COLOMBO'S TOP*

by

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(Received March 19, 1987; accepted October 1, 1987)

ABSTRACT

We analyse in details a simple dynamical system proposed by G. Colombo for the description of the rotational state of planets and satellites. We show that the derivatives of the critical areas are simple analytical functions of the parameters of the problem. These quantities are instrumental in computing the probabilities of capture of the precession of the spin axis in resonance with the precession of the orbit.

* dedicated to the memory of G. Colombo.

Celestial Mechanics **40** (1987), 345–366.

I. INTRODUCTION

In 1693, Cassini described the rotation of the moon by his three famous empirical laws. The first one notes that the rotation rate is synchroneous with the orbital mean motion such that one side always faces the Earth. The other two describe the position of the spin axis of the moon with respect to the normal to the orbital plane and the normal to the ecliptic plane : The spin axis of the Moon maintains a constant inclination to the ecliptic plane and the spin axis, the orbit normal and the ecliptic normal remain coplanar.

Colombo (1966) showed that Cassini's second and third laws were generalizable for an axially symmetric planet with an arbitrary spin angular velocity. They correspond to an equilibrium (called Cassini's state by Peale, 1969) of the following simple dynamical system.

Consider an axially symmetric oblate planet the spin axis of which is directed along its symmetry axis. The averaged effect of the Sun on its equatorial bulge can be modelized as the effect of a ring of material at distance r from the center of mass of the planet. Let us assume further that the plane of motion of the Sun (or equivalently the plane of the ring of material) is not fixed in space but precess uniformly. Such a dynamical system may be described by the Hamiltonian function :

$$\mathbf{H}_C = -\frac{1}{2}(Z - b)^2 - a\sqrt{1 - Z^2}\cos h \qquad (1)$$

where $h - \pi/2$ and $\arccos Z$ are the spherical coordinates of the angular momentum of the planet (in a frame moving with the ring of material).

As Colombo was the first to point out the importance of this simple model in the study of the obliquity of planets and satellites and to describe its main features, we like to call it Colombo's Top.

Subsequent analysis by Peale (1969, 1974), Ward (1975, 1979) and others have shown that this model (or some refinement of it) can be used as the basis for most investigations about the rotation of Celestial Bodies. Beletskii (1972) gives an account of the Russian investigations of similar problems especially in connection with the attitude of Artificial Satellites.

Observe also that when a is small $(a = \varepsilon^{3/2} a')$ and b close to one $(b = 1 + b'\varepsilon)$, the dynamical system can be approximated in the neighborhood of $Z = 1$ (with $Z = 1 + \varepsilon I$) by

$$\frac{1}{\varepsilon^2} \mathbf{H}_C = \frac{1}{2} (I - b')^2 - a'\sqrt{2I} \cos h + \theta(\varepsilon)$$

which is what we have called the second fundamental model of resonance (Henrard-Lemaître, 1983) because of its frequent occurence in problems of resonance in Celestial Mechanics.

The second fundamental model of resonance is a good model for physically interesting systems only for I small enough, while Colombo's Top is valid for any values of Z in its domain of definition $(-1 \leqq Z \leqq 1)$. In this sense, Colombo's Top may be though of as an extension of the second fundamental model on a nice compact manifold : the sphere. This makes it the more interesting from the mathematical point of view.

This dynamical system (and some refinement of it) has been studied extensively mainly by Colombo himself (1966) and by Peale (1969, 1974) and Ward (1975, 1979). Both of the last two authors have studied it in connection with the Adiabatic Invariant theory (when a and b changes slowly with time).

We would like to show, and this is the main point of our paper, that the quantities instrumental in evaluating the probabilities of capture in resonance (Henrard, 1982) and the coefficient of diffusion describing the chaotic motion related to periodic separatrix crossing (Escande, 1985) can be evaluated in this problem by simple analytical expressions.

In section 2, we define more precisely Colombo's Top and develop its Hamiltonian function. Our treatment of this differs from previous one by the use of Andoyer's variables.

In section 3, we follow previous authors in describing the phase space trajectories as intersections of parabolic cylinders and the sphere, and in defining the Cassini's states, the equilibria of the dynamical system. We add to this discussion a simple criterium (eq. 20) by which one can separate the cases where there are four Cassini's states from the case where there are only two. A similar criterion was also discussed by Beletskii (1972).

Section 4 is devoted to the description of the two homoclinic orbits which divide the sphere into three domains, each associated with one stable Cassini's states.

In section 5, we compute analytically the areas of these three domains as functions of the parameters of the problem.

These areas, or rather their derivatives with respect to the parameters are instrumental in computing the probabilities of transition from one domain to another one when the parameters vary slowly with the time.

These derivatives are computed in section 5 and turns out to be very simple expressions (equation 37 and 38).

As a check of these formulae we recompute in section 6 the probability of capture of the spin axis of Mars in domain 3 which was computed numerically in (Henrard, 1982).

2. COLOMBO'S TOP

We develop in this section the Hamiltonian function for the simple dynamical system we like to call Colombo's Top.

Our exposition of this development does not depart significantly from previous ones (we follow mainly Peale's notations and terminology) except for the use of Andoyer's canonical variables to describe the motion of a body around its center of mass.

These canonical variables have not been widely used in this context, although we believe that they are as useful for this problem as Delaunay's variables are for the Keplerian problem as it has been shown in related context by Deprit (1967), Kinoshita (1975, 1977), Henrard and Moons (1978), Borderies (1980), Moons (1982, 1984).

In order to describe the Andoyer's variables let us consider two frames of reference, one $(0 , \underline{I} , \underline{J} , \underline{K})$ fixed in space and the other one $(0 , \underline{i} , \underline{j} , \underline{k})$ fixed in the body (for simplicity we shall assume that $\underline{i}$, $\underline{j}$, $\underline{k}$ are the principal axis of inertia, although it is not necessary).

Consider further the plane π_2 perpendicular to the angular momentum $\underline{\mathbf{L}}$ of the body. The orientation of the body frame with respect to the inertial frame can be expressed in terms of the two sets of Eulerian angles (h , K , ℓ) and (ℓ , J , g) as in Figure 1.

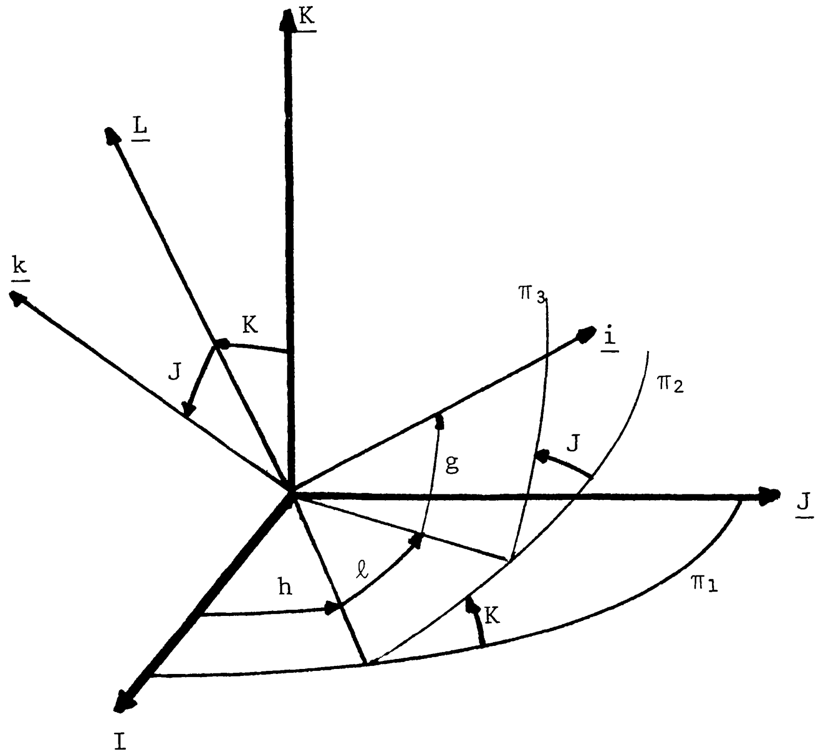


Figure 1 : **The two sets of Eulerian angles. The plane π_1 is fixed in space, π_2 is perpendicular to the angular momentum $\underline{L}$ and π_3 is the equatorial plane of the body.**

It can be shown (e.g. Deprit, 1967) that the quantities :

$$\begin{array}{ll} \ell & L = \|\underline{L}\| \\ g & G = \|\underline{L}\| \cos J \\ h & H = \|\underline{L}\| \cos K \end{array} \tag{2}$$

are canonical variables, the momenta L , G , H being conjugated respectively to the angular variables ℓ , g , h . By this we mean that there exist a canonical transformation from the usual Eulerian angles ϕ , θ , ψ and their conjugated momentum to the set $(\ell , g , h , L , G , H)$.

The Kinetic energy of the body (which is the Hamiltonian function for the Euler-Poinsot problem, the free rotation of the body) is written :

$$H_{free} = \frac{1}{2} (\underline{L} \mid \underline{\omega}) = \frac{1}{2} \frac{L^2}{C} [1 + \sin^2 J \ (\gamma + \delta \cos 2g)] \tag{3}$$

where $A \leqq B \leqq C$ are the principal moment of inertia, $\underline{\omega}$ the instantaneous rotation vector of $(\underline{i}, \underline{j}, \underline{k})$ with respect to $(\underline{I}, \underline{J}, \underline{K})$ and

$$\gamma = \frac{1}{2}\left[\frac{C-A}{A} + \frac{C-B}{B}\right] \qquad \delta = \frac{1}{2}\left[\frac{C-B}{B} - \frac{C-A}{A}\right] . \tag{4}$$

Let us consider a mass M standing at a distance r from the center of mass 0 of the body. The part of the gravitational potential which depends upon the body's orientation is

$$V = -\frac{3}{2}\frac{k^2 M}{r^5}\left[(C - A)\xi_1^2 + (C - B)\xi_2^2\right] \tag{5}$$

when truncated at order two in its expansion in spherical harmonics. The quantities (ξ_1, ξ_2) are the first two coordinates of the position vector of the perturbing mass M in the frame $(0, \underline{i}, \underline{j}, \underline{k})$ of the principal axis of inertia of the body.

Let us assume that the perturbing mass M is on a circular orbit at the distance r in the fixed plane $(0, \underline{I}, \underline{J})$ with a constant angular velocity n $(n^2 = k^2 M/r^3)$ and that we are interested only in the <u>averaged</u> effect of this perturbation upon the orientation of the body. This situation could be modelized by considering a circular ring of material in the plane $(0, \underline{I}, \underline{J})$.

Let us assume further that the body possesses an axial symmetry $(A = B)$ and that the angular momentum $\underline{L}$ lies along the third axis of inertia $\underline{k}$ (i.e. $J = 0$). This can be shown to be a stable partial equilibrium. Furthermore it is one toward which the system would evolve if some dissipation of energy takes place inside the body.

Then the Hamiltonian

$$\mathbf{H}_i = \mathbf{H}_{free} + V \tag{6}$$

reduces to

$$\mathbf{H}_i = \frac{1}{2}\frac{L^2}{C} - \frac{3}{4}n^2(C - A)\cos^2 K \tag{7}$$

The solution of the corresponding dynamical system can be described as a uniform rotation

$$\dot{\ell} + \cos K\,\dot{h} = \frac{L}{C} = \omega \tag{8}$$

and a uniform precession of the angular momentum :

$$\dot{h} = -\frac{3}{2}\frac{n^2}{\omega}\frac{C-A}{C}\cos K \ . \tag{9}$$

The distinctive feature of Colombo's Top is introduced now. Let us assume that the plane $(0\ ,\ \underline{I}\ ,\ \underline{J})$ of motion of the perturbing body is no longer fixed in space but precessing with an constant instantaneous rotation vector $\underline{\mu}$ (perpendicular to the "invariable plane" π_0).

To take into account this non-inertial effect, a term $\mathbf{H}_{n.i.}$ should be added to the Hamiltonian function:

$$\mathbf{H}_{n.i.} = -(\dot{\vec{\mu}}\ |\ \vec{L}) \ . \tag{10}$$

If we choose the vector $\underline{I}$ along the ascending node of the orbit along the invariable plane (as in Figure 2) this term is expressed in terms of Andoyer's variables as

$$\mathbf{H}_{n.i.} = \mu\, L\, [\cos i \cos K - \sin i \sin K \cos h] \tag{11}$$

where μ is the absolute value of the angular velocity of the precession.

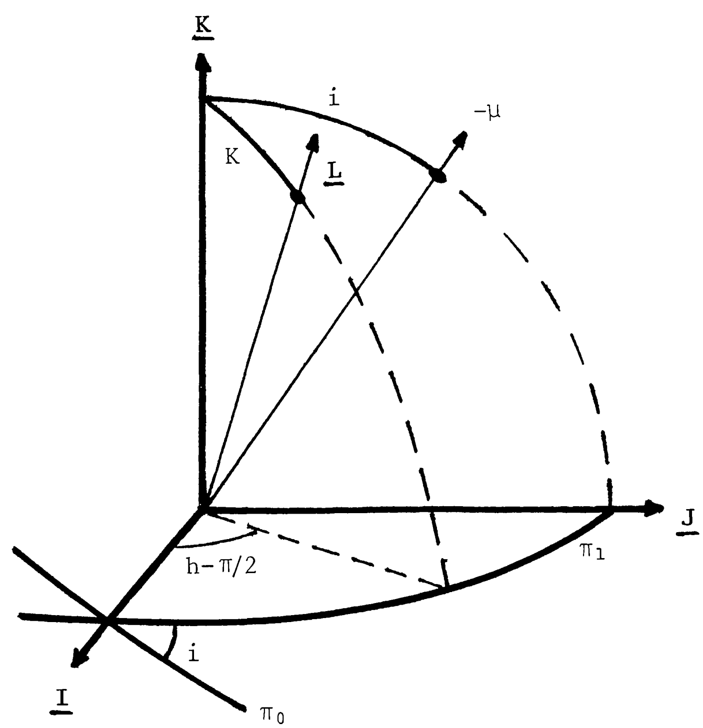


Figure 2 : Geometry of the precession. Notice that the third axis of the coordinate system associated with the invariable plane π_0 is taken along the vector $(-\ \underline{\mu})$ to take into account the usual regression of the nodal line.

The norm $L = C\omega$ of the angular momentum remains constant for the dynamical system described by $\mathbf{H}_i + \mathbf{H}_{n.i.}$. We can thus consider it as a one degree of freedom (h,H) dynamical system and drop the constant $L^2/2C$ in $\mathbf{H}_i$. A further scaling of the momentum and time

$$H = L\,Z \qquad\qquad \tau = \frac{3}{2}\,\frac{n^2}{\omega}\,\frac{C-A}{C}\,t \tag{11}$$

completes the descrption of the model, the canonical variables of which are (h,Z) :

$$\mathbf{H}_C = -\frac{1}{2}(Z-b)^2 - a\sqrt{1-Z^2}\cos h \tag{12}$$

where

$$a = \frac{2}{3}\,\frac{\omega\mu}{n^2}\,\frac{C}{C-A}\sin i \qquad\qquad b = \frac{2}{3}\,\frac{\omega\mu}{n^2}\,\frac{C}{C-A}\cos i\ . \tag{13}$$

We have defined the parameter a and b of the model in such a way that they are positive for most of the applications. If they were not, it would be easy to make them so either by a translation $(h \to h + \pi)$ which would reverse the sign of a or by the involution $(h,Z) \to (-h,-Z)$, which reverses the sign of b.

In what follows, we shall assume that a and b are positive.

3. THE PHASE SPACE AND THE CASSINI'S STATES

The Hamiltonian function for our model problem, Colombo's Top, is then

$$\mathbf{H}_C = -\frac{1}{2}(Z-b)^2 - a\sqrt{1-Z^2}\cos h \tag{14}$$

where Z is the momentum conjugated to the variable h and the parameters a and b are positive.

From Figure 2, we notice that h and $Z = \cos K$ are related to the spherical coordinates of the unit vector along the angular momentum $\underline{L}$. If we introduce the cartesian coordinates (in the frame 0 , $\underline{I}$, $\underline{J}$, $\underline{K}$) of this vector :

$$X = \sqrt{1-Z^2}\sin h \quad ; \quad Y = -\sqrt{1-Z^2}\cos h \quad ; \quad Z = Z \tag{15}$$

the Hamiltonian function (14) reads

$$H_C = - \frac{1}{2} (Z - b)^2 + aY \tag{16}$$

and it is then apparent that the curves H_C = constant (which are the trajectories of the dynamical system described by (14)) are the intersections of the parabolic cylinders (16) and the unit sphere, in the three dimensional space (X , Y , Z).

Indeed the topology of the unit sphere is the proper topology associated with the phase space of Colombo's Top. The singularities $Z = \pm 1$ introduced in (14) by the use of spherical coordinates (or Euler angles) are removed in (16).

As shown in Figure 3a as many as four parabolic cylinders of the family (16) can be tangent to the sphere. The points of tangency correspond to equilibria of the dynamical system.

When there are four equilibria, one of them is unstable as the intersection of the corresponding parabolic cylinder and the sphere is not reduced to the tangency point but contains the stable and unstable manifolds of the equilibrium.

As usual in a two dimensional phase space, the stable and unstable manifolds merge and form two closed loops, the homoclinic orbits. The equilibria were called "Cassini's states" by Peale (1969) because in some problems appropriate dissipative processes which we do not consider here can drive the system to one of them. Cassini's laws for the Moon describe such an equilibrium.

The location of these equilibria, numbered 1 to 4 as in Figure 3, are given by :

$$X_i = 0 \quad ; \quad Y_i = - \frac{aZ_i}{(Z_i - b)} \tag{17}$$

and Z_i , root of

$$(Z_i - b)^2 (1 - Z_i^2) = a^2 Z_i^2 . \tag{18}$$

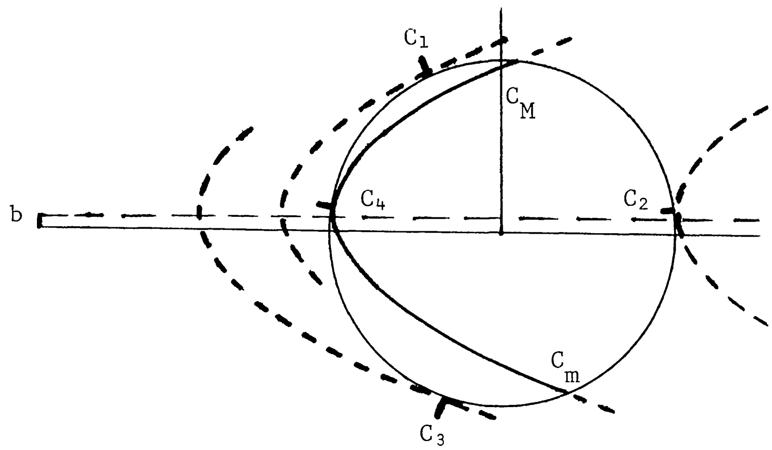
One checks easily that the fourth degree polynomial (18) possesses a double root Z_D whenever a and b are such that

$$a = - Y_D^3 \quad ; \quad b = Z_D^3 . \tag{19}$$

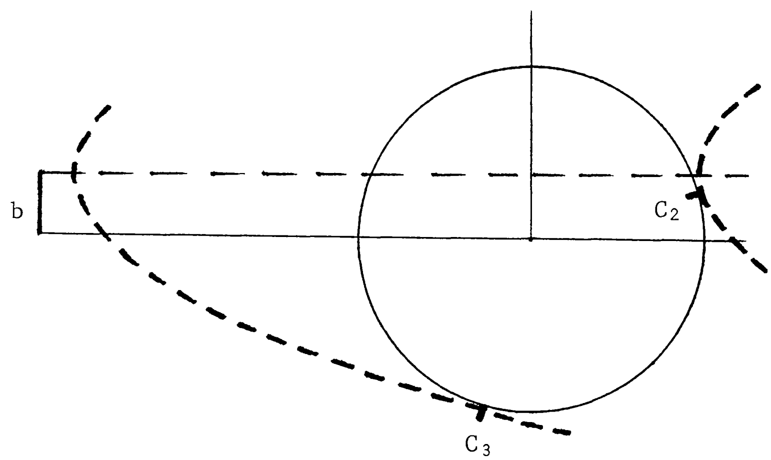
Hence the Colombo's top possesses four Cassini's states whenever

$$a^{2/3} + b^{2/3} \leq 1 \tag{20}$$

and only two for larger values of a and b . In the latter case, the Cassini's states (1) and (4) have merged together and disappeared, leaving only the two stables Cassini's states (2) and (3) as in Figure 3b.



(A) a = 0.4 , b = 0.1



(B) a = 0.4 , b = 0.4

Figure 3 : The parabolic cylinders tangent to the sphere and the Cassini's states.

In order to give a better view of what the trajectories of the dynamical system look like, we show in Figure 4 their projections on the plane (Y,Z) for various values of the parameters a and b . Also shown in Figure 5 are three other projections of the trajectories for one typical set of values of the parameters. These trajectories are of course, as mentioned earlier, the intersections of the parabolic cylinders (16) with the unit sphere.

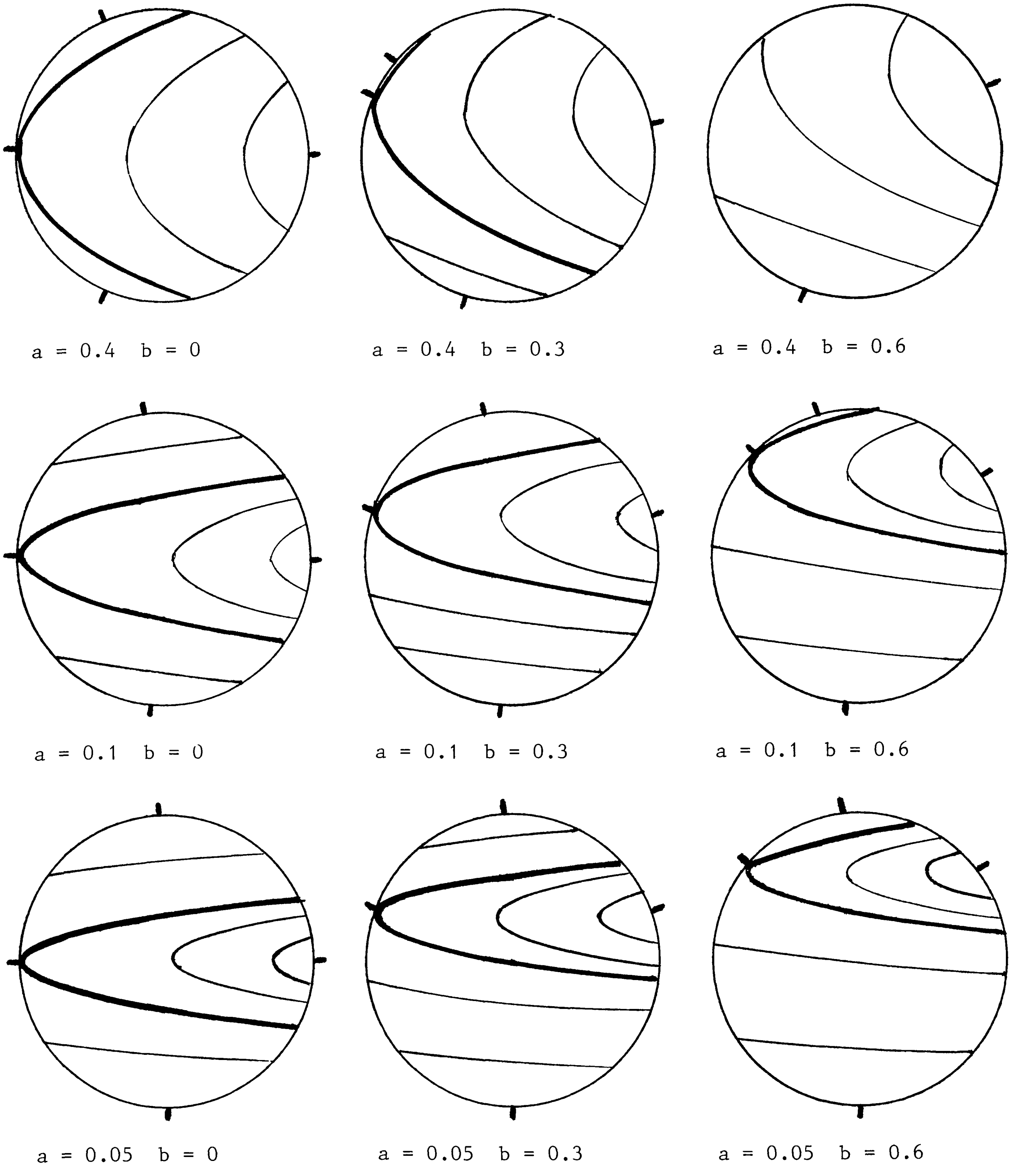


Figure 4 : Projections upon the plane (Y,Z) of the trajectories of the dynamical system (14) for various values of the parameters a and b .

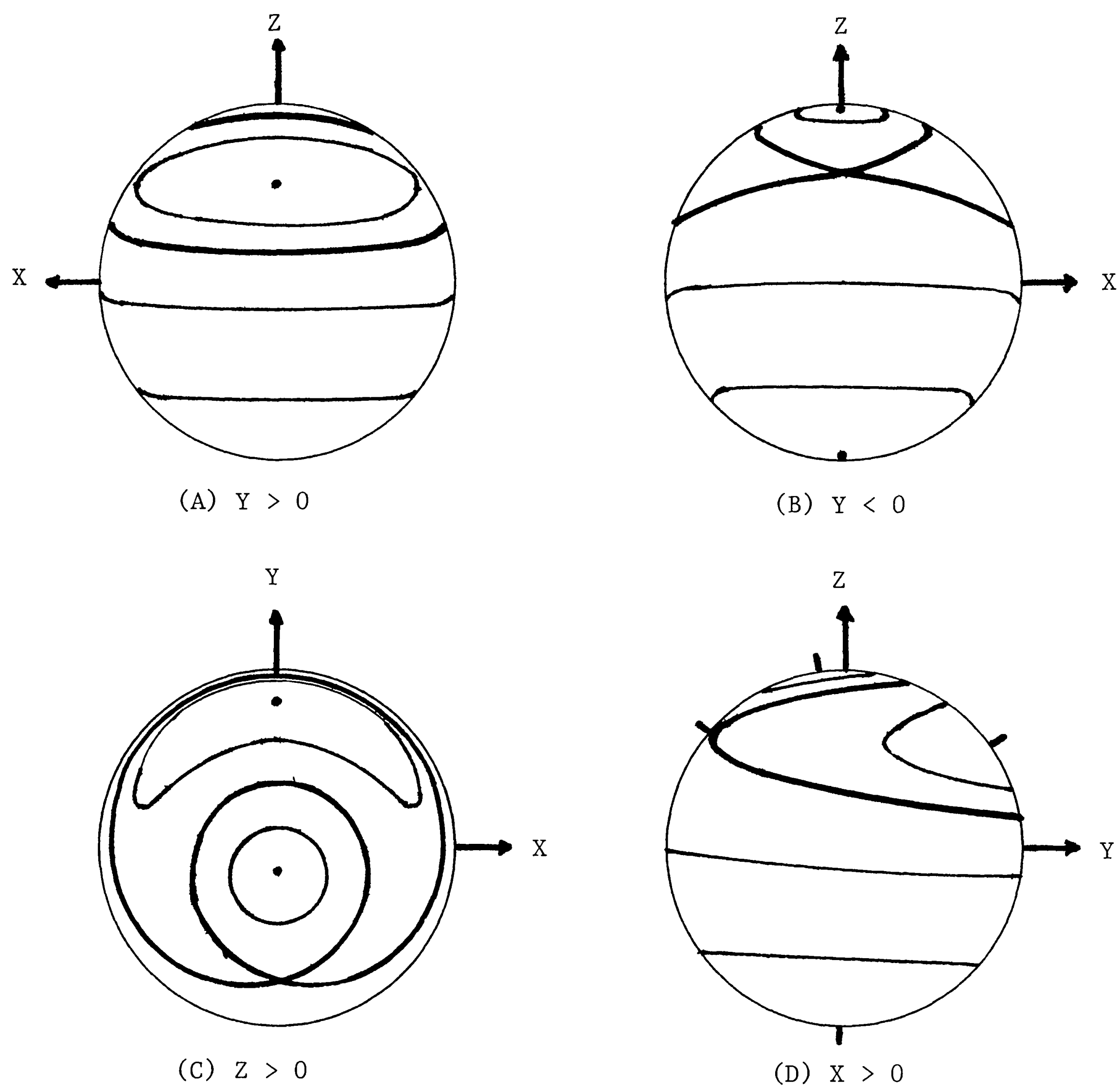


Figure 5 : Different projections of the last case (a = 0.05 , b = 0.6) of Figure 4. (A) Projection upon the plane (X,Z) of the half-sphere Y > 0 . (B) Projection upon the plane (X,Z) of the half-sphere Y < 0 . (C) Projection upon the plane (X,Y) of the half-sphere Z > 0 . (D) Projection upon the plane (Y,Z) of the half-sphere X > 0 .

4. THE HOMOCLINIC ORBITS

When a and b are small enough (see condition 20), the two homoclinic orbits associated with the unstable Cassini's state (Cassini's state n°4) divide the sphere into three domains, each of them associated with one of the three stable Cassini's state (see Figure 6).

Notice that the numbering of the domains and the numbering of the Cassini's states they contain, do not correspond. This comes from a conflict of "traditions". The traditional numbering of the Cassini's states and the traditional numbering of the domains of definition of the action-angle variables. (Number 3 being reserved for the domain touching both homoclinic orbits).

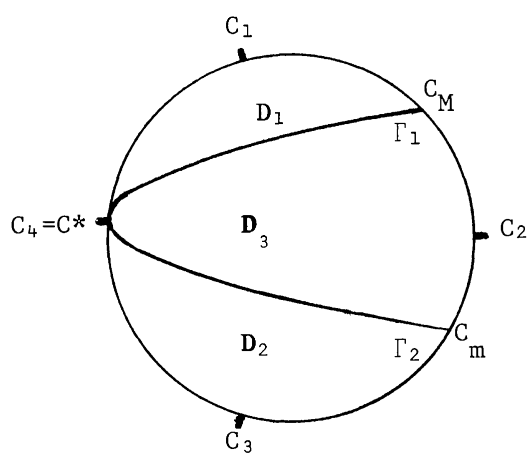


Figure 6 : The three domains D_i $(1 \leq i \leq 3)$ defined on the sphere by the homoclinic orbits Γ_1 and Γ_2.

In each of these domains, action-angle variables can be introduced. These action-angle variables are singular on the homoclinic orbits on the boundaries of the domains; they cannot be extended smoothly from one domain to another one.

As the action-angle variables are the proper setting for the adiabatic invariant theory, the analysis of these singularities forms the backbone of the analysis of the transition from one domain to another one when the parameters a and b vary slowly (see for instance Yoder, 1979; Henrard, 1982, 1987).

It is thus convenient to study in more details those particular trajectories.

In order to simplify this analysis we introduce a new parametrization of Colombo's Top. Let us define implicitely two angular parameters α $(0 \leqq \alpha \leqq \pi/2)$ and β $(0 \leqq \beta \leqq \alpha)$ such that

$$a = \sin\alpha \sin^2\beta \qquad\qquad b = \cos\alpha\cos^2\beta \tag{21}$$

With this parametrization the inequality (20) is always verified, the equality corresponding to the upper limit of the range of β.

One checks that the coordinates of the unstable equilibrium $C^* = C_4$, and the coordinates of the middle points C_m, C_M of the homoclinic orbits are given by (The coordinates of the middle points are found as the intersection of the circle $x = 0$ with the homoclinic orbits) :

$$\begin{aligned}
&C^* : X^* = 0 \qquad Y^* = -\sin\alpha \qquad Z^* = \cos\alpha \\
&C_m : X_m = 0 \qquad Y_m = \sin(\alpha + 2\beta) \qquad Z_m = \cos(\alpha + 2\beta) \\
&C_M : X_M = 0 \qquad Y_M = \sin(\alpha - 2\beta) \qquad Z_M = \cos(\alpha - 2\beta) .
\end{aligned} \tag{22}$$

In other words, the parameter α describes the angular position of the unstable equilibrium C^* on the circle $X = 0$, and the parameter 4β the angular separation of the two points C_m and C_M. Notice that the angular position of the middle point between C_m and C_M (along the circle $X = 0$) is also given by α.

With these notations, and after some algebra, we find that **along** the homoclinic curves, the differential equations corresponding to (14) are given by

$$\begin{aligned}
\mathring{h} &= \frac{(Z^* - Z)}{2(1 - Z^2)}\,[2\cos^2\beta - Z^*Z - Z^2] \\
\mathring{Z} &= -\frac{1}{2}\,\mathrm{sign}\,(X)\,[(Z^* - Z)^2\,(Z - Z_m)\,(Z_M - Z)]^{1/2} .
\end{aligned} \tag{23}$$

The variational equations corresponding to the unstable equilibrium have for Hamiltonian function the expansion of the Hamiltonian function (14) around the equilibrium truncated at degree 2. This Hamiltonian is

$$\mathbf{H}_V = \frac{1}{2} \sin^2 \alpha \sin^2 \beta \, . \, (\delta h)^2 - \frac{1}{2} \frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 \alpha} (\delta Z)^2 \, . \tag{24}$$

The solutions of the variational equations are, of course, of the exponential type

$$\begin{aligned} \delta h &= c_1 \exp (\rho t) + c_2 \exp (-\rho t) \\ \delta Z &= c_3 \exp (\rho t) + c_4 \exp (-\rho t) \, . \end{aligned} \tag{25}$$

The quantity ρ , the time constant of the variational equations is given by

$$\rho = [\sin^2 \alpha - \sin^2 \beta]^{1/2} \, . \sin \beta \tag{26}$$

which is always real and positive. It goes to zero on the boundary of our parameter space.

4. THE CRITICAL AREAS

The action variable J of a one-degree of freedom hamiltonian system is usually defined as :

$$J = \frac{1}{2\pi} \oint Z \, d \, h \tag{27}$$

where the path integral is taken along the periodic orbits of the Hamiltonian system.

When the periodic orbit of system (14) tends toward the limit of one of the domains defined on the sphere by the homoclinic orbits, the action variable J tends toward one of the values

$$J_i^* = \frac{1}{2\pi} \oint Z^{(i)} \, d \, h \tag{28}$$

where this time the path integral is taken along the upper homoclinic orbit for $i = 1$, the lower homoclinic orbit for $i = 2$ and both of them for $i = 3$.

As the action variable J is related to the area enclosed by the periodic orbit, so the critical values of J (the J_i^*) are related to the areas of the three domains. Taking into account the orientation of the trajectories on the sphere, we find that

$$A_1 = 2\pi(1 + J_1^*)$$

$$A_2 = 2\pi(1 + J_2^*) \tag{29}$$

$$A_3 = -2\pi J_3^* = -2\pi(J_1^* + J_2^*)$$

where the A_i are the areas of the three domains on the sphere.

Formulae (23) are instrumental in computing analytically the critical values J_i^*. Indeed if we take for instance J_1 we find

$$J_1^* = \frac{1}{2\pi} \oint Z \frac{\mathring{h}}{\mathring{Z}} dZ = \frac{1}{\pi} \int_{Z*}^{Z_M} \frac{Z(Z^2 - ZZ* - 2\cos^2\beta)}{(1 - Z^2)[(Z - Z_m)(Z_M - Z)]^{1/2}} dZ \tag{30}$$

This last integral is an excellent exercice in elementary calculus. After a considerable amount of algebra we find

$$\begin{aligned} J_1^* = &- b + \frac{2b}{\pi} \text{arc sin} (\text{tg}\,\beta / \text{tg}\,\alpha) \\ &- \frac{2}{\pi} \sin\beta [\sin^2\alpha - \sin^2\beta]^{1/2} \\ &- \frac{1}{\pi} \text{arc tg} \left\{\frac{2\sin\beta[\sin^2\alpha - \sin^2\beta]^{1/2}}{\sin^2\alpha - 2\sin^2\beta}\right\} . \end{aligned} \tag{31}$$

The expression (31) may be simplified further by changing again the parametrization. Instead of the angular parameters α and β, let us introduce the following quantities

$$z = \cos\alpha \qquad x = [\sin^2\alpha - \sin^2\beta]^{1/2} / \sin\beta$$
$$0 \leqq z \leqq 1 \qquad 0 \leqq x \leqq \infty . \tag{32}$$

We check that the original parameters a and b are now given by :

$$a = \frac{(1 - z^2)^{3/2}}{1 + x^2} \qquad b = \frac{z(x^2 + z^2)}{1 + x^2} . \tag{33}$$

The critical values of the action variable are now given by

$$J_1^* = - b + J_3^*/2$$

$$J_2^* = b + J_3^*/2 \tag{34}$$

$$J_3^* = \frac{4b}{\pi} \text{ arc sin } S - \frac{4}{\pi} \rho - \frac{2}{\pi} \text{ arc tg } T$$

where we have

$$S = z / (x^2 + z^2)^{1/2}$$

$$\rho = x(1 - z^2) / (1 + x^2) \tag{35}$$

$$T = 2x / (x^2 - 1)$$

where incidently the quantity ρ is the time constant of the variational equations defined in (26).

5. PROBABILITIES OF TRANSITIONS

The probability of transition from domain (i) to domain (j) when the parameter a and b vary slowly with the time, is given by

$$\Pr (i \to j) = \text{sign } \frac{\partial J_j^* / \partial t}{\partial J_i^* / \partial t} \tag{36}$$

(see Henrard, 1982, 1987, Escande, 1985), where "sign" is equal to -1 if (i,j) = (1,2) or (2,1) and is equal to +1 in the other cases. The partial derivatives $\partial J_i^*/\partial t$ are also instrumental in evaluating the Adiabatic Invariant changes due to separatrix crossing and thus in evaluating the coefficient of diffusion describing the chaotic motion related to periodic separatrix crossing (Escande, 1985; Henrard, 1987).

The variations $(\mathring{a},\mathring{b})$ of the parameters with respect to the time being given, it is enough, according to formulae (34), to evaluate $\partial J_3^*/\partial a$ and $\partial J_3^*/\partial b$ in order to evaluate the $\partial J_i^*/\partial t$.

It is remarkable that these latter quantities are very simple to evaluate analytically. We find after some algebra

$$\frac{\partial J_3^*}{\partial a} = -\frac{4\rho}{\pi a} \tag{37}$$

$$\frac{\partial J_3^*}{\partial b} = \frac{4}{\pi} \text{ arc sin } S \tag{38}$$

where ρ and S are given by (35), or equivalently in terms of the parameter α and β introduced in (21), by

$$\rho = \sin\beta\,[\sin^2\alpha - \sin^2\beta]^{1/2} \qquad S = \operatorname{tg}\beta \,/\, \operatorname{tg}\alpha\,. \tag{39}$$

It is convenient to express α and β in functions of a and b rather than the converse. From (21) we find that α is the largest of the two roots of

$$\sin 2\alpha = 2a\cos\alpha + 2b\sin\alpha \tag{40}$$

in the interval $0 \leqq \alpha \leqq \pi/2$. Using the notation of equation (13), the equality (40) can also be written :

$$\sin 2\alpha = \frac{4}{3}\,\frac{\omega\mu}{n^2}\left(\frac{C}{C-A}\right)\sin(\alpha + i)\,. \tag{41}$$

The smallest root of (40) actually gives the location of the Cassini's states number 1.

Knowing α, the value of β is easily found by using $\sin^2\beta = a/\sin\alpha$.

6. AN APPLICATION : THE OBLIQUITY OF MARS

Ward et al (1979) have suggested that the variation of the oblateness of Mars due to the Tharsis uplift and its partial compensation could have driven the precession of Mars through resonance with one or two terms in the expansion of the precession of its orbit.

The building of the shield volcanoes in the Tharsis region may have changed the ratio $(C - A)/C$ and thus the angular velocity of the precession of the angular momentum (see equation 9) by as much as 10°/° . In doing so this angular velocity may have come close to the angular velocity of two terms (classified n°2 and n°26 in the analysis of Bretagnon, 1974) in the expansion of the inclination and node of the orbit of Mars.

The parameters a and b (see equation 13) corresponding to each of these precessional terms independently can thus be of the order of unity implying large perturbations of the precession of the spin axis. The possibility of large oscillations of the obliquity of Mars at some time in the past is an important piece in the debate concerning the climatic history of Mars and related questions regarding the origin of several of the planet's surface features (see e.g. Squyres, 1984).

An intriguing facet to the history of Mars obliquity, pointed out by Ward et al but not fully explored is the possibility of the capture of Mars spin axis in the domain 3 (see Figure 6).

The capture can occur only during a period of increase of the oblateness and coming from domain 2 (see Figure 6).

In a previous paper (Henrard, 1982) we had computed by numerical integration the probabilities of such captures and found them to be respectively 0.0785 (for the term n°2 in Bretagnon's classification) and 0.0805 (for the term n°26).

We shall recompute the first one here as a check of our analytical formulae. The problem is characterized by values of a and b at the time of transition (see Henrard 1982) :

$$b = 0.883 \qquad\qquad a = 0.001 \ . \tag{42}$$

For a small value of i such as this, equation 41 possesses one root close to zero (corresponding to Cassini's state n°1) and one root close to

$$\alpha \sim \text{arc cos } (b) = 0.4886 \ . \tag{43}$$

A Newton-Raphson procedure enables us to compute that $\alpha = 0.4845$ from which it follows that $\beta = 0.04636$. The values of the derivatives of J_3^* are evaluated from equations 37, 38 and 39.

$$\frac{\partial J_3^*}{\partial a} = - 27.343 \qquad\qquad \frac{\partial J_3^*}{\partial b} = 0.11235 \tag{44}$$

on the other hand the variations of a and b with the time are given by

$$\mathring{a} = - \varepsilon a \qquad\qquad \mathring{b} = - \varepsilon b \tag{45}$$

with

$$\varepsilon = [\frac{C}{C - A}] \frac{d}{dt} [\frac{C - A}{C}] \tag{46}$$

Hence the variations of J_i^* are given by

$$\frac{dJ_1^*}{dt} = + 0.847 \varepsilon \quad ; \quad \frac{dJ_2^*}{dt} = - 0.919 \varepsilon \quad ; \quad \frac{dJ_3^*}{dt} = - 0.0719 \varepsilon \tag{47}$$

and the probability of transition from domain 2 to domain 3 by

$$Pr(2 \rightarrow 3) = \frac{\partial J_3^*}{\partial t} \Big/ \frac{\partial J_2^*}{\partial t} = 0.0782 \tag{48}$$

as found previously.

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