If the dimension of the Kernel of T is not one, I propose to add a Cauchy matrix below A instead of adding something to T. We are still adding some constraints to ensure that x is unique and x relains a solution.

Assume $\mathsf{A} = [\,\mathsf{A}_r \;\; \mathsf{B}\,]$ with $\det(\mathsf{A}_r) \neq 0$ and consider

$$\mathsf{A}' = \left[\begin{array}{c|c} \mathsf{A}_r & \mathsf{B} \\ \hline \left(\begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix} [\, l_1 \;\; \ldots \;\; l_r \,]\right) \odot \mathsf{C}' & \left(\begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix} [\, l_{r+1} \;\; \ldots \;\; l_{r+s} \,]\right) \odot \mathsf{C} \end{array}\right]$$

where $c_1, \ldots, c_s, l_1, \ldots, l_{r+s}$ are indeterminates, and C, C' are Cauchy matrices for good vectors u, v_0 and u, v_1 (u can be chosen freely, while v_0,v_1 depend on the Cauchy structure of respectively A_r and B).

The principal leading minor A_{r+1} is a non zero polynomial in $k[c_1, \ldots, c_s, l_1, \ldots, l_{r+s}]$ because its term in $c_1\, l_{r+1}$ is $c_1\, l_{r+1} \det(\mathsf{A}_r) \det(\mathsf{C}_1)$ which is non zero since the principal leading minor C_1 is non zero.

Now the principal leading minor A_{r+2} is a non zero polynomial because its term in $c_1\, c_2\, l_{r+1}\, l_{r+2}$ is $c_1\, c_2\, l_{r+1}\, l_{r+2} \det(\mathsf{A}_r) \det(\mathsf{C}_2) \neq 0$ since $\det(\mathsf{C}_2) \neq 0$. Indeed the presence of $l_{r+1}\, l_{r+2}$ in $\det(\mathsf{A}_{r+2})$ means that we chose a permutation that involves two terms of the bottom right part of A', and the other terms must come from A_r.