves of different entities
- Order of data does not matter: no time structure

Time Series Data

Time series data are very common in financial econometrics and central banking. They entail specific estimation methods to do the **time-dependence**.

- Data for a single entity (person, bank, country, etc.) collected at multiple time periods. Repeated observations of the same variables (interest rate, GDP, prices, etc.)
- Order of data is important!
- The observations are typically not independent over time
- In this case, the notion of population corresponds to the **Data Generating Process (DGP)**

Panel Data

Also called longitudinal data. They contain the most information and allow for more complex estimation and analysis.

- Data for multiple entities (individuals, firms, countries, banks, etc.) in which outcomes and characteristics of each entity are observed at multiple points in time
- Combine cross-sectional and time-series information
- Present several advantages with respect to cross-sectional and time series data, depending on the topic at hands

Econometric Model

Definition: Econometric Model

An econometric model specificies the statistical relationship between different economic variables, that are expected to be stable over time

1. **Parametric model:** fully characterization of the relationship by a **set of parameters** θ and a **link function** f supposed to be known; the specification can be linear or non linear, and includes some randomness ϵ

$$Y = f(X; \theta) + \epsilon$$

2. **Non-parametric and semi-parametric models**: the link function can not be described using a finite number of parameters. The link function is assumed to be unknown and has to be estimated

Empirical Strategy

The general approach of (financial) econometrics is as follows:

1. Specification of the model

2. Estimation of the parameters

3. Diagnostic tests
    - Significance tests
    - Specification tests
    - Backtesting tests
    - etc.

4. Interpretation and use of the model (forecasting, historical studies, etc.)

Refresher: Random Variables

- Mathematicians are formalizing and modeling randomness via the concept of **random variables**.
- Pay attention: a random variable is neither random (it is formalized via laws and distributions), not a variable (it is a function)
- A **random variable** is a function $f : \Omega \mapsto \mathcal{R}$ that assigns to a set of outcome Ω a **value**, often a real number.
- The probability of an outcome is equal to its **measure** divided by the measure of all possible outcomes
  - Example: obtaining an even number by rolling a dice: $\{2, 4, 6\}$
  - Probability to obtain an even number by rolling a dice: $m(\{2, 4, 6\})/m(\{1, 2, 3, 4, 5, 6\}) = \frac{1}{2}$ *(here, the measure simply "counts" the outcomes with equal weights)*

Random Variables (II)

- Random variables are the "building block" of statistics:
  - Random variables are characterized by their distribution (generating function, moments, quantiles, etc.)
  - The behavior of two or more random variables can be characterized by their dependence/independence, matrix of variance-covariance, joint distribution, etc.
  - The main theorems in statistics (law of large numbers, central limit theorem, etc.) leverage the properties of random variables

Refresher: Probability Distributions for Continuous Variables

Definition: Probability Distribution

The probability distribution of a random variable describes how the probabilities of the outcomes are distributed. How more likely is one outcome compared to another? Is there more upside risk or downside risk? Etc.

Distributions are equivalently represented by:

1. **The probability density function (pdf)**
2. **The cumulative density function (pdf)**

- **The probability density function (pdf)** usually denoted $f_X(x0)$ (remember than X stands for the random variable, while $x0$ stands for a realization of the random variable) represents the relatively likelihood that the random variable X will fall within a small neighborhood of $x0$ (infinitesimal concept). It is easier to conceptualize the pdf via the cdf
- **The cumulative density function (cdf)** usually denoted $F_X(x0)$ represents the probability that the random variable will be lower than $x0$. It cumulates the pdf ("all the small neighborhoods") such that:

$$F_X(x0) \equiv \mathbb{P}[X \leqslant x0] = \int_{-infty}^{x0} f_X(h)dh$$

Moments: Overview

	Formula	Interpretation
Mean	$\mathbb{E}[X_t] = \mu$	Central tendency
Variance	$\mathbb{V}[X] = \mathbb{E}\left[(X_t - \mu)^2\right] = \sigma^2$	Dispersion around μ
Skewness	$\mathbb{S}[X] = \mathbb{E}\left[(X_t - \mu)^3\right] = \mathrm{sk}$	Symmetry
Kurtosis	$\mathbb{K}[X] = \mathbb{E}\left[(X_t - \mu)^4\right] = \kappa$	Tail heaviness

Moments: In Practice

- The moments allow to characterize the shape of the returns distribution
- However, the theoretical moments are **unobservable** and need to be estimated
- Assume that we have a sample $\{x_1, \ldots, x_T\}$ realizations of the sequence of X_t

Estimator

Definition: Estimator

An **estimator** is any function $F(x1, \ldots, x_t)$ of a sample. Note that any descriptive statistics is an estimator (a simple one)

Example: Sample Mean

The sample mean (or overage) of a sample is an estimator of the (theoretical) mean $\mathbb{E}[X_t] = \mu$.
The estimator is simply: $\hat{\mu_t} \equiv \bar{X}_t = \frac{1}{T}\sum_{t=1}^{T} x_t$

Example: Variance

Example: Sample Mean

Assume that the observations are drawn from $i.i.d$ random variables.
The **sample variance** $\hat{sigma}_T^2 = \frac{1}{T-1}\sum_{t=1}^{T}(x_t - \bar{x}_t)^2$

Note: the denominator is equal to *T-1* as to define a sample variance corrected for the small sample bias.

Sampling Distribution

Fact

An estimator $\hat{\theta}$ is a **random variable**

Therefore, $\hat{\theta}$ has a (marginal or conditional) **probability distribution**. This sampling distribution is characterized by a probability distribution function (pdf) $f_{\hat{\theta}}(u)$

Definition: Sampling Distributions

The probability distribution of an estimator is called the **sampling distribution**

The sampling distribution is described by its moments, such as expectations, variance, skewness, etc.

Point Estimate

Definition: Estimate

An estimate is the realized value of an estimator (e.g. a number, in a case of a point estimate) that is obtained when a sample is actually taken.
For an estimator $\hat{\theta}$, it can be denoted by $\hat{\theta}(y)$

Estimate of a linear regression

- We are interested in the following DGP: $Y = \alpha + \beta * X + \epsilon$, where we observe a joint sample $y1, \ldots, y_T, x1, \ldots, x_T$.
- We have an estimator (for instance an OLS) of $\hat{\alpha}, \hat{\beta}$
- Then, for any value of $X = x_0$, we can simply project the **conditional expected estimate** $y_0 = \hat{\alpha} + \hat{\beta} * x_o + \hat{\epsilon}$
- If the estimator is unbiased, the fitted residuals $\hat{\epsilon} = y_t - \hat{\alpha} - \hat{\beta} * x_t$ are centered on **average**:$\mathbb{E}\left[\hat{\epsilon}\right] = 0$. This is why the residual disappear from the estimate of the conditional expected estimate in an OLS... but no bias doesn't mean no variance !

What Constitutes a Good Estimator?

The idea is to study the properties of the **sampling distribution** of the estimator $\hat{\theta}$, and especially its moments, such as:

- $\mathbb{E}[\hat{\theta}]$ for the biais
- $\mathbb{V}[\hat{\theta}]$ for the precision
- $\mathbb{S}[\hat{\theta}]$ for the symmetry
- $\mathbb{K}[\hat{\theta}]$ for the tail-risks
- etc.

Estimators Properties

Estimators are compared on the basis of a variety of attributes

1. **Finite sample properties** (or finite sample distributions) investigate how the estimator behave when the observational sample is relatively limited (a few hundreds to a few thousands of observations max)

2. However, these properties rely on a distributional assumption (usually normality or gaussianity), that may be difficult to test.

3. When the normality assumption is no longer valid (and the finite sample distribution is unknown), estimators are evaluated on the basis on their **large sample**, or **asymptotic properties**

Finite Sample Theorem

Theorem: Finite Sample Distributions

If we assume that, with a sample of size T, generated from a stochastic process with i.i.d random variables $X_1, X_2, \ldots, X_T$ with $X_t \sim \mathcal{N}(\mu, \sigma^2)$, then the estimators of the sample mean $\hat{\mu_T}$ and the estimator of the sample variance/population variance $(T-1)\frac{\hat{\sigma_T^2}}{\sigma^2}$ have a **finite sample distribution**

$$\hat{\mu_T} = \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right) \qquad \forall\ T\ \in\ \mathbb{N}$$

$$\frac{T-1}{\sigma^2}\hat{\sigma^2} \sim \chi^2(T-1) \qquad \forall T\ \geqslant\ 2$$

Example: Finite Sample Distribution

Under the Gaussianity assumption, with a sample size of $T = 10$, then:

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{10}\right) \qquad 9 \times \frac{sig\hat{m}a_T^2}{\sigma^2} \sim \chi^2(9)$$

Difficulties with Finite Sample Inference

In most cases, it is impossible to derive the **exact/finite sample distribution** for the estimator (or any transformation of the estimator).

1. In some cases, the exact distribution of $\{X1, X2, \ldots, X_T\}$ is known, but the estimator function $S()$ (also called the "statistics") is too complicated

$$\hat{\theta} = S(X_1, X_2, \ldots, X_T) \sim ??? \qquad \forall\ T \ \in \mathbb{N}$$

2. In most cases, the distribution of the stochastic process $\{X1, X2, \ldots, X_T\}$ is unknown

$$\hat{\theta} = S(X_1, X_2, \ldots, X_T) \sim ??? \qquad \forall\ T \ \in \mathbb{N}$$

Asymptotic Properties

What is the behavior of the estimator $\hat{\theta_T}$ when the sample size T tends to infinity?

Definition: Asymptotic Theory

Asymptotic or **large sample theory** consists in the study of the distribution of the estimator when the sample size is sufficiently large (usually, more than 10k)

The asymptotic theory is fundamentally based on the notion of **convergence**

Convergence in Probability

There are many types of convergence, but typically applied statisticians are interested in:

1. Convergence in probability
2. Convergence in distribution

- **Convergence in probability** (also called "mean squared convergence" or "almost sure convergence")

$$\hat{\theta_T} \xrightarrow{p} \theta \qquad \text{for } \theta \in \mathbb{R}$$

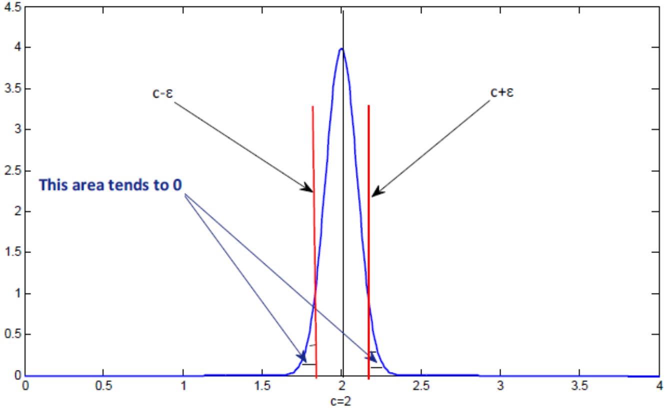
- Technically, for a stochastic process $\theta_T{}_{-\infty}^{+\infty}$

$$\hat{\theta_T} \xrightarrow{p} \theta \quad \Leftrightarrow \quad \lim_{T \to +\infty} \mathbb{P}\left[|\theta_T - \theta| > \epsilon\right] = 0$$

- The convergence in probability is used to derive the **consistency property** of estimators

Illustration: Distribution in Probability

$$\hat{X}_T \xrightarrow{p} c \text{ if } \Leftrightarrow \quad \lim_{T\to+\infty}\mathbb{P}\left[|X_T - c| > \epsilon\right] = 0$$



Interpretation

1. The distribution of the sample moments is highly concentrated around the true value (unknown) of the population moment when the sample size T tends to infinity

2. In other words, over a very large sample of data (for instance, hundred of thousands of observations) the **moments estimated in the sample** are then very "close" to the true value of **the moments in the population**

Convergence in Distribution

- **Convergence in distribution** is interested in the distribution of the bias (the distance between the estimator and the true value)

$$\sqrt{T}\left(\hat{\theta_T} - \theta_0\right) \sim \mathcal{N}(\mu_., \sigma_.) \qquad \mathcal{N} \text{ is the most common}$$

- Technically, for a stochastic process ${X_T}_{-\infty}^{+\infty}$, with a cdf $F_T(.)$; X_T is said to **converge in distribution** to a random variable X if:

$$X_t \xrightarrow{d} X \quad \Leftrightarrow \quad F_T(x) = F(x) \qquad \forall\ x \in \mathbb{R}$$

- The convergence in distribution is used to derive the **asymptotic distribution** of the estimators and to make **inference** (tests) about the true value of the parameters

Stylized Properties

Fan and Yao (2015, the Elements of Financial Econometrics) identify 8 main "stylized facts"

1. Stationarity
2. Absence of autocorrelations
3. Heavy tails
4. Asymmetry
5. Volatility clustering
6. Aggregational Gaussianity
7. Long range dependence
8. Leverage effect

Stationarity

Example

In general, prices are non-stationary but returns are stationary

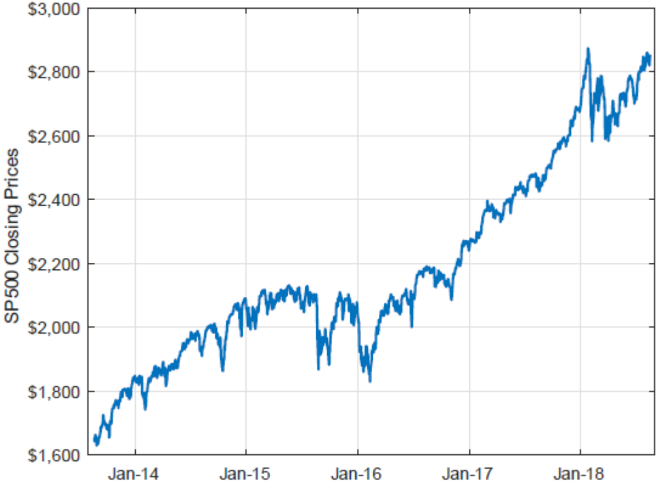
Definition: Weak Stationarity (Second Order Stationarity)

A stochastic process $X_{t t\in\mathbb{Z}}$ is weakly stationary if and only if:

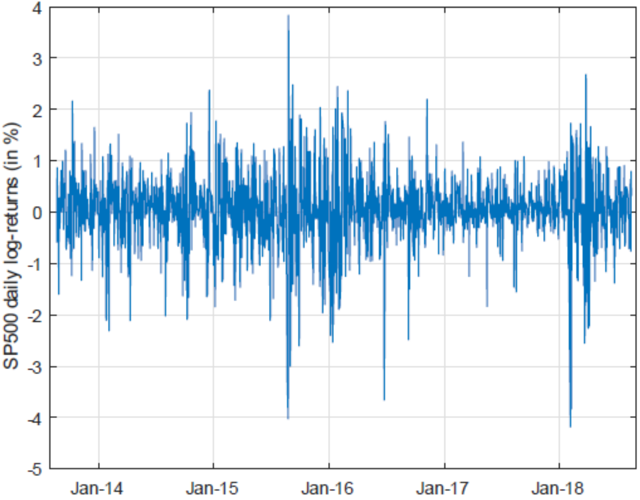
- $\mathbb{E}(X_t^2) < \infty \quad \forall\, t \in \mathbb{Z}$
- $\mathbb{E}(X_t) = \mu \quad \forall\, t \in \mathbb{Z}$ doesn't depend on t
- $\mathrm{Cov}(x_t, x_{t+h}) = \mathbb{E}[(x_{t+h}-m)(x_t-m)] = \gamma_h \quad \forall\,(t,h) \in \mathbb{Z}^2$ desn't depend on t

Weak stationarity means that the stochastic process oscillates around a constant level, is not trended, and their correlation over time is independent from the period studied. Technically, we say that the **first two moments are time-invariant**.

Prices are Usually Non-Stationary...



... But Returns Are !



Absence of autocorrelations

Fact: Common absence of autocorrelations

The autocorrelations of assets returns (in general) are often insignificant, except for very small intraday time scales (around 20 minutes) for which the microstructure effects come into play

Note: The fact that returns hardly show any serial correlation does not mean that they are independent

Definition: Autocorrelation

The autocorrelation, denoted $\rho(k)$ of a weak stationary process X_t is the correlation between the values of the process at different times:

$$\rho_k \;=\; \mathrm{Corr}(X_t, X_{t-k}) \;=\; \frac{\mathbb{E}\left[(X_t-\mu)(X_{t-k}-\mu)\right]}{\mathbb{V}(X_t)} = \frac{\gamma_k}{\sigma^2}$$

with $\mu = \mathbb{E}[X_t]$, $\sigma^2 = \mathbb{V}(X_t),\ \forall\ t$ and γ_k the autocovariance of order k

Measuring Sample Autocorrelation

Definition: Sample Autocorrelation

The **sample autocorrelation**, denoted $\hat{\rho}(k)$ of a weak stationary process X is an estimator of $\rho(k)$ defined as:

$$\hat{\rho}_k = \operatorname{corr}(X_t, X_{t-k}) = \frac{1}{(T-k)\hat{\sigma^2}} \sum_{t=k+1}^{R} (X_t - \hat{mu})(X_{t-k} - \hat{mu})$$

where $\hat{\sigma^2}$ and $\hat{\mu}$ are consistent estimators of $\mu = \mathbb{E}(X_t)$ and $\sigma^2 = \mathbb{V}(X_t) \qquad \forall t$

Example: Autocorrelation of the SP500

The **Autocorrelation Function (ACF)** (or **correlogram**) represents the sample autocorrelation for different lags, from $k = 1$ to a maximum lag order, for instance $k = 15$

Correlogram of SP500_RETURNS

Date: 09/09/18 Time: 11:25
Sample: 8/19/2013 8/17/2018
Included observations: 1259

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.019	-0.019	0.4455	0.504
		2	-0.054	-0.054	4.1067	0.128
		3	0.024	0.022	4.8464	0.183
		4	-0.043	-0.045	7.1488	0.128
		5	-0.028	-0.027	8.1122	0.150
		6	-0.008	-0.014	8.1869	0.225
		7	0.013	0.011	8.3965	0.299
		8	-0.037	-0.039	10.165	0.254
		9	-0.054	-0.057	13.887	0.126
		10	-0.019	-0.028	14.324	0.159
		11	0.002	-0.003	14.330	0.215
		12	-0.001	-0.004	14.330	0.280
		13	-0.015	-0.022	14.617	0.332
		14	-0.020	-0.027	15.102	0.371
		15	-0.073	-0.079	21.926	0.110