Fact to explain: It is the same transformations to derive $S^{(i+j)}$ from $S^{(j)}$ than to derive $S^{(i)}$ from $S^{(0)} = A \in \mathbb{K}^{m \times n}$?

Let $p = \min(m, n)$ and consider the matrix

$$R^{(0)} = \begin{pmatrix} A \\ I_{p,n} \end{pmatrix} = \begin{pmatrix} S^{(0)} \\ I_{p,n} \end{pmatrix} \in \mathbb{K}^{(m+p) \times n}$$

where $I_{p,n} \in \mathbb{K}^{p \times n}$ is the rectangular matrix with ones on the diagonal. For $i \in \{0, \ldots, p\}$, write A as

$$A = \begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \\ A_{1,0}^{(i)} & A_{1,1}^{(i)} \end{pmatrix}$$

where $A_{0,0}^{(i)} \in \mathbb{K}^{i \times i}$.

Now assume that the principal minor $A_{0,0}^{(i)}$ is invertible. Using column operations, we want to transform $\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix}$ to $\begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix}$. When we apply this transformation to $R^{(0)}$, we get

$$R^{(i)} := \begin{pmatrix} \mathrm{Id}_i & 0 \\ A_{1,0}^{(i)} A_{0,0}^{(i)\,-1} & A_{1,1} - A_{1,0}^{(i)} A_{0,0}^{(i)\,-1} A_{0,1}^{(i)} \\ A_{0,0}^{(i)\,-1} & -A_{0,0}^{(i)\,-1} A_{0,1}^{(i)} \\ 0 & I_{p-i,n-i} \end{pmatrix} = \begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \\ A_{1,0}^{(i)} & A_{1,1}^{(i)} \\ \mathrm{Id}_i & 0 \\ 0 & I_{p-i,n-i} \end{pmatrix} \times \begin{pmatrix} A_{0,0}^{(i)\,-1} & -A_{0,0}^{(i)\,-1} A_{0,1}^{(i)} \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix}$$

Define $S^{(i)}$ as the central part of $R^{(i)}$, so that

$$R^{(i)} = \begin{pmatrix} \mathrm{Id}_i \,|\, 0 \\ S^{(i)} \\ 0 \,|\, I_{p-i,n-i} \end{pmatrix}.$$

Composition rule. We want to prove that it is the same transformations to derive $S^{(i+j)}$ from $S^{(j)}$ than to derive $S^{(i)}$ from $S^{(0)} = A \in \mathbb{K}^{m \times n}$. Put it differently, we can compute $S^{(i+j)}$ as the composition of one transformation of order i and one of order j.

Proof. The transformation that we apply to $R^{(0)}$ to get $R^{(i)}$ is uniquely determined by the first i rows $\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix}$ of A. It consists of a column reduction that maps $\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix}$ to $\begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix}$ using $A_{0,0}^{(i)}$ as pivot in the following manner

$$\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix} \times \begin{pmatrix} U & V \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix},$$

for $U = A_{0,0}^{(i)\,-1} \in \mathbb{K}^{i \times i}$ and $V = -A_{0,0}^{(i)\,-1} A_{0,1}^{(i)} \in \mathbb{K}^{i \times n-i}$. The matrices U and V are uniquely determined by last equation ; it can be seen easily on the following implied equation

$$\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} \times \begin{pmatrix} U & V \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_i & 0 \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} U & V \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix}^{-1}.$$

To see that this transformation of order i coincides with the composition of column reduction of order k and $(i-k)$ for any $k \in \{0, \ldots, i\}$, it remains to prove that the composition performs

$$\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix} \times \begin{pmatrix} U & V \\ 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix},$$

for some $U \in \mathbb{K}^{i\times i}$ and $V \in \mathbb{K}^{i\times n-i}$. Let us write

$$\begin{pmatrix} A_{0,0}^{(i)} & A_{0,1}^{(i)} \end{pmatrix} = \begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \end{pmatrix}$$

where $B_{0,0} \in \mathbb{K}^{k\times k}$ and $B_{1,1} \in \mathbb{K}^{(i-k)\times(i-k)}$. The first column reduction of order k performs

$$\begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \end{pmatrix} \times \begin{pmatrix} * & * & * \\ 0 & \mathrm{Id}_{i-k} & 0 \\ 0 & 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ C_{1,0} & C_{1,1} & C_{1,2} \end{pmatrix}.$$

It is followed by the following column reduction of order $(i-k)$ on the last $(i-k)$ rows using $C_{1,1}$ as pivot

$$\begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ C_{1,0} & C_{1,1} & C_{1,2} \end{pmatrix} \times \begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ * & * & * \\ 0 & 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ 0 & \mathrm{Id}_{i-k} & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} * & * & * \\ 0 & \mathrm{Id}_{i-k} & 0 \\ 0 & 0 & \mathrm{Id}_{n-i} \end{pmatrix} \times \begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ * & * & * \\ 0 & 0 & \mathrm{Id}_{n-i} \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \mathrm{Id}_{n-i} \end{pmatrix}$$

the composed operation verifies the same assumption and coincide with the transformation of order i by uniqueness. □

Link with the inverse. An important fact with the matrices $R^{(i)}$ and $S^{(i)}$ is that, if $A \in \mathbb{K}^{n\times n}$ is invertible, then

$$R^{(n)} = \begin{pmatrix} \mathrm{Id}_n \\ A^{-1} \end{pmatrix} \text{ and } S^{(n)} = A^{-1}.$$

More generally, if $r = \mathrm{rank}(A)$ and $A_{0,0}^{(r)}$ is invertible, then

$$R^{(r)} := \begin{pmatrix} \mathrm{Id}_i & 0 \\ A_{1,0}^{(r)} A_{0,0}^{(r)-1} & 0 \\ A_{0,0}^{(r)-1} & -A_{0,0}^{(r)-1} A_{0,1}^{(r)} \\ 0 & I_{p-r,n-r} \end{pmatrix}$$

Iterative inversion. The link between the matrices $R^{(i)}$ and the inverse, together with the composition rule, are the keys of Cardinal's paper for Cauchy structured matrix inversion. It must also be linked to Strassen's matrix inversion formula.